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Research Article

Condensing Mappings and Best Proximity Point Results

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Best proximity pair results are proved for noncyclic relatively u-continuous condensing mappings. In addition, best proximity points of upper semicontinuous mappings are obtained which are also fixed points of noncyclic relatively u-continuous condensing mappings. It is shown that relative u-continuity of $\mathfrak{T}$ is a necessary condition that cannot be omitted. Some examples are given to support our results.

1. Introduction

The concept of measure of noncompactness was first introduced by Kuratowski [1]. However, the interest in the concept was revived in 1955 when Darbo [2] proved a generalization of Schauder's fixed point theorem using this concept. Sadovskii [3], in 1967, defined condensing mappings and extended Darbo's theorem. Since then a lot of work has been done using this concept, and several interesting results have appeared, see, for instance, [4–9].

Let (W, Z) be a nonempty pair in a Banach space (that is, both W and Z are nonempty sets). A mapping $\mathfrak{T}\colon W \cup Z \longrightarrow W \cup Z$ is called noncyclic provided $\mathfrak{T}(W) \subseteq W$ and $\mathfrak{T}(Z) \subseteq Z$. If there exists $(w, z) \in W \times Z$ which satisfies $w = \mathfrak{T}(w)$, $z = \mathfrak{T}(z)$, and $\|w - z\| = \operatorname{dist}(W, Z)$, then we say that the noncyclic mapping $\mathfrak{T}$ has a best proximity pair. For a multivalued nonself mapping $S\colon W \longrightarrow 2^Z$, a point $w \in W$ is called a fixed point of S if $w \in S(w)$. The necessary condition for the existence of a fixed point for such S is $W \cap Z \neq \varnothing$. If $W \cap Z = \varnothing$, then $\operatorname{dist}(w, S(w)) > 0$ for each $w \in W$. Best proximity point theorems provide sufficient conditions for the existence of at least one solution for the minimization problem, $\min_{w \in W} \operatorname{dist}(w, S(w))$. If $\operatorname{dist}(w, S(w)) = \operatorname{dist}(W, Z)$, the point w is called a best proximity point of S. The existence results of best proximity points for multivalued mappings were obtained in [10–14] and [15]. Best proximity point theorems for relatively nonexpansive and relatively u-continuous were established by Elderd et al. in [16, 17] and by Markin and Shahzad in [18]. In recent years, the topics of best proximity points of single-valued and multivalued mappings have attracted the attention of many researchers, see, for example, the work in [6, 7, 19, 20] and the references cited therein. In this paper, we prove best proximity pair theorems for noncyclic relatively u-continuous condensing mappings. In addition, we obtain best proximity points of upper semicontinuous mappings which are fixed points of noncyclic relatively u-continuous condensing mappings. Also, we give examples to support our results and show by giving an example that relative u-continuity of $\mathfrak{T}$ is a necessary condition that cannot be omitted. Our results extend and complement results of [6, 7, 11].

2. Preliminaries

In this section, we present some notions and known results which will be used in the sequel.

Definition 1. Let K be a bounded set in a metric space X. The Kuratowski noncompactness measure $\alpha(K)$ (or simply, measure of noncompactness) is defined as follows:

$$\alpha(K) = \inf\left\{\eta > 0\colon K \subseteq \bigcup_{l=1}^{m} A_l\colon \operatorname{diam}(A_l) \le \eta, \quad \forall 1 \le l \le m < \infty\right\}. \tag{1}$$

Theorem 1. *Let X be a metric space. Then, for any nonempty bounded pair (C_1, C_2) in X (that is, both C_1 and C_2 are nonempty and bounded sets), the following hold:*

(1) $\alpha(C_1) = 0$ if and only if C_1 is relatively compact

(2) $C_1 \subseteq C_2$ implies $\alpha(C_1) \le \alpha(C_2)$

(3) $\alpha(\overline{C_1}) = \alpha(C_1)$, where $\overline{C_1}$ denotes the closure of C_1

(4) $\alpha(C_1 \cup C_2) = max\{\alpha(C_1), \alpha(C_2)\}$

(5) For a normed space X:

(i) $\alpha(C_1 + x) = \alpha(C_1)$

(ii) $\alpha(C_1 + C_2) \le \alpha(C_1) + \alpha(C_2)$

(iii) $\alpha(\lambda C_1) = |\lambda|\alpha(C_1)$, for any number λ

(iv) $\alpha(\overline{con}(C_1)) = \alpha(con(C_1)) = \alpha(C_1)$, where $con(C_1)$ represents the convex hull of C_1

Theorem 2. *Let $\{F_j\}$ be a decreasing sequence of nonempty closed subsets of a complete metric space X. If $\alpha(F_j) \longrightarrow 0$ as $j \longrightarrow \infty$, then $\cap_{j\in\mathbb{N}} F_j \ne \varnothing$.*

For more details about the measure of noncompactness, see [4].

Definition 2. Let (W, Z) be a nonempty pair in Banach space X and $\mathfrak{T}\colon W \cup Z \longrightarrow W \cup Z$ a mapping. Then, $\mathfrak{T}$ is said to be noncyclic relatively u-continuous. If $\mathfrak{T}$ is noncyclic and for each $\epsilon > 0$, there is $\gamma > 0$ such that

$$\begin{aligned} \|\mathfrak{T}(w) - \mathfrak{T}(z)\| < \varepsilon + \operatorname{dist}(W, Z) \text{ whenever } \|w - z\| \\ < \gamma + \operatorname{dist}(W, Z), \end{aligned} \tag{2}$$

for each $w \in W$ and $z \in Z$.

Definition 3. Let (W, Z) be a nonempty convex pair in Banach space X. A mapping $\mathfrak{T}\colon W \cup Z \longrightarrow W \cup Z$ is said to be affine if for each $\alpha, \beta \in [0, 1]$ with $\alpha + \beta = 1$ and $x_1, x_2 \in W$ (respectively, $x_1, x_2 \in Z$),

$$\mathfrak{T}(\alpha x_1 + \beta x_2) = \alpha\mathfrak{T}(x_1) + \beta\mathfrak{T}(x_2). \tag{3}$$

Definition 4. Let (W, Z) be a nonempty pair in Banach space X and $S\colon W \longrightarrow 2^Z$ a multivalued mapping on W, then S is said to be upper semicontinuous if for each closed subset B in Z, $S^{-1}(B) = \{w \in W\colon S(w) \cap B \ne \varnothing\}$ is closed in W.

Lemma 1. *(see [21]). Let Y be a nonempty, convex, and compact subset of a Banach space X. If $f\colon Y \longrightarrow 2^Y$ can be written as a finite composition of upper semicontinuous multivalued mappings of nonempty, compact, and convex values, then f has a fixed point.*

Definition 5. Let $\mathfrak{T}\colon W \cup Z \longrightarrow W \cup Z$ be a noncyclic relatively u-continuous mapping and $S\colon W \longrightarrow KC(Z)$ be an upper semicontinuous multivalued mapping (here, $KC(Z)$ denotes the collection of all nonempty, convex, and compact subsets of Z), then by the commutativity of $\mathfrak{T}$ and S, we mean that $\mathfrak{T}(S(w)) \subseteq S(\mathfrak{T}(w))$ holds for each $w \in W$.

Given (W, Z), a nonempty pair in Banach space, its proximal pair (W_0, Z_0) is given by

$$\begin{aligned} W_0 &= \left\{w \in W\colon \|w - z^*\| = \operatorname{dist}(W, Z) \text{ for some } z^* \in Z\right\}, \\ Z_0 &= \left\{z \in Z\colon \|w^* - z\| = \operatorname{dist}(W, Z) \text{ for some } w^* \in W\right\}. \end{aligned} \tag{4}$$

Moreover, if (W, Z) is a nonempty, convex, and compact pair in X, then (W_0, Z_0) is also a nonempty, convex, and compact pair.

Definition 6. Let X be a normed space. For a nonempty subset C of X, the metric projection operator $P_C\colon X \longrightarrow 2^C$ is given by

$$P_C(u) := \{v \in C\colon \|u - v\| = \operatorname{dist}(u, C)\}. \tag{5}$$

For a nonempty, convex, and compact subset C of a strictly convex Banach space, P_C is a single-valued mapping. Furthermore, for a nonempty, convex, and compact subset C of a Banach space X, P_C is upper semicontinuous with nonempty, convex, and compact values.

Lemma 2. *(see [11]). Let (W, Z) be a nonempty, convex, and compact pair in a strictly convex Banach space X. Let $\mathfrak{T}\colon W \cup Z \longrightarrow W \cup Z$ be a noncyclic relatively u-continuous and $P\colon W \cup Z \longrightarrow W \cup Z$ be a mapping given by*

$$P(u) = \begin{cases} P_Z(u), & \text{if } u \in W, \\ P_W(u), & \text{if } u \in Z. \end{cases} \tag{6}$$

Then, $\mathfrak{T}(P(u)) = P(\mathfrak{T}(u))$ for each $u \in W_0 \cup Z_0$.

Theorem 3. *(see [18]). Let (W, Z) be a nonempty, convex, and compact pair in a strictly convex Banach space X. If $\mathfrak{T}\colon W \cup Z \longrightarrow W \cup Z$ is a noncyclic relatively u-continuous mapping. Then, $\mathfrak{T}$ has best proximity pair.*

In [6], Gabeleh and Markin introduced the class of noncyclic condensing operators.

Recall that a nonempty pair (W, Z) in a Banach space X is called proximinal if $W = W_0$ and $Z = Z_0$.

Definition 7. Let (W, Z) be a nonempty convex pair in a strictly convex Banach space X. A mapping $\mathfrak{T}\colon W \cup Z \longrightarrow W \cup Z$ is called noncyclic condensing operator provided that, for any nonempty, bounded, closed, convex, proximinal, and $\mathfrak{T}$-invariant pair $(H_1, H_2) \subseteq (W, Z)$ with $\operatorname{dist}(H_1, H_2) = \operatorname{dist}(W, Z)$, there exists $k \in (0, 1)$ such that

$$\alpha(\mathfrak{T}(H_1) \cup \mathfrak{T}(H_2)) \le k\alpha(H_1 \cup H_2). \tag{7}$$

Lemma 3. *(see [11]). Let (W, Z) be a nonempty, convex, and compact pair in a strictly convex Banach space X. If*

$\mathfrak{T}$: $W\cup Z \longrightarrow W\cup Z$ is a noncyclic relatively u-continuous mapping, then T is continuous on W_0 and Z_0.

3. Main Results

Throughout this paper, we will assume that X is a strictly convex Banach space and α is the measure of non-compactness on X.

Remark 1. Let $\mathfrak{T}$: $W \longrightarrow W$ be condensing in the sense of Definition 7 with $k\in(0,1)$. Then, for any bounded subset H of W, $\mathfrak{T}$ satisfies

$$\alpha(\mathfrak{T}(H))\le k\alpha(H). \tag{8}$$

To see this, in (7), set $W = Z$ and $H_1 = H_2 = H$. Since $H\subseteq\overline{\text{con}}(H)$, then

$$\alpha(\mathfrak{T}(H))\le\alpha(\mathfrak{T}(\overline{\text{con}}(H)))\le k\alpha(\overline{\text{con}}(H)) = k\alpha(H). \tag{9}$$

Theorem 4. *Let (W,Z) be a nonempty, convex, and closed pair in X such that W is bounded and W_0 is nonempty. Suppose $\mathfrak{T}$: $W\cup Z\longrightarrow W\cup Z$ is a noncyclic relatively u-continuous, affine, and condensing mapping. Then, there exists $(u_0,v_0)\in W\times Z$ such that $\mathfrak{T}(u_0)=u_0$, $\mathfrak{T}(v_0)=v_0$ and $\|u_0-v_0\| = \text{dist}(W,Z)$. Moreover, if S: $W\longrightarrow KC(Z)$ is an upper semicontinuous multivalued mapping, $\mathfrak{T}$ and S commute, and for each $x\in W_0$, $S(x)\cap Z_0\neq\varnothing$, there exists $w\in W$ such that $\mathfrak{T}(w)=w$ and $\text{dist}(w,S(w)) = \text{dist}(W,Z)$.*

Proof. We follow [6, 11]. Clearly, (W_0,Z_0) is a nonempty, closed, convex, proximinal, and $\mathfrak{T}$-invariant pair. Let $(w_0,z_0)\in W_0\times Z_0$ be such that $\|w_0-z_0\| = \text{dist}(W,Z)$. Suppose $\mathscr{F}$ is a family of nonempty, closed, convex, proximinal, and $\mathfrak{T}$-invariant pairs $(C,D)\subseteq(W,Z)$ such that $(w_0,z_0)\in(C,D)$, then $\mathscr{F}$ is nonempty. Set $(F_1,F_2) = \cap_{(C,D)\in\mathscr{F}}(C,D)$, $G_1 = \overline{\text{con}}(\mathfrak{T}(F_1)\cup\{w_0\})$, and $G_2 = \overline{\text{con}}(\mathfrak{T}(F_2)\cup\{z_0\})$. So, $(w_0,z_0)\in G_1\times G_2$ and $(G_1,G_2)\subseteq(F_1,F_2)$. Furthermore, $\mathfrak{T}(G_1)\subseteq G_1$ and $\mathfrak{T}(G_2)\subseteq G_2$, that is, $\mathfrak{T}$ is noncyclic on $G_1\cup G_2$. Also, for $x\in G_1$, $x=\sum_{\mathfrak{l}=1}^{m-1}c_{\mathfrak{l}}\mathfrak{T}(w_{\mathfrak{l}})+c_m w_0$, where for all $\mathfrak{l}\in\{1,2,\ldots,m-1\}$ with $c_{\mathfrak{l}}\ge 0$ and $\sum_{\mathfrak{l}=1}^{m}c_{\mathfrak{l}}=1$, $w_{\mathfrak{l}}\in F_1$. Since (F_1,F_2) is proximinal, there is $z_{\mathfrak{l}}\in F_2$ such that $\|w_{\mathfrak{l}}-z_{\mathfrak{l}}\| = \text{dist}(W,Z)$, for each $\mathfrak{l}\in\{1,2,\ldots,m-1\}$. Set $y=\sum_{\mathfrak{l}=1}^{m-1}c_{\mathfrak{l}}\mathfrak{T}(z_{\mathfrak{l}})+c_m z_0$. Then, $y\in G_2$. Moreover,

$$\begin{aligned}\|x-y\| &= \left\|\left(\sum_{\mathfrak{l}=1}^{m-1}c_{\mathfrak{l}}\mathfrak{T}(w_{\mathfrak{l}})+c_m w_0\right)-\left(\sum_{\mathfrak{l}=1}^{m-1}c_{\mathfrak{l}}\mathfrak{T}(z_{\mathfrak{l}})+c_m z_0\right)\right\|\\ &\le\sum_{\mathfrak{l}=1}^{m-1}c_{\mathfrak{l}}\|\mathfrak{T}(w_{\mathfrak{l}})-\mathfrak{T}(z_{\mathfrak{l}})\|+c_m\|w_0-z_0\|\\ &= \text{dist}(W,Z).\end{aligned} \tag{10}$$

So, one can conclude that $(G_1)_0 = G_1$. Similarly, $(G_2)_0 = G_2$, and hence, $(G_1,G_2)\in\mathscr{F}$, that is, $G_1 = F_1$ and $G_2 = F_2$. Notice that

$$\begin{aligned}\alpha(G_1\cup G_2) &= \max\{\alpha(G_1),\alpha(G_2)\} = \max\{\alpha(\overline{\text{con}}(\mathfrak{T}(F_1)\cup\{w_0\})),\alpha(\overline{\text{con}}(\mathfrak{T}(F_2)\cup\{z_0\}))\}\\ &= \max\{\alpha(\mathfrak{T}(F_1)),\alpha(\mathfrak{T}(F_2))\} = \alpha(\mathfrak{T}(F_1)\cup\mathfrak{T}(F_2)) = \alpha(\mathfrak{T}(G_1)\cup\mathfrak{T}(G_2))\le k\alpha(G_1\cup G_2).\end{aligned} \tag{11}$$

But $k\in(0,1)$, so $\alpha(G_1\cup G_2)=0$. We conclude that (G_1,G_2) is a nonempty, compact, and convex pair with $\text{dist}(G_1,G_2) = \text{dist}(W,Z)$. By Theorem 3, there exists $(u_0,v_0)\in W\times Z$ such that $\mathfrak{T}(u_0)=u_0$, $\mathfrak{T}(v_0)=v_0$ and $\|u_0-v_0\| = \text{dist}(W,Z)$.

Now, let $\text{Fix}(\mathfrak{T}) = \{x\in W\cup Z\text{: }\mathfrak{T}(x)=x\}$, $\text{Fix}_W(\mathfrak{T}) = \text{Fix}(\mathfrak{T})\cap W_0$, and $\text{Fix}_Z(\mathfrak{T}) = \text{Fix}(\mathfrak{T})\cap Z_0$. By the above part, the pair $(\text{Fix}_W(\mathfrak{T}),\text{Fix}_Z(\mathfrak{T}))$ is nonempty. Also, it is a convex pair. Indeed, for α ,$\beta\in[0,1]$, with $\alpha+\beta=1$ and $x,y\in\text{Fix}_W(\mathfrak{T})$ (respectively, $\text{Fix}_Z(\mathfrak{T})$):

$$\mathfrak{T}(\alpha x+\beta y) = \alpha\mathfrak{T}(x)+\beta\mathfrak{T}(y) = \alpha x+\beta y, \tag{12}$$

and by convexity of W_0 (respectively, Z_0), we conclude that $\alpha x+\beta y\in\text{Fix}_W(\mathfrak{T})$ (respectively, $\text{Fix}_Z(\mathfrak{T})$). Furthermore, since $\mathfrak{T}$ is condensing,

$$\begin{aligned}\alpha(\text{Fix}_W(\mathfrak{T})\cup\text{Fix}_Z(\mathfrak{T})) &= \alpha(\mathfrak{T}(\text{Fix}_W(\mathfrak{T}))\cup\mathfrak{T}(\text{Fix}_Z(\mathfrak{T})))\\ &\le k\alpha(\text{Fix}_W(\mathfrak{T})\cup\text{Fix}_Z(\mathfrak{T})),\end{aligned} \tag{13}$$

which implies that the pair $(\text{Fix}_W(\mathfrak{T}),\text{Fix}_Z(\mathfrak{T}))$ is compact.

For $x\in Fix_W(\mathfrak{T})$ and $u\in S(x)$, we have

$$\mathfrak{T}(u)\in\mathfrak{T}(S(x))\subseteq S(\mathfrak{T}(x)) = S(x), \tag{14}$$

that is, $S(x)$ is invariant under $\mathfrak{T}$. So, by the invariance of Z_0 under $\mathfrak{T}$, $S(x)\cap Z_0\neq\varnothing$ is invariant under $\mathfrak{T}$. So, in view of Remark 3.1, Darbo's fixed point theorem guarantees the existence of a fixed point for the continuous mapping $\mathfrak{T}$: $S(x)\cap Z_0\longrightarrow S(x)\cap Z_0$. Thus, $S(x)\cap\text{Fix}_Z(\mathfrak{T})\neq\varnothing$, for $x\in\text{Fix}_W(\mathfrak{T})$. Define f: $\text{Fix}_W(\mathfrak{T})\longrightarrow 2^{\text{Fix}_Z(\mathfrak{T})}$ by $f(x) = S(x)\cap\text{Fix}_Z(\mathfrak{T})$, for each $x\in\text{Fix}_W(\mathfrak{T})$. Then, f is an upper semicontinuous multivalued mapping with nonempty, compact, and convex values. Moreover, P_W: $\text{Fix}_Z(\mathfrak{T})\longrightarrow\text{Fix}_W(\mathfrak{T})$ is well-defined. Indeed, for $y\in\text{Fix}_Z(\mathfrak{T})$, there is $x\in W$ such that $\|x-y\| = \text{dist}(W,Z)$. So,

$$y = P_Z(x)\text{ and }x = P_W(y). \tag{15}$$

By relative u-continuity of $\mathfrak{T}$, we conclude that $\|\mathfrak{T}(x)-\mathfrak{T}(y)\| = \text{dist}(W,Z)$. Thus, $\mathfrak{T}(y) = P_Z(\mathfrak{T}(x))$ and $\mathfrak{T}(x) = P_W(\mathfrak{T}(y))$, by (15), $\mathfrak{T}(x) = \mathfrak{T}(P_W(y)) = P_W(\mathfrak{T}(y)) = P_W(y)$. Then, $P_W(y)\in\text{Fix}_W(\mathfrak{T})$. Consider $P_W\circ\text{f}$: $\text{Fix}_W(\mathfrak{T})\longrightarrow 2^{\text{Fix}_W(\mathfrak{T})}$, by Lemma 1, there is $w\in\text{Fix}_W(\mathfrak{T})$ such that $w\in(P_W\circ\text{f})(\text{w})$, that is, $\mathfrak{T}(w)=w$ and $w\in(P_W(f(w)))$. So, there is $z\in f(w)\subset S(w)\cap Z_0$ such that $w = P_W(z)$. We conclude that $\|z-w\| = \text{dist}(z,W)$. But since

$z \in Z_0$, there is $w^* \in W$ such that $\|w^* - z\| = \text{dist}(W, Z)$. Thus,

$$\begin{aligned}\text{dist}(W,Z) &\le \text{dist}(w, S(w)) \le \|w - z\| = \text{dist}(z, W)\\ &\le \left\|z - w^*\right\| = \text{dist}(W, Z).\end{aligned} \tag{16}$$

Hence, $\text{dist}(w, S(w)) = \text{dist}(W, Z)$. ◻

Example 1. Consider the Hilbert space $X = \ell_2$ over $\mathbb{R}$ with the basis $\{e_n\colon n \in \mathbb{N}\}$ (the canonical basis) and let

$$\begin{aligned}W &= \{\zeta_1 e_1 + \zeta_2 e_2\colon \zeta_1 \in [0,4], \zeta_2 = -1\} \text{ and } Z\\ &= \{\zeta_1 e_1 + \zeta_2 e_2\colon \zeta_1 \le 0, \zeta_2 = 1\}.\end{aligned} \tag{17}$$

Then, (W, Z) be a nonempty, convex, and closed pair of X such that W is bounded. Furthermore, $\text{dist}(W, Z) = 2$ and

$$W_0 = \{-e_2\} \text{ and } Z_0 = \{e_2\}. \tag{18}$$

Define the mapping $\mathfrak{T}\colon W \cup Z \longrightarrow W \cup Z$ by $\mathfrak{T}(\zeta_1 e_1 + \zeta_2 e_2) = (3\zeta_1/4)e_1 + \zeta_2 e_2$, for each $\zeta_1 e_1 + \zeta_2 e_2 \in W \cup Z$. Then, $\mathfrak{T}$ is a noncyclic relatively u-continuous, affine, and condensing mapping. Now, define $S\colon W \longrightarrow KC(Z)$ by $S(\zeta e_1 - e_2) = \{-\zeta e_1 + e_2\}$; then, S is an upper semicontinuous multivalued mapping, $\mathfrak{T}$ and S commute, and for each $x \in W_0$, $S(x) \cap Z_0 \neq \varnothing$. For $w = -e_2 \in W$, we have $\mathfrak{T}(w) = w$ and $\text{dist}(w, S(w)) = \text{dist}(W, Z)$.

Example 2. Consider the Hilbert space $X = \ell_2$ over $\mathbb{R}$ with the basis $\{e_n\colon n \in \mathbb{N}\}$ and let

$$\begin{aligned}W &= \{\zeta_1 e_1 + \zeta_2 e_2\colon \zeta_1 \in [0,4], \zeta_2 \in [1,5]\} \text{ and}\\ Z &= \{\zeta_1 e_1 + \zeta_2 e_2\colon \zeta \ge 0, \zeta_2 = 0\}.\end{aligned} \tag{19}$$

Then, (W, Z) be a nonempty, convex, and closed pair of X such that W is bounded with $\text{dist}(W, Z) = 1$ and

$$\begin{aligned}W_0 &= \{\zeta_1 e_1 + e_2\colon \zeta_1 \in [0,4]\},\\ Z_0 &= \{\zeta e_1\colon \zeta \in [0,4]\}.\end{aligned} \tag{20}$$

Define the mapping $\mathfrak{T}\colon W \cup Z \longrightarrow W \cup Z$ by $\mathfrak{T}(\zeta_1 e_1 + \zeta_2 e_2) = ((2\zeta_1 + 1)/3)e_1 + \zeta_2 e_2$ for each $\zeta_1 e_1 + \zeta_2 e_2 \in W \cup Z$. Then, $\mathfrak{T}$ is a noncyclic relatively u-continuous, affine, and condensing mapping. Furthermore, for $(u_0, v_0) = (e_2, 0) \in W \times Z$, we have $\mathfrak{T}(u_0) = u_0$, $\mathfrak{T}(v_0) = v_0$, and $\|u_0 - v_0\| = \text{dist}(W, Z)$. Now, let $S\colon W \longrightarrow KC(Z)$ given by $S(\zeta_1 e_1 + \zeta_2 e_2) = \{\gamma e_1\colon \gamma \in [\zeta_1, 4]\}$, then S is an upper semicontinuous multivalued mapping, $\mathfrak{T}$ and S commute, and for each $x \in W_0$, $S(x) \cap Z_0 \neq \varnothing$. For $w = e_1 + e_2 \in W$, we have $\mathfrak{T}(w) = w$ and $\text{dist}(w, S(w)) = \text{dist}(W, Z)$.

Remark 2. The relative u-continuity of $\mathfrak{T}$ is necessary in Theorem 4.

To see this, consider the Hilbert space $X = \ell_2$ over $\mathbb{R}$ with the basis $\{e_n\colon n \in \mathbb{N}\}$ and let $W = \{x \in X\colon \|x\| \le 1\}$, $Z = \{\zeta e_2\colon \zeta \in [2,3]\}$. Then, (W, Z) is a nonempty, convex, and closed pair in X such that W is bounded. Obviously, $\text{dist}(W, Z) = 1$ and

$$\begin{aligned}W_0 &= \{e_2\},\\ Z_0 &= \{2e_2\}.\end{aligned} \tag{21}$$

Define the mapping $\mathfrak{T}\colon W \cup Z \longrightarrow W \cup Z$ by

$$\mathfrak{T}(x) = \left\{ \sum_{i=1}^{\infty} \frac{\zeta_i}{2} e_i, \text{for } x \in W, \left(\frac{\zeta_2}{2} + 1\right) e_2 \text{for } x \in Z, \right. \tag{22}$$

for $x = (\zeta_1, \zeta_2, \zeta_3, \ldots) \in W \cup Z$. Then, $\mathfrak{T}$ is a noncyclic, affine, and condensing mapping. Let $S\colon W \longrightarrow KC(Z)$ given by $S((\zeta_1, \zeta_2, \zeta_3, \ldots)) = \{(2 + |\zeta_1|)e_2\}$. Then, S is an upper semicontinuous multivalued mapping, $\mathfrak{T}$ and S commute, and for each $x \in W_0$, $S(x) \cap Z_0 \neq \varnothing$. Here, $w = (0,0,0,\ldots)$ is the only fixed point of $\mathfrak{T}$ in W, but $\text{dist}(w, S(w)) > \text{dist}(W, Z)$. Note that $\|e_2 - 2e_2\| < \text{dist}(W, Z) + \delta$ for all $\delta > 0$ but $\|T(e_2) - T(2e_2)\| > \text{dist}(W, Z) + (1/4)$.

The following corollary follows immediately from Theorem 4.

Corollary 1. *Let (W, Z) be a nonempty, convex, and closed pair in X such that W is bounded and W_0 is nonempty. Suppose $\mathfrak{T}\colon W \longrightarrow W$ is a continuous, affine, and condensing mapping. If $S\colon W \longrightarrow KC(W)$ is an upper semicontinuous multivalued mapping, $\mathfrak{T}$ and S commute, and then there is $w \in W$ which satisfies $w \in Fix(\mathfrak{T}) \cap Fix(S)$.*

Theorem 5. *Let (W, Z) be a nonempty, convex, and closed pair in X such that W is bounded and W_0 is nonempty. If $\mathfrak{T}_1, \mathfrak{T}_2\colon W \cup Z \longrightarrow W \cup Z$ are commuting, noncyclic relatively u-continuous, affine, and condensing mappings, then there exists $(u_0, v_0) \in W \times Z$ such that $\mathfrak{T}_1(u_0) = u_0 = \mathfrak{T}_2(u_0)$, $\mathfrak{T}_1(v_0) = v_0 = \mathfrak{T}_2(v_0)$, and $\|u_0 - v_0\| = dist(W, Z)$.*

Proof. Since W_0 is nonempty and by relative u-continuity of $\mathfrak{T}_1$, for $w_0 \in W_0$, there exists $z_0 \in Z$ such that $\|w_0 - z_0\| = \text{dist}(W, Z)$. Consequently, $\|\mathfrak{T}_1(w_0) - \mathfrak{T}_1(z_0)\| = \text{dist}(W, Z)$. That is, W_0 is invariant under $\mathfrak{T}_1$. Thus, Darbo's fixed point theorem guarantees that there is $u \in W_0$ such that $\mathfrak{T}_1(u) = u$. Notice $\mathfrak{T}_1(\text{Fix}_W(\mathfrak{T}_1)) = \text{Fix}_W(\mathfrak{T}_1)$ and so $\alpha(\text{Fix}_W(\mathfrak{T}_1)) = \alpha(\mathfrak{T}_1(\text{Fix}_W(\mathfrak{T}_1))) \le k\alpha(\text{Fix}_W(\mathfrak{T}_1))$. Thus, $\alpha(\text{Fix}_W(\mathfrak{T}_1)) = 0$, and thus, $Fix_W(\mathfrak{T}_1)$ is compact. Furthermore, $\mathfrak{T}_1(\mathfrak{T}_2(u)) = \mathfrak{T}_2(\mathfrak{T}_1(u)) = \mathfrak{T}_2(u)$. So, $\mathfrak{T}_2\colon \text{Fix}_W(\mathfrak{T}_1) \longrightarrow \text{Fix}_W(\mathfrak{T}_1)$ is a continuous mapping on a compact convex set. By Schauder's fixed point theorem, there is $u_0 \in \text{Fix}_W(\mathfrak{T}_1)$ such that $\mathfrak{T}_2(u_0) = u_0$, that is, $u_0 \in \text{Fix}_W(\mathfrak{T}_1) \cap \text{Fix}_W(\mathfrak{T}_2)$. Let v_0 in Z_0 be the unique closest point to u_0. By relative u-continuity of $\mathfrak{T}_1$ and $\mathfrak{T}_2$, we infer that, since $\|u_0 - v_0\| = \text{dist}(W, Z)$, $\|\mathfrak{T}_1(u_0) - \mathfrak{T}_1(v_0)\| = \text{dist}(W, Z)$ and $\|\mathfrak{T}_2(u_0) - \mathfrak{T}_2(v_0)\| = \text{dist}(W, Z)$. Hence, $\mathfrak{T}_1(u_0) = u_0 = \mathfrak{T}_2(u_0)$, $\mathfrak{T}_1(v_0) = v_0 = \mathfrak{T}_2(v_0)$. ◻

Lemma 4. *Let (W, Z) be a nonempty, convex, and closed pair in X such that W is bounded and W_0 is nonempty. Let $\mathscr{C}$ be the collection of the commuting, noncyclic relatively u-continuous, affine, and condensing mappings on $W \cup Z$.*

Then, the mappings in $\mathscr{C}$ have common fixed points $u_0 \in W_0$ and $v_0 \in Z_0$.

Proof. For each $\mathfrak{T} \in \mathscr{C}$, consider $\mathrm{Fix}(\mathfrak{T}), \mathrm{Fix}_W(\mathfrak{T})$, and $\mathrm{Fix}_Z(\mathfrak{T})$ defined previously. Then, $\mathrm{Fix}_W(\mathfrak{T})$ is nonempty, compact, and convex. Let $\mathfrak{T}_1, \mathfrak{T}_2, \ldots, \mathfrak{T}_k$ be a finite subcollection of $\mathscr{C}$. Assume $F = \cap_{1\le i\le k}\mathrm{Fix}_W(\mathfrak{T}_i) \neq \varnothing$, $F_1 = F \cap \mathrm{Fix}_W(\mathfrak{T}_{k+1}) = \cap_{1\le i\le k+1}\mathrm{Fix}_W(\mathfrak{T}_i)$, and $F_{n+1} = F_n \cap \mathrm{Fix}_W(\mathfrak{T}_{k+n+1}) = \cap_{1\le i\le k+n+1}\mathrm{Fix}_W(\mathfrak{T}_i)$, for $n \in \mathbb{N}$. Then, $\{F_n\}$ is a decreasing sequence of compact subsets of X. Furthermore, $F_n \neq \varnothing$ for each $n \in \mathbb{N}$. Indeed, for $w \in F$ and each $m \in \{1, 2, \ldots, k\}$, then $\mathfrak{T}_m(\mathfrak{T}_{k+1}(w)) = \mathfrak{T}_{k+1}(\mathfrak{T}_m(w)) = \mathfrak{T}_{k+1}(w)$, and this implies that $\mathfrak{T}_{k+1}(w) \in F$. Thus, F is invariant under $\mathfrak{T}_{k+1}$. By Schauder's fixed point theorem, we get that $F_1 \neq \varnothing$. Now, for each $n \in \mathbb{N}$ and $m \in \{1, 2, \ldots, k+n\}$, pick $x \in F_n$:

$$\mathfrak{T}_m(\mathfrak{T}_{k+n+1}(x)) = \mathfrak{T}_{k+n+1}(\mathfrak{T}_m(x)) = \mathfrak{T}_{k+n+1}(x), \tag{23}$$

that is, $\mathfrak{T}_{k+n+1}(x) \in F_n$. So, $\mathfrak{T}_{k+n+1}\colon F_n \longrightarrow F_n$ is continuous on F_n, and then there is $y \in F_n$ such that $\mathfrak{T}_{k+n+1}(y) = y$. Therefore, $y \in F_{n+1} \neq \varnothing$. By Theorem 2, $\cap_{n\in\mathbb{N}} F_n \neq \varnothing$. Hence, $\cap_{\mathfrak{T}\in\mathscr{C}}\mathrm{Fix}_W(\mathfrak{T}) \neq \varnothing$. Similarly, we can show that $\cap_{\mathfrak{T}\in\mathscr{C}}\mathrm{Fix}_Z(\mathfrak{T}) \neq \varnothing$. □

Theorem 6. *Let (W, Z) be a nonempty, convex, and closed pair in X such that W is bounded and W_0 is nonempty. Let $\mathscr{C}$ be the collection of the commuting, noncyclic, relatively u-continuous, affine, and condensing mappings on $W \cup Z$. Then, there is $(u_0, v_0) \in W \times Z$ such that, for each $\mathfrak{T} \in \mathscr{C}$: $\mathfrak{T}(u_0) = u_0$, $\mathfrak{T}(v_0) = v_0$, and $\|u_0 - v_0\| = \mathrm{dist}(W, Z)$.*

Proof. Based on the previous lemma, the mappings in $\mathscr{C}$ have a fixed point in common $u_0 \in W$, that is, $\mathfrak{T}(u_0) = u_0$, for each $\mathfrak{T} \in \mathscr{C}$. Let $v_0 \in Z$ be the unique closest point to u_0. By relative u-continuity of $\mathfrak{T}$, since $\|u_0 - v_0\| = \mathrm{dist}(W, Z)$,

$$\begin{aligned}\left\|u_0 - \mathfrak{T}(v_0)\right\| &= \left\|\mathfrak{T}(u_0) - \mathfrak{T}(v_0)\right\| \\ &= \mathrm{dist}(W, Z), \quad \text{for each } \mathfrak{T} \in \mathscr{C}.\end{aligned} \tag{24}$$

Hence, $\mathfrak{T}(v_0) = v_0$. □

Theorem 7. *Let (W, Z) be a nonempty, convex, and closed pair in X such that W is bounded and W_0 is nonempty. Let $\mathscr{C}$ be the collection of the commuting, noncyclic relatively u-continuous, affine, and condensing mappings on $W \cup Z$. If $S\colon W \longrightarrow KC(Z)$ is an upper semicontinuous multivalued mapping such that, for each $x \in W_0$: $S(x) \cap Z_0 \neq \varnothing$. If $\mathscr{C}$ commutes with S, then there exists $w \in W$ such that*

$$\mathfrak{T}(w) = w \text{ and } \mathrm{dist}(w, S(w)) = \mathrm{dist}(W, Z). \tag{25}$$

Proof. By Lemma 4, $(\cap_{\mathfrak{T}\in\mathscr{C}}\mathrm{Fix}_W(\mathfrak{T}), \cap_{\mathfrak{T}\in\mathscr{C}}\mathrm{Fix}_Z(\mathfrak{T}))$ is a nonempty compact convex pair. Also, in view to the proof of Theorem 4, for $\mathfrak{T} \in \mathscr{C}$ and for each $x \in \mathrm{Fix}_W(\mathfrak{T})$, we have $S(x)$ and Z_0 are invariant under $\mathfrak{T}$. So, $S(x) \cap (\cap_{\mathfrak{T}\in\mathscr{C}}\mathrm{Fix}_Z(\mathfrak{T})) \neq \varnothing$.

Define $f\colon \cap_{\mathfrak{T}\in\mathscr{C}}\mathrm{Fix}_W(\mathfrak{T}) \longrightarrow 2^{\cap_{\mathfrak{T}\in\mathscr{C}}\mathrm{Fix}_Z(\mathfrak{T})}$ by $f(x) = S(x) \cap (\cap_{\mathfrak{T}\in\mathscr{C}}\mathrm{Fix}_Z(\mathfrak{T}))$, for $x \in \cap_{\mathfrak{T}\in\mathscr{C}}\mathrm{Fix}_W(\mathfrak{T})$. Then, f is an upper semicontinuous multivalued mapping with nonempty, compact, and convex values. Moreover, $P_W\colon \cap_{\mathfrak{T}\in\mathscr{C}}\mathrm{Fix}_Z(\mathfrak{T}) \longrightarrow \cap_{\mathfrak{T}\in\mathscr{C}}\mathrm{Fix}_W(\mathfrak{T})$ is well-defined. Indeed, for $y \in \cap_{\mathfrak{T}\in\mathscr{C}}\mathrm{Fix}_Z(\mathfrak{T})$, there exists $x \in W$ such that $\|x - y\| = \mathrm{dist}(W, Z)$. So,

$$y = P_Z(x) \text{ and } x = P_W(y), \tag{26}$$

By relative u-continuity of $\mathfrak{T}$, one can conclude that $\|\mathfrak{T}(x) - \mathfrak{T}(y)\| = \mathrm{dist}(W, Z)$. Thus, $\mathfrak{T}(y) = P_Z(\mathfrak{T}(x))$ and $\mathfrak{T}(x) = P_W(\mathfrak{T}(y))$, and by (26), $\mathfrak{T}(x) = \mathfrak{T}(P_W(y)) = P_W(\mathfrak{T}(y)) = P_W(y)$. Thus, $P_W(y) \in \cap_{\mathfrak{T}\in\mathscr{C}}\mathrm{Fix}_W(\mathfrak{T})$. Note that $P_W{\circ}f\colon \cap_{\mathfrak{T}\in\mathscr{C}}\mathrm{Fix}_W(\mathfrak{T}) \longrightarrow 2^{\mathrm{Fix}_W(\mathfrak{T})}$, and by Lemma 1, there is $w \in \cap_{\mathfrak{T}\in\mathscr{C}}\mathrm{Fix}_W(\mathfrak{T})$ such that $w \in (P_W{\circ}f)(\mathrm{w})$, that is, for $\mathfrak{T} \in \mathscr{C}$, we have $\mathfrak{T}(w) = w$ and $w \in (P_W(f(w)))$. So, there is $z \in f(w) = S(w) \cap (\cap_{\mathfrak{T}\in\mathscr{C}}\mathrm{Fix}_Z(\mathfrak{T}))$ such that $w = P_W(z)$. We infer that $\|z - w\| = \mathrm{dist}(z, W)$. But $z \in Z_0$, then there is $w^* \in W$ such that $\|w^* - z\| = \mathrm{dist}(W, Z)$. Then,

$$\begin{aligned}\mathrm{dist}(W, Z) &\le \mathrm{dist}(w, S(w)) \le \|w - z\| \le \left\|z - w^*\right\| \\ &= \mathrm{dist}(W, Z).\end{aligned} \tag{27}$$

Hence, $\mathrm{dist}(w, S(w)) = \mathrm{dist}(W, Z)$. □

Example 3. Let $X = \ell_2$ over $\mathbb{R}$ with the basis $\{e_n\colon n \in \mathbb{N}\}$ and let

$$\begin{aligned}W &= \{\zeta_1 e_1 + \zeta_2 e_2\colon \zeta_1 \in [-3, -1], \zeta_2 \in [-8, 8]\} \\ &\text{and } Z = \{\zeta_1 e_1 + \zeta_2 e_2\colon \zeta_1 \in [0, 3], \zeta_2 \in \mathbb{R}\}.\end{aligned} \tag{28}$$

Then, (W, Z) be a nonempty, convex, and closed pair in X such that W is bounded. Furthermore, $\mathrm{dist}(W, Z) = 1$ and

$$W_0 = \{-e_1 + \zeta_2 e_2\colon \zeta_2 \in [-8, 8]\} \text{ and } Z_0 = \{\zeta_2 e_2\colon \zeta_2 \in [-8, 8]\}. \tag{29}$$

Consider $\mathfrak{T}_1, \mathfrak{T}_2\colon W \cup Z \longrightarrow W \cup Z$ given by

$$\begin{aligned}\mathfrak{T}_1(\zeta_1 e_1 + \zeta_2 e_2) &= \zeta_1 e_1 + \frac{\zeta_2}{2} e_2 \text{ and } \mathfrak{T}_2(\zeta_1 e_1 + \zeta_2 e_2) \\ &= \zeta_1 e_1 + \frac{\zeta_2}{4} e_2,\end{aligned} \tag{30}$$

for each $\zeta_1 e_1 + \zeta_2 e_2 \in W \cup Z$. Then, $\mathfrak{T}_1$ and $\mathfrak{T}_2$ are noncyclic, affine, and condensing mappings. Furthermore, $\mathfrak{T}_1$ and $\mathfrak{T}_2$ commute.

Define $S\colon W \longrightarrow KC(Z)$ by $S(\zeta_1 e_1 + \zeta_2 e_2) = \{\gamma e_1 + \zeta_2 e_2\colon \gamma \in [0, -\zeta_1]\}$, then S is an upper semicontinuous multivalued mapping that commutes with $\mathfrak{T}_1$ and $\mathfrak{T}_2$ and satisfies that, for each $x \in W_0$: $S(x) \cap Z_0 \neq \varnothing$. For $w = -e_1$ and $z = \mathbf{0}$, $\mathfrak{T}_1(w) = \mathfrak{T}_2(w) = w$ and $\mathfrak{T}_1(z) = \mathfrak{T}_2(z) = z$. Furthermore, $\|w - z\| = \mathrm{dist}(W, Z)$ and $\mathrm{dist}(w, S(w)) = \mathrm{dist}(W, Z)$.

4. Conclusion

We have proved some best proximity pair theorems for noncyclic relatively u-continuous and condensing mappings. We have also obtained best proximity points of upper semicontinuous mappings which are fixed points of noncyclic relatively u-continuous condensing mappings.

Moreover, we have given some examples to support our results. It has been shown that relative u-continuity of $\mathfrak{T}$ is a necessary condition that cannot be omitted. We have extended recent results of [6, 11].

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

References

[1] K. Kuratowski, *Introduction to Set Theory and Topology*, Pergamon Press, Oxford, England, 1956.

[2] G. Darbo, "Punti uniti in transformazioni a condomino non compatto," *Rendiconti del Seminario Matematico della Università di Padova*, vol. 24, pp. 84–92, 1955.

[3] B. N. Sadovskii, "On a fixed point principle," *Functional Analysis and Its Applications*, vol. 4, pp. 74–76, 1967.

[4] M. Ayerbe Toledano, T. Dominguez Benavides, and G. Lopez Acedo, "Measures of non-compactness in metric fixed point theory," *Operator Theory: Advances and Applications*, vol. 99, 1997.

[5] L. Cai and J. Liang, "New generalizations of Darbo's fixed point theorem," *Journal of Fixed Point Theory and Applications*, vol. 20158 pages, 2015.

[6] M. Gabeleh and J. Markin, "Optimum solutions for a system of differential equations via measure of noncompactness," *Indagationes Mathematicae*, vol. 29, no. 3, pp. 895–906, 2018.

[7] M. Gabeleh, S. P. Moshokoa, and C. Vetro, "Cyclic (noncyclic) ϕ-condensing operator and its application to a system of differential equations," *Nonlinear Analysis: Modelling and Control*, vol. 24, pp. 985–1000, 2019.

[8] J. Liang, J. H. Liu, M. V. Nguyen, and T. J. Xiao, "Periodic solutions of impulsive differential equations with infinite delay in Banach spaces," *Journal of Nonlinear Functional Analysis*, vol. 201918 pages, 2019.

[9] R. Pietkun, "On some generalizations of the M onch, Sadovskii and Darbo fixed point theorems," *J. Fixed Point Theory Appl.*vol. 20, p. 11, 2018.

[10] A. Abkar and M. Gabeleh, "The existence of best proximity points for multivalued non-self-mappings," *Revista de la Real Academia de Ciencias Exactas, Fisicas y Naturales. Serie A. Matematicas*, vol. 107, no. 2, pp. 319–325, 2013.

[11] G. AlNemer, J. Markin, and N. Shahzad, "On best proximity points of upper semicontinuous multivalued mappings," *Journal of Fixed Point Theory and Applications*, vol. 201510 pages, 2015.

[12] M. A. Al-Thagafi and N. Shahzad, "Best proximity pairs and equilibrium pairs for Kakutani multimaps," *Nonlinear Analysis: Theory, Methods & Applications*, vol. 70, no. 3, pp. 1209–1216, 2009.

[13] A. Amini-Harandi, "Best proximity pair and coincidence point theorems for nonexpansive set-valued maps in Hilbert spaces," *Bulletin of the Iranian Mathematical Society*, vol. 37, pp. 229–234, 2011.

[14] M. De la Sen, "Some results on fixed and best proximity points of multivalued cyclic self-mappings with a partial order," *Abstract and Applied Analysis*, vol. 2013, Article ID 968492, 11 pages, 2013.

[15] W. A. Kirk, S. Reich, and P. Veeramani, "Proximinal retracts and best proximity pair theorem," *Numerical Functional Analysis and Optimization*, vol. 24, no. 7-8, pp. 851–862, 2003.

[16] A. A. Elderd, V. S. Raj, and P. Veeramani, "On best proximity pair theorems for relatively u-continuous mapping," *Nonlinear Analysis*, vol. 74, pp. 3870–3875, 2011.

[17] A. A. Eldred, W. A. Kirk, and P. Veeramani, "Proximal normal structure and relatively nonexpansive mappings," *Studia Mathematica*, vol. 171, no. 3, pp. 283–293, 2005.

[18] J. Markin and N. Shahzad, "Best proximity points for relatively u-continuous mappings in Banach and hyperconvex spaces," *Abstract and Applied Analysis*, vol. 2013, Article ID 680186, 5 pages, 2013.

[19] E. Naraghirad, "Bregman best proximity points for Bregman asymptotic cyclic contraction mappings in Banach spaces," *Journal of Nonlinear Functional Analysis*, vol. 3, pp. 27–44, 2019.

[20] N. Shahzad, M. A. Alghamdi, and G. AlNemer, "On best proximity points of multivalued nonexpansive mappings," *Dynamic Systems and Applications*, vol. 30, pp. 411–424, 2021.

[21] M. Lassonde, "Fixed points for Kakutani factorizable multifunctions," *Journal of Mathematical Analysis and Applications*, vol. 152, no. 1, pp. 46–60, 1990.