Research Article

Condensing Mappings and Best Proximity Point Results

Sarah O. Alshehri,1,2 Hamed H. Alsulami,1 and Naseer Shahzad1

1King Abdulaziz University, Department of Mathematics, P.O. Box 80203, Jeddah 2159, Saudi Arabia
2University of Jeddah, College of Science, Department of Mathematics, Jeddah, Saudi Arabia

Correspondence should be addressed to Naseer Shahzad; nshahzad@kau.edu.sa

Received 22 January 2021; Revised 8 February 2021; Accepted 23 February 2021; Published 21 April 2021

Academic Editor: Xiaolong Qin

Copyright © 2021 Sarah O. Alshehri et al. This is an open access article distributed under the Creative Commons Attribution License, which permits unrestricted use, distribution, and reproduction in any medium, provided the original work is properly cited.

Best proximity pair results are proved for noncyclic relatively u-continuous condensing mappings. In addition, best proximity points of upper semicontinuous mappings are obtained which are also fixed points of noncyclic relatively u-continuous condensing mappings. It is shown that relative u-continuity of \( T \) is a necessary condition that cannot be omitted. Some examples are given to support our results.

1. Introduction

The concept of measure of noncompactness was first introduced by Kuratowski [1]. However, the interest in the concept was revived in 1955 when Darbo [2] proved a generalization of Schauder’s fixed point theorem using this concept. Sadovskii [3], in 1967, defined condensing mappings and extended Darbo’s theorem. Since then a lot of work has been done using this concept, and several interesting results have appeared, see, for instance, [4–9].

Let \((W, Z)\) be a nonempty pair in a Banach space (that is, both \(W\) and \(Z\) are nonempty sets). A mapping \( \mathcal{T} : W \cup Z \to W \cup Z \) is called noncyclic provided \( \mathcal{T}(W) \subseteq W \) and \( \mathcal{T}(Z) \subseteq Z \). If there exists \((w, z) \in W \times Z\) which satisfies \( w = \mathcal{T}(w) \), \( z = \mathcal{T}(z) \), and \( \|w - z\| = \text{dist}(W, Z) \), then we say that the noncyclic mapping \( \mathcal{T} \) has a best proximity pair. For a multivalued nonself mapping \( S : W \to 2^Z \), a point \( w \in W \) is called a fixed point of \( S \) if \( w \in S(w) \). The necessary condition for the existence of a fixed point for such \( S \) is \( W \cap Z \neq \emptyset \). If \( W \cap Z = \emptyset \), then \( \text{dist}(w, S(w)) > 0 \) for each \( w \in W \). Best proximity point theorems provide sufficient conditions for the existence of at least one solution for the minimization problem, \( \min_{w \in W} \text{dist}(w, S(w)) \). If \( \text{dist}(w, S(w)) = \text{dist}(W, Z) \), the point \( w \) is called a best proximity point of \( S \). The existence results of best proximity points for multivalued mappings were obtained in [10–14] and [15]. Best proximity point theorems for relatively nonexpansive and relatively u-continuous were established by Elderd et al. in [16, 17] and by Markin and Shahzad in [18]. In recent years, the topics of best proximity points of single-valued and multivalued mappings have attracted the attention of many researchers, see, for example, the work in [6, 7, 19, 20] and the references cited therein. In this paper, we prove best proximity pair theorems for noncyclic relatively u-continuous condensing mappings. In addition, we obtain best proximity points of upper semicontinuous mappings which are fixed points of noncyclic relatively u-continuous condensing mappings. Also, we give examples to support our results and show by giving an example that relative u-continuity of \( \mathcal{T} \) is a necessary condition that cannot be omitted. Our results extend and complement results of [6, 7, 11].

2. Preliminaries

In this section, we present some notions and known results which will be used in the sequel.

Definition 1. Let \( K \) be a bounded set in a metric space \( X \). The Kuratowski noncompactness measure \( \alpha(K) \) (or simply, measure of noncompactness) is defined as follows:
\[ \alpha(k) = \inf \left\{ \eta > 0 : \text{for each } A_i, \text{ diam}(A_i) \leq \eta, \quad \forall 1 \leq i \leq m < \infty \right\}. \] (1)

**Theorem 1.** Let be a metric space. Then, for any nonempty bounded pair \((C_1, C_2)\) in \(X\) (that is, both \(C_1\) and \(C_2\) are nonempty and bounded sets), the following hold:

1. \(\alpha(C_1) = 0\) if and only if \(C_1\) is relatively compact
2. \(C_1 \subseteq C_2\) implies \(\alpha(C_1) \leq \alpha(C_2)\)
3. \(\alpha(C_1) = \alpha(C_1^c)\), where \(C_1^c\) denotes the closure of \(C_1\)
4. \(\alpha(C_1 \cup C_2) = \max\{\alpha(C_1), \alpha(C_2)\}\)

**Definition 2.** Let \(\{F_i\}\) be a decreasing sequence of nonempty closed subsets of a complete metric space \(X\). If \(\alpha(F_j) \to 0\) as \(j \to \infty\), then \( \cap_{j \in \mathbb{N}} F_j \neq \emptyset \).

For more details about the measure of noncompactness, see [4].

**Definition 2.** Let \((W, Z)\) be a nonempty pair in Banach space \(X\) and \(\mathcal{L} : W \cup Z \to W \cup Z\) a mapping. Then, \(\mathcal{L}\) is said to be noncyclic relatively \(u\)-continuous. If \(\mathcal{L}\) is noncyclic and for each \(\varepsilon > 0\), there is \(\gamma > 0\) such that

\[ \|\mathcal{L}(u) - \mathcal{L}(z)\| < \varepsilon + \text{dist}(W, Z) \quad \text{whenever} \quad \|u - z\| < \gamma + \text{dist}(W, Z), \] (2)

for each \(w \in W\) and \(z \in Z\).

**Definition 3.** Let \((W, Z)\) be a nonempty convex pair in Banach space \(X\). A mapping \(\mathcal{L} : W \cup Z \to W \cup Z\) is said to be affine if for each \(\alpha, \beta \in [0, 1]\) with \(\alpha + \beta = 1\) and \(x_1, x_2 \in W\) (respectively, \(x_1, x_2 \in Z\)),

\[ \mathcal{L}(\alpha x_1 + \beta x_2) = \alpha \mathcal{L}(x_1) + \beta \mathcal{L}(x_2). \] (3)

**Definition 4.** Let \((W, Z)\) be a nonempty convex pair in Banach space \(X\) and \(S : W \to 2^Z\) a multivalued mapping on \(W\), then \(S\) is said to be upper semicontinuous if for each closed subset \(B\) in \(Z\), \(S^{-1}(B) = \{w \in W : S(w) \cap B \neq \emptyset\}\) is closed in \(W\).

**Lemma 1.** (see [21]). Let \(Y\) be a nonempty, convex, and compact subset of a Banach space \(X\). If \(f : Y \to 2^Y\) can be written as a finite composition of upper semicontinuous multivalued mappings of nonempty, compact, and convex values, then \(f\) has a fixed point.

**Definition 5.** Let \(\mathcal{L} : W \cup Z \to W \cup Z\) be a noncyclic relatively \(u\)-continuous mapping and \(S : W \to KC(Z)\) be an upper semicontinuous multivalued mapping (here, \(KC(Z)\) denotes the collection of all nonempty, convex, and compact subsets of \(Z\), then by the commutativity of \(\mathcal{L}\) and \(S\), we mean that \(S(S(w)) \subseteq S(\mathcal{L}(u))\) holds for each \(w \in W\).

Given \((W, Z)\), a nonempty pair in Banach space, its proximal pair \((W_0, Z_0)\) is given by

\[ W_0 = \{w \in W : \|w - z^*\| = \text{dist}(W, Z) \text{ for some } z^* \in Z\}, \]
\[ Z_0 = \{z \in Z : \|w^* - z\| = \text{dist}(W, Z) \text{ for some } w^* \in W\}. \] (4)

Moreover, if \((W, Z)\) is a nonempty, convex, and compact pair in \(X\), then \((W_0, Z_0)\) is also a nonempty, convex, and compact pair.

**Definition 6.** Let \(X\) be a normed space. For a nonempty subset \(C\) of \(X\), the metric projection operator \(P_C : X \to 2^C\) is given by

\[ P_C(u) = \{v \in C : \|u - v\| = \text{dist}(u, C)\}. \] (5)

For a nonempty, convex, and compact subset \(C\) of a strictly convex Banach space, \(P_C\) is a single-valued mapping. Furthermore, for a nonempty, convex, and compact subset \(C\) of a Banach space \(X\), \(P_C\) is upper semicontinuous with nonempty, convex, and compact values.

**Lemma 2.** (see [11]). Let \((W, Z)\) be a nonempty, convex, and compact pair in a strictly convex Banach space \(X\). Let \(\mathcal{L} : W \cup Z \to W \cup Z\) be a noncyclic relatively \(u\)-continuous and \(P : W \cup Z \to W \cup Z\) be a mapping given by

\[ P(u) = \begin{cases} P_Z(u), & \text{if } u \in W, \\ P_W(u), & \text{if } u \in Z. \end{cases} \] (6)

Then, \(\mathcal{L}(P(u)) = P(\mathcal{L}(u))\) for each \(u \in W_0 \cup Z_0\).

**Theorem 3.** (see [18]). Let \((W, Z)\) be a nonempty, convex, and compact pair in a strictly convex Banach space \(X\). If \(\mathcal{L} : W \cup Z \to W \cup Z\) is a noncyclic relatively \(u\)-continuous mapping. Then, \(\mathcal{L}\) has best proximity pair.

In [6], Gabeleh and Markin introduced the class of noncyclic condensing operators. Recall that a nonempty pair \((W, Z)\) in a Banach space \(X\) is called proximinal if \(W = W_0\) and \(Z = Z_0\).

**Definition 7.** Let \((W, Z)\) be a nonempty convex pair in a strictly convex Banach space \(X\). A mapping \(\mathcal{L} : W \cup Z \to W \cup Z\) is called noncyclic condensing operator provided that, for any nonempty, bounded, closed, convex, proximinal, and \(\mathcal{L}\)-invariant pair \((H_1, H_2) \subseteq (W, Z)\) with \(\text{dist}(H_1, H_2) = \text{dist}(W, Z)\), there exists \(k \in (0, 1)\) such that

\[ \alpha(\mathcal{L}(H_1) \cup \mathcal{L}(H_2)) \leq k \alpha(H_1 \cup H_2). \] (7)

**Lemma 3.** (see [11]). Let \((W, Z)\) be a nonempty, convex, and compact pair in a strictly convex Banach space \(X\). If
$\mathbf{T}: W \cup Z \rightarrow W \cup Z$ is a noncyclic relatively $u$-continuous mapping, then $T$ is continuous on $W_0$ and $Z_0$.

3. Main Results

Throughout this paper, we will assume that $X$ is a strictly convex Banach space and $\alpha$ is the measure of noncompactness on $X$.

**Remark 1.** Let $\mathbf{T}: W \rightarrow W$ be condensing in the sense of Definition 7 with $k \in (0, 1)$. Then, for any bounded subset $H$ of $W$, $\mathbf{T}$ satisfies

$$\alpha(\mathbf{T}(H)) \leq k\alpha(H).$$

(8)

To see this, in (7), set $W = Z$ and $H_1 = H_2 = H$. Since $H \subseteq \overline{\text{co}}(H)$, then

$$\alpha(\mathbf{T}(H)) \leq \alpha(\mathbf{T}(\overline{\text{co}}(H))) \leq k\alpha(\overline{\text{co}}(H)) = k\alpha(H).$$

(9)

**Theorem 4.** Let $(W, Z)$ be a nonempty, convex, and closed pair in $X$ such that $W$ is bounded and $W_0$ is nonempty. Suppose $\mathbf{T}: W \cup Z \rightarrow W \cup Z$ is a noncyclic relatively $u$-continuous, affine, and condensing mapping. Then, there exists $(u_0, v_0) \in W \times Z$ such that $\mathbf{T}(u_0) = u_0$, $\mathbf{T}(v_0) = v_0$ and $\|u_0 - v_0\| = \text{dist}(W, Z)$. Moreover, if $S: W \rightarrow KC(Z)$ is an upper semicontinuous multivalued mapping, $\mathbf{T}$ and $S$ commute, and for each $x \in W_0$, $S(x) \cap Z_0 \neq \emptyset$, there exists $w \in W$ such that $\mathbf{T}(w) = w$ and $\text{dist}(w, S(w)) = \text{dist}(W, Z)$.

Proof. We follow [6, 11]. Clearly, $(W_0, Z_0)$ is a nonempty, closed, convex, proximinal, and $\mathbf{T}$-invariant pair. Let $(w_0, z_0) \in W_0 \times Z_0$ be such that $\|w_0 - z_0\| = \text{dist}(W, Z)$. Suppose $\mathcal{F}$ is a family of nonempty, closed, convex, proximinal, and $\mathbf{T}$-invariant pairs $(C, D) \subseteq (W, Z)$ such that $(w_0, z_0) \in (C, D)$, then $\mathcal{F}$ is nonempty. Set $F_1 = \cap (C, D) \subseteq (C, D)$, $G_1 = \overline{\text{co}}(\mathbf{T}(F_1) \cup \{w_0\})$, and $G_2 = \overline{\text{co}}(\mathbf{T}(F_2) \cup \{z_0\})$. So, $(w_0, z_0) \in G_1 \times G_2$ and $(G_1, G_2) \subseteq (F_1, F_2)$. Furthermore, $\mathbf{T}(G_1) \subseteq G_1$ and $\mathbf{T}(G_2) \subseteq G_2$, that is, $\mathbf{T}$ is noncyclic on $G_1 \cup G_2$. Also, for $x \in G_1$, $x = \sum_{i=1}^{m-1} c_i \mathbf{T}(u_i) + c_m w_0$, where for all $i \in \{1, 2, \ldots, m - 1\}$ with $c_i \geq 0$ and $\sum_{i=1}^{m} c_i = 1$, $u_i \in F_1$. Since $(F_1, F_2)$ is proximinal, there is $z_i \in F_2$ such that $\|w_0 - z_i\| = \text{dist}(W, Z)$, for each $i \in \{1, 2, \ldots, m - 1\}$. Set $y = \sum_{i=1}^{m-1} c_i \mathbf{T}(z_i) + c_m z_0$. Then, $y \in G_2$. Moreover, $\|x - y\| = \|\sum_{i=1}^{m-1} c_i \mathbf{T}(u_i) + c_m w_0 - \left(\sum_{i=1}^{m-1} c_i \mathbf{T}(z_i) + c_m z_0\right)\|$

$$\leq \sum_{i=1}^{m-1} c_i \|\mathbf{T}(u_i) - \mathbf{T}(z_i)\| + c_m \|w_0 - z_0\| = \text{dist}(W, Z).$$

(10)

So, one can conclude that $(G_1, G_2) = (G_1)$. Similarly, $(G_2, G_0) = (G_2)$, and hence, $(G_1, G_2) \in \mathcal{F}$, that is, $G_1 = F_1$ and $G_2 = F_2$. Notice that

$$\alpha(G_1 \cup G_2) = \max\{\alpha(G_1), \alpha(G_2)\} = \max\{\alpha(\overline{\text{co}}(\mathbf{T}(F_1) \cup \{w_0\})), \alpha(\overline{\text{co}}(\mathbf{T}(F_2) \cup \{z_0\}))\}$$

$$= \max\{\alpha(\mathbf{T}(F_1)), \alpha(\mathbf{T}(F_2))\} = \alpha(\mathbf{T}(F_1) \cup \mathbf{T}(F_2)) = \alpha(\mathbf{T}(G_1) \cup \mathbf{T}(G_2)) \leq k\alpha(G_1 \cup G_2).$$

(11)

But $k \in (0, 1)$, so $\alpha(G_1 \cup G_2) = 0$. We conclude that $(G_1, G_2)$ is a nonempty, compact, and convex pair with $\text{dist}(G_1, G_2) = \text{dist}(W, Z)$. By Theorem 3, there exists $(u_0, v_0) \in W \times Z$ such that $\mathbf{T}(u_0) = u_0$, $\mathbf{T}(v_0) = v_0$ and $\|u_0 - v_0\| = \text{dist}(W, Z)$. Now, let $\text{Fix}(\mathbf{T}) = \{x \in W \cup Z: \mathbf{T}(x) = x\}$, $\text{Fix}_W(\mathbf{T}) = \text{Fix}(\mathbf{T}) \cap W_0$, and $\text{Fix}_Z(\mathbf{T}) = \text{Fix}(\mathbf{T}) \cap Z_0$. By the above part, the pair $(\text{Fix}_W(\mathbf{T}), \text{Fix}_Z(\mathbf{T}))$ is nonempty. Also, it is a convex pair. Indeed, for $\alpha, \beta \in [0, 1]$, with $\alpha + \beta = 1$ and $x, y \in \text{Fix}_W(\mathbf{T})$ (respectively, $\text{Fix}_Z(\mathbf{T})$) and convexity of $W_0$ (respectively, $Z_0$), we conclude that $\alpha \beta y \in \text{Fix}_W(\mathbf{T})$ (respectively, $\text{Fix}_Z(\mathbf{T})$). Furthermore, since $\mathbf{T}$ is condensing,

$$\alpha(\text{Fix}_W(\mathbf{T}) \cup \text{Fix}_Z(\mathbf{T})) = \alpha(\text{Fix}_W(\mathbf{T}) \cup \text{Fix}_Z(\mathbf{T}))$$

$$\leq k\alpha(\text{Fix}_W(\mathbf{T}) \cup \text{Fix}_Z(\mathbf{T})),$$

(13)

which implies that the pair $(\text{Fix}_W(\mathbf{T}), \text{Fix}_Z(\mathbf{T}))$ is compact.

For $x \in \text{Fix}_W(\mathbf{T})$ and $u \in S(x)$, we have

$$\mathbf{T}(u) \in C(S(x)) \subseteq S(\mathbf{T}(x)) = S(x),$$

(14)

that is, $S(x)$ is invariant under $\mathbf{T}$. So, by the invariance of $Z_0$ under $\mathbf{T}$, $S(x) \cap Z_0 \neq \emptyset$ is invariant under $\mathbf{T}$. So, in view of Remark 3.1, Darbo’s fixed point theorem guarantees the existence of a fixed point for the continuous mapping $\mathbf{T}: S(x) \cap Z_0 \rightarrow S(x) \cap Z_0$. Thus, $S(x) \cap \text{Fix}_Z(\mathbf{T}) \neq \emptyset$, for $x \in \text{Fix}_W(\mathbf{T})$. Define $\text{Fix}_W(\mathbf{T}) \rightarrow 2^{\text{Fix}_W(\mathbf{T})}$ by $f(x) = S(x) \cap \text{Fix}_Z(\mathbf{T})$, for each $x \in \text{Fix}_W(\mathbf{T})$. Then, $f$ is an upper semicontinuous multivalued mapping with nonempty, compact, and convex values. Moreover, $P_W: \text{Fix}_W(\mathbf{T}) \rightarrow \text{Fix}_W(\mathbf{T})$ is well-defined. Indeed, for $y \in \text{Fix}_W(\mathbf{T})$, there is $x \in W$ such that $\|x - y\| = \text{dist}(W, Z)$. So,

$$y = P_Z(x) \text{ and } x = P_W(y).$$

(15)

By relative u-continuity of $\mathbf{T}$, we conclude that $\|\mathbf{T}(x) - \mathbf{T}(y)\| = \text{dist}(W, Z)$, thus, $\mathbf{T}(y) = P_Z(\mathbf{T}(x))$ and $\mathbf{T}(x) = P_W(\mathbf{T}(y))$. By (15), $\mathbf{T}(x) = (P_W(\mathbf{T}(y)) = P_W(\mathbf{T}(y)) = P_W(y)$. Then, $P_W(\mathbf{T}) \in \text{Fix}_W(\mathbf{T})$. Consider $P_W^*:\text{Fix}_W(\mathbf{T}) \rightarrow 2^{\text{Fix}_W(\mathbf{T})}$, by Lemma 1, there is $w \in \text{Fix}_W(\mathbf{T})$ such that $w \in (P_W^*f)(w)$, that is, $\mathbf{T}(w) = w$ and $w \in (P_W^*(f))(w)$. So, there is $z \in f(w) \subseteq S(w) \cap Z_0$ such that $w = P_W(z)$. We conclude that $\|z - w\| = \text{dist}(z, W)$. But since
there is \( w^* \in W \) such that \( \|w^* - z\| = \text{dist}(W, Z) \). Thus,
\[
\text{dist}(W, Z) \leq \text{dist}(w, S(w)) \leq \|w - z\| = \text{dist}(z, W)
\]
(16)

Hence, \( \text{dist}(w, S(w)) = \text{dist}(W, Z) \).

**Example 1.** Consider the Hilbert space \( X = \ell_2 \) over \( \mathbb{R} \) with the basis \( \{e_n; n \in \mathbb{N}\} \) (the canonical basis) and let
\[
W = \{\tilde{\zeta}_1 e_1 + \tilde{\zeta}_2 e_2; \tilde{\zeta}_1 \in [0, 4], \tilde{\zeta}_2 = -1\} \quad \text{and} \quad Z = \{\tilde{\zeta}_1 e_1 + \tilde{\zeta}_2 e_2; \tilde{\zeta}_1 \leq 0, \tilde{\zeta}_2 = -1\}.
\]

Then, \( (W, Z) \) is a nonempty, convex, and closed pair of \( X \) such that \( W \) is bounded. Furthermore, \( \text{dist}(W, Z) = 2 \) and
\[
W_0 = \{e_2\} \quad \text{and} \quad Z_0 = \{2e_2\}.
\]

Defining the mapping \( \mathcal{U}: W \cup Z \rightarrow W \cup Z \) by
\[
\mathcal{U}(\zeta_1 e_1 + \zeta_2 e_2) = \left(\frac{1}{2}\zeta_1 + 1\right)e_2 \quad \text{for } \zeta_1, \zeta_2 \geq 0.
\]

Then, \( (W, Z) \) is a noncyclic relatively \( u \)-continuous, affine, and condensing mapping. Furthermore, \( S: W \rightarrow KC(Z) \) by \( S(\zeta_1 e_1 + \zeta_2 e_2) = -(\zeta_1 e_1 + \zeta_2 e_2) \); thus, \( S \) is an upper semicontinuous multivalued mapping. \( \mathcal{U} \) and \( S \) commute, and for each \( x \in W_0 \), \( S(x) \cap Z_0 \neq \emptyset \). Here, \( w = e_1 \in W \) we have
\[
\mathcal{U}(w) = w \quad \text{and} \quad \text{dist}(w, S(w)) = \text{dist}(W, Z).
\]

**Example 2.** Consider the Hilbert space \( X = \ell_2 \) over \( \mathbb{R} \) with the basis \( \{e_n; n \in \mathbb{N}\} \) and let
\[
W = \{\tilde{\zeta}_1 e_1 + \tilde{\zeta}_2 e_2; \tilde{\zeta}_1 \in [0, 4], \tilde{\zeta}_2 \in [1, 5]\} \quad \text{and} \quad Z = \{\tilde{\zeta}_1 e_1 + \tilde{\zeta}_2 e_2; \tilde{\zeta}_1 \geq 0, \tilde{\zeta}_2 = 0\}.
\]

Then, \( (W, Z) \) is a nonempty, convex, and closed pair of \( X \) such that \( W \) is bounded with \( \text{dist}(W, Z) = 1 \) and
\[
W_0 = \{\tilde{\zeta}_1 e_1 + e_2; \tilde{\zeta}_1 \in [0, 4]\}
\]
\[
Z_0 = \{e_2; \tilde{\zeta}_2 \in [0, 4]\}.
\]

Defining the mapping \( \mathcal{U}: W \cup Z \rightarrow W \cup Z \) by
\[
\mathcal{U}(\zeta_1 e_1 + \zeta_2 e_2) = \left(\frac{1}{2}\zeta_1 + 1\right)e_2 \quad \text{for } \zeta_1, \zeta_2 \geq 0.
\]

Then, \( \mathcal{U} \) is a noncyclic relatively \( u \)-continuous, affine, and condensing mapping. Furthermore, for \( (u_0, v_0) = (e_2, 0) \in W \times Z \), we have \( \mathcal{U}(u_0) = u_0 \), \( \mathcal{U}(v_0) = v_0 \), and \( \|u_0 - v_0\| = \text{dist}(W, Z) \). Now, let \( S: W \rightarrow KC(Z) \) given by \( S(\zeta_1 e_1 + \zeta_2 e_2) = \{y e_2; y \in [1, 4]\} \); then \( S \) is an upper semicontinuous multivalued mapping. \( \mathcal{U} \) and \( S \) commute, and for each \( x \in W_0 \), \( S(x) \cap Z_0 \neq \emptyset \). For \( w = e_1 + e_2 \in W \), we have
\[
\mathcal{U}(w) = w \quad \text{and} \quad \text{dist}(w, S(w)) = \text{dist}(W, Z).
\]

**Remark 2.** The relative \( u \)-continuity of \( \mathcal{U} \) is necessary in Theorem 4.

**Corollary 1.** Let \( (W, Z) \) be a nonempty, convex, and closed pair in \( X \) such that \( W \) is bounded and \( W_0 \) is nonempty. Suppose \( \mathcal{U}: W \rightarrow W \) is a continuous, affine, and condensing mapping. Then \( \mathcal{U} \) is a noncyclic relatively \( u \)-continuous, affine, and condensing mapping. Furthermore, \( W \) is an upper semicontinuous multivalued mapping. \( \mathcal{U} \) and \( S \) commute, and then there exists \( (u_0, v_0) \in W \times Z \) such that \( \mathcal{U}(u_0) = u_0 = \mathcal{U}_2(u_0) \), \( \mathcal{U}_1 (v_0) = v_0 = \mathcal{U}_2 (v_0) \), and \( \|u_0 - v_0\| = \text{dist}(W, Z) \).

**Theorem 5.** Let \( (W, Z) \) be a nonempty, convex, and closed pair in \( X \) such that \( W \) is bounded and \( W_0 \) is nonempty. If \( \mathcal{U}_1, \mathcal{U}_2: W \cup Z \rightarrow W \cup Z \) are commuting, noncyclic relatively \( u \)-continuous, affine, and condensing mappings, then there exists \( (u_0, v_0) \in W \times Z \) such that \( \mathcal{U}_1(u_0) = u_0 = \mathcal{U}_2(u_0) \), \( \mathcal{U}_1 (v_0) = v_0 = \mathcal{U}_2 (v_0) \), and \( \|u_0 - v_0\| = \text{dist}(W, Z) \).

**Proof.** Since \( W_0 \) is nonempty and by relative \( u \)-continuity of \( \mathcal{U}_1 \), for \( u_0 \in W_0 \), there exists \( z_0 \in Z \) such that \( \|u_0 - z_0\| = \text{dist}(W, Z) \). Consequently, \( \|\mathcal{U}_1(u_0) - \mathcal{U}_1(z_0)\| = \text{dist}(W, Z) \). That is, \( W_0 \) is invariant under \( \mathcal{U}_1 \). Thus, Darbo’s fixed point theorem guarantees that there is \( u \in W_0 \) such that \( \mathcal{U}_1(u) = u \). Notice \( \mathcal{U}_1(\text{Fix}_W(\mathcal{U}_1)) = \text{Fix}_W(\mathcal{U}_1) \) and so \( \alpha(\text{Fix}_W(\mathcal{U}_1)) \leq \alpha(\text{Fix}_W(\mathcal{U}_1)) \leq \alpha(\text{Fix}_W(\mathcal{U}_1)) \) and thus, \( \text{Fix}_W(\mathcal{U}_1) \) is compact. Furthermore, \( \mathcal{U}_1(\mathcal{U}_2(\mathcal{U}_1(u))) = \mathcal{U}_1(\mathcal{U}_2(u)) \), and \( \text{Fix}_W(\mathcal{U}_1(\mathcal{U}_2(u))) \) is a continuous mapping on a compact convex set. By Schauder’s fixed point theorem, there is \( u_0 \in \text{Fix}_W(\mathcal{U}_1) \) such that \( \mathcal{U}_2(u_0) = u_0 \), that is, \( u_0 \in \text{Fix}_W(\mathcal{U}_1(\mathcal{U}_2(u))) \cap \text{Fix}_W(\mathcal{U}_2(u)) \). Let \( v_0 \in Z \) be the unique closest point to \( u_0 \). By relative \( u \)-continuity of \( \mathcal{U}_1 \) and \( \mathcal{U}_2 \), we infer that, since \( \|u_0 - v_0\| = \text{dist}(W, Z) \), \( \|\mathcal{U}_1(u_0) - \mathcal{U}_2(v_0)\| = \text{dist}(W, Z) \) and \( \|\mathcal{U}_2(u_0) - \mathcal{U}_2(v_0)\| = \text{dist}(W, Z) \). Hence, \( \mathcal{U}_1(u_0) = u_0 = \mathcal{U}_2(u_0) \), \( \mathcal{U}_1 (v_0) = v_0 = \mathcal{U}_2 (v_0) \).

**Lemma 4.** Let \( (W, Z) \) be a nonempty, convex, and closed pair in \( X \) such that \( W \) is bounded and \( W_0 \) is nonempty. Let \( \mathcal{C} \) be the collection of the commuting, noncyclic relatively \( u \)-continuous, affine, and condensing mappings on \( W \cup Z \).
Then, the mappings in $\mathcal{C}$ have common fixed points $u_0 \in W_0$ and $v_0 \in Z_0$.

Proof. For each $x \in \mathcal{C}$, consider $\text{Fix}(x) \cap \text{Fix}_W(x)$. Then, $\text{Fix}_W(x)$ is nonempty, compact, and convex. Let $\mathcal{T}_1, \mathcal{T}_2, \ldots, \mathcal{T}_k$ be a finite sub-collection of $\mathcal{C}$. Assume $F = \cap_{1 \leq k \leq n} \text{Fix}_W(x) \neq \emptyset$. Then, $F_1 = F \cap \text{Fix}_W(x)_{1 \leq k \leq n}$, and $F_{m+1} = F \cap \text{Fix}_W(x)_{1 \leq k \leq n}$, for $n \in \mathbb{N}$. Then, $\{F_n\}$ is a decreasing sequence of compact subsets of $X$. Furthermore, $F_i \neq \emptyset$ for each $n \in \mathbb{N}$. Indeed, for $w \in F$ and each $m \in \{1, \ldots, k\}$, then $\mathcal{T}_m(x_{k+1}) = \mathcal{T}_m(x_k) = x_{k+1}$ continuous in $F_k$, and then there is $y \in F_k$ such that $F_{k+1} = F_k \cap \mathcal{T}_m(x_{k+1}) = \mathcal{T}_m(x_k)$, and this implies that $\mathcal{T}_m(x_k) = x_k$. F. Thus, $F$ is invariant under $\mathcal{T}_k$. By Schauder’s fixed point theorem, we get that $F_1 \neq \emptyset$. Now, for each $n \in \mathbb{N}$ and $m \in \{1, \ldots, k+n\}$, pick $x \in F_n$:

$$
\mathcal{T}_m(x_{k+1}(x)) = \mathcal{T}_{k+1}((m(x)) = \mathcal{T}_{k+1}(x), \quad (23)
$$

that is, $\mathcal{T}_m(x_{k+1}(x)) = \mathcal{T}_{k+1}(x_m(x)) = \mathcal{T}_{x_{k+1}(x)}$. By Theorem 2, $\mathcal{T}_m(x_{k+1}(x)) \neq \emptyset$. Similarly, we can show that $\mathcal{T}_m(x_{k+1}(x)) \neq \emptyset$.

Theorem 6. Let $(W, Z)$ be a nonempty, convex, and closed pair in $X$ such that $W$ is bounded and $W_0$ is nonempty. Let $\mathcal{C}$ be the collection of the commuting, noncyclic, relatively $u$-continuous, affine, and condensing mappings on $W \cup Z$. Then, there is $(u_0, v_0) \in W \times Z$ such that, for each $x \in \mathcal{C}$, $\mathcal{T}(u_0, v_0) = (u_0, v_0)$, and $\|u_0 - v_0\| = \text{dist}(W, Z)$.

Proof. Based on the previous lemma, the mappings in $\mathcal{C}$ have a fixed point in common $u_0 \in W$, that is, $\mathcal{T}(u_0) = u_0$, for each $x \in \mathcal{C}$. Let $v_0 \in Z$ be the unique closest point to $u_0$. By relative $u$-continuity of $\mathcal{C}$, since $\|u_0 - v_0\| = \text{dist}(W, Z)$,

$$
\|u_0 - \mathcal{T}(v_0)\| = \|\mathcal{T}(u_0) - \mathcal{T}(v_0)\| = \text{dist}(W, Z), \quad (24)
$$

Hence, $\mathcal{T}(v_0) = v_0$.

Theorem 7. Let $(W, Z)$ be a nonempty, convex, and closed pair in $X$ such that $W$ is bounded and $W_0$ is nonempty. Let $\mathcal{C}$ be the collection of the commuting, noncyclic relatively $u$-continuous, affine, and condensing mappings on $W \cup Z$. If $S: W \rightarrow KC(Z)$ is an upper semicontinuous multivalued mapping such that, for each $x \in W$, $S(x) \cap Z_0 \neq \emptyset$. If $\mathcal{C}$ commutes with $S$, then there exists $w \in W$ such that

$$
\mathcal{T}(w) = w \text{ and dist}(w, S(w)) = \text{dist}(W, Z). \quad (25)
$$

Proof. By Lemma 4, $(\cap_{\mathcal{C} \mathcal{C}} \text{Fix}_W(\mathcal{C}), \cap_{\mathcal{C} \mathcal{C}} \text{Fix}_Z(\mathcal{C}))$ is a nonempty compact convex pair. Also, in view to the proof of Theorem 4, for $x \in \mathcal{C}$ and for each $x \in \text{Fix}_W(x)$, we have $S(x) \cap Z_0 \neq \emptyset$. Hence, $S(x) \cap (\cap_{\mathcal{C} \mathcal{C}} \text{Fix}_Z(\mathcal{C})) \neq \emptyset$.

Define $f: \cap_{\mathcal{C} \mathcal{C}} \text{Fix}_W(\mathcal{C}) \rightarrow 2^{\cap_{\mathcal{C} \mathcal{C}} \text{Fix}_Z(\mathcal{C})}$ by $f(x) = S(x) \cap (\cap_{\mathcal{C} \mathcal{C}} \text{Fix}_Z(\mathcal{C}))$, for $x \in \cap_{\mathcal{C} \mathcal{C}} \text{Fix}_W(\mathcal{C})$. Then, $f$ is an upper semicontinuous multivalued mapping with nonempty, compact, and convex values. Moreover, $P_W: \cap_{\mathcal{C} \mathcal{C}} \text{Fix}_Z(\mathcal{C}) \rightarrow \cap_{\mathcal{C} \mathcal{C}} \text{Fix}_Z(\mathcal{C})$ is well-defined. Indeed, for $y \in \cap_{\mathcal{C} \mathcal{C}} \text{Fix}_Z(\mathcal{C})$, there exists $x \in W$ such that $\|x - y\| = \text{dist}(W, Z)$.

$$
y = P_w(x) \text{ and } x = P_w(y) \quad (26)
$$

By relative $u$-continuity of $\mathcal{C}$, one can conclude that $\|\mathcal{T}(x) - \mathcal{T}(y)\| = \text{dist}(W, Z)$. Thus, $\mathcal{T}(y) = P_w(\mathcal{T}(x))$ and $\mathcal{T}(x) = P_w(\mathcal{T}(y))$, and by (26), $\mathcal{T}(x) = \mathcal{T}(P_w(y)) = P_w(\mathcal{T}(y)) = P_w(y)$.

\section{Conclusion}

We have proved some best proximity pair theorems for noncyclic relatively $u$-continuous and condensing mappings. We have also obtained best proximity points of upper semicontinuous mappings which are fixed points of noncyclic relatively $u$-continuous condensing mappings.
Moreover, we have given some examples to support our results. It has been shown that relative u-continuity of $\mathcal{F}$ is a necessary condition that cannot be omitted. We have extended recent results of [6, 11].

**Data Availability**

No data were used to support this study.

**Conflicts of Interest**

The authors declare that they have no conflicts of interest.

**References**


