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Research Article

Bounds on Co-Independent Liar's Domination in Graphs

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A set $S \subseteq V$ of a graph $G = (V, E)$ is called a co-independent liar's dominating set of G if (i) for all $v \in V$, $|N_G[v] \cap S| \geq 2$, (ii) for every pair $u, v \in V$ of distinct vertices, $|(N_G[u] \cup N_G[v]) \cap S| \geq 3$, and (iii) the induced subgraph of G on $V - S$ has no edge. The minimum cardinality of vertices in such a set is called the co-independent liar's domination number of G, and it is denoted by $\gamma_{\text{coi}}^{LR}(G)$. In this paper, we introduce the concept of co-independent liar's domination number of the middle graph of some standard graphs such as path and cycle graphs, and we propose some bounds on this new parameter.

1. Introduction

For notations and nomenclature, we refer [1]. Specifically, let $G = (V, E)$ be a graph with vertex set V of order $p = |V|$ and edge set E of size $q = |E|$. The diameter of G is the greatest distance between any two vertices of G. The middle graph $M(G)$ is the derived graph obtained from G by inserting a new vertex into every edge of G and then joining these new vertices by edges which lie on the adjacent edges of G [2]. Haynes et al. introduced the concept of domination in graphs [3].

A topological index is a real number related to a graph, which must be a structural invariant. The topological indices are a vital tool for quantitative structure activity relationship and quantitative structure property relationship. For more work on topological indices of a graph, refer recent papers [4, 5].

The concept of liar's domination was introduced by Panda and Paul in [6]. A graph $G = (V, E)$ admits a liar's dominating set if each of its connected components has at least three vertices. Several different domination parameters were studied in [7–12]. For references on liar's domination, see, for instance, [2, 13]. A subset $S \subseteq V$ of a graph $G = (V, E)$ is called a co-independent liar's dominating set of G if (i) for all $v \in V$, $|N_G[v] \cap S| \geq 2$, (ii) for every pair $u, v \in V$ of distinct vertices, $|(N_G[u] \cup N_G[v]) \cap S| \geq 3$, and (iii) the induced subgraph of G on $V - S$ has no edge. The minimum cardinality of vertices in such a set is called the co-independent liar's domination number of G, and it is denoted by $\gamma_{\text{coi}}^{LR}(G)$. In this paper, we initiate the study of co-independent liar's domination in graphs.

2. Co-Independent Liar's Domination in Graphs

In this section, we first strengthen the co-independent liar's domination number of the middle graphs of some standard graphs. Eventually, some bounds will be obtained.

Theorem 1. *Let $M(P_p)$ be the middle graph of a path graph P_p of order p. Then,*

$$\gamma_{\text{coi}}^{LR}(M(P_p)) \leq p + 1. \quad (1)$$

Proof. Let $u_1, u_2, u_3, \ldots, u_p$ be the vertices of P_p and also the vertices in $V(M(P_p) - P_p)$ be $u_{p+1}, u_{p+2}, u_{p+3}, \ldots, u_{2p-1}$. Let $u \in V(M(P_p))$. We prove that all the vertices of $M(P_p)$ get a co-independent liars dominating set arising in four cases:

(1) Case (i): let $u = u_1$. Recall that $\deg(u_1) = \deg(u_p) = 1$. So, $N[u_1] = \{u_1, u_{p+1}\}$ in $M(P_p)$ and $|N[u_1]| = 2$. Therefore, $|N[u] \cap S| \geq 2$, for all $u \in V(M(P_p))$, and

every set consisting of a vertex of $N[u_1]$ should be a component of S and $\{u_1, u_{p+1}\} \in S$. So, $\{u_{2p-1}, u_p\} \in S$ for $u = u_p$. Therefore, $\{u_1, u_{p+1}, u_{2p-1}, u_p\} \in S$. Hence, $u_2 \in V - S$, and it is independent.

(2) Case (ii): let u be an element of $\{u_2, u_3, u_4, \ldots, u_{p-1}\}$. Then, $\deg(u_2) = \deg(u_3) = \cdots = \deg(u_{p-1}) = 2$ and $N[u_i] = \{u_i, u_{p+i}, u_{p+i-1}\}$ for $i = 2, 3, \ldots, p-1$. Let $N[u_j] \cap N[u_k] = S_{j,k}$ for $k = 2, 3, \ldots, p-2$ and $j = k+1$. Then,

$$\bigcup_{j,k=2}^{p-1} S_{j,k} = \{u_{p+2}, u_{p+3}, u_{p+4}, \ldots, u_{2p-2}\} \in S. \tag{2}$$

Therefore, $\{u_1, u_{p+1}, u_{p+2}, u_{p+3}, u_{p+4}, \ldots, u_{2p-2}, u_{2p-1}, u_p\} \in S$, and also, we get $|N[u_j] \cap S| \geq 2$ for all $u_j, j = 1, 2, 3, \ldots, p$. Next, we demonstrate that $|(N[u] \cup N[v]) \cap S| \geq 3$ for every pair of distinct vertices. Note that $|N[u_j] \cap S| = 2$, $|N[u_k] \cap S| \geq 3$, and $|N[u_j] \cap N[u_k]| \leq 1$. So, we have $|(N[u_j] \cup N[u_k]) \cap S| \geq 3$ for $j, k = 1, 2, \ldots, p$. Therefore, $\{u_2, u_3, u_4, \ldots, u_{p-1}\} \in V - S$, and no two elements are adjacent in $V - S$.

(3) Case (iii): let $u = u_{p+1}$. Then, $\deg(u_{p+1}) = 3$, and we can write $N[u_{p+1}] = \{u_1, u_2, u_{p+1}, u_{p+2}\}$. We have $N[u_{p+1}] \cap S = \{u_1, u_{p+1}, u_{p+2}\}$, and therefore, $|N[u_{p+1}] \cap S| = 3$. Let $u = u_{2p-1}$. Then, $\deg(u_{2p-1}) = 3$ and $N[u_{2p-1}] = \{u_{2p-1}, u_{p-1}, u_p, u_{2p-2}\}$. We get $N[u_{2p-1}] \cap S = \{u_{2p-2}, u_{2p-1}, u_p\}$ and similarly $|N[u_{2p-1}] \cap S| = 3$.

(4) Case (iv): let u be an element of $\{u_{p+2}, u_{p+3}, \ldots, u_{2p-2}\}$ and $\deg(u_{p+2}) = \deg(u_{p+3}) = \cdots = \deg(u_{2p-2}) = 4$. Then,

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$$\begin{aligned} N[u_{p+2}] &= \{u_{p+2}, u_{p+1}, u_2, u_3, u_{p+3}\}, \\ N[u_{p+3}] &= \{u_{p+3}, u_{p+2}, u_3, u_4, u_{p+4}\}, \\ &\vdots \\ N[u_{2p-2}] &= \{u_{2p-2}, u_{2p-3}, u_{p-2}, u_{p-1}, u_{2p-1}\}, \end{aligned} \tag{3}$$

and we obtain $|N[u_j] \cap S| \geq 3$ for $j = p+2, p+3, \ldots, 2p-2$. Therefore, $\{u_2, u_3, u_4, \ldots, u_{p-1}\} \in V - S$, and for all $uv \in V - S$, $uv \notin E$. Hence, $S = \{u_1, u_{p+1}, u_{p+2}, u_{p+3}, \ldots, u_{2p-2}, u_{2p-1}, u_p\}$ is a co-independent liar's dominating set of $M(P_p)$ and $\gamma_{\text{coi}}^{LR}(M(P_p)) \leq p+1$. □

Theorem 2. *Let $M(C_p)$ be the middle graph of a cycle graph. Then,*

$$\gamma_{\text{coi}}^{LR}(M(C_p)) \leq p. \tag{4}$$

Proof. Let the vertices of C_p be $u_1, u_2, \ldots, u_p$ and $u_{p+1}, u_{p+2}, \ldots, u_{2p}$ be the vertices of $V(M(C_p) - C_p)$. We investigate all vertices of $M(C_p)$ to get a co-independent liar's dominating set in two cases:

(1) Case (i): let $\deg(u_i) = 2$ where $i = 1, 2, 3, \ldots, p$, and let u be an element of $\{u_1, u_2, \ldots, u_p\}$. Then, S contains at least two vertices of $N[u_i]$ and it double dominates u_i for $i = 1, 2, 3, \ldots, p$. Hence,

$$\begin{aligned} N[u_1] &= \{u_1, u_{p+1}, u_{p+2}\}, \\ N[u_2] &= \{u_2, u_{p+2}, u_{p+3}\}, \\ &\vdots \\ N[u_{p-1}] &= \{u_{p-1}, u_{2p-1}, u_{2p}\}, \\ N[u_p] &= \{u_p, u_{2p}, u_{p+1}\}. \end{aligned} \tag{5}$$

If $uv \in E(C_p)$, then we consider $N[u] \cap N[v]$. Let $N[u_j] \cap N[u_k] = S_{j,k}$, where $j, k = 1, 2, 3, \ldots, p$. So, we have

$$S_1 = \bigcup_{j,k=1}^{p} S_{j,k} = \{u_{p+1}, u_{p+2}, \ldots, u_{2p-2}, u_{2p-1}, u_{2p}\} \in S. \tag{6}$$

Therefore, S_1 double dominates u_i for $i = 1, 2, \ldots, p$. Clearly, we have $|(N[u_j] \cup N[u_k]) \cap S_1| \geq 3$ as $|N[u_j] \cap S_1| = 2$ and $|N[u_j] \cap N[u_k]| = 1$ for the vertices u_j and u_k, where $u_j u_k \in E(C_p)$ for $j, k = 1, 2, \ldots, p$. Likewise, we have $|(N[u_j] \cup N[u_k]) \cap S_1| = 4 \geq 3$ as $|N[u_j] \cap S_1| = 2$ and $|N[u_j] \cap N[u_k]| = 0$ for the vertices u_j and u_k, where $u_j u_k \notin E(C_p)$ for $j, k = 1, 2, 3, \ldots, p$. Therefore, S_1 triple dominates u_i for $i = 1, 2, 3, \ldots, p$. Hence, $\{u_1, u_2, \ldots, u_p\} \in V - S$, and no two elements in $V - S$ can form an edge.

(2) Case (ii): let $\deg(u_j) = 4$, where $j = p+1, p+2, \ldots, 2p$, and let u be an element of $\{u_{p+1}, \ldots, u_{2p}\}$.

$$\begin{aligned} N[u_{p+1}] &= \{u_{p+1}, u_p, u_1, u_{2p}, u_{p+2}\}, \\ N[u_{p+2}] &= \{u_{p+2}, u_1, u_2, u_{p+1}, u_{p+3}\}, \\ &\vdots \\ N[u_{2p-1}] &= \{u_{2p-1}, u_{p-2}, u_{p-1}, u_{2p-2}, u_{2p}\}, \\ N[u_{2p}] &= \{u_{2p}, u_{p-1}, u_p, u_{2p-1}, u_{p+1}\}. \end{aligned} \tag{7}$$

We obtain $|N[u_j] \cap S_1| = 3$ for $j = p+1, p+2, \ldots, 2p$. Therefore, S_1 triple dominates u_j for $j = p+1, p+2, \ldots, 2p$. So, $\{u_1, u_2, \ldots, u_p\} \in V - S$. For every $u, v \in V - S$, $uv \notin E$.

Hence, $S = S_1 = \{u_{p+1}, u_{p+2}, \ldots, u_{2p}\}$ is a co-independent liar's dominating set of $M(C_p)$ and $\gamma_{\text{coi}}^{LR}(M(C_p)) \leq p$. □

Note 1. Let $M(W_p)$ be the middle graph of a wheel graph, then a co-independent liar's dominating set does not exist.

We have the following lower bound on co-independent liar's domination number in terms of the diameter of a graph.

Theorem 3. *Let G be a graph of order $p \geq 4$. Then,*

$$\gamma_{\text{coi}}^{LR}(G) \geq \frac{3}{4}(\text{diam}(G) + 2). \tag{8}$$

Proof. Let S be a $\gamma_{\text{coi}}^{LR}(G)$ set. We employ an induction on the number r of components of $G[S]$ to indicate that $\gamma_{\text{coi}}^{LR}(G) \geq \text{diam}(G) - r + 2$, and hence the result will follow: suppose that $G[S]$ has precisely one component, that is, $G[S]$ is connected and $|S| \geq 3$. We show that the distance between any pair of vertices in G is atmost $|S| - 1$. Let y_1 and y_2 be two distinct vertices of G. If $y_1, y_2 \in S$, then $d_G(y_1, y_2) \leq \text{diam}(G[S]) \leq |S| - 1$. Next, assume that $y_1 \notin S$ and $y_2 \notin S$. It may be verified that $|N[y_1] \cap S| \geq 2$. Since $y_1 \notin S$, there are at least two vertices y_i and y_j in $N(y_1) \cap S$. Similarly, it may be concluded that there are two vertices y_i' and y_j' in $N(y_2) \cap S$. If $\{y_i, y_j\} \cap \{y_i', y_j'\} \neq \phi$, then $d(y_1, y_2) \leq |S| - 1$ since $|S| \geq 3$. Suppose that $\{y_i, y_j\} \cap \{y_i', y_j'\} = \phi$. Assume, without loss of generality, that $d_{G[S]}(y_i, y_i') = \min\{d_{G[S]}(v, w)\colon v \in \{y_i, y_j\}, w \in \{y_i', y_j'\}\}$. It is easy to verify that $d_{G[S]}(y_i, y_i') \leq |S| - 3$. Now,

$$\begin{aligned} d_G(y_1, y_2) &\leq d_G(y_1, y_i) + d_{G[S]}(y_i, y_i') + d_G(y_i', y_2) \\ &\leq 1 + (|S| - 3) + 1 = |S| - 1. \end{aligned} \tag{9}$$

Next, we take $y_1 \notin S$ and $y_2 \in S$. As before, there are two vertices y_i and y_j in $N(y_1) \cap S$. If $\{y_i, y_j\} \cap \{y_2\} \neq \phi$, then $d_G(y_1, y_2) = 1 \leq |S| - 1$ as $|S| \geq 3$. Let $\{y_i, y_j\} \cap \{y_2\} = \phi$. Without loss of generality, we might assume that $d_{G[S]}(y_i, y_2) = \min\{d_{G[S]}(v, y_2)\colon v \in (y_i, y_j)\}$. Then, $d_G(y_1, y_2) \leq d_G(y_1, y_i) + d_{G[S]}(y_i, y_2) \leq 1 + (|S| - 2) = |S| - 1$. Suppose that the result is true for the number of components of $G[S]$ which are less than r. Let $S = \cup_{i=1}^{r} S_i$, where $G[S_i]$ is the component of $G[S]$ for $i = 1, 2, 3, \ldots, r$. Let V_i be the set of all vertices of $V(G) - S$ with at least two neighbors in L_i and $G_i = G[V_i \cup S_i]$. If for every $m, n \in V - S$, then $mn \notin E(G)$. In order to maximize the diameter, without loss of generality, we may assume that, for $i = 1, 2, \ldots, r - 1$, $|N(G_i) \cap N(G_{i+1})| = 1$ and for every $j > i + 1$, $|N(G_i) \cap N(G_j)| = 0$. Let $N(G_i) \cap N(G_{i+1}) = \{v_i\}$ for $i = 1, 2, \ldots, r - 1$. Let p, q be two distinct vertices of $V(G)$ with $d_G(p, q) = \text{diam}(G)$. Then,

$$\begin{aligned} d_G(p, q) &\leq \sum_{i=1}^{r} \text{diam}(G_i) + d_G(G_1, v_1) + d_G(v_1, G_2) + d_G(G_2, v_2) + \cdots + d_G(v_{r-1}, G_r) \\ &= \sum_{i=1}^{r} \text{diam}(G_i) + 2(r - 1). \end{aligned} \tag{10}$$

By induction, we find that $\text{diam}(G_i) \leq |S_i| - 1, \forall\, 1 \leq i \leq r$. Hence,

$$d_G(p, q) \leq \sum_{i=1}^{r} \left(|S_i| - 1\right) + 2(r - 1). \tag{11}$$

Hence, $d_G(p, q) \leq |S| + r - 2$. Therefore, $\gamma_{\text{coi}}^{LR}(G) \geq \text{diam}(G) - r + 2$.

We now study some bounds on co-independent liar's domination number in terms of the components of a cut vertex deleted graph. □

Theorem 4. *Let x be a cut vertex of a graph G and $T_1, T_2, T_3, \ldots, T_m$ be the components of $G - x$. If $|T_j| \geq 2$ and $G_j = T_j \cup \{x\}$ for $1 \leq j \leq m$, then*

$$\sum_{j=1}^{m} \gamma_{\text{coi}}^{LR}(G_j) - (2m - 1) \leq \gamma_{\text{coi}}^{LR}(G) \leq \sum_{j=1}^{m} \gamma_{\text{coi}}^{LR}(G_j). \tag{12}$$

Proof. First, we show that $\sum_{j=1}^{m} \gamma_{\text{coi}}^{LR}(G_j) - (2m - 1) \leq \gamma_{\text{coi}}^{LR}(G)$. Let S be a $\gamma_{\text{coi}}^{LR}(G)$ set and $S_j = S \cap V(G_j)$ for $j = 1, 2, 3, \ldots, m$. If $x \in S$, we have $\sum_{j=1}^{m} |S_j| = \gamma_{\text{coi}}^{LR}(G) + m$ and if $x \notin S$, then $\sum_{j=1}^{m} |S_j| = \gamma_{\text{coi}}^{LR}(G)$. Clearly, for every vertex, $u \in V(G_j)/\{x\}$, $(j = 1, 2, 3, \ldots, m)$, $N_{G_j}[u] \cap S_j = N_G[u] \cap S$, and $|N_{G_j}[u] \cap S_j| \geq 2$. Moreover, for any pair $u, v \in V(G_j)/\{x\}$, $(j = 1, 2, 3, \ldots, m)$, $|(N[u] \cup N[v]) \cap S_j| = |(N[u] \cup N[v]) \cap S| \geq 3$, and also the set $V(G_j) - S_j$ is independent. We have to consider following two cases:

Case (i): let $x \in S$. For any $1 \leq j \leq m$, there is at least one vertex $r_j \in N(x) \cap S_j$, so $|N[x] \cap S_j| \geq 2$. Since each vertex $z \in V(G_j)/(N_{G_j}(x) \cap N(r_j))$ should be double dominated by S_j, there is at least a vertex $w \in N(z) \cap S_j$, $w \neq r_j$. Therefore, for each vertex $z \in V(G_j) - (N_{G_j}(x) \cap N(r_j))$,

$$\left|\left(N_{G_j}[x] \cup N[z] \cap S_j\right)\right| \geq \left|\{x, r_j, w\}\right| = 3. \tag{13}$$

Hence, let $y \in N_{G_j}(x) \cap N(r_j)$ and $S'_j = S_j \cup \{z\}$. So, for every vertex $y \in N_{G_j}(x) \cap N(r_j)$,

$$\left|\left(N_{G_j}[x] \cup N[y] \cap S'_j\right)\right| \geq \left|\left\{x, r_j, z\right\}\right| = 3. \quad (14)$$

and for every $r, s \in V(G_j) - S_j$, $rs \notin E(G_j)$. Hence, S'_j is a co-independent liar's dominating set for each G_j, and therefore,

$$\gamma_{\text{coi}}^{LR}\left(G_j\right) \leq \left|S'_j\right| = \left|S_j\right| + 1. \quad (15)$$

For each vertex $t \in N(x) \cap V(G_j)$, $t \notin S_j$. Repeating these, each vertex $z \in V(G_j)/\{x\}$ should be double dominated by S_j. In such a component, there is at least one vertex $z\prime \in N(z) \cap S_j$. Let $S'_j = S_j \cup \{t\}$ for each vertex $t \in N(x) \cap V(G_j)$, $t \notin S_j$. Hence, for every vertex $z \in V(G_j)/\{x\}$,

$$\left|\left(N_{G_j}[x] \cup N[z]\right) \cap S'_j\right| \geq \left|\left\{x, t, z\prime\right\}\right| = 3. \quad (16)$$

Also,

$$\left|N_{G_j}[x] \cap S'_j\right| \geq |\{x, t\}| = 2. \quad (17)$$

and no two vertices can be adjacent in $V(G_j) - S_j$. There is at least one vertex $r_j \in N(x) \cap S_j$. As before, for $z \in V(G_j)/(N_{G_j}(x) \cap N(r_j))$, we have

$$\left|\left(N_{G_j}[x] \cup N[z]\right) \cap S_j\right| \geq \left|\left\{x, r_j, w\right\}\right| = 3. \quad (18)$$

Let $S'_j = S_j \cup \{z\}$. For each vertex $y \in N_{G_j}(x) \cap N(r_j)$, we have

$$\left|\left(N_{G_j}[x] \cup N[y]\right) \cap S'_j\right| \geq \left|\left\{x, r_j, z\right\}\right| = 3. \quad (19)$$

Thus, for each pair of vertices in $V(G_j) - S_j$ for $j = 1, 2, 3, \ldots, m$, S'_j is a co-independent liar's dominating set and

$$\gamma_{\text{coi}}^{LR}(G) \leq \left|S'_j\right| = \left|S_j\right| + 1. \quad (20)$$

Thus, in any case, as $x \in S$, S' is a co-independent liar's dominating set for $G_j\,(1 \leq j \leq m)$. Thus,

$$\begin{aligned}\sum_{j=1}^{m} \gamma_{\text{coi}}^{LR}\left(G_j\right) &\leq \sum_{j=1}^{m} \left|S'_j\right| \\ &\leq \sum_{j=1}^{m}\left(\left|S_j\right| + 1\right) \\ &= \sum_{j=1}^{m}\left|S_j\right| + m \\ &= \gamma_{\text{coi}}^{LR}(G) + m - 1 + m.\end{aligned} \quad (21)$$

So, $\sum_{j=1}^{m} \gamma_{\text{coi}}^{LR}(G_j) - (2m-1) \leq \gamma_{\text{coi}}^{LR}(G)$.

Case (ii): let $x \notin S$. Because $|N[x] \cap S| \geq 2$, there is at least one component. Without loss of generality, we might assume that $r_1 \in N(x) \cap S_1$. Let $S'_j = S_j \cup \left\{x, r_j\right\}$, where r_j is an arbitrary vertex in $N(x) \cap V(G_j)$, $(j = 1, 2, 3, \ldots, m)$. As for every vertex $z \in V(G_j)/\{x\}$, $|N[z] \cap S_j| \geq 2$, there are at least two vertices $w, y \in N[z] \cap S_j$. So, for each $z \in V(G_j)/\{x\}$, we have

$$\left|\left(N[z] \cup N_{G_j}[x]\right) \cap S'_j\right| \geq |\{w, y, x\}| = 3, \quad (22)$$

and $|N[x] \cap S'_j| \geq |\left\{x, r_j\right\}| = 2$, and for every $r, s \in V(G_j) - S_j$, $rs \notin E(G_j)$. Therefore,

$$\begin{aligned}\sum_{j=1}^{m} \gamma_{\text{coi}}^{LR}\left(G_j\right) &\leq \sum_{j=1}^{m} \left|S'_j\right| \\ &\leq \sum_{j=1}^{m}\left(\left|S_j\right| + 2\right) - 1 \\ &= \sum_{j=1}^{m}\left|S_j\right| + 2m - 1 \\ &= \gamma_{\text{coi}}^{LR}(G) + 2m - 1.\end{aligned} \quad (23)$$

Similarly, the second inequality can be proven.

We now characterize the graphs according to the sensitivity of a co-independent liar's dominating set versus a cut edge. □

Theorem 5. *Let $e = xy$ be a cut edge (bridge) in a graph G and G_1 and G_2 be the components of $G - e$. If $|V(G_1)| \geq 4$ and $|V(G_2)| \geq 4$, then*

$$\gamma_{\text{coi}}^{LR}(G_1) + \gamma_{\text{coi}}^{LR}(G_2) - 2 \leq \gamma_{\text{coi}}^{LR}(G) \leq \gamma_{\text{coi}}^{LR}(G_1) + \gamma_{\text{coi}}^{LR}(G_2). \quad (24)$$

Proof. Let S be a co-independent liar's dominating set and $S_1 = S \cap G_1$, $S_2 = S \cap G_2$. When $x, y \notin S$, deleting the edge xy does not change the sizes of co-independent liar's dominating sets of G_1 and G_2. Assume that $x \in S$ and $y \notin S$. Then, clearly, for every component of S having a minimum of four vertices, we can assume that there are two vertices $x' \in N(x) \cap S_1$ and $x'' \in (N(x) \cap S_1) \cup (N(x\prime) \cap S_1)/\{x, x'\}$ and $V(G_1) - S_1$ has no edge. Similarly, there are two vertices $y' \in N(y) \cap S_2$, $y'' \in (N(y) \cap S_2) \cup ((N(y\prime) \cap S_2)/\{y, y'\})$, and also, the subgraph induced by $V(G_2) - S_2$ is independent. Let $S'_1 = S_1$ and $S'_2 = S_2 \cup \{y\}$. Then, S'_1 and S'_2 form co-independent liar's dominating sets for G_1 and G_2 and$|S'_1| + |S'_2| - 1 \leq |S|$.

$$\left|S'_1\right| + \left|S'_2\right| - 1 \leq |S|. \quad (25)$$

Next, suppose that both x and $y \in S$. Then, there is a vertex $x' \in N(x) \cap S_1$ or $y\prime \in N(y) \cap S_2$. Without loss of generality, we might assume that it is $x\prime$. Let $S'_1 = S_1 \cup \{x''\}$, where $x'' \in (N(x) \cap G_1 \cup (N(x\prime) \cap G_1/\left\{x, x\prime\right\}$ and

$S_2' = S_2 \cup \{y\prime, y''\}$, where $y\prime \in N(y) \cap G_2$ and $y'' \in (N(y) \cap G_2) \cup ((N(y\prime) \cap G_2)/\{y, y'\})$. Therefore, S_1', S_2' form co-independent liar's dominating sets of G_1 and G_2 and

$$|S_1'| + |S_2'| - 2 \leq |S|. \quad (26)$$

The right side inequality follows from the fact that the union of the $\gamma_{coi}^{LR}(G_1)$ set and the $\gamma_{coi}^{LR}(G_2)$ set forms a co-independent liar's dominating set for G. □

Theorem 6. *Let G and $\overline{G}$ be connected and $|V(G)| = p$, $p \geq 5$, then $16 \leq \gamma_{coi\_lr}(G)\gamma_{coi\_lr}(\overline{G}) \leq p^2$.*

Proof. The upper bound is obvious. We established the lower bound. Let G be a connected graph of order p such that $\overline{G}$ is connected. Clearly, $\gamma_{coi\_lr}(G) \geq 3$ and $\gamma_{coi\_lr}(\overline{G}) \geq 3$. The result is obvious if $\min\{\gamma_{coi\_lr}(G), \gamma_{coi\_lr}(\overline{G})\} \geq 4$. Hence, let $\min\{\gamma_{coi\_lr}(G), \gamma_{coi\_lr}(\overline{G})\} = 4$. Without loss of generality, assume that $\gamma_{coi\_lr}(G) = 4$. We show that $\gamma_{coi\_lr}(\overline{G}) \geq 4$. Let $S = \{u_1, u_2, u_3, u_4\}$ be a $\gamma_{coi\_lr}(G)$ set and $S\prime$ be a $\gamma_{coi\_lr}(\overline{G})$ set. We partition the set $V(G)/S$ into two sets $M = \{y \in V(G)/S\colon |N(y) \cap S| = 3\}$ and $N = \{y \in V(G)/S\colon |N(y) \cap S| = 4\}$. Note that $|M| = 4$. Since $\overline{G}$ is connected and none of the vertices in N are adjacent to any vertex in S in $\overline{G}$, we can deduce that $|M| \geq 4$. Hence, $|M| = 4$. Let $M = \{v_1, v_2, v_3, v_4\}$, where v_j is not adjacent u_j in G for $j = 1, 2, 3, 4$. Since $G[S]$ is connected, we may think that $\{u_1, u_3\} \subseteq N(u_2)$. Now, the only vertex adjacent to u_2 in $\overline{G}$ is v_2, so $\{v_2, u_2\} \in S'$. In addition, according to the co-independent liar's domination which has been discussed, $|\{u_1, v_1, u_3, v_3\} \cap S\prime| \geq 3$. Therefore, $\gamma_{coi\_lr}(\overline{G}) = |S\prime| \geq 4$. □

Theorem 7. *Let $M(T)$ be the middle graph of a tree with $p \geq 3$. Then, $\gamma_{coi\_lr}(M(T)) \leq p + 2$.*

Proof. Let the vertices of T be $u_1, u_2, u_3, \ldots, u_p$ and $u_{p+1}, u_{p+2}, u_{p+3}, \ldots, u_{2p-1}$ be the vertices of $V(M(T))$. We prove that all vertices of $M(T)$ arise in four cases to get a co-independent liars dominating set:

(i) Case (i): if $\deg(u_i) = 1$, that is, u_i is a pendant vertex in $V(M(T))$, then $N[u_i] = \{u_i, u_{p+i}\}$ in $M(T)$ and $|N[u_i]| = 2$. Therefore, $|N[u_i] \cap S| \geq 2$, $\forall u_i \in V(M(T))$, and all the vertices of $N[u_i]$ should be components of S, $\{u_i, u_{p+i}\} \in S$. So, $\{u_{2p-1}, u_p\} \in S$ for $u = u_p$. Therefore, $\{u_i, u_{p+i}, u_{2p-i}, u_p\} \in S$.

(ii) Case (ii): if $\deg(u_i) = 2$ in $V(M(T))$, then $N[u_i] = \{u_i, u_{p+i}, u_{p+i-1}\}$ for $i = 2, 3, \ldots, p-1$. Let $N[u_j] \cap N[u_k] = S_{j,k}$, where $k = 2, 3, \ldots, p-2$ and $j = k+1$. Then, $\cup_{j,k=2}^{p-1} S_{j,k} = \{u_{p+2}, u_{p+3}, u_{p+4}, \ldots, u_{2p-2}\} \in S$. Therefore, $\{u_1, u_{p+1}, u_{p+2}, u_{p+3}, u_{p+4}, \ldots, u_{2p-2}, u_{2p-1}, u_p\} \in S$ and $|N[u_j] \cap S| \geq 2, \forall u_j, j = 1, 2, 3, \ldots, p$. Next, we demonstrate that $|(N[u] \cup N[v]) \cap S| \geq 3$ for every pair of distinct vertices. We see that $|N[u_j] \cap S| = 2$, $|N[u_k] \cap S| \geq 3$, and $|N[u_j] \cap N[u_k]| \leq 1$. So, we have $|(N[u_j] \cup N[u_k]) \cap S| \geq 3$, where $j, k = 1, 2, \ldots, p$. Therefore, $\{u_2, u_3, u_4, \ldots, u_{p-1}\} \in V - S$, and no two elements are adjacent in $V - S$.

(iii) Case (iii): if $\deg(u_i) = 3$ in $V(M(T))$, then $N[u_{p+i}] = \{u_1, u_2, u_{p+i}, u_{p+i+1}\}$. We have $N[u_{p+i}] \cap S = \{u_1, u_{p+i}, u_{p+i+1}\}$ and $|N[u_{p+i}] \cap S| = 3$. Let u be u_{2p-1}. Then, $\deg(u_{2p-1}) = 3$ and $N[u_{2p-1}] = \{u_{2p-1}, u_{p-1}, u_p, u_{2p-2}\}$. We get $N[u_{2p-1}] \cap S = \{u_{2p-2}, u_{2p-1}, u_p\}$ and $|N[u_{2p-1}] \cap S| = 3$.

(iv) Case (iv): if $\deg(u_i) = 4$ in $V(M(T))$, then

$$\begin{aligned} N[u_{p+2}] &= \{u_{p+2}, u_{p+1}, u_2, u_3, u_{p+3}\}, \\ N[u_{p+3}] &= \{u_{p+3}, u_{p+2}, u_3, u_4, u_{p+4}\}, \\ &\vdots \\ N[u_{2p-2}] &= \{u_{2p-2}, u_{2p-3}, u_{p-2}, u_{p-1}, u_{2p-1}\}. \end{aligned} \quad (27)$$

We obtain $|N[u_j] \cap S| \geq 3$, where $j = p+2, p+3, \ldots, 2p-2$. Therefore, $\{u_2, u_3, u_4, \ldots, u_{p-1}\} \in V - S$, and for all $uv \in V - S$, $uv \notin E$. Hence, $S = \{u_1, u_{p+1}, u_{p+2}, u_{p+3}, \ldots, u_{2p-2}, u_{2p-1}, u_p\}$ is the co-independent liar's dominating set of $M(T)$ and $\gamma_{coi\_lr}(M(T)) \leq p + 2$. □

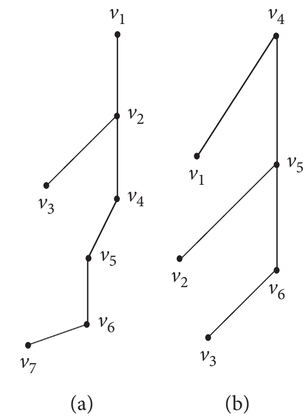

Figure 1: Trees for Remark 1.

Remark 1. (i) A tree is a co-independent liar's dominating set. Indeed in Figure 1(a), let $V(G) = \{v_1, v_2, v_3, v_4, v_5, v_6, v_7\}$. If we take $S = \{v_1, v_2, v_3, v_5, v_6, v_7\}$, then $V - S = \{v_4\}$ which is a co-independent liar's dominating set. (ii) Co-independent liar's dominating set need not exist for all trees. For example, in Figure 1(b), let $V(G) = \{v_1, v_2, v_3, v_4, v_5, v_6\}$.

Suppose we take the co-independent liar's dominating set $S = \{v_1, v_2, v_3, v_4, v_5, v_6\}$. But $V - S = \phi$ which satisfies conditions (i) and (ii) but not (iii).

(i) For all $v \in V$, $|N_G[v] \cap S| \geq 2$

(ii) For every pair $u, v \in V$ of distinct vertices, $|(N_G[u] \cup N_G[v]) \cap S| \geq 3$

(iii) The induced subgraph of G on $V - S$ has no edge

3. Conclusion

In this paper, the co-independent liar's domination number of the middle graphs of some graph classes such as path and cycle graphs is calculated. Also, some general results and bounds on the co-independent liar's domination number of graphs are obtained. It has been shown that no general result can be obtained for trees, unicyclic, bicyclic, and tricyclic graphs in terms of co-independent liar's domination number.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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