

Research Article

Null Controllability of a Four Stage and Age-Structured Population Dynamics Model

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This paper is devoted to study the null controllability properties of a population dynamics model with age structuring and nonlocal boundary conditions. More precisely, we consider a four-stage model with a second derivative with respect to the age variable. The null controllability is related to the extinction of eggs, larvae, and female population. Thus, we estimate a time T to bring eggs, larvae, and female subpopulation density to zero. Our method combines fixed point theorem and Carleman estimate. We end this work with numerical illustrations.

1. Introduction

Let $u_i(t, a)$, $1 \leq i \leq 4$, be, respectively, the distribution of eggs, larvae, and female and male individuals of age a at time t ; A_i , $1 \leq i \leq 4$, is the life expectancy of an i -stage individual

and T is a positive constant. We consider the following population dynamics model based on Fokker-Planck or Kolmogorov-type equations which is written as

$$\begin{cases} \partial_t u_1(t, a) + \partial_a [v_1(\zeta(t), a)u_1(t, a)] - k\partial_a^2 u_1(t, a) = -[\mu_1(\zeta(t), a) + \beta_1(\zeta(t), a)]u_1(t, a) + m(a)w_1(t, a), \\ \partial_t u_2(t, a) + \partial_a [v_2(\zeta(t), a)u_2(t, a)] - k\partial_a^2 u_2(t, a) = -[\mu_2(\zeta(t), a) + \beta_2(\zeta(t), a)]u_2(t, a) + m(a)w_2(t, a), \\ \partial_t u_3(t, a) + \partial_a [v_3(\zeta(t), a)u_3(t, a)] - k\partial_a^2 u_3(t, a) = -\mu_3(\zeta(t), a)u_3(t, a) + m(a)w_3(t, a), \\ \partial_t u_4(t, a) + \partial_a [v_4(\zeta(t), a)u_4(t, a)] - k\partial_a^2 u_4(t, a) = -\mu_4(\zeta(t), a)u_4(t, a), \end{cases} \quad (1)$$

where $(t, a) \in \Omega_i = (0, T) \times (0, A_i)$ and $w_1, w_2,$ and w_3 are, respectively, the control function of eggs, larvae, and females. Here, $m(a)$ is the characteristic function of

$\omega = (0, a^*)$ with $0 < a^* < \min\{A_1, A_2, A_3\}$. To complete the system, the boundary conditions are stated as follows:

$$\left\{ \begin{array}{l} [v_1(\zeta(t), a)u_1(t, a) - k\partial_a u_1(t, a)]_{a=0} = \int_0^{A_3} \beta_3(\zeta(t), a)u_3(t, a)da, \\ [v_2(\zeta(t), a)u_2(t, a) - k\partial_a u_2(t, a)]_{a=0} = \int_0^{A_1} \beta_1(\zeta(t), a)u_1(t, a)da, \\ [v_3(\zeta(t), a)u_3(t, a) - k\partial_a u_3(t, a)]_{a=0} = \int_0^{A_2} \sigma\beta_2(\zeta(t), a)u_2(t, a)da, \\ [v_4(\zeta(t), a)u_4(t, a) - k\partial_a u_4(t, a)]_{a=0} = \int_0^{A_2} (1 - \sigma)\beta_2(\zeta(t), a)u_2(t, a)da, \\ u_i(0, a) = u_i^0(a), a \in (0, A_i), \quad 1 \leq i \leq 4, \\ u_i(t, A_i) = 0, \quad t \in (0, T), 1 \leq i \leq 4. \end{array} \right. \quad (2)$$

Here, $\mu_i(\zeta(t), a), \beta_i(\zeta(t), a),$ and $v_i(\zeta(t), a)$ are, respectively, the i -stage mortality rate, the i -stage age specific

transition functions, and the i -stage growth rate of age a at temperature $\zeta(t)$ and at time t .

Finally,

$$\int_0^{A_i} \sigma_i \beta_i(\zeta(t), a)u_i(t, a)da, \quad 1 \leq i \leq 4, \quad \sigma_1 = 1, \sigma_2 = \sigma, \sigma_3 = 1, \sigma_4 = 1 - \sigma. \quad (3)$$

Denote, respectively, the distribution of the newborns of eggs, larvae, females, and males. The constant σ denotes the sex ratio and $k > 0$ (diffusion coefficient) corresponds to small deviations of order 2 which represent the dispersion effects of individuals during their development. Our aim is to study the null controllability of (1) and (2).

Given a fixed $T > 0$, we look for controls $w_1, w_2,$ and w_3 such that $u_1, u_2,$ and u_3 satisfy

$$u_1(T, a) = u_2(T, a) = u_3(T, a) = 0. \quad (4)$$

In practice, this study applies to the dynamics cowpea pest insects (*Callosobruchus maculatus*). These insects live in hot areas, namely, tropical and subtropical regions [1] and cause important damage to the seeds of cowpeas and other legumes. This damage is exclusively due to the larvae. Indeed, females lay their eggs on the pods or on stocked seeds and then larvae grow by feeding on the cotyledons, see [2]. Twenty five days and a temperature varying between twenty-seven and thirty-one degrees Celsius are necessary for the growth of the larvae.

The life cycle of *Callosobruchus maculatus* can be divided into three development stages: egg, larva + pupa (growing stages), and adult (reproduction ability). Besides, during their growth, beetle larvae transform nitrogen to toxic uric acid that accumulates into cowpea and make it unfit for consumption.

Several methods of combating these insects have been considered. Among them is the method of preservation in

hermetically sealed containers, thus preventing the development of the larvae.

The method of control we are considering is to remove or eliminate individuals. This can be done with the help of pesticides. We recall that the optimal and exact control problems are widely investigated for age-structured population dynamics by many researchers. Most of these studies are focused on optimal control problems [3–5] and the references therein. One can also refer to [6, 7]. Lebeau and Robbiano establish in [8] the null controllability of the linear heat equation. Next, the exact controllability for age dependent linear and nonlinear single-species population models with spatial diffusion was investigated by Ainseba et al. [9, 10]. Barbu et al. also considered the exact controllability of the linear Lotka–McKendrick model without spatial structure by establishing an observability inequality for the backward adjoint system [11]. Later on, He and Ainseba investigate the exact null controllability of a stage and age-structured population dynamics system in [12] and the exact null controllability of the Lobesia Botrana model with spatial diffusion in [13].

Moreover, in [14], the authors investigated a semilinear problem: the null controllability of the heat equation with a Fourier boundary condition over the entire boundary. In our work, the Fourier condition contains a nonlocal term and is set only on a part of the domain. This induces additional difficulties in establishing the Carleman inequality.

In [15], Hegoburu and Anita study in a very practical way the null controllability of a nonlinear model of

population dynamics by means of a feedback control. This interesting result cannot be applied easily in our case since we study a four-stage model via three controls. More precisely, in our case, we control only three stages. So, it seems difficult here to compute explicitly the feedback control as it was done in [15].

The fundamental difference with [13] lies in the fact that our model combines a transport effect with the presence of a first-order derivation with respect to the age variable and a diffusion along the same variable. Thus, the inequalities guaranteeing the null controllability obtained in [13] become inoperative in our case.

As far as we know, there are no results dealing with the null controllability for an age and stage-dependent system involving a second derivative with respect to the age variable. The second derivative with respect to the physiological age variable corresponds to small deviations of order two and represents the dispersion effects of individuals during their development due to, for example, the temperature, the humidity, and the quality of food. This article is structured as follows. In Section 2, the assumptions and the main result are stated. We study the null controllability of some associated auxiliary model in Section 3. Section 4 is devoted to the proof of the main result, and some numerical illustrations are given in Section 5.

2. Assumptions and Main Result

For the sequel, we make the following assumptions on the demographic parameters:

(A1) The transition functions $\beta_k(\zeta(\cdot, \cdot))$ are positive and $\beta_k \in L^\infty((0, T) \times (0, A_k))$, $1 \leq k \leq 3$.

(A2) The mortality functions $\mu_k(\zeta(\cdot, \cdot))$ are positive and $\mu_k \in L^\infty((0, T) \times (0, A_k))$, $1 \leq k \leq 4$.

(A3) The initial conditions $u_k^0(a) \geq 0$ a.e. $a \in (0, A)$, $1 \leq k \leq 4$.

(A4) The growth functions $v_i(\zeta(\cdot, \cdot))$, $1 \leq i \leq 4$ are positive and bounded, that is,

$$0 < v_i^{\min} < v_i(\zeta(t), a) < v_i^{\max}, \quad \forall (t, a) \in (0, T) \times (0, A_i). \tag{5}$$

In addition, for all $t \in [0, T]$, $v_i(\zeta(t), \cdot) \in C^1([0, A])$, and there is a positive constant C_i such that

$$\|\partial_a v_i(\zeta(t), a)\|_\infty \leq C_i, \quad \forall (t, a) \in (0, T) \times (0, A_i). \tag{6}$$

(A5) The temperature function $\zeta(\cdot)$ is measurable.

These assumptions are biologically meaningful, see [3, 16–18].

Setting $\hat{u}_i = e^{-\lambda t} u_i$, $1 \leq i \leq 4$, the vector $(\hat{u}_1, \hat{u}_2, \hat{u}_3, \hat{u}_4)$ verifies the system:

$$\left\{ \begin{array}{l} \partial_t \hat{u}_1 + \partial_a (v_1 \hat{u}_1) - k \partial_a^2 \hat{u}_1 = -(\mu_1 + \beta_1 + \lambda) \hat{u}_1 + m(a) \hat{w}_1, \\ \partial_t \hat{u}_2 + \partial_a (v_2 \hat{u}_2) - k \partial_a^2 \hat{u}_2 = -(\mu_2 + \beta_2 + \lambda) \hat{u}_2 + m(a) \hat{w}_2, \\ \partial_t \hat{u}_3 + \partial_a (v_3 \hat{u}_3) - k \partial_a^2 \hat{u}_3 = -(\mu_3 + \lambda) \hat{u}_3 + m(a) \hat{w}_3, \\ \partial_t \hat{u}_4 + \partial_a (v_4 \hat{u}_4) - k \partial_a^2 \hat{u}_4 = -(\mu_4 + \lambda) \hat{u}_4, \end{array} \right. \left\{ \begin{array}{l} [v_1(\zeta(t), a) \hat{u}_1(t, a) - k \partial_a \hat{u}_1(t, a)]_{a=0} = \int_0^{A_3} \beta_3(\zeta(t), a) \hat{u}_3(t, a) da, \\ [v_2(\zeta(t), a) \hat{u}_2(t, a) - k \partial_a \hat{u}_2(t, a)]_{a=0} = \int_0^{A_1} \beta_1(\zeta(t), a) \hat{u}_1(t, a) da, \\ [v_3(\zeta(t), a) \hat{u}_3(t, a) - k \partial_a \hat{u}_3(t, a)]_{a=0} = \int_0^{A_2} \sigma \beta_2(\zeta(t), a) \hat{u}_2(t, a) da, \\ [v_4(\zeta(t), a) \hat{u}_4(t, a) - k \partial_a \hat{u}_4(t, a)]_{a=0} = \int_0^{A_2} (1 - \sigma) \beta_2(\zeta(t), a) \hat{u}_2(t, a) da, \\ \hat{u}_i(0, a) = \hat{u}^0(a), a \in (0, A_i) \text{ and } \hat{u}_i(t, A_i) = 0, t \in (0, T), \quad 1 \leq i \leq 4. \end{array} \right. \tag{7}$$

Now, we consider the following auxiliary system:

$$\begin{cases} \partial_t \widehat{u}_1 + \partial_a (v_1 \widehat{u}_1) - k \partial_a^2 \widehat{u}_1 = -(\mu_1 + \beta_1 + \lambda) \widehat{u}_1 + m(a) \widehat{w}_1, \\ \partial_t \widehat{u}_2 + \partial_a (v_2 \widehat{u}_2) - k \partial_a^2 \widehat{u}_2 = -(\mu_2 + \beta_2 + \lambda) \widehat{u}_2 + m(a) \widehat{w}_2, \\ \partial_t \widehat{u}_3 + \partial_a (v_3 \widehat{u}_3) - k \partial_a^2 \widehat{u}_3 = -(\mu_3 + \lambda) \widehat{u}_3 + m(a) \widehat{w}_3, \\ \partial_t \widehat{u}_4 + \partial_a (v_4 \widehat{u}_4) - k \partial_a^2 \widehat{u}_4 = -(\mu_4 + \lambda) \widehat{u}_4, \end{cases} \quad (8)$$

$$\begin{cases} [v_1(\zeta(t), a) \widehat{u}_1(t, a) - k \partial_a \widehat{u}_1(t, a)]_{a=0} = \int_0^{A_3} \beta_3(\zeta(t), a) \phi_3(t, a) da, \\ [v_2(\zeta(t), a) \widehat{u}_2(t, a) - k \partial_a \widehat{u}_2(t, a)]_{a=0} = \int_0^{A_1} \beta_1(\zeta(t), a) \phi_1(t, a) da, \\ [v_3(\zeta(t), a) \widehat{u}_3(t, a) - k \partial_a \widehat{u}_3(t, a)]_{a=0} = \int_0^{A_2} \sigma \beta_2(\zeta(t), a) \phi_2(t, a) da, \\ [v_4(\zeta(t), a) \widehat{u}_4(t, a) - k \partial_a \widehat{u}_4(t, a)]_{a=0} = \int_0^{A_2} (1 - \sigma) \beta_2(\zeta(t), a) \phi_2(t, a) da, \\ \widehat{u}_i(0, a) = \widehat{u}^0(a), \quad a \in (0, A_i) \text{ and } \widehat{u}_i(t, A_i) = 0, t \in (0, T), 1 \leq i \leq 4. \end{cases} \quad (9)$$

For the existence of the solutions to problems similar to equations (8) and (9), we refer, for instance, to [19] and the references therein.

Let $A = \max\{A_1, A_2, A_3, A_4\}$, we can state the main result.

Theorem 1. *Assume that (A1) – (A5) hold. Let $A > 0$ and $T > 0$ be given. Then, for all $(u_1^0, u_2^0, u_3^0) \in (L^2(0, A))^3$, there*

exist a control $w = (w_1, w_2, w_3) \in (L^2((0, T) \times \omega))^3$ such that the associated solution of equations (1) and (2) verify (4).

3. Null Controllability of an Auxiliary System

This section is devoted to the null controllability of the following auxiliary system obtained from equations (1) and (2):

$$\begin{cases} \partial_t u_1(t, a) + \partial_a [v_1(\zeta(t), a) u_1(t, a)] - k \partial_a^2 u_1(t, a) = -[\mu_1(\zeta(t), a) + \beta_1(\zeta(t), a)] u_1(t, a) + m(a) w_1(t, a), \\ \partial_t u_2(t, a) + \partial_a [v_2(\zeta(t), a) u_2(t, a)] - k \partial_a^2 u_2(t, a) = -[\mu_2(\zeta(t), a) + \beta_2(\zeta(t), a)] u_2(t, a) + m(a) w_2(t, a), \\ \partial_t u_3(t, a) + \partial_a [v_3(\zeta(t), a) u_3(t, a)] - k \partial_a^2 u_3(t, a) = -\mu_3(\zeta(t), a) u_3(t, a) + m(a) w_3(t, a), \\ \partial_t u_4(t, a) + \partial_a [v_4(\zeta(t), a) u_4(t, a)] - k \partial_a^2 u_4(t, a) = -\mu_4(\zeta(t), a) u_4(t, a), \end{cases} \quad (10)$$

$$\begin{cases} [v_1(\zeta(t), a) u_1(t, a) - k \partial_a u_1(t, a)]_{a=0} = h_1(t), \\ [v_2(\zeta(t), a) u_2(t, a) - k \partial_a u_2(t, a)]_{a=0} = h_2(t), \\ [v_3(\zeta(t), a) u_3(t, a) - k \partial_a u_3(t, a)]_{a=0} = h_3(t), \\ [v_4(\zeta(t), a) u_4(t, a) - k \partial_a u_4(t, a)]_{a=0} = h_4(t), \\ u_i(0, a) = u_i^0(a), \quad a \in (0, A), 1 \leq i \leq 4, \\ u_i(t, a) = 0, \quad t \in (0, T), 1 \leq i \leq 4. \end{cases} \quad (11)$$

Here, $h_i = \int_0^A \sigma_i \beta_j(\zeta(t), a) \phi_j(t, a) da$, $(i, \sigma_i, j) \in \{(1, 1, 3), (2, 1, 1), (3, \sigma, 2), (4, 1 - \sigma, 2)\}$. We first established a

Carleman-type inequality for the following adjoint system of equations (10) and (11):

$$\begin{cases} -\partial_t q_1 - \partial_a(v_1 q_1) - k\partial_a^2 q_1 + (\mu_1 + \beta_1 + \partial_a v_1)q_1 = 0, \\ -\partial_t q_2 - \partial_a(v_2 q_2) - k\partial_a^2 q_2 + (\mu_2 + \beta_2 + \partial_a v_2)q_2 = 0, \\ -\partial_t q_3 - \partial_a(v_3 q_3) - k\partial_a^2 q_3 + (\mu_3 + \partial_a v_3)q_3 = 0, \\ -\partial_t q_4 - \partial_a(v_4 q_4) - k\partial_a^2 q_4 + (\mu_4 + \partial_a v_4)q_4 = 0, \end{cases} \quad (12)$$

with $(t, a) \in Q$ and the following boundary conditions

$$\begin{cases} q_i(t, A) = \partial_a q_i(t, 0) = 0, \quad t \in (0, T), 1 \leq i \leq 4; \\ q_i(T, a) = \frac{u_i(T, a)}{\varepsilon}, \quad a \in (0, A), 1 \leq i \leq 3; \\ q_4(T, a) = 0, \quad a \in (0, A). \end{cases} \quad (13)$$

The system equations (12) and (13) admit a unique solution.

Let $\omega_0 \subset \subset \omega$ be a nonempty bounded subset of $\Omega = (0, A)$, and $\psi \in C^2(\bar{\Omega})$ satisfying (see Lemma 1.1 in [20])

$$\begin{aligned} \psi(a) > 0; \quad & \text{for any } a \in \Omega, \psi(a) = 0, \\ & \text{for any } a \in \partial\Omega, |\nabla\psi(a)| > 0, \\ & \text{for any } a \in \overline{\Omega} \setminus \omega_0. \end{aligned} \quad (14)$$

Now, we define the following weight functions, which are constructed using the function ψ (see [21] or [20]):

$$\begin{cases} \varphi(t, a) = \frac{e^{\lambda\psi(a)}}{t(T-t)}, \alpha(t, a) = \frac{-e^{2\lambda\|\psi\|_{C(\bar{\Omega})}} + e^{\lambda\psi(a)}}{t(T-t)}, \\ \tilde{\varphi}(t, a) = \frac{e^{-\lambda\psi(a)}}{t(T-t)}, \tilde{\alpha}(t, a) = \frac{-e^{2\lambda\|\psi\|_{C(\bar{\Omega})}} + e^{-\lambda\psi(a)}}{t(T-t)}, \end{cases} \quad (15)$$

where λ is an appropriate positive constant.

Remark 1. Using the definition of $\varphi, \tilde{\varphi}, \alpha$, and $\tilde{\alpha}$, we deduce that there is a positive constant c such that

- (1) $\partial_a \varphi = \lambda \partial_a \psi \varphi, \partial_a \tilde{\varphi} = -\lambda \partial_a \psi \tilde{\varphi}, \partial_a \alpha = \lambda \partial_a \psi \varphi$, and $\partial_a \tilde{\alpha} = -\lambda \partial_a \psi \tilde{\varphi}$
- (2) $|\partial_t \varphi| \leq c\varphi^2, |\partial_t \tilde{\varphi}| \leq c\tilde{\varphi}^2, |\partial_t \alpha| \leq c\varphi^2, |\partial_t \tilde{\alpha}| \leq c\tilde{\varphi}^2, |\partial_t^2 \varphi| \leq c\varphi^3, |\partial_t^2 \tilde{\varphi}| \leq c\tilde{\varphi}^3, |\partial_t^2 \alpha| \leq c\varphi^3, |\partial_t^2 \tilde{\alpha}| \leq c\tilde{\varphi}^3, |\partial_t \partial_a \alpha| \leq c\varphi^2$, and $|\partial_t \partial_a \tilde{\alpha}| \leq c\tilde{\varphi}^2$

Lemma 1. Let $\varphi, \alpha, \tilde{\varphi}$, and $\tilde{\alpha}$ be defined as in equation (15). Then, there exists a number $\hat{\lambda} > 1$ such that, for any $\lambda > \hat{\lambda}$, there exists a number $s_0(\lambda)$ such that, for every $s \geq s_0(\lambda)$, we have the following inequalities:

$$\begin{aligned} \int_Q \left[\frac{1}{s\varphi} \left(|\partial_t q_i|^2 + |\partial_a^2 q_i|^2 \right) + s\varphi |\partial_a q_i|^2 + s^3 \varphi^3 q_i^2 \right] \left(e^{2s\alpha} + e^{2s\tilde{\alpha}} \right) dt dx \leq C \\ \cdot \int_{[0, T] \times \omega} s^3 \varphi^3 q_i^2 \left(e^{2s\alpha} + e^{2s\tilde{\alpha}} \right) dt da, \end{aligned} \quad (16)$$

for all $q_i \in L^2((0, T) \times (0, A))$ solution of (12) and (13), $i = 1, 2, 3$, and

$$\begin{aligned} \int_Q \varphi^3 (q_1^2 + q_2^2 + q_3^2) \left(e^{2s\alpha} + e^{2s\tilde{\alpha}} \right) dt da \leq C \\ \cdot \int_{[0, T] \times \omega} \varphi^3 (q_1^2 + q_2^2 + q_3^2) \left(e^{2s\alpha} + e^{2s\tilde{\alpha}} \right) dt da. \end{aligned} \quad (17)$$

Remark 2. Here, we have to control the boundary conditions which is not the case studied in [20].

Proof of Lemma 1. Let us consider the operator

$$\hat{L}q_i = -\partial_t q_i - k\partial_a^2 q_i, \quad 1 \leq i \leq 3. \quad (18)$$

We set

$$\tilde{g}_i(t, a) = -(\mu_i + \beta_i)q_i + v_i \partial_a q_i, \quad (\beta_3 = \beta_4 = 0), 1 \leq i \leq 3. \quad (19)$$

Let $w_i = e^{s\alpha} q_i$ and $\tilde{w}_i = e^{s\tilde{\alpha}} q_i, (1 \leq i \leq 3)$, where s is a positive constant. Put $\Sigma_0 = [0, T] \times \{A\}, \Sigma_1 = [0, T] \times \{0\}$, and $\Sigma = \Sigma_0 \cup \Sigma_1$.

From (15), we have

$$w_i(T, a) = \tilde{w}_i(T, a) = w_i(0, a) = \tilde{w}_i(0, a) = 0 \text{ on } [0, A]. \quad (20)$$

Let us define the operators P and \tilde{P} as follows: $Pw_i = e^{s\alpha} \hat{L}e^{-s\alpha} w_i$ and $\tilde{P}w_i = e^{s\tilde{\alpha}} \hat{L}e^{-s\tilde{\alpha}} w_i$.

It follows from equations (12), (18), and (19) that

$$\begin{aligned} Pw_i &= e^{s\alpha} \hat{L}e^{-s\alpha} w_i = e^{s\alpha} \tilde{g}_i, \quad \text{on } Q, \\ \tilde{P}w_i &= e^{s\tilde{\alpha}} \hat{L}e^{-s\tilde{\alpha}} w_i = e^{s\tilde{\alpha}} \tilde{g}_i, \quad \text{on } Q. \end{aligned} \quad (21)$$

Explicitly, we have

$$Pw_i = -\partial_t w_i - k\partial_a^2 w_i + s\partial_t \alpha w_i + 2ks\lambda\partial_a \psi \varphi \partial_a w_i + ks\lambda\partial_a^2 \psi \varphi w_i + ks\lambda^2 (\partial_a \psi)^2 \varphi w_i - ks^2 \lambda^2 (\partial_a \psi)^2 \varphi^2 w_i, \quad (22)$$

$$\tilde{P}\tilde{w}_i = -\partial_t \tilde{w}_i - k\partial_a^2 \tilde{w}_i + s\partial_t \tilde{\alpha} \tilde{w}_i - 2ks\lambda\partial_a \psi \tilde{\varphi} \partial_a \tilde{w}_i - ks\lambda\partial_a^2 \psi \tilde{\varphi} \tilde{w}_i + ks\lambda^2 (\partial_a \psi)^2 \tilde{\varphi} \tilde{w}_i - ks^2 \lambda^2 (\partial_a \psi)^2 \tilde{\varphi}^2 \tilde{w}_i. \quad (23)$$

We also consider the operators L_1 , L_2 , \tilde{L}_1 , and \tilde{L}_2 defined as follows:

$$\begin{cases} L_1 w_i = -k\partial_a^2 w_i - ks^2 \lambda^2 (\partial_a \psi)^2 \varphi^2 w_i + s\partial_t \alpha w_i, \\ L_2 w_i = -\partial_t w_i + 2ks\lambda\partial_a \psi \varphi \partial_a w_i + 2ks\lambda^2 (\partial_a \psi)^2 \varphi w_i, \\ \tilde{L}_1 \tilde{w}_i = -k\partial_a^2 \tilde{w}_i - ks^2 \lambda^2 (\partial_a \psi)^2 \tilde{\varphi}^2 \tilde{w}_i + s\partial_t \tilde{\alpha} \tilde{w}_i, \\ \tilde{L}_2 \tilde{w}_i = -\partial_t \tilde{w}_i - 2ks\lambda\partial_a \psi \tilde{\varphi} \partial_a \tilde{w}_i + 2ks\lambda^2 (\partial_a \psi)^2 \tilde{\varphi} \tilde{w}_i. \end{cases} \quad (24)$$

Using equations (19), (22), and (24), we obtain

$$\begin{cases} L_1 w_i + L_2 w_i = f_s, & \text{in } Q, \\ \tilde{L}_1 \tilde{w}_i + \tilde{L}_2 \tilde{w}_i = \tilde{f}_s, & \text{in } Q, \end{cases} \quad (25)$$

where

$$\begin{cases} f_s = \tilde{g}_i e^{s\alpha} - ks\lambda\varphi\partial_a^2 \psi w_i + ks\lambda^2 \varphi (\partial_a \psi)^2 w_i, \\ \tilde{f}_s = \tilde{g}_i e^{s\tilde{\alpha}} + ks\lambda\tilde{\varphi}\partial_a^2 \psi \tilde{w}_i + ks\lambda^2 \tilde{\varphi} (\partial_a \psi)^2 \tilde{w}_i. \end{cases} \quad (26)$$

Taking the L_2 -norm of (25), we obtain

$$\begin{cases} \|f_s\|_{L^2(Q)}^2 = \|L_1 w_i\|_{L^2(Q)}^2 + \|L_2 w_i\|_{L^2(Q)}^2 + 2(L_1 w_i, L_2 w_i)_{L^2(Q)}, \\ \|\tilde{f}_s\|_{L^2(Q)}^2 = \|\tilde{L}_1 \tilde{w}_i\|_{L^2(Q)}^2 + \|\tilde{L}_2 \tilde{w}_i\|_{L^2(Q)}^2 + 2(\tilde{L}_1 \tilde{w}_i, \tilde{L}_2 \tilde{w}_i)_{L^2(Q)}. \end{cases} \quad (27)$$

By using (24), we have the following equalities:

$$\begin{aligned} 2(L_1 w_i, L_2 w_i)_{L^2(Q)} &= 2k^2 s^3 \lambda^4 \int_Q (w_i)^2 \varphi^3 (\partial_a \psi)^4 dt da + 6k^2 s \lambda^2 \int_Q (\partial_a w_i)^2 \varphi (\partial_a \psi)^2 dt da \\ &\quad - 2ks\lambda \int_0^T [w_i \partial_t w_i \varphi (\partial_a \psi)]_{a=0} dt + 4k^2 s^2 \lambda^3 \int_0^T [(w_i)^2 \varphi^2 (\partial_a \psi)^3]_{a=0} dt \\ &\quad + 2k^2 s^3 \lambda^3 \int_0^T [(w_i)^2 \varphi^3 (\partial_a \psi)^3]_{a=0} dt - 2k^2 s \lambda \int_0^T [(\partial_a w_i)^2 \varphi (\partial_a \psi)]_0^A dt \\ &\quad - 2ks^2 \lambda \int_0^T [(w_i)^2 \varphi (\partial_a \psi) \partial_t \alpha]_{a=0} dt + X_1, \end{aligned} \quad (28)$$

$$\begin{aligned} 2(\tilde{L}_1 \tilde{w}_i, \tilde{L}_2 \tilde{w}_i)_{L^2(Q)} &= 2k^2 s^3 \lambda^4 \int_Q (\tilde{w}_i)^2 \tilde{\varphi}^3 (\partial_a \psi)^4 dt da + 6k^2 s \lambda^2 \int_Q (\partial_a \tilde{w}_i)^2 \tilde{\varphi} (\partial_a \psi)^2 dt da \\ &\quad + 2ks\lambda \int_0^T [\tilde{w}_i \partial_t \tilde{w}_i \tilde{\varphi} (\partial_a \psi)]_{a=0} dt - 4k^2 s^2 \lambda^3 \int_0^T [(\tilde{w}_i)^2 \tilde{\varphi}^2 (\partial_a \psi)^3]_{a=0} dt \\ &\quad - 2k^2 s^3 \lambda^3 \int_0^T [(\tilde{w}_i)^2 \tilde{\varphi}^3 (\partial_a \psi)^3]_{a=0} dt + 2k^2 s \lambda \int_0^T [(\partial_a \tilde{w}_i)^2 \tilde{\varphi} (\partial_a \psi)]_0^A dt \\ &\quad + 2ks^2 \lambda \int_0^T [(\tilde{w}_i)^2 \tilde{\varphi} (\partial_a \psi) \partial_t \tilde{\alpha}]_{a=0} dt + X_2, \end{aligned} \quad (29)$$

with

$$|X_1| \leq c_1 \int_Q [(s^3 \lambda^3 \varphi^3 + s^2 \lambda^4 \varphi^3)(w_i)^2 + (s\lambda\varphi + 1)|\partial_a w_i|^2] dt da, \quad \forall s \geq 1, \lambda \geq 1, \quad (30)$$

$$|X_2| \leq c_2 \int_Q [(s^3 \lambda^3 \tilde{\varphi}^3 + s^2 \lambda^4 \tilde{\varphi}^3)(\tilde{w}_i)^2 + (s\lambda\tilde{\varphi} + 1)|\partial_a \tilde{w}_i|^2] dt da, \quad \forall s \geq 1, \lambda \geq 1. \quad (31)$$

Considering equations (27)–(29) and using the fact that $\psi(a) = 0$ on $\partial\Omega$, it follows that

$$\begin{aligned} \|f_s\|_{L^2(Q)}^2 + \|\tilde{f}_s\|_{L^2(Q)}^2 &= \|L_1 w_i\|_{L^2(Q)}^2 + \|L_2 w_i\|_{L^2(Q)}^2 + \|\tilde{L}_1 \tilde{w}_i\|_{L^2(Q)}^2 + \|\tilde{L}_2 \tilde{w}_i\|_{L^2(Q)}^2 \\ &\quad + 2X_1 + 2X_2 + 2k^2 s^3 \lambda^4 \int_Q [(w_i)^2 \varphi^3 (\partial_a \psi)^4 + (\tilde{w}_i)^2 \tilde{\varphi}^3 (\partial_a \psi)^4] dt da \\ &\quad + 6k^2 s \lambda^2 \int_Q [(\partial_a w_i)^2 \varphi (\partial_a \psi)^2 + (\partial_a \tilde{w}_i)^2 \tilde{\varphi} (\partial_a \psi)^2] dt da, \end{aligned} \tag{32}$$

and we deduce from (32)

$$\begin{aligned} \|f_s\|_{L^2(Q)}^2 + \|\tilde{f}_s\|_{L^2(Q)}^2 &\geq \|L_1 w_i\|_{L^2(Q)}^2 + \|L_2 w_i\|_{L^2(Q)}^2 + \|\tilde{L}_1 \tilde{w}_i\|_{L^2(Q)}^2 + \|\tilde{L}_2 \tilde{w}_i\|_{L^2(Q)}^2 \\ &\quad + X_1 + X_2 + 2k^2 s^3 \lambda^4 \int_Q [(w_i)^2 \varphi^3 (\partial_a \psi)^4 + (\tilde{w}_i)^2 \tilde{\varphi}^3 (\partial_a \psi)^4] dt da \\ &\quad + 2k^2 s \lambda^2 \int_Q [(\partial_a w_i)^2 \varphi (\partial_a \psi)^2 + (\partial_a \tilde{w}_i)^2 \tilde{\varphi} (\partial_a \psi)^2] dt da. \end{aligned} \tag{33}$$

Note that there exists $\delta > 0$ such that

$$|\partial_a \psi(a)| > \delta > 0, \quad \forall a \in \overline{\Omega \setminus \omega_0}. \tag{34}$$

After these calculations, we can now use the same arguments developed in [21] or [20] to get (16).

By summing with respect to i , one deduces easily inequality (17). We can now establish the observability inequality of the system equations (12) and (13) and then the null controllability result of the system equations (10) and (11). We start by proving our observability inequality by means of Carleman estimate. \square

Lemma 2. *We have the following inequality:*

$$\begin{aligned} \int_0^A \sum_{i=1}^3 |q_i(0, a)|^2 da + \int_0^T \sum_{i=1}^3 |q_i(0, a)|^2 dt \\ \leq K \int_{[0, T] \times \omega} \sum_{i=1}^3 |q_i|^2 dt da, \end{aligned} \tag{35}$$

for all solution, $q = (q_1, q_2, q_3) \in (L^2((0, T) \times (0, A)))^3$ of equations (12) and (13).

Let λ be a positive constant and $\hat{q}_i(t, a) = e^{\lambda t} q_i(t, a)$, $1 \leq i \leq 4$. Then, equations (12) and (13) become

$$\begin{cases} -\partial_t \hat{q}_1 - \partial_a (v_1 \hat{q}_1) - k \partial_a^2 \hat{q}_1 + (\lambda + \mu_1 + \beta_1 + \partial_a v_1) \hat{q}_1 = 0, \\ -\partial_t \hat{q}_2 - \partial_a (v_2 \hat{q}_2) - k \partial_a^2 \hat{q}_2 + (\lambda + \mu_2 + \beta_2 + \partial_a v_2) \hat{q}_2 = 0, \\ -\partial_t \hat{q}_3 - \partial_a (v_3 \hat{q}_3) - k \partial_a^2 \hat{q}_3 + (\lambda + \mu_3 + \partial_a v_3) \hat{q}_3 = 0, \\ -\partial_t \hat{q}_4 - \partial_a (v_4 \hat{q}_4) - k \partial_a^2 \hat{q}_4 + (\lambda + \mu_4 + \partial_a v_4) \hat{q}_4 = 0, \end{cases} \tag{36}$$

$$\begin{cases} \hat{q}_i(t, A) = \partial_a \hat{q}_i(t, 0) = 0, & t \in (0, T), 1 \leq i \leq 4; \\ \hat{q}_i(T, a) = \frac{e^{\lambda T} u_i(T, a)}{\varepsilon}, & a \in (0, A) \quad 1 \leq i \leq 3; \\ \hat{q}_4(T, a) = 0, & a \in (0, A). \end{cases} \tag{37}$$

For the proof of Lemma 2, we need an auxiliary result. This result is given in the following proposition.

Proposition 1. *Let $\hat{q} = (\hat{q}_1, \hat{q}_2, \hat{q}_3) \in (L^2((0, T) \times (0, A)))^3$ be solution of (36) and (37). Then,*

$$\begin{aligned} \int_0^A |\hat{q}_1(0, a)|^2 + |\hat{q}_2(0, a)|^2 + |\hat{q}_3(0, a)|^2 da \\ \leq K' \int_{[0, T] \times \omega} |\hat{q}_1|^2 + |\hat{q}_2|^2 + |\hat{q}_3|^2 dt da. \end{aligned} \tag{38}$$

Proof of Proposition 1. Remark that

$$\varphi^3 e^{2s\tilde{\alpha}} \geq \frac{e^{-2C(1+(1/T)^2)}}{T^6}, \quad t \in [(T/4), (3T/4)], \tag{39}$$

$$\varphi^3 e^{2s\alpha} \leq B_1 e^{3C}, \quad t \in [0, T]. \tag{40}$$

Using (39), (40), and (16), it follows that

$$\int_{\Omega \times (T/4), (3T/4)} (\hat{q}_i)^2 dt da \leq K_1 \int_{[0, T] \times \omega} (\hat{q}_i)^2 dt da. \tag{41}$$

Let $\theta \in C^2([0, T])$ be a cut-off function such that

$$\begin{cases} \theta(t) = 1 & t \in [0, (T/4)], \\ 0 \leq \theta(t) \leq 1 & t \in [(T/4), (3T/4)], \\ \theta(t) = 0 & t > (3T/4). \end{cases} \quad (42)$$

Multiplying the equation i of (36) by $\theta \hat{q}_i$ and integrating by parts on Q , we obtain

$$\begin{aligned} & \int_0^A (\hat{q}_i(0, a))^2 da + (2\lambda - C_i) \int_Q (\hat{q}_i)^2 \theta dt da \\ & \leq \int_Q -\partial_t \theta (\hat{q}_i)^2 dt da. \end{aligned} \quad (43)$$

From the definition of θ , it follows that

$$\int_0^A (\hat{q}_i(0, a))^2 da \leq K_2 \int_{\Omega \times [(T/4), (3T/4)]} (\hat{q}_i)^2 dt da. \quad (44)$$

Next, (41) and (44) give

$$\int_0^A (\hat{q}_i(0, a))^2 da \leq K' \int_{[0, T] \times \omega} (\hat{q}_i)^2 dt da. \quad (45)$$

Finally, (45) leads to the desired result by summing with respect to i , $1 \leq i \leq 3$. \square

Proof of Lemma 2.

$$\text{Minimize } \left\{ \frac{1}{2} \int_Q (|w_1(t, a)|^2 + |w_2(t, a)|^2 + |w_3(t, a)|^2) dt da + \frac{1}{2\varepsilon} \int_0^A |u_1(T, a)|^2 + |u_2(T, a)|^2 + |u_3(T, a)|^2 da \right\}, \quad (48)$$

where $Q = (0, T) \times (0, A)$, $w_i \in L^2(Q)$, and u_i ($1 \leq i \leq 3$) are solutions of (10) and (11).

$$J_\varepsilon(w) = \frac{1}{2} \int_Q (|w_1(t, a)|^2 + |w_2(t, a)|^2 + |w_3(t, a)|^2) dt da + \frac{1}{2\varepsilon} \int_0^A |u_1(T, a)|^2 + |u_2(T, a)|^2 + |u_3(T, a)|^2 da, \quad (49)$$

with $w = (w_1, w_2, w_3)$.

The functional J_ε is continuous, convex, and coercive. Hence, it admits a unique minimizer $w_\varepsilon = (w_{1,\varepsilon}, w_{2,\varepsilon}, w_{3,\varepsilon})$, and we have after easy computations

$$w_{i,\varepsilon}(t, a) = m(a)q_{i,\varepsilon}(t, a), \quad a.e. (t, a) \in Q, 1 \leq i \leq 3. \quad (50)$$

$$\int_Q q_{i,\varepsilon} m(a) w_{i,\varepsilon} dt da + \frac{1}{\varepsilon} \int_0^A u_{i,\varepsilon}^2(T, a) da + \int_0^A q_{i,\varepsilon}(0, a) u_i^0(a) da + \int_0^T q_{i,\varepsilon}(t, 0) h_i(t) dt = 0. \quad (51)$$

Note that $q_i(t, 0) = -\int_0^A (\partial_a q_i) da$. Using Cauchy-Schwarz and Young inequalities and integrating on $[0, T]$, we obtain

$$\int_0^T |q_i(t, 0)|^2 dt \leq \frac{A}{\rho(T/2)} \int_0^T \int_0^A \rho (\partial_a q_i)^2 da dt. \quad (46)$$

By using (16), (40), and (46), it follows that

$$\int_0^T |q_i(t, 0)|^2 dt \leq K_s \int_{[0, T] \times \omega} q_i^2 dt da, \quad (47)$$

with $K_s = s^3 B_1 e^{3C}$. Now, as $\hat{q}_i(0, a) = q_i(0, a)$, $\hat{q}_i^2 \leq e^{2\lambda T} q_i^2$, using (47) and Proposition 1, we get (35).

With our observability inequality, we are ready to prove the null controllability result. \square

Theorem 2. *Assume that (A1) – (A5) hold. Let $A > 0$ and $T > 0$ be given.*

Then, for all $(u_1^0, u_2^0, u_3^0) \in (L^2(0, A))^3$, there exists a control $w = (w_1, w_2, w_3) \in (L^2((0, T) \times \omega))^3$ such that the associated solution of (10) and (11) verifies (4).

Proof of Theorem 2.

We consider systems (10) and (11) and the optimal control problem:

Let

Note that $(q_{1,\varepsilon}, q_{2,\varepsilon}, q_{3,\varepsilon})$ is the solution of (12) and (13) and $u_\varepsilon = (u_{1,\varepsilon}, u_{2,\varepsilon}, u_{3,\varepsilon})$ is the solution of (10) and (11) associated to w_ε .

Multiplying equation number i of (12) and (13) by $u_{i,\varepsilon}$ and integrating on Q , it follows that

From (51) and (50), we obtain

$$\int_{[0,T] \times \omega} w_{i,\varepsilon}^2 dt da + \frac{1}{\varepsilon} \int_0^A u_{i,\varepsilon}^2(T, a) da = - \int_0^A q_{i,\varepsilon}(0, a) u_i^0(a) da - \int_0^T q_{i,\varepsilon}(t, 0) h_i(t) dt. \tag{52}$$

Young inequality in (52) leads to

$$\int_{[0,T] \times \omega} w_{i,\varepsilon}^2 dt da + \frac{1}{\varepsilon} \int_0^A u_{i,\varepsilon}^2(T, a) da \leq \frac{1}{2K} \left(\int_0^A q_{i,\varepsilon}^2(0, a) da + \int_0^T q_{i,\varepsilon}^2(t, 0) dt \right) + 2K \left(\int_0^A (u_i^0)^2(a) da + \int_0^T h_i^2(t) dt \right). \tag{53}$$

Thus,

$$\int_{[0,T] \times \omega} \sum_{i=1}^3 w_{i,\varepsilon}^2 dt da + \frac{1}{\varepsilon} \int_0^A \sum_{i=1}^3 u_{i,\varepsilon}^2(T, a) da \leq \frac{1}{2K} \left(\int_0^A \sum_{i=1}^3 q_{i,\varepsilon}^2(0, a) da + \int_0^T \sum_{i=1}^3 q_{i,\varepsilon}^2(t, 0) dt \right) + 2K \left(\int_0^A \sum_{i=1}^3 (u_i^0)^2(a) da + \int_0^T \sum_{i=1}^3 h_i^2(t) dt \right). \tag{54}$$

Therefore, inequalities (35) and (54) imply

$$\frac{1}{2} \int_{[0,T] \times \omega} \sum_{i=1}^3 w_{i,\varepsilon}^2 dt da + \frac{1}{\varepsilon} \int_0^A \sum_{i=1}^3 u_{i,\varepsilon}^2(T, a) da \leq 2K \left(\int_0^A \sum_{i=1}^3 (u_i^0)^2(a) da + \int_0^T \sum_{i=1}^3 h_i^2(t) dt \right). \tag{55}$$

From (55), we deduce

$$\begin{cases} \|w_{i,\varepsilon}\|_{L^2([0,T] \times \omega)}^2 \leq 4K \left(\int_0^A \sum_{i=1}^3 (u_i^0)^2(a) da + \int_0^T \sum_{i=1}^3 h_i^2(t) dt \right), & 1 \leq i \leq 3, \\ \int_0^A u_{i,\varepsilon}^2(T, a) da \leq 2\varepsilon K \left(\int_0^A \sum_{i=1}^3 (u_i^0)^2(a) da + \int_0^T \sum_{i=1}^3 h_i^2(t) dt \right), & 1 \leq i \leq 3. \end{cases} \tag{56}$$

Then, one can extract subsequences also denoted by $w_{i,\varepsilon}, u_{i,\varepsilon}$ ($1 \leq i \leq 3$) such that $w_{i,\varepsilon} \rightharpoonup w_i$ weakly in $L^2((0, T) \times \omega)$ and $u_{i,\varepsilon} \rightharpoonup u_i$ weakly in $L^2((0, T) \times (0, A))$. Moreover, $u = (u_1, u_2, u_3)$ is the unique solution of (10) and (11) associated to $w = (w_1, w_2, w_3)$ and verifies (4). \square

4. Proof of the Main Result

Now, we consider a real number $\lambda_0 > 0$ and we set $\tilde{u}_i = e^{-\lambda_0 t} u_i$ ($1 \leq i \leq 4$). Then, $\tilde{u} = (\tilde{u}_1, \tilde{u}_2, \tilde{u}_3, \tilde{u}_4)$ is solution of the following system:

$$\begin{cases} \partial_t \tilde{u}_1 + \partial_a (v_1 \tilde{u}_1) - k \partial_a^2 \tilde{u}_1 + (\lambda_0 + \mu_1 + \beta_1) \tilde{u}_1 = m(a) \tilde{w}_1, \\ \partial_t \tilde{u}_2 + \partial_a (v_2 \tilde{u}_2) - k \partial_a^2 \tilde{u}_2 + (\lambda_0 + \mu_2 + \beta_2) \tilde{u}_2 = m(a) \tilde{w}_2, \\ \partial_t \tilde{u}_3 + \partial_a (v_3 \tilde{u}_3) - k \partial_a^2 \tilde{u}_3 + (\lambda_0 + \mu_3) \tilde{u}_3 = m(a) \tilde{w}_3, \\ \partial_t \tilde{u}_4 + \partial_a (v_4 \tilde{u}_4) - k \partial_a^2 \tilde{u}_4 + (\lambda_0 + \mu_4) \tilde{u}_4 = 0, \end{cases} \tag{57}$$

with $(t, a) \in \Omega = (0, T) \times (0, A)$, $\tilde{w}_i = e^{-\lambda_0 t} w_i$, and

$$\left\{ \begin{aligned} (v_1 \tilde{u}_1 - k \partial_a \tilde{u}_1)_{a=0} &= \int_0^A \beta_3 \tilde{u}_3(t, a) da, \\ (v_2 \tilde{u}_2 - k \partial_a \tilde{u}_2)_{a=0} &= \int_0^A \beta_1 \tilde{u}_1(t, a) da, \\ (v_3 \tilde{u}_3 - k \partial_a \tilde{u}_3)_{a=0} &= \int_0^A \beta_2 \tilde{u}_2(t, a) da, \\ (v_4 \tilde{u}_4 - k \partial_a \tilde{u}_4)_{a=0} &= \int_0^A (1 - \sigma) \beta_2 \tilde{u}_2(t, a) da, \\ \tilde{u}_i(0, a) &= \tilde{u}_i^0(a), \quad a \in \Omega_A = (0, A), 1 \leq i \leq 4, \\ \tilde{u}_i(t, A) &= 0, \quad t \in \Omega_T = (0, T), 1 \leq i \leq 4. \end{aligned} \right. \quad (58)$$

Let us consider the following system:

$$\left\{ \begin{aligned} \partial_t \tilde{u}_1 + \partial_a (v_1 \tilde{u}_1) - k \partial_a^2 \tilde{u}_1 + (\lambda_0 + \mu_1 + \beta_1) \tilde{u}_1 &= m(a) \tilde{w}_1, \\ \partial_t \tilde{u}_2 + \partial_a (v_2 \tilde{u}_2) - k \partial_a^2 \tilde{u}_2 + (\lambda_0 + \mu_2 + \beta_2) \tilde{u}_2 &= m(a) \tilde{w}_2, \\ \partial_t \tilde{u}_3 + \partial_a (v_3 \tilde{u}_3) - k \partial_a^2 \tilde{u}_3 + (\lambda_0 + \mu_3) \tilde{u}_3 &= m(a) \tilde{w}_3, \\ \partial_t \tilde{u}_4 + \partial_a (v_4 \tilde{u}_4) - k \partial_a^2 \tilde{u}_4 + (\lambda_0 + \mu_4) \tilde{u}_4 &= 0, \end{aligned} \right. \quad (59)$$

and

$$\left\{ \begin{aligned} (v_1 \tilde{u}_1 - k \partial_a \tilde{u}_1)_{a=0} &= h_1(t), \\ (v_2 \tilde{u}_2 - k \partial_a \tilde{u}_2)_{a=0} &= h_2(t), \\ (v_3 \tilde{u}_3 - k \partial_a \tilde{u}_3)_{a=0} &= h_3(t), \\ (v_4 \tilde{u}_4 - k \partial_a \tilde{u}_4)_{a=0} &= h_4(t), \\ \tilde{u}_i(0, a) &= \tilde{u}_i^0(a), \quad a \in \Omega_A = (0, A), 1 \leq i \leq 4, \\ \tilde{u}_i(t, A) &= 0, \quad t \in \Omega_T = (0, T), 1 \leq i \leq 4, \end{aligned} \right. \quad (60)$$

with $h_1, h_2, h_3, h_4 \in L^2(\Omega_T)$ is defined as in (58).

Note that, from Section 3, systems (59) and (60) are null controllable. Now, we will prove the null controllability of (57) and (58).

Let $\mathcal{M} = \prod_{i=1}^4 L^2(\Omega_T)$, and we define the following set:

$$\begin{aligned} \mathcal{A}_\gamma(R_1, R_2, R_3, R_4) &= \left\{ \tilde{w} = (\tilde{w}_1, \tilde{w}_2, \tilde{w}_3) \in \prod_{i=1}^3 L^2(q), \tilde{u} \text{ is solution of (59) - (60) and } (\tilde{u}_1, \tilde{u}_2, \tilde{u}_3) \text{ verifies (4); } h_i \right. \\ &= R_i \text{ and } \tilde{w} \text{ satisfies (56)} \left. \right\}. \end{aligned} \quad (61)$$

We introduce a multivalued function as follows:

$$\Lambda_\gamma: (R_1, R_2, R_3, R_4) \in \mathcal{M} \mapsto \Lambda_\gamma(R_1, R_2, R_3, R_4) \in 2^{\mathcal{M}}, \quad (62)$$

such that

$$\begin{aligned} \Lambda_\gamma(R_1, R_2, R_3, R_4) &= \left\{ \left(\int_0^A \beta_3 \tilde{u}_{3,\tilde{w}}(t, a) da, \int_0^A \beta_1 \tilde{u}_{1,\tilde{w}}(t, a) da, \int_0^A \sigma \beta_2 \tilde{u}_{2,\tilde{w}}(t, a) da, \int_0^A (1 - \sigma) \beta_2 \tilde{u}_{2,\tilde{w}}(t, a) da \right) : \right. \\ &\left. (\tilde{u}_{1,\tilde{w}}, \tilde{u}_{2,\tilde{w}}, \tilde{u}_{3,\tilde{w}}) \text{ is solution of (59) - (60) and } \tilde{w} \in \mathcal{A}_\gamma(R_1, R_2, R_3, R_4) \right\}. \end{aligned} \quad (63)$$

The goal is to prove that the multivalued function Λ_γ admits a fixed point. To this end, we will use the generalization of Leray–Schauder fixed point theorem. Consider

$$\begin{aligned} \mathcal{N}_\gamma &= \{(R_1, R_2, R_3, R_4) \in \mathcal{M}: \exists \rho \in (0, 1), (R_1, R_2, R_3, R_4) \\ &\in \rho \Lambda_\gamma(R_1, R_2, R_3, R_4)\}. \end{aligned} \quad (64)$$

The fixed point existence of the multivalued function Λ_γ is an immediate consequence of the following proposition.

Proposition 2. (i) \mathcal{N}_γ is bounded on \mathcal{M}

(ii) For all $(R_1, R_2, R_3, R_4) \in \mathcal{M}$, $\Lambda_\gamma(R_1, R_2, R_3, R_4)$ is a nonempty closed and convex subset of \mathcal{M}

(iii) $\Lambda_\gamma: \mathcal{M} \rightarrow 2^{\mathcal{M}}$ is compact

(iv) The multivalued mapping Λ_γ is upper semi-continuous on \mathcal{M}

Proof of Proposition 2.

Let us set $(\tilde{u}_{1,\tilde{w}}, \tilde{u}_{2,\tilde{w}}, \tilde{u}_{3,\tilde{w}}, \tilde{u}_{4,\tilde{w}}) = (\tilde{u}_1, \tilde{u}_2, \tilde{u}_3, \tilde{u}_4)$. (i) Let $(R_1, R_2, R_3, R_4) \in \mathcal{N}_\gamma$. There is $\rho \in (0, 1)$ such that $(1/\rho)(R_1, R_2, R_3, R_4) \in \Lambda_\gamma(R_1, R_2, R_3, R_4)$. Hence, there is $(\tilde{u}_1, \tilde{u}_2, \tilde{u}_3) \in \prod_{i=1}^3 L^2(\Omega)$ associated to $\tilde{w} \in \prod_{i=1}^3 L^2(q)$ such that

$$\begin{aligned}
 R_1 &= \int_0^A \beta_3 \tilde{u}_3 da, & \eta_n^1 &= \int_0^A \beta_3 \tilde{u}_{3,n} da, \\
 R_2 &= \int_0^A \beta_1 \tilde{u}_1 da, & \eta_n^2 &= \int_0^A \beta_1 \tilde{u}_{1,n} da, \\
 R_3 &= \int_0^A \beta_2 \tilde{u}_2 da, & \eta_n^3 &= \int_0^A \sigma \beta_2 \tilde{u}_{2,n} da, \\
 R_4 &= \int_0^A \beta_2 \tilde{u}_2 da, & \eta_n^4 &= \int_0^A (1 - \sigma) \beta_2 \tilde{u}_{2,n} da.
 \end{aligned} \tag{65}$$

with $(\tilde{u}_1, \tilde{u}_2, \tilde{u}_3)$ solution of (57) and (58) associated to \tilde{w} satisfying (56). Then, we have

$$\sum_{i=1}^4 \|R_i\|_{L^2(\Omega_T)}^2 \leq \sum_{i=1}^4 \xi_i \|\tilde{u}_i\|_{L^2(\Omega)}^2 \quad \xi_i = A \|\beta_i\|_{L^\infty([0,A])} \text{ and } \xi_4 = 0. \tag{66}$$

Multiplying the equation number i of (57) and (58) by \tilde{u}_i and integrating by parts on Ω , we obtain after summing with respect to i ($1 \leq i \leq 4$):

$$\begin{aligned}
 \sum_{i=1}^4 (2\lambda_0 - C_i - 1) \|\tilde{u}_i\|_{L^2(\Omega)}^2 &\leq \sum_{i=1}^4 \|\tilde{u}_i^0\|_{L^2([0,A])}^2 + \sum_{i=1}^4 \|R_i\|_{L^2(\Omega_T)}^2 \\
 &+ \sum_{i=1}^4 \|\tilde{w}_i\|_{L^2(q)}^2, \quad (w_4 \equiv 0).
 \end{aligned} \tag{67}$$

In virtue of (66), (67), and (56), it follows that

$$\begin{aligned}
 \sum_{i=1}^4 (2\lambda_0 - C_i - 1) \|\tilde{u}_i\|_{L^2(\Omega)}^2 &\leq (4K + 1) \sum_{i=1}^4 \|\tilde{u}_i^0\|_{L^2([0,A])}^2 \\
 &+ (4K + 1) \sum_{i=1}^4 \xi_i \|\tilde{u}_i\|_{L^2(\Omega)}^2.
 \end{aligned} \tag{68}$$

Inequality (68) leads to

$$\sum_{i=1}^4 (2\lambda_0 - C_i - 1 - (4K + 1)\xi_i) \|\tilde{u}_i\|_{L^2(\Omega)}^2 \leq (4K + 1) \sum_{i=1}^4 \|\tilde{u}_i^0\|_{L^2([0,A])}^2. \tag{69}$$

Considering again (66) and (69), we get the desired result. (ii) $(R_1, R_2, R_3, R_4) \in \mathcal{M}$, $\Lambda_\gamma(R_1, R_2, R_3, R_4)$ is non-empty. In fact, systems (57) and (58) are equivalent to systems (10) and (11) which admits a solution. Also, \tilde{w} satisfies (56).

Since the function $(R_1, R_2, R_3, R_4) \mapsto (\tilde{u}_1, \tilde{u}_2, \tilde{u}_3, \tilde{u}_4)$ is affine, it follows that $\Lambda_\gamma(R_1, R_2, R_3, R_4)$ is convex.

Let $(\eta_n^1, \eta_n^2, \eta_n^3, \eta_n^4) \in \Lambda_\gamma(R_1, R_2, R_3, R_4)$ such that $(\eta_n^1, \eta_n^2, \eta_n^3, \eta_n^4) \rightarrow (\eta^1, \eta^2, \eta^3, \eta^4)$ in \mathcal{M} . Thus, for all n , there exists \tilde{w}_n which satisfies (56), such that

Here, $(\tilde{u}_{1,n}, \tilde{u}_{2,n}, \tilde{u}_{3,n})$ is solution of (59) and (60), with (R_1, R_2, R_3, R_4) instead of (h_1, h_2, h_3, h_4) and $(\tilde{u}_{1,n}, \tilde{u}_{2,n}, \tilde{u}_{3,n})$ which verify (69). From (66), (69), and (56), one can extract sequences still denoted by $((\tilde{u}_{1,n}, \tilde{u}_{2,n}, \tilde{u}_{3,n}))_n$ and $(\tilde{w}_n)_n$ that converge weakly towards $(\tilde{u}_1, \tilde{u}_2, \tilde{u}_3)$ and \tilde{w} in $(L^2(\Omega))^3$ and in $(L^2(q))^3$, respectively.

So,

$$\begin{aligned}
 \eta^1 &= \int_0^A \beta_3 \tilde{u}_3 da, \\
 \eta^2 &= \int_0^A \beta_1 \tilde{u}_1 da, \\
 \eta^3 &= \int_0^A \sigma \beta_2 \tilde{u}_2 da, \\
 \eta^4 &= \int_0^A (1 - \sigma) \beta_2 \tilde{u}_2 da.
 \end{aligned} \tag{71}$$

Moreover, $(\tilde{u}_1, \tilde{u}_2, \tilde{u}_3)$ is solution of (59) and (60), with $(R_1, R_2, R_3, R_4) = (h_1, h_2, h_3, h_4)$. Also, \tilde{w} verifies (56) and $(\tilde{u}_1, \tilde{u}_2, \tilde{u}_3)$ verifies (4). Then, $(\eta^1, \eta^2, \eta^3, \eta^4) \in \Lambda_\gamma(R_1, R_2, R_3, R_4)$. (iii) Let $(R_1, R_2, R_3, R_4) \in K$ a bounded subset of \mathcal{M} . Then, there exist $r_i > 0$ ($1 \leq i \leq 4$) such that

$$\|R_i\|_{L^2(Q_T)} \leq r_i, \quad 1 \leq i \leq 4. \tag{72}$$

Consider $((\eta_n^1, \eta_n^2, \eta_n^3, \eta_n^4))_n \in \Lambda_\gamma(R_1, R_2, R_3, R_4)$; for all $n \in \mathbb{N}$, there is $(\tilde{u}_{1,n}, \tilde{u}_{2,n}, \tilde{u}_{3,n}) \in (L^2(\Omega))^3$ and a control function $\tilde{w}_n \in (L^2(q))^3$ such that

$$\begin{aligned}
 \eta_n^1 &= \int_0^A \beta_3 \tilde{u}_{3,n} da, \\
 \eta_n^2 &= \int_0^A \beta_1 \tilde{u}_{1,n} da, \\
 \eta_n^3 &= \int_0^A \sigma \beta_2 \tilde{u}_{2,n} da, \\
 \eta_n^4 &= \int_0^A (1 - \sigma) \beta_2 \tilde{u}_{2,n} da,
 \end{aligned} \tag{73}$$

where $(\tilde{u}_{1,n}, \tilde{u}_{2,n}, \tilde{u}_{3,n})$ is solution of (59) and (60) with $(R_{1,n}, R_{2,n}, R_{3,n}, R_{4,n})$ instead of (h_1, h_2, h_3, h_4) and \tilde{w}_n verifies (56). Using (56), it follows that

$$\int_q \sum_{i=1}^3 \tilde{w}_{i,n}^2(t, a) dt da \leq 4K \left(\int_0^A \sum_{i=1}^4 (\tilde{u}_i^0)^2 da + \sum_{i=1}^4 r_i^2 \right). \tag{74}$$

Therefore, (\tilde{w}_n) is bounded, and we can extract a sequence still denoted by (\tilde{w}_n) that weakly converges towards \tilde{w} in $L^2(Q)$.

Now, multiplying the equation number i of (59) and (60) by $\tilde{u}_{i,n}$, integrating by parts on Q and using (72), we get after summing

$$\int_Q \sum_{i=1}^4 k((\tilde{u}_{i,n})_a)^2(t, a) + \tilde{u}_{i,n}^2(t, a) dt da \leq (4K + 1) \left(\int_0^A \sum_{i=1}^4 (\tilde{u}_i^0)^2 da + \sum_{i=1}^4 r_i^2 \right). \tag{75}$$

Consequently, the sequence $(\tilde{u}_{1,n}, \tilde{u}_{2,n}, \tilde{u}_{3,n})$ is bounded in $(L^2(Q))^3$. One can extract a sequence denoted $((\tilde{u}_{1,n_k}, \tilde{u}_{2,n_k}, \tilde{u}_{3,n_k}))$ that converges weakly towards $(\tilde{u}_1, \tilde{u}_2, \tilde{u}_3)$ in $(L^2(Q))^3$.

As

$$\begin{aligned} \eta_{n_k}^1 &= \int_0^A \beta_3 \tilde{u}_{3,n_k} da, \\ \eta_{n_k}^2 &= \int_0^A \beta_1 \tilde{u}_{1,n_k} da, \\ \eta_{n_k}^3 &= \int_0^A \sigma \beta_2 \tilde{u}_{2,n_k} da, \\ \eta_{n_k}^4 &= \int_0^A (1 - \sigma) \beta_2 \tilde{u}_{2,n_k} da, \end{aligned} \tag{76}$$

using (75), we obtain

$$\sum_{i=1}^4 \|\eta_{n_k}^i\|_{L^2(Q_T)} \leq K' \left(\int_0^A \sum_{i=1}^4 (\tilde{u}_i^0)^2 da + \sum_{i=1}^4 r_i^2 \right). \tag{77}$$

From (77), one can extract a sequence which also denoted $((\eta_{n_k}^1, \eta_{n_k}^2, \eta_{n_k}^3, \eta_{n_k}^4))$ that converges weakly to $(\eta^1, \eta^2, \eta^3, \eta^4)$ on $L^2(Q_T)$.

So,

$$\int_{Q_T} \phi \eta_{n_j}^i dt \longrightarrow \int_{Q_T} \phi \eta^i dt, \quad \forall \phi \in L^2(Q_T), 1 \leq i \leq 4. \tag{78}$$

Moreover, the sequence $((\tilde{u}_{1,n_k}, \tilde{u}_{2,n_k}, \tilde{u}_{3,n_k}))$ converges weakly towards $(\tilde{u}_1, \tilde{u}_2, \tilde{u}_3)$ on $(L^2(Q))^3$. Thus, the sequence $((\tilde{u}_{1,n_k}, \tilde{u}_{2,n_k}, \tilde{u}_{3,n_k}))$ associated to $((\eta_{n_k}^1, \eta_{n_k}^2, \eta_{n_k}^3, \eta_{n_k}^4))$ converges weakly towards $(\tilde{u}_1, \tilde{u}_2, \tilde{u}_3)$. Since $\phi \beta_k \in L^2(Q_T)$, for all $\phi \in L^2(Q_T)$ and for all $k \in \{1, 2, 3\}$, it follows that

$$\left\{ \begin{aligned} \int_{Q_T} \phi \eta_{n_k}^1 dt &\longrightarrow \int_{Q_T} \phi \left(\int_0^A \beta_3 \tilde{u}_3 da \right) dt, \\ \int_{Q_T} \phi \eta_{n_k}^2 dt &\longrightarrow \int_{Q_T} \phi \left(\int_0^A \beta_1 \tilde{u}_1 da \right) dt, \\ \int_{Q_T} \phi \eta_{n_k}^3 dt &\longrightarrow \int_{Q_T} \phi \left(\int_0^A \sigma \beta_2 \tilde{u}_2 da \right) dt, \\ \int_{Q_T} \phi \eta_{n_k}^4 dt &\longrightarrow \int_{Q_T} \phi \left(\int_0^A (1 - \sigma) \beta_2 \tilde{u}_2 da \right) dt. \end{aligned} \right. \tag{79}$$

Equations (78) and (79) imply

$$\left\{ \begin{aligned} \int_{Q_T} \phi \left(\eta^1 - \int_0^A \beta_3 \tilde{u}_3 da \right) dt &= 0, \quad \forall \phi \in L^2(Q_T), \\ \int_{Q_T} \phi \left(\eta^2 - \int_0^A \beta_1 \tilde{u}_1 da \right) dt &= 0, \quad \forall \phi \in L^2(Q_T), \\ \int_{Q_T} \phi \left(\eta^3 - \int_0^A \sigma \beta_2 \tilde{u}_2 da \right) dt &= 0, \quad \forall \phi \in L^2(Q_T), \\ \int_{Q_T} \phi \left(\eta^4 - \int_0^A (1 - \sigma) \beta_2 \tilde{u}_2 da \right) dt &= 0, \quad \forall \phi \in L^2(Q_T). \end{aligned} \right. \tag{80}$$

Consequently,

$$\left\{ \begin{aligned} \eta^1 &= \int_0^A \beta_3 \tilde{u}_3 da, \quad a.e. t \in Q_T, \\ \eta^2 &= \int_0^A \beta_1 \tilde{u}_1 da, \quad a.e. t \in Q_T, \\ \eta^3 &= \int_0^A \sigma \beta_2 \tilde{u}_2 da, \quad a.e. t \in Q_T, \\ \eta^4 &= \int_0^A (1 - \sigma) \beta_2 \tilde{u}_2 da, \quad a.e. t \in Q_T. \end{aligned} \right. \tag{81}$$

Then, $(\tilde{u}_1, \tilde{u}_2, \tilde{u}_3, \tilde{u}_4)$ is solution of (59) and (60) with (R_1, R_2, R_3, R_4) instead of (h_1, h_2, h_3, h_4) ; $(\tilde{u}_1, \tilde{u}_2, \tilde{u}_3)$ and \tilde{w} satisfy, respectively, (4) and (56). (iv) Let K be a closed subset of \mathcal{M} . We will prove that $\Lambda_\gamma^{-1}(K)$ is a closed subset of \mathcal{M} . Remark that $\Lambda_\gamma^{-1}(K) = \{(R_1, R_2, R_3, R_4) \in \mathcal{M}; \Lambda_\gamma(R_1, R_2, R_3, R_4) \cap K \neq \emptyset\}$.

Suppose that $(R_{1,n}, R_{2,n}, R_{3,n}, R_{4,n}) \longrightarrow (R_1, R_2, R_3, R_4)$ (strongly) in \mathcal{M} . Then, the sequence $(R_{1,n}, R_{2,n}, R_{3,n}, R_{4,n})$ is bounded. There also exists a sequence $((\eta_n^1, \eta_n^2, \eta_n^3, \eta_n^4))_n \in K$, such that, for all $n \geq 1$, $(\eta_n^1, \eta_n^2, \eta_n^3, \eta_n^4) \in \Lambda_\gamma(R_{1,n}, R_{2,n},$

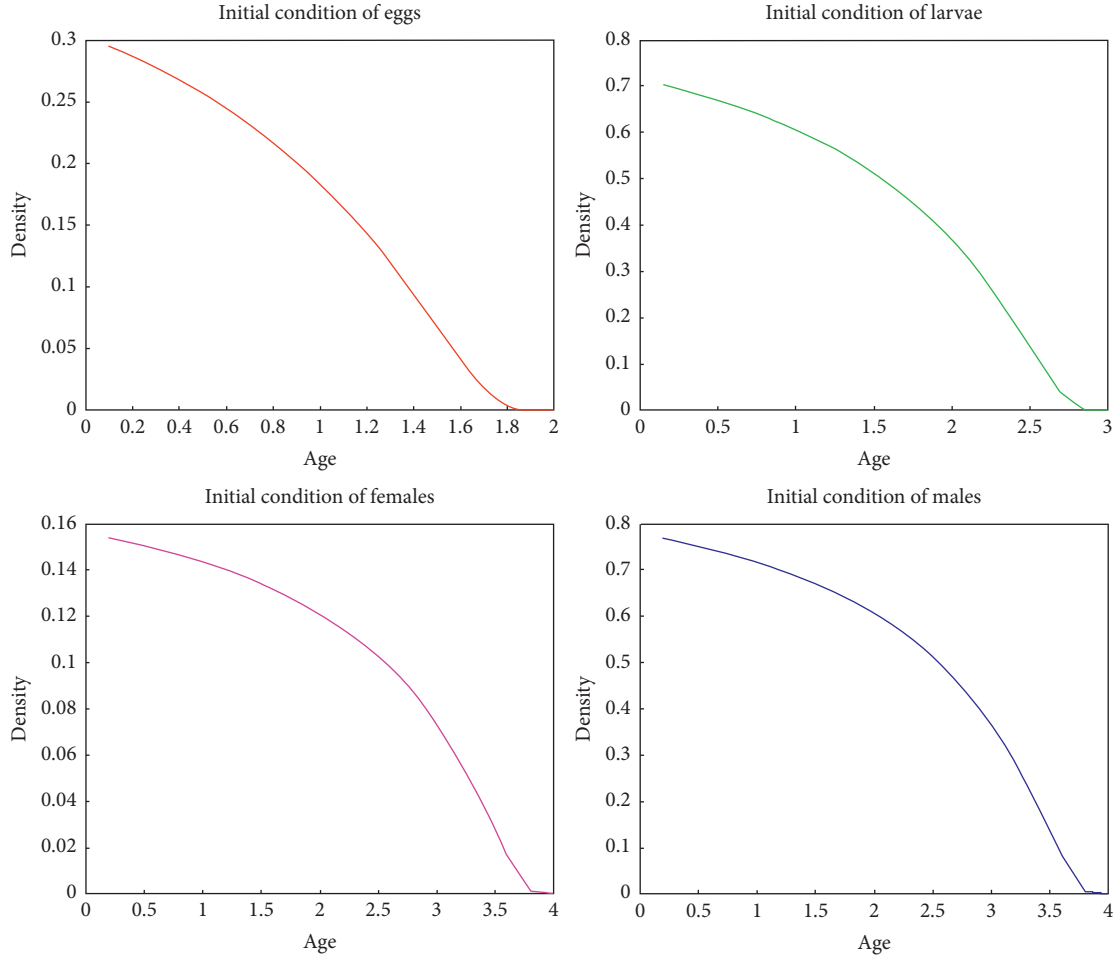


FIGURE 1: Initial conditions.

$R_{3,n}, R_{4,n}$). So, there exist $(\tilde{u}_{1,n}, \tilde{u}_{2,n}, \tilde{u}_{3,n}) \in (L^2(\Omega))^3$ and a control function $\tilde{w}_n \in (L^2(q))^3$ such that

$$\begin{aligned}
 \eta_n^1 &= \int_0^A \beta_3 \tilde{u}_{3,n} da, \\
 \eta_n^2 &= \int_0^A \beta_1 \tilde{u}_{1,n} da, \\
 \eta_n^3 &= \int_0^A \sigma \beta_2 \tilde{u}_{2,n} da, \\
 \eta_n^4 &= \int_0^A (1 - \sigma) \beta_2 \tilde{u}_{2,n} da,
 \end{aligned} \tag{82}$$

with $\tilde{w}_n, (\tilde{u}_{1,n}, \tilde{u}_{2,n}, \tilde{u}_{3,n})$ verifying (56) and $(\tilde{u}_{1,n}, \tilde{u}_{2,n}, \tilde{u}_{3,n})$ is solution of (59) and (60) with $(R_{1,n}, R_{2,n}, R_{3,n}, R_{4,n})$ instead of (h_1, h_2, h_3, h_4) , and $(\tilde{u}_{1,n}, \tilde{u}_{2,n}, \tilde{u}_{3,n})$ verifies (4). From (66), (69), and (56), it follows that (\tilde{w}_n) and $(\tilde{u}_{1,n}, \tilde{u}_{2,n}, \tilde{u}_{3,n})$ are bounded, respectively, in $(L^2(q))^3$ and $(L^2(\Omega))^3$. Thus, there are sequences also denoted by $(\tilde{w}_n)_n$ and $((\tilde{u}_{1,n}, \tilde{u}_{2,n}, \tilde{u}_{3,n}))_n$ that converge weakly towards \tilde{w} and $(\tilde{u}_1, \tilde{u}_2, \tilde{u}_3)$, respectively, in $(L^2(q))^3$ and $(L^2(\Omega))^3$. Moreover, \tilde{w} verifies (56) and

$$\begin{aligned}
 \eta^1 &= \int_0^A \beta_3 \tilde{u}_3 da, \\
 \eta^2 &= \int_0^A \beta_1 \tilde{u}_1 da, \\
 \eta^3 &= \int_0^A \sigma \beta_2 \tilde{u}_2 da, \\
 \eta^4 &= \int_0^A (1 - \sigma) \beta_2 \tilde{u}_2 da.
 \end{aligned} \tag{83}$$

Here, $(\tilde{u}_1, \tilde{u}_2, \tilde{u}_3)$ is solution of (59) and (60) with (R_1, R_2, R_3, R_4) instead of (h_1, h_2, h_3, h_4) , and $(\tilde{u}_1, \tilde{u}_2, \tilde{u}_3)$ verifies (4). Therefore,

$$(\eta^1, \eta^2, \eta^3, \eta^4) \in \Lambda_\gamma(R_1, R_2, R_3, R_4). \tag{84}$$

In addition, from (77) and Lions–Aubin lemma, one can extract a sequence $((\eta_n^1, \eta_n^2, \eta_n^3, \eta_n^4))_n$ that converges strongly to $(\eta^1, \eta^2, \eta^3, \eta^4)$ in \mathcal{M} . Since K is closed, then $(\eta^1, \eta^2, \eta^3, \eta^4) \in K$.

Finally, from (84), we conclude that $(R_1, R_2, R_3, R_4) \in \Lambda_\gamma^{-1}(K)$. \square

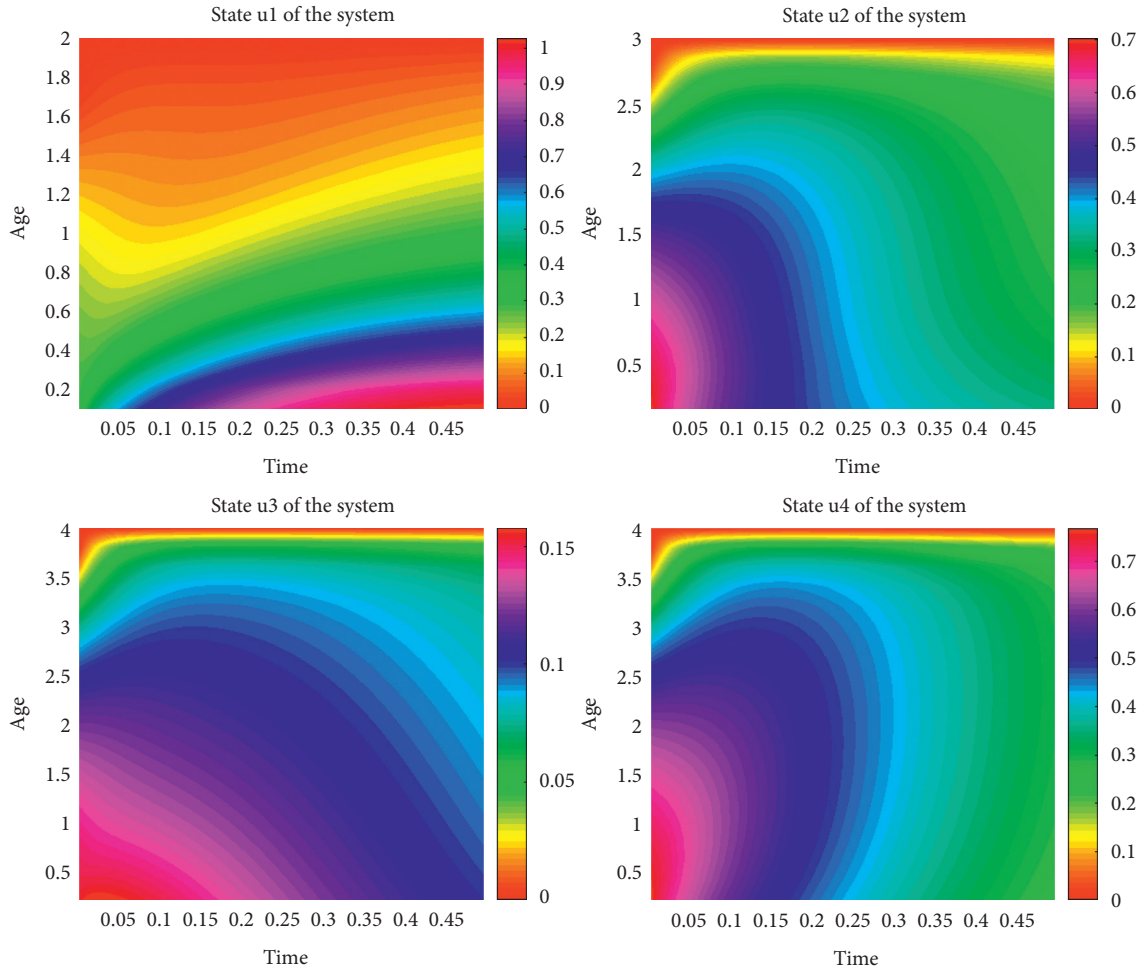


FIGURE 2: Uncontrolled solution at final time $T = 0.5$ and $A = 4$.

Proof. of Theorem 1.

This result is a consequence of Proposition 2. In fact, from Proposition 2, there is $(R_1, R_2, R_3, R_4) \in \mathcal{M}$ such that $(R_1, R_2, R_3, R_4) \in \Lambda_\gamma(R_1, R_2, R_3, R_4)$, see Theorem 1.3 in [22]. \square

5. Numerical Simulations

In this section, we display the numerical simulations of problems (1) and (2). The first part is to reduce the PDE to a finite dimensional system of the form: $\dot{X} = AX + BY$, where A and B are matrices. We construct the control problem, which consists in minimizing the functional and we choose the classical Hum functional and the control matrix $B = \chi_\Theta$, where $\Theta = (0, A) \times (0, T)$.

Example 1. For the simulation, we take $k = 0.25$, $A_1 = 2$, $A_2 = 3$, $A_3 = A_4 = 4$, and $\Delta a = 1/20$. Moreover, the transition functions β_1 , β_2 , and β_3 are given by

$$\begin{aligned}
 \beta_1(a) &= \begin{cases} 0, & \text{if } a = 0, \\ \frac{7(11a - 4)^4 e^{-0.52(10a-4)}}{\Gamma(5)}, & \text{if } a \neq 0, \end{cases} \\
 \beta_2(a) &= \begin{cases} 0, & \text{if } a = 0, \\ \frac{7(50a - 4)^4 e^{-0.52(10a-4)}}{\Gamma(5)}, & \text{if } a \neq 0, \end{cases} \\
 \beta_3(a) &= \begin{cases} 0, & \text{if } a = 0, \\ \frac{7(12 \cdot 10^6 a - 4)^3 e^{-0.52(10a-4)}}{\Gamma(5)}, & \text{if } a \neq 0, \end{cases}
 \end{aligned} \tag{85}$$

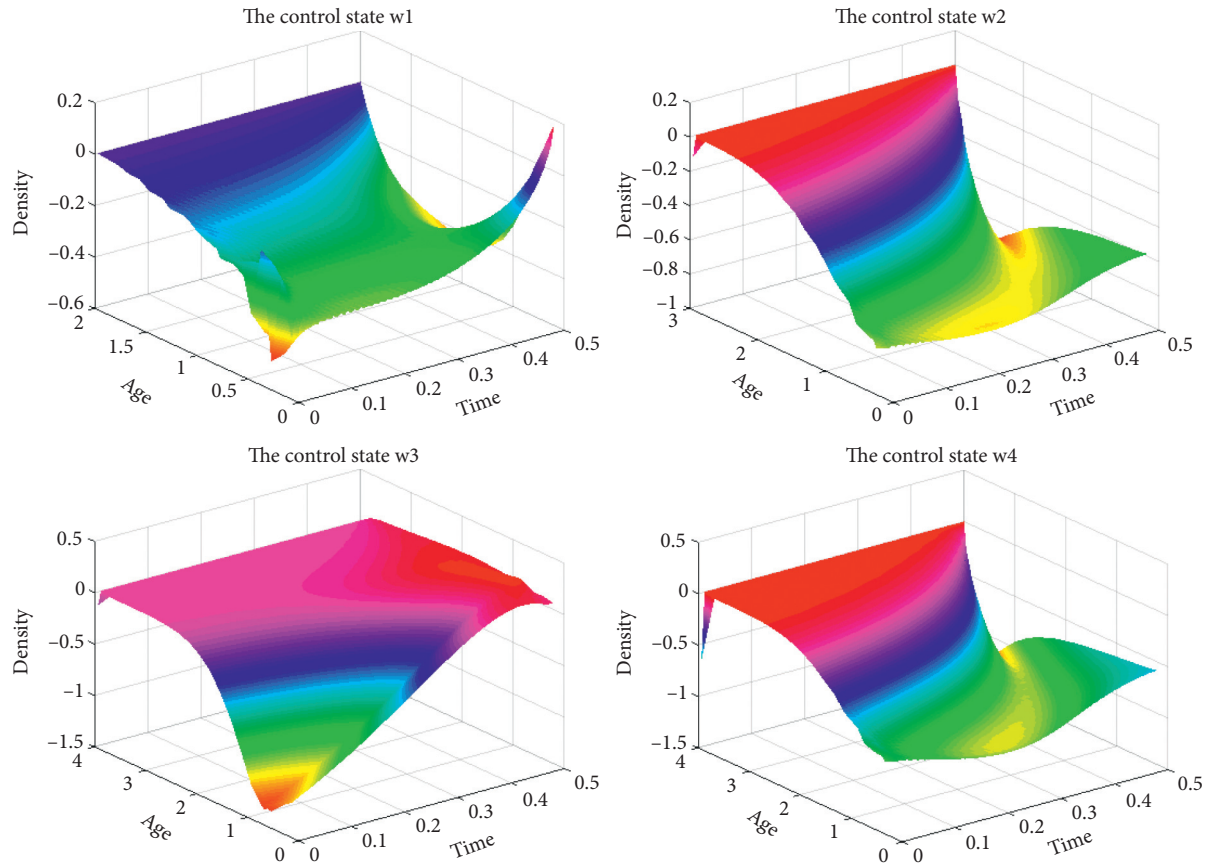


FIGURE 3: Control state at final time $T = 0.5$ and $A = 4$.

and the initial population distributions $u_i^0(a) = k_i e^{-(1/A_i - a)}$, ($1 \leq i \leq 4$, with $k_1 = 0.5, k_2 = k_4 = 1$ and $k_3 = 0.2$) will be used in the computation. The growth rates are given by

$$\begin{aligned}
 v_1(a) &= \begin{cases} 0.2, & \text{if } a = 0, \\ a, & \text{if } a \neq 0, \end{cases} \\
 v_2(a) &= \begin{cases} 0.3, & \text{if } a = 0, \\ a, & \text{if } a \neq 0, \end{cases} \\
 v_3(a) &= \begin{cases} 0.4, & \text{if } a = 0, \\ a, & \text{if } a \neq 0, \end{cases} \\
 v_4(a) &= \begin{cases} 0.4, & \text{if } a = 0, \\ a, & \text{if } a \neq 0. \end{cases}
 \end{aligned} \tag{86}$$

The following numerical results were obtained.

We notice that we have the positivity of the initial conditions, see Figure 1. Moreover, the colors green, blue, and yellow show that the uncontrolled solutions are not null

at the final time $T = 0.5$ and the color red shows that the uncontrolled solutions are null at the final age $A = 4$, see Figure 2.

We use ODE 45 for the simulations of the uncontrolled system.

Example 2. In this example, we are going back to the data from Example 1.

The control states and the controlled states results are obtained for $T = 0.5$ and $A = 4$.

We notice that numerically the control covers all the domains. However, we could not take a positivity constraint in our simulations, see Figure 3.

Moreover, the colors red and yellow show that the controlled states are zero at the final time $T = 0.5$, see Figure 4. This confirms the result of null controllability obtained theoretically.

The CaSadi toolbox is used to simulate the control (by the minimization of functional: Hum Method).

Below, we have the numerical simulations results.

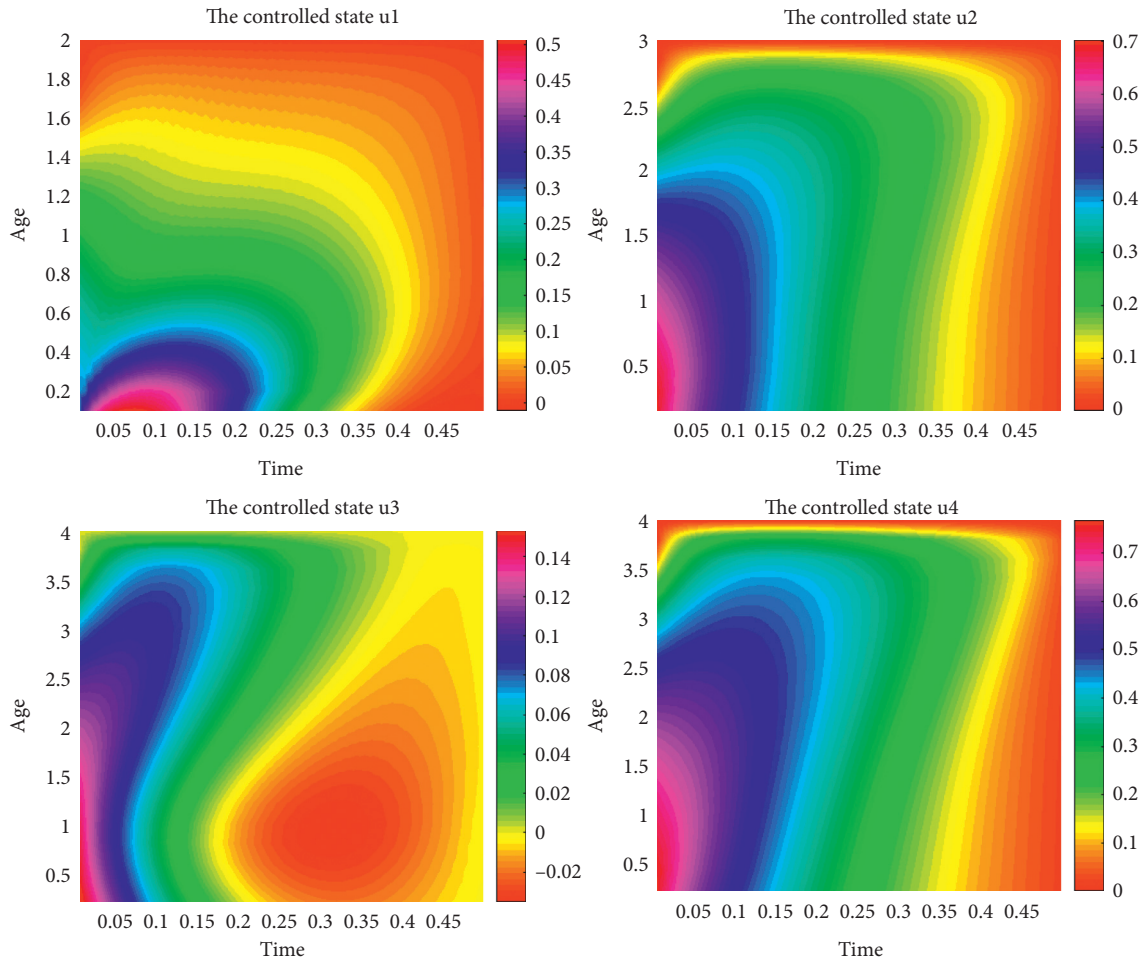


FIGURE 4: Controlled solution at final time $T = 0.5$ and $A = 4$.

6. Conclusion

In this paper, we have proved the null controllability of a four-stage and age-structured population dynamics model. We consider three controls acting on eggs, larvae, and females. We end with numerical illustrations to confirm the theoretical results. In order to consolidate the result obtained in this article, we are going to think about studying the controllability with less controls (two or one) and also to establish algorithms allowing to calculate the control from its characterization.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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