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Retraction

Retracted: (m, n)-Ideals in Semigroups Based on Int-Soft Sets

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This article has been retracted by Hindawi following an investigation undertaken by the publisher [1]. This investigation has uncovered evidence of one or more of the following indicators of systematic manipulation of the publication process:

(1) Discrepancies in scope
(2) Discrepancies in the description of the research reported
(3) Discrepancies between the availability of data and the research described
(4) Inappropriate citations
(5) Incoherent, meaningless and/or irrelevant content included in the article
(6) Peer-review manipulation

The presence of these indicators undermines our confidence in the integrity of the article's content and we cannot, therefore, vouch for its reliability. Please note that this notice is intended solely to alert readers that the content of this article is unreliable. We have not investigated whether authors were aware of or involved in the systematic manipulation of the publication process.

Wiley and Hindawi regrets that the usual quality checks did not identify these issues before publication and have since put additional measures in place to safeguard research integrity.

We wish to credit our own Research Integrity and Research Publishing teams and anonymous and named external researchers and research integrity experts for contributing to this investigation.

The corresponding author, as the representative of all authors, has been given the opportunity to register their agreement or disagreement to this retraction. We have kept a record of any response received.

References

[1] G. Muhiuddin and A. M. Alanazi, "(m, n)-Ideals in Semigroups Based on Int-Soft Sets," *Journal of Mathematics*, vol. 2021, Article ID 5546596, 10 pages, 2021.

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Research Article

(m, n)-Ideals in Semigroups Based on Int-Soft Sets

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Algebraic structures play a prominent role in mathematics with wide ranging applications in many disciplines such as theoretical physics, computer sciences, control engineering, information sciences, coding theory, and topological spaces. This provides sufficient motivation to researchers to review various concepts and results from the realm of abstract algebra in the broader framework of fuzzy setting. In this paper, we introduce the notions of int-soft (m, n)-ideals, int-soft $(m, 0)$-ideals, and int-soft $(0, n)$-ideals of semigroups by generalizing the concept of int-soft bi-ideals, int-soft right ideals, and int-soft left ideals in semigroups. In addition, some of the properties of int-soft (m, n)-ideal, int-soft $(m, 0)$-ideal, and int-soft $(0, n)$-ideal are studied. Also, characterizations of various types of semigroups such as (m, n)-regular semigroups, $(m, 0)$-regular semigroups, and $(0, n)$-regular semigroups in terms of their int-soft (m, n)-ideals, int-soft $(m, 0)$-ideals, and int-soft $(0, n)$-ideals are provided.

1. Introduction

Soft set theory of Molodtsov [1] is an important mathematical tool to dealing with uncertainties and fuzzy or vague objects and has huge applications in real-life situations. In soft sets, the problems of uncertainties deal with enough numbers of parameters which make it more accurate than other mathematical tools. Thus, the soft sets are better than the other mathematical tools to describe the uncertainties. Aktaş and Çağman [2] show that the soft sets are more accurate tools to deal the uncertainties by comparing the soft sets to rough and fuzzy sets. The decision-making problem in soft sets had been considered by Maji et al. [3]. In [4], Maji et al. investigated several operations on soft sets. The notions of soft sets introduced in different algebraic structures had been applied and studied by several authors, for example, Aktaş and Çağman [2] for soft groups, Feng et al. [5] for soft semirings, and Naz and Shabir [6, 7] for soft semi-hypergroups.

Song [8] introduced the notions of int-soft semigroups, int-soft left (resp. right) ideals, and int-soft quasi-ideals. Afterthat, Dudek and Jun [9] studied the properties of int-soft left (resp. right) ideals, and characterizations of these int-soft ideal are obtained. Moreover, they introduced the concept of int-soft (generalized) bi-ideals, and characterizations of (int-soft) generalized bi-ideals and int-soft bi-ideals are obtained. Dudek and Jun [9] introduced and characterized the notion of soft interior ideals of semigroups. The concept of union-soft semigroups, union-soft l-ideals, union-soft r-ideals, and union-soft semiprime soft sets have been considered by [10]. In addition, Muhiuddin et al. studied the soft set theory on various aspects (see, for example, [11–21]). For more related concepts, the readers are referred to [22–31].

The results of this paper are arranged as follows. Section 2 summarises some concepts and properties related to semigroups, soft sets, and int-soft ideals that are required to establish our key results, while Section 3 presents the principle of int-soft (m, n)-ideals. We prove that the int-soft bi-ideals are int-soft (m, n)-ideals for each positive integer m, n, but the converse is not necessarily valid. Then, we prove that the A subset of the S semigroup is (m, n)-ideal of S if and only if $(\widehat{\chi A}, S)$ over U is an int-soft (m, n)-ideal over U. Also, we prove that a soft set $(\widetilde{\mathscr{K}}, S)$ over U is an int-soft (m, n)-ideal over U if and only if $(\widetilde{\mathscr{K}}^m \circ \widehat{\chi_S} \circ \widetilde{\mathscr{K}}^n, S) \subseteq (\widetilde{\mathscr{K}}, S)$. Moreover, we characterize (m, n) regular semigroups in terms of int-soft (m, n)-ideals over U. In this respect, we prove that a semigroup S is (m, n)-regular if and only if $(\widetilde{\mathscr{K}}, S) = (\widetilde{\mathscr{K}}^m \circ \widehat{\chi_S} \circ \widetilde{\mathscr{K}}^n, S)$ for each int-soft (m, n)-ideal

$(\widehat{\mathscr{K}}, S)$ over U. In Section 4, first, we present the idea of int-soft $(m, 0)$-ideal and $(0, n)$-ideal over U. After that, we obtain some analogues' results to the previous section. Furthermore, we prove that a semigroup S is (m, n)-regular if and only if $(\widehat{\mathscr{K}} \cap \widehat{\mathscr{G}}, S) = (\widehat{\mathscr{K}}^m \circ \widehat{\mathscr{G}} \cap \widehat{\mathscr{K}} \circ \widehat{\mathscr{G}}^n, S)$ for each int-soft $(m, 0)$-ideal $(\widehat{\mathscr{K}}, S)$ and for each int-soft $(0, n)$-ideal $(\widehat{\mathscr{G}}, S)$ over U. At the end of this section, we provide the existence theorem for int-soft (m, n)-ideal over U and for the minimality of int-soft (m, n)-ideal over U. We also provide a conclusion in Section 5 that contains the direction for certain potential work.

2. Preliminaries

Let S be a semigroup. For $(\varnothing \neq)\Omega, \mho \subseteq S$, $\Omega\mho$ is defined as $\Omega\mho = \{v\hbar | v \in \Omega, \hbar \in \mho\}$. A subset $(\varnothing \neq)\Omega$ of S is called a sub-semigroup of S if $v\hbar \in \Omega\, \forall v, \hbar \in \Omega$. A subset $(\varnothing \neq)\Omega$ of S is called a left (resp. right) ideal of S if $S\Omega \subseteq \Omega$ (resp. $\Omega S \subseteq \Omega$) and is called an ideal of S if Ω is both left and right ideal of S. A sub-semigroup $\mho$ of S is called a bi-ideal of S if $\mho S \mho \subseteq \mho$.

Let U be a universal set and let E be a set of parameters. Let $\mathscr{P}(U)$ denote the power set of U and let $\Omega \subseteq E$. A pair $(\widehat{\mathscr{K}}, \Omega)$ is called a soft set (over U) [32] if $F\colon \Omega \longrightarrow \mathscr{P}(U)$ is a mapping. We denote the set of all soft sets over U with parameter set S by $\mathscr{S}_S(U)$.

Let $(\widehat{\mathscr{K}}, \Omega)$ and $(\widehat{\mathscr{G}}, \mho)$ be soft sets over U. Then, $(G, \mho)$ is called a soft subset of $(\widehat{\mathscr{K}}, \Omega)$ if $\mho \subseteq \Omega$ and $\widehat{\mathscr{G}}(v) \subseteq \widehat{\mathscr{K}}(v)$, $\forall v \in \mho$.

Let $(\widehat{\mathscr{K}}, \Omega)$ and $(\widehat{\mathscr{G}}, \Omega)$ be two soft sets. Then, for each $v \in \Omega$, the union and intersection are defined as

$$\begin{aligned} (\widehat{\mathscr{K}} \Cup \widehat{\mathscr{G}})(v) &= \widehat{\mathscr{K}}(v) \cup \widehat{\mathscr{G}}(v), \\ (\widehat{\mathscr{K}} \Cap \widehat{\mathscr{G}})(v) &= \widehat{\mathscr{K}}(v) \cap \widehat{\mathscr{G}}(v). \end{aligned} \tag{1}$$

For any two soft sets $(\widehat{\mathscr{K}}, \Omega)$ and $(\widehat{\mathscr{G}}, \Omega)$ of S, the int-soft product $\widehat{\mathscr{K}} \circ \widehat{\mathscr{G}}$ is defined as

$$(\widehat{\mathscr{K}} \circ \widehat{\mathscr{G}})(v) = \begin{cases} \bigcup\limits_{v = \hbar\kappa} \left\{\widehat{\mathscr{K}}(\hbar) \cap \widehat{\mathscr{G}}(\kappa)\right\}, & \text{if there exist } \hbar,\ \kappa \in S \text{ such that } v = \hbar\kappa, \\ \varnothing, & \text{otherwise.} \end{cases} \tag{2}$$

A soft set $(\widehat{\mathscr{K}}, S)$ over U is called an int-soft right (resp. Left) ideal over U if $\widehat{\mathscr{K}}(v\kappa) \supseteq \widehat{\mathscr{K}}(v)$ (resp. $\widehat{\mathscr{K}}(v\kappa) \supseteq \widehat{\mathscr{K}}(\kappa)$) for all $v, \kappa \in S$. It is called an int-soft ideal over U if it is both int-soft left and int-soft right ideal over U. An int-soft sub-semigroup $(\widehat{\mathscr{K}}, S)$ over U is called an int-soft bi-ideal over U if $\widehat{\mathscr{K}}(v\kappa\hbar) \supseteq \widehat{\mathscr{K}}(v) \cap \widehat{\mathscr{K}}(\hbar)$ for all $v, \kappa, \hbar \in S$. The set of all int-soft left (resp. Right) ideals and int-soft bi-ideals over U will be denoted by $\mathscr{I}_L(U)$ (resp. $\mathscr{I}_R(U)$) and $\mathscr{I}_B(U)$.

More concepts related to our study in different aspects have been studied in [33–39].

For $(\varnothing \neq)\Omega \subseteq S$, the characteristic soft set over U is denoted by $(\widehat{\chi_\Omega}, S)$ and defined as

$$\widehat{\chi_\Omega}(v) = \begin{cases} U, & \text{if } v \in \Omega, \\ \varnothing, & \text{if } v \notin \Omega. \end{cases} \tag{3}$$

Let $(\varnothing \neq)\Omega, \mho \subseteq S$. Then, we have (1) $\chi_\Omega \circ \chi_\mho = \chi_{\Omega\mho}$ and (2) $\chi_\Omega \cap \chi_\mho = \chi_{\Omega \cap \mho}$.

The concept of (m, n)-ideals of semigroups was introduced by Lajos [40] as follows. Let S be a semigroup and m, n be nonnegative integers. Then, a sub-semigroup Ω of S is said to be an (m, n)-ideal of S if $\Omega^m S \Omega^n \subseteq \Omega$. After that, the concept of (m, n)-ideals in various algebraic structures such as ordered semigroups, LA-semigroups, and fuzzy semigroups had been studied by, for instance, Akram et al. [41], Bussaban and Changphas [42], Changphas [43], Mahboob et al. [44], and many others.

We denote by $[v]_{(m,n)}$ the principal (m, n)-ideal, $[v]_{(m,0)}$ the principal $(m, 0)$-ideal, and $[v]_{(0,n)}$ the principal $(0, n)$-ideal generated by an element v of S, respectively. They were given by Krgovic [45] as follows:

$$\begin{aligned} [v]_{(m,n)} &= \bigcup_{i=1}^{m+n} v^i \cup v^m S v^n, \\ [v]_{(m,0)} &= \bigcup_{i=1}^{m} v^i \cup v^m S, \\ [v]_{(0,n)} &= \bigcup_{i=1}^{n} v^i \cup S v^n. \end{aligned} \tag{4}$$

In whatever follows, $\mathscr{M}_{(m,n)}$, $\mathscr{M}_{(m,0)}$, and $\mathscr{M}_{(0,n)}$ denote the set of all (m, n)-ideals, $(m, 0)$-ideals, and $(0, n)$-ideals of S.

3. Int-Soft (m, n)-Ideals

Definition 1. An int-soft sub-semigroup $(\widehat{\mathscr{K}}, S)$ over U is called an int-soft (m, n)-ideal over U if

$$\widehat{\mathscr{K}}(\hbar_1\hbar_2, \ldots, \hbar_m \kappa v_1 v_2, \ldots, v_m) \supseteq \widehat{\mathscr{K}}(\hbar_1) \cap \widehat{\mathscr{K}}(\hbar_2) \cap \cdots \widehat{\mathscr{K}}(\hbar_m) \cap \widehat{\mathscr{K}}(v_1) \cap \widehat{\mathscr{K}}(v_2) \cap \cdots \cap \widehat{\mathscr{K}}(v_n), \tag{5}$$

for all $\hbar_1, \hbar_2, \ldots, \hbar_n, \kappa, v_1, v_2, \ldots, v_m \in S$.

The set of all int-soft (m,n)-ideals over U will be denoted by $\mathscr{I}_{(m,n)}(U)$.

Example 1. Let $S=\{0,v,\hbar\}$. Define the binary operation '$\cdot$' on S as follows.

$\cdot$	0	v	$\hbar$
0	0	0	0
v	0	0	0
$\hbar$	0	0	v

Then, $(S,\cdot)$ is a semigroup. Define $(\widehat{\mathscr{K}},S)\in\mathscr{S}_S(U)$ as

$$\widehat{\mathscr{K}}(\kappa)=\begin{cases}U_1, & \text{if } \kappa\in\{0,v\},\\ U_2, & \text{if } \kappa=\hbar,\end{cases}\tag{6}$$

where $U_1,U_2\subseteq U$ such that $U_2\subseteq U_1$. It is straightforward to verify that $(\widehat{\mathscr{K}},S)\in\mathscr{I}_{(m,n)}(U)$.

Lemma 1. *In S, $(\widehat{\mathscr{K}},S)\in\mathscr{I}_B(U)\Rightarrow(\widehat{\mathscr{K}},S)\in\mathscr{I}_{(m,n)}(U)$.*

Proof (straightforward). □

Remark 1. In general, in a semigroup S, $(\widehat{\mathscr{K}},S)\in\mathscr{I}_{(m,n)}(U)\nRightarrow(\widehat{\mathscr{K}},S)\in\mathscr{I}_B(U)$.

Example 2. Let $S=\{0,v,\hbar,\kappa\}$. Define the binary operation '$\cdot$' on S as follows.

$\cdot$	0	v	$\hbar$	κ
0	0	0	0	0
v	v	v	v	v
$\hbar$	$\hbar$	$\hbar$	$\hbar$	$\hbar$
κ	0	0	v	0

Then, S is a semigroup. Define $(\widehat{\mathscr{K}},S)\in\mathscr{S}_S(U)$ as

$$\widehat{\mathscr{K}}(\omega)=\begin{cases}U, & \text{if } \omega\in\{0,\kappa\},\\ \varnothing, & \text{if } \omega\in\{v,\hbar\}.\end{cases}\tag{7}$$

Then, $(\widehat{\mathscr{K}},S)\in\mathscr{I}_{(m,n)}(U)$, $\forall m,n\geq 2$, but $\widehat{\mathscr{K}}\notin\mathscr{I}_B(U)$ because $\varnothing=\widehat{\mathscr{K}}(v)=\widehat{\mathscr{K}}(\kappa\hbar 0)\nsupseteq\widehat{\mathscr{K}}(\kappa)\cap\widehat{\mathscr{K}}(0)=U$.

Theorem 1. *Let $(\widehat{\mathscr{K}},S),(\widehat{\mathscr{F}},S)\in\mathscr{I}_{(m,n)}(U)$. Then, $(\widehat{\mathscr{K}}\widetilde{\cap}\widehat{\mathscr{F}},S)\in\mathscr{I}_{(m,n)}(U)$.*

Proof. Let $v,\hbar\in S$. We have

$$(\widehat{\mathscr{K}}\widetilde{\cap}\widehat{\mathscr{F}})(v\hbar)=\widehat{\mathscr{K}}(v\hbar)\cap\widehat{\mathscr{F}}(v\hbar),=\widehat{\mathscr{K}}(v)\cap\widehat{\mathscr{K}}(\hbar)\cap\widehat{\mathscr{F}}(v)\cap\widehat{\mathscr{F}}(\hbar)=(\widehat{\mathscr{K}}\widetilde{\cap}\widehat{\mathscr{F}})(v)\cap(\widehat{\mathscr{K}}\widetilde{\cap}\widehat{\mathscr{F}})(\hbar).\tag{8}$$

Let $v_1,v_2,\ldots,v_m,\kappa,\hbar_1,\hbar_2,\ldots,\hbar_n\in S$. Now, we have

$$\begin{aligned}&(\widehat{\mathscr{K}}\widetilde{\cap}\widehat{\mathscr{F}})(v_1v_2,\ldots,v_m\kappa\hbar_1\hbar_2,\ldots,\hbar_n)\supseteq\widehat{\mathscr{K}}(v_1v_2,\ldots,v_m\kappa\hbar_1\hbar_2,\ldots,\hbar_n)\cap\widehat{\mathscr{F}}(v_1v_2,\ldots,v_m\kappa\hbar_1\hbar_2,\ldots,\hbar_n),\\&\supseteq\widehat{\mathscr{K}}(v_1)\cap\widehat{\mathscr{K}}(v_2)\cap\cdots\cap\widehat{\mathscr{K}}(v_m)\cap\widehat{\mathscr{K}}(\hbar_1)\cap\widehat{\mathscr{K}}(\hbar_2)\cap\cdots\cap\widehat{\mathscr{K}}(\hbar_n)\cap\widehat{\mathscr{F}}(v_1)\cap\widehat{\mathscr{F}}(v_2)\\&\cap\cdots\cap\widehat{\mathscr{F}}(v_m)\cap\widehat{\mathscr{F}}(\hbar_1)\cap\widehat{\mathscr{F}}(\hbar_2)\cap\cdots\cap\widehat{\mathscr{F}}(\hbar_n)\\&\supseteq(\widehat{\mathscr{K}}\widetilde{\cap}\widehat{\mathscr{F}})(v_1)\cap(\widehat{\mathscr{K}}\widetilde{\cap}\widehat{\mathscr{F}})(v_2)\cap\cdots\cap(\widehat{\mathscr{K}}\widetilde{\cap}\widehat{\mathscr{F}})(v_m)\cap(\widehat{\mathscr{K}}\widetilde{\cap}\widehat{\mathscr{F}})(\hbar_1)\cap(\widehat{\mathscr{K}}\widetilde{\cap}\widehat{\mathscr{F}})(\hbar_2)\cap\cdots\cap(\widehat{\mathscr{K}}\widetilde{\cap}\widehat{\mathscr{F}})(\hbar_n).\end{aligned}\tag{9}$$

Therefore, $(\widehat{\mathscr{K}}\widetilde{\cap}\widehat{\mathscr{F}},S)\in\mathscr{I}_{(m,n)}(U)$. □

Theorem 2. *Let $(\varnothing\neq)\Omega\subseteq S$. Then, $(\varnothing\neq)\Omega\in\mathscr{M}_{(m,n)}\Leftrightarrow(\widehat{\chi_\Omega},S)\in\mathscr{I}_{(m,n)}(U)$.*

Proof. $(\Rightarrow)$ Let $v_1,v_2,\ldots,v_m,\kappa,\hbar_1,\hbar_2,\ldots,\hbar_n\in S$. Below are the cases we have:

Case 1. If $x_k\notin\Omega$ for some $k\in\{1,2,\ldots,m\}$, then

$$\widehat{\chi_\Omega}(v_1v_2,\ldots,v_m\kappa\hbar_1\hbar_2,\ldots,\hbar_n)\supseteq\widehat{\chi_\Omega}(v_1)\cap\widehat{\chi_\Omega}(v_2)\cap\cdots\cap\widehat{\chi_\Omega}(v_m)\cap\widehat{\chi_\Omega}(\hbar_1)\cap\widehat{\chi_\Omega}(\hbar_2)\cap\cdots\cap\widehat{\chi_\Omega}(\hbar_n)\}.\tag{10}$$

Case 2. If $y_l\notin\Omega$ for some $l\in\{1,2,\ldots,n\}$, then

$$\widehat{\chi_\Omega}(v_1v_2,\ldots,v_m\kappa\hbar_1\hbar_2,\ldots,\hbar_n)\supseteq\widehat{\chi_\Omega}(v_1)\cap\widehat{\chi_\Omega}(v_2)\cap\cdots\cap\widehat{\chi_\Omega}(v_m)\cap\widehat{\chi_\Omega}(\hbar_1)\cap\widehat{\chi_\Omega}(\hbar_2)\cap\cdots\cap\widehat{\chi_\Omega}(\hbar_n).\tag{11}$$

When $x_k \notin \Omega$ and $y_l \notin \Omega$ for $k \in \{1, 2, \ldots, m\}$ and $l \in \{1, 2, \ldots, n\}$ are used in previous cases.

Case 3. If $x_k, y_l \in \Omega$, $\forall k \in \{1, 2, \ldots, m\}$ and $l \in \{1, 2, \ldots, n\}$, then $v_1 v_2, \ldots, v_m z \hbar_1 \hbar_2, \ldots, \hbar_n \in \Omega^m S \Omega^n \subseteq \Omega$. Therefore,

$$\begin{aligned} &\widehat{\chi_\Omega}(v_1 v_2, \ldots, v_m c \hbar_1 \hbar_2, \ldots, \hbar_n) = 1, \\ &\supseteq \widehat{\chi_\Omega}(v_1) \cap \widehat{\chi_\Omega}(v_2) \cap \cdots \cap \widehat{\chi_\Omega}(v_m) \cap \widehat{\chi_\Omega}(\hbar_1) \cap \widehat{\chi_\Omega}(\hbar_2) \cap \cdots \cap \widehat{\chi_\Omega}(\hbar_n)\}. \end{aligned} \tag{12}$$

Hence, $(\widehat{\chi_\Omega}, S) \in \mathcal{I}_{(m,n)}(U)$.

($\Leftarrow$) Let $v_1, v_2, \ldots, v_m, \hbar_1, \hbar_2, \ldots, \hbar_n \in \Omega$ and $\kappa \in S$. Then, $\widehat{\chi_\Omega}(v_1 v_2, \ldots, v_m c \hbar_1 \hbar_2, \ldots, \hbar_n) \supseteq \widehat{\chi_\Omega}(v_1) \cap \widehat{\chi}_\Omega(v_2) \cap \cdots \cap \widehat{\chi_\Omega}(v_m) \cap \widehat{\chi_\Omega}(\hbar_1) \cap \widehat{\chi_\Omega}(\hbar_2) \cap \cdots \cap \widehat{\chi_\Omega}(\hbar_n) = 1$ implies $\widehat{\chi_\Omega}(v_1 v_2, \ldots, v_m c \hbar_1 \hbar_2, \ldots, \hbar_n) = 1$. Therefore, $v_1 v_2, \ldots, v_m \kappa \hbar_1 \hbar_2, \ldots, \hbar_n \in \Omega$. Thus, $\Omega^m S \Omega^n \subseteq \Omega$, as required. □

Theorem 3. *Let* $(\widehat{\mathcal{K}}, S) \in \mathcal{S}_S(U)$. *Then,* $(\widehat{\mathcal{K}}, S) \in \mathcal{I}_{(m,n)}(U) \Leftrightarrow (\widehat{\mathcal{K}}^m \circ \widehat{\chi}_S \circ \widehat{\mathcal{K}}^n, S) \subseteq (\widehat{\mathcal{K}}, S)$.

Proof. ($\Rightarrow$) Let $a \in S$. If $(\widehat{\mathcal{K}}^m \circ \widehat{\chi}_S \circ \widehat{\mathcal{K}}^n)(a) = \varnothing$, then $(\widehat{\mathcal{K}}^m \circ \widehat{\chi}_S \circ \widehat{\mathcal{K}}^n, S) \subseteq (\widehat{\mathcal{K}}, S)$. In the other case, when $(f^m \circ S \circ f^n)(a) \neq \varnothing$, then there exist elements $r, s \in S$ such that $a = rs$, $(\widehat{\mathcal{K}}^m \circ \widehat{\chi}_S)(r) \neq \varnothing$ and $\widehat{\mathcal{K}}^n(s) \neq \varnothing$. As $(\widehat{\mathcal{K}}^m \circ \widehat{\chi}_S)(s) \neq \varnothing$, there exist $u_1, v_1 \in S$ such that $x = u_1 v_1$, $\widehat{\mathcal{K}}^m(u_1) \neq \varnothing$ and $\widehat{\chi}_S(v_1) = U$. It is easy to show that there exist $u_2, v_2, \ldots, u_m, v_m \in S$ such that, for any $l \in \{2, \ldots, m\}$, we have $u_{l-1} = u_l v_l$, $\widehat{\mathcal{K}}(u_l) \neq 0$ and $\widehat{\mathcal{K}}^{m-l+1}(v_l) \neq \varnothing$. As $\widehat{\mathcal{K}}^n(y) \neq \varnothing$, there exist $u_1', v_1' \in S$ such that $y = u_1' v_1'$, $\widehat{\mathcal{K}}(u_1') \neq \varnothing$ and $\widehat{\mathcal{K}}^{n-1}(v_1') \neq \varnothing$. Similarly, there exist $u_2', v_2', \ldots, u_{n-1}', v_{n-1}' \in S$ such that, for $l \in \{2, \ldots, n-1\}$, we have $u_{l-1} = u_l' v_l', \widehat{\mathcal{K}}(u_l') \neq \varnothing$ and $\widehat{\mathcal{K}}^{n-l}(v_l') \neq \varnothing$. Now, we have

$$\begin{aligned} \left(\widehat{\mathcal{K}}^m \circ \widehat{\chi}_S \circ \widehat{\mathcal{K}}^n\right)(a) &= \bigcup_{a=rs} \left\{\widehat{\mathcal{K}}^m \circ \widehat{\chi}_S\right)(r) \cap \widehat{\mathcal{K}}^n(s)\}, \\ &= \bigcup_{a=rs} \left\{ \bigcup_{r=u_1 v_1} \left\{\widehat{\mathcal{K}}^m(u_1) \cap \widehat{\chi}_S(v_1)\right\} \cap \bigcup_{s=u_1' v_1'} \left\{\widehat{\mathcal{K}}(u_1') \cap \widehat{\mathcal{K}}^{n-1}(v_1')\right\} \right\} \\ &= \bigcup_{a=rs} \bigcup_{r=u_1 v_1} \bigcup_{s=u_1' v_1'} \left\{\widehat{\mathcal{K}}^m(u_1) \cap S(v_1) \cap \widehat{\mathcal{K}}(u_1') \cap \widehat{\mathcal{K}}^{n-1}(v_1')\right\} \\ &= \bigcup_{a=rs} \bigcup_{r=u_1 v_1} \bigcup_{s=u_1' v_1'} \left\{\widehat{\mathcal{K}}^m(u_1) \cap \widehat{\mathcal{K}}(u_1') \cap \widehat{\mathcal{K}}^{n-1}(v_1')\right\} \\ &= \bigcup_{a=rs} \bigcup_{x=u_1 v_1} \bigcup_{y=u_1' v_1'} \left\{ \bigcup_{u_1=u_2 v_2} \left\{\widehat{\mathcal{K}}(u_2) \cap \widehat{\mathcal{K}}^{m-1}(v_2)\right\} \cap \bigcup_{u_1'=u_2' v_2'} \left\{\widehat{\mathcal{K}}(u_2') \cap \widehat{\mathcal{K}}^{n-2}(v_2')\right\} \cap \widehat{\mathcal{K}}(v_1') \right\} \\ &= \bigcup_{a=xy} \bigcup_{x=u_1 v_1} \bigcup_{y=u_1' v_1'} \bigcup_{u_1=u_2 v_2} \bigcup_{u_1'=u_2' v_2'} \left\{\widehat{\mathcal{K}}(u_2) \cap \widehat{\mathcal{K}}^{m-1}(v_2) \cap \widehat{\mathcal{K}}(u_2') \cap \widehat{\mathcal{K}}^{n-2}(v_2') \cap \widehat{\mathcal{K}}(v_1')\right\} \\ &= \bigcup_{a=xy} \bigcup_{x=u_1 v_1} \bigcup_{y=u_1' v_1'} \bigcup_{u_1=u_2 v_2} \bigcup_{u_1'=u_2' v_2'} \cdots \bigcup_{u_{m-1}=u_m, v_m} \bigcup_{u_{n-2}'=u_{n-1}' v_{n-1}'} \\ &\quad \left\{\widehat{\mathcal{K}}(u_2) \cap \widehat{\mathcal{K}}(u_3) \cap \cdots \cap \widehat{\mathcal{K}}(u_m) \cap \widehat{\mathcal{K}}(v_m) \cap \widehat{\mathcal{K}}(u_{n-1}') \cap \widehat{\mathcal{K}}(v_{n-1}') \cap \cdots \cap \widehat{\mathcal{K}}(v_2') \cap \widehat{\mathcal{K}}(v_1')\right\} \\ &\subseteq \bigcup_{a=xy} \left\{\widehat{\mathcal{K}}(u_2 u_3, \ldots, u_m v_m v_1 u_1' u_2', \ldots, u_{n-1}' v_{n-1}')\right\} \\ &= \bigcup_{a=xy} \left\{\widehat{\mathcal{K}}(xy)\right\}, \quad (\text{since } x = u_2 u_3, \ldots, u_m v_m v_1 \text{ and } y = u_1' u_2', \ldots, u_{n-1}' v_{n-1}') \\ &= \widehat{\mathcal{K}}(a). \end{aligned} \tag{13}$$

($\Leftarrow$) For any $v_1, v_2, \ldots, v_m, \kappa, \hbar_1, \hbar_2, \ldots, \hbar_n \in S$, let $a = v_1 v_2, \ldots, v_m \kappa \hbar_1 \hbar_2, \ldots, \hbar_n$. Since $(\widehat{\mathcal{K}}^m \circ \widehat{\chi}_S \circ \widehat{\mathcal{K}}^n, S) \subseteq (\widehat{\mathcal{K}}, S)$, we have

$$\begin{aligned}
\widehat{\mathscr{K}}\left(v_1v_2,\dots,v_m\kappa\hbar_1\hbar_2,\dots,\hbar_n\right) &= \widehat{\mathscr{K}}(a),\\
&\supseteq \left(\widehat{\mathscr{K}}^m\circ\widehat{\chi}_S\circ\widehat{\mathscr{K}}^n\right)(a)\\
&= \bigcup_{a=pq}\left\{\left(\widehat{\mathscr{K}}^m\circ\widehat{\chi}_S\right)(p)\cap\widehat{\mathscr{K}}^n(q)\right\}\\
&\supseteq \left\{\left(\widehat{\mathscr{K}}^m\circ\widehat{\chi}_S\right)(v_1v_2,\dots,v_m\kappa)\cap\widehat{\mathscr{K}}^n(\hbar_1\hbar_2,\dots,\hbar_n)\right\}\\
&\supseteq \left\{\bigcup_{v_1v_2,\dots,v_m\kappa=uv}\left\{\widehat{\mathscr{K}}^m(u)\cap\widehat{\chi}_S(v)\right\}\cap\bigcup_{\hbar_1\hbar_2,\dots,\hbar_n=u'v'}\left\{\widehat{\mathscr{K}}(u')\cap\widehat{\mathscr{K}}^{n-1}(v')\right\}\right\}\\
&\supseteq \left\{\left\{\widehat{\mathscr{K}}^m(v_1v_2,\dots,v_m)\cap\widehat{\chi}_S(\kappa)\right\}\cap\left\{\widehat{\mathscr{K}}(\hbar_1)\cap\widehat{\mathscr{K}}^{n-1}(\hbar_2,\dots,\hbar_{n-1}\hbar_n)\right\}\right\}\\
&\supseteq \left\{\widehat{\mathscr{K}}^m(v_1v_2,\dots,v_m)\cap\widehat{\mathscr{K}}^{n-1}(\hbar_1\hbar_2,\dots,\hbar_{n-1})\cap\widehat{\mathscr{K}}(\hbar_n)\right\}\\
&\vdots\\
&\supseteq \left\{\widehat{\mathscr{K}}(v_1)\cap\widehat{\mathscr{K}}(v_2)\cap\cdots\cap\widehat{\mathscr{K}}(v_m)\cap\widehat{\mathscr{K}}(\hbar_1)\cap\widehat{\mathscr{K}}(\hbar_2)\cap\cdots\cap\widehat{\mathscr{K}}(\hbar_n)\right\}.
\end{aligned}\tag{14}$$

Hence, $(\widehat{\mathscr{K}},S)\in\mathscr{I}_{(m,n)}(U)$. □

Definition 2. A semigroup S is called the (m,n)-regular if, $\forall a\in S\exists x\in S$ such that $a=a^mxa^n$.

Lemma 2. *If S is (m,n)-regular, $(\widehat{\mathscr{K}},S)\in\mathscr{I}_{(m,n)}(U)\nRightarrow(\widehat{\mathscr{K}},S)\in\mathscr{I}_B(U)$.*

Proof. Suppose that $(\widehat{\mathscr{K}},S)\in\mathscr{I}_{(m,n)}(U)$ and $v,\kappa,\hbar\in S$. Since S is (m,n)-regular, $v\kappa\hbar=v^mpv^n\kappa\hbar^mq\hbar^n$ for some $p,q\in S$. Therefore,

$$\begin{aligned}
\widehat{\mathscr{K}}(v\kappa\hbar) &= \widehat{\mathscr{K}}\left(v^mpv^n\kappa\hbar^mq\hbar^n\right),\\
&= \widehat{\mathscr{K}}\left(v^m\left(pv^n\kappa\hbar^mq\right)\hbar^n\right)\\
&\supseteq \left\{\widehat{\mathscr{K}}(v)\cap\widehat{\mathscr{K}}(\hbar)\right\},
\end{aligned}\tag{15}$$

as required. □

Lemma 3. *Let $(\widehat{\mathscr{K}},S)\in\mathscr{S}_S(U)$. Then, $\widehat{\mathscr{K}}(v)\subseteq\widehat{\mathscr{K}}^l(v^l)$, $\forall l\in\mathbb{Z}^+$ and $v\in S$.*

Proof. Let $v\in S$. As $v^l=vv^{l-1}$, we have

$$\begin{aligned}
\widehat{\mathscr{K}}^l\left(v^l\right) &= \bigcup_{v^l=\hbar\kappa}\left\{\widehat{\mathscr{K}}(\hbar)\cap\widehat{\mathscr{K}}^{l-1}(\kappa)\right\},\\
&\supseteq \widehat{\mathscr{K}}(v)\cap\widehat{\mathscr{K}}^{l-1}\left(v^{l-1}\right)\\
&= \widehat{\mathscr{K}}(v)\cap\bigcup_{v^{l-1}=\hbar'\kappa'}\left\{\widehat{\mathscr{K}}(\hbar')\cap\widehat{\mathscr{K}}^{l-2}(\kappa')\right\}\\
&\supseteq \widehat{\mathscr{K}}(v)\cap\widehat{\mathscr{K}}(v)\cap\widehat{\mathscr{K}}^{l-2}\left(v^{l-2}\right)\\
&\vdots\\
&\supseteq \widehat{\mathscr{K}}(v)\cap\cdots\cap\widehat{\mathscr{K}}(v)\cap\widehat{\mathscr{K}}(v)\\
&= \widehat{\mathscr{K}}(v).
\end{aligned}\tag{16}$$

□

Theorem 4. *S is (m,n)-regular $\Leftrightarrow(\widehat{\mathscr{K}},S)\subseteq(\widehat{\mathscr{K}}^m\circ\widehat{\chi}_S\circ\widehat{\mathscr{K}}^n,S)$, $\forall(\widehat{\mathscr{K}},S)\in\mathscr{S}_S(U)$.*

Proof. ($\Rightarrow$) Let $v\in S$. Then, $v=v^mxv^n$ for some $x\in S$. We have

$$\begin{aligned}
\left(\widehat{\mathscr{K}}^m\circ\widehat{\chi}_S\circ\widehat{\mathscr{K}}^n\right)(v) &= \bigcup_{v=rs}\left\{\left(\widehat{\mathscr{K}}^m\circ\widehat{\chi}_S\right)(r)\cap\widehat{\mathscr{K}}^n(s)\right\},\\
&\supseteq \left(\widehat{\mathscr{K}}^m\circ\widehat{\chi}_S\right)\left(v^mx\right)\cap\widehat{\mathscr{K}}^n\left(v^n\right)\\
&= \bigcup_{v^mx=pq}\left\{\widehat{\mathscr{K}}^m(p)\cap\widehat{\chi}_S(q)\right\}\cap\widehat{\mathscr{K}}^n\left(v^n\right)\\
&\supseteq \widehat{\mathscr{K}}^m\left(v^m\right)\cap\widehat{\chi}_S(x)\cap\widehat{\mathscr{K}}^n\left(v^n\right)\\
&= \widehat{\mathscr{K}}^m\left(v^m\right)\cap\widehat{\mathscr{K}}^n\left(v^n\right)\supseteq\widehat{\mathscr{K}}(v)\cap\widehat{\mathscr{K}}(v),\quad\text{by Lemma 3}\\
&= \widehat{\mathscr{K}}(v).
\end{aligned}\tag{17}$$

Therefore, $(\widehat{\mathscr{K}}, S) \subseteq (\widehat{\mathscr{K}}^{m} \circ \widehat{\chi}_S \circ \widehat{\mathscr{K}}^{n}, S)$.

($\Leftarrow$) Let $v \in S$. Since $(\widehat{\chi}_v, S) \in \mathscr{S}_S(U)$, so by Theorem 2, $(\widehat{\chi}_v, S) \subseteq (\widehat{\chi}_v^{m} \circ \widehat{\chi}_S \circ \widehat{\chi}_v^{n}, S)$. Therefore, $\widehat{\chi}_v(x) \subseteq \widehat{\chi}_v^{m} \circ \widehat{\chi}_S \circ \widehat{\chi}_v^{n}(x) = \chi_{v^m S v^n}(x)$. It follows that $v \in v^m S v^n$, and so, S is (m, n)-regular. □

Theorem 5. *S is (m, n)-regular $\Leftrightarrow (\widehat{\mathscr{K}}, S) = (\widehat{\mathscr{K}}^{m} \circ \widehat{\chi}_S \circ \widehat{\mathscr{K}}^{n}, S) \forall (\widehat{\mathscr{K}}, S) \in \mathscr{I}_{(m,n)}(U)$.*

Proof. ($\Rightarrow$) Suppose that S is (m, n)-regular and $(\widehat{\mathscr{K}}, S) \in \mathscr{I}_{(m,n)}(U)$. Then, by Theorems 3 and 4, $(\widehat{\mathscr{K}}^{m} \circ \widehat{\chi}_S \circ \widehat{\mathscr{K}}^{n}, S) \subseteq (\widehat{\mathscr{K}}, S)$ and $(\widehat{\mathscr{K}}, S) \subseteq (\widehat{\mathscr{K}}^{m} \circ \widehat{\chi}_S \circ \widehat{\mathscr{K}}^{n}, S)$. Hence, $(\widehat{\mathscr{K}}, S) = (\widehat{\mathscr{K}}^{m} \circ \chi_S \circ \widehat{\mathscr{K}}^{n}, S)$.

($\Leftarrow$) Suppose that $\omega \in S$. As $[\omega]_{(m,n)} \in \mathscr{M}_{(m,n)}$, by Theorem 2, $(\chi_{[\omega]_{(m,n)}}, S) \in \mathscr{I}_{(m,n)}(U)$. Thus, by hypothesis, we have

$$\chi_{[\omega]_{(m,n)}} = \chi^{m}_{[\omega]_{(m,n)}} \circ \widehat{\chi}_S \circ \chi^{n}_{[\omega]_{(m,n)}} = \chi_{([\omega]_{(m,n)})^{m} S ([\omega]_{(m,n)})^{n}}. \tag{18}$$

Therefore, $[\omega]_{(m,n)} = ([\omega]_{(m,n)})^{m} S ([\omega]_{(m,n)})^{n}$. By Lemma 1 in [4], $[\omega]_{(m,n)} = \omega^m S \omega^n$. Thus, $\omega \in \omega^m S \omega^n$, as required. □

Lemma 4. *If $(\widehat{\mathscr{K}}, S) \in \mathscr{I}_{(m,n)}(U)$ and $(\widehat{\mathscr{F}}, S)$ is an int-soft sub-semigroup over U, such that*

$$\left(\widehat{\mathscr{K}}^{m} \circ \widehat{\chi}_S \circ \widehat{\mathscr{K}}^{n}, S\right) \subseteq (\widehat{\mathscr{F}}, S) \subseteq (\widehat{\mathscr{K}}, S), \tag{19}$$

then $(\widehat{\mathscr{F}}, S) \in \mathscr{I}_{(m,n)}(U)$.

Proof. As $(\widehat{\mathscr{F}}, S)$ is an int-soft sub-semigroup over U, by Theorem 3, it is sufficient to show that $(\widehat{\mathscr{F}}^{m} \circ \widehat{\chi}_S \circ \widehat{\mathscr{F}}^{n}, S) \subseteq (\widehat{\mathscr{F}}, S)$. Now,

$$\left(\widehat{\mathscr{F}}^{m} \circ \widehat{\chi}_S \circ \widehat{\mathscr{F}}^{n}\right)(v) \subseteq \left(\widehat{\mathscr{F}}^{m} \circ \widehat{\chi}_S \circ \widehat{\mathscr{F}}^{n}\right)(v) \subseteq \widehat{\mathscr{F}}(v). \tag{20}$$

Hence, $(\widehat{\mathscr{F}}, S) \in \mathscr{I}_{(m,n)}(U)$. □

Lemma 5. *Let $(\widehat{\mathscr{K}}, S) \in \mathscr{I}_{(m,n)}(U)$ and $(\widehat{\mathscr{F}}, S) \in \mathscr{S}_S(U)$. If $(\widehat{\mathscr{K}} \circ \widehat{\mathscr{F}}, S) \subseteq (\widehat{\mathscr{K}}, S)$ or $(\widehat{\mathscr{F}} \circ \widehat{\mathscr{K}}, S) \subseteq (\widehat{\mathscr{K}}, S)$, then*

(1) $(\widehat{\mathscr{K}} \circ \widehat{\mathscr{F}}, S) \in \mathscr{I}_{(m,n)}(U)$

(2) $(\widehat{\mathscr{F}} \circ \widehat{\mathscr{K}}, S) \in \mathscr{I}_{(m,n)}(U)$

Proof. When $(\widehat{\mathscr{K}} \circ \widehat{\mathscr{F}}, S) \subseteq (\widehat{\mathscr{K}}, S)$, then we have

$$\begin{aligned} ((\widehat{\mathscr{K}} \circ \widehat{\mathscr{F}}) \circ (\widehat{\mathscr{K}} \circ \widehat{\mathscr{F}}))(v) &\subseteq (\widehat{\mathscr{K}} \circ (\widehat{\mathscr{K}} \circ \widehat{\mathscr{F}}))(v) \\ &= (\widehat{\mathscr{K}} \circ \widehat{\mathscr{K}} \circ \widehat{\mathscr{F}})(v) \\ &\subseteq (\widehat{\mathscr{K}} \circ \widehat{\mathscr{F}})(v). \end{aligned} \tag{21}$$

It follows that $(\widehat{\mathscr{K}} \circ \widehat{\mathscr{F}}, S)$ is an int-soft sub-semigroup over U. Also, we have

$$\begin{aligned} \left((\widehat{\mathscr{K}} \circ \widehat{\mathscr{F}})^{m} \circ \widehat{\chi}_S \circ (\widehat{\mathscr{K}} \circ \widehat{\mathscr{F}})^{n}\right)(v) &= \left((\widehat{\mathscr{K}} \circ \widehat{\mathscr{F}})^{m} \circ \widehat{\chi}_S \circ (\widehat{\mathscr{K}} \circ \widehat{\mathscr{F}})^{n-1} \circ (\widehat{\mathscr{K}} \circ \widehat{\mathscr{F}})\right)(v), \\ &\subseteq \left(\widehat{\mathscr{K}}^{m} \circ \widehat{\chi}_S \circ \widehat{\mathscr{K}}^{n-1} \circ (\widehat{\mathscr{K}} \circ \widehat{\mathscr{F}})\right)(v) \\ &\subseteq \left(\widehat{\mathscr{K}}^{m} \circ \widehat{\chi}_S \circ \widehat{\mathscr{K}}^{n} \circ \widehat{\mathscr{F}}\right)(v) \subseteq (\widehat{\mathscr{K}} \circ \widehat{\mathscr{F}})(v). \end{aligned} \tag{22}$$

Thus, $(\widehat{\mathscr{K}} \circ \widehat{\mathscr{F}}, S) \in \mathscr{I}_{(m,n)}(U)$. Similarly, when $(\widehat{\mathscr{F}} \circ \widehat{\mathscr{K}}, S) \subseteq (\widehat{\mathscr{K}}, S)$, then $(\widehat{\mathscr{K}} \circ \widehat{\mathscr{F}}, S) \in \mathscr{I}_{(m,n)}(U)$. Similar to (1), it can be verified. □

4. Int-Soft $(m, 0)$-Ideals and Int-Soft $(0, n)$-Ideals

Definition 3. An int-soft sub-semigroup $(\widehat{\mathscr{K}}, S)$ over S is called an int-soft $(m, 0)$-ideal over U if

$$\widehat{\mathscr{K}}(v_1 v_2, \ldots, v_m \kappa) \supseteq \widehat{\mathscr{K}}(v_1) \cap \widehat{\mathscr{K}}(v_2) \cap \cdots \cap \widehat{\mathscr{K}}(v_m), \tag{23}$$

for all $v_1, v_2, \ldots, v_m, \kappa \in S$.

An int-soft $(0, n)$-ideal can be described dually.

Whatever follows, we denote the set of all int-soft $(m, 0)$-ideals and $(0, n)$-ideals over U by $\mathscr{I}_{(m,0)}(U)$ and $\mathscr{I}_{(0,n)}(U)$.

Example 3. Let $S = \{0, v, \hbar, \kappa\}$. Define the binary operation '$\cdot$' on S as follows.

$\cdot$	0	v	$\hbar$	κ
0	0	0	0	0
v	0	0	κ	0
$\hbar$	0	0	0	0
κ	0	0	0	0

Then, S is a semigroup. Define $(\widehat{\mathscr{K}}, S), (\widehat{\mathscr{F}}, S) \in \mathscr{S}_S(U)$ as

$$\begin{aligned} \widehat{\mathscr{K}}(\omega) &= \begin{cases} U, & \text{if } \omega \in \{0, \hbar\}, \\ \varnothing, & \text{if } \omega \in \{v, \kappa\}, \end{cases} \\ \widehat{\mathscr{F}}(\omega) &= \begin{cases} V, & \text{if } \omega \in \{0, v\}, \\ \varnothing, & \text{if } \omega \in \{\hbar, \kappa\}. \end{cases} \end{aligned} \tag{24}$$

It is straightforward to verify that $(\widehat{\mathscr{K}}, S) \in \mathscr{I}_{(m,0)}(U)$ and $(\widehat{\mathscr{F}}, S) \in \mathscr{I}_{(0,n)}(U)$.

Lemma 6. *In S, $(\widehat{\mathscr{K}}, S) \in \mathscr{I}_R(U)$ (resp. $(\widehat{\mathscr{K}}, S) \in \mathscr{I}_L(U)$) $\Rightarrow (\widehat{\mathscr{K}}, S) \in \mathscr{I}_{(m,0)}(U)$ (resp. $(\widehat{\mathscr{K}}, S) \in \mathscr{I}_{(0,n)}(U)$).*

Proof (straightforward). □

Remark 2. In general, $(\widehat{\mathscr{K}}, S) \in \mathscr{I}_{(m,0)}(U)$ (resp. $(\widehat{\mathscr{K}}, S) \in \mathscr{I}_{(0,n)}(U)) \nRightarrow (\widehat{\mathscr{K}}, S) \in \mathscr{I}_R(U)$ (resp. $(\widehat{\mathscr{K}}, S) \in \mathscr{I}_L(U)$).

Example 4. In Example 3, $(\widehat{\mathscr{K}}, S) \in \mathscr{S}_S(U) \Rightarrow (\widehat{\mathscr{K}}, S) \in \mathscr{I}_{(m,0)}(U), \mathscr{I}_{(0,n)}(U)\, \forall m, n \geq 2$, but $(\widehat{\mathscr{K}}, S) \notin \mathscr{I}_R(U), \mathscr{I}_L(U)$.

Definition 4. A semigroup S is called the $(m,0)$-regular (resp. $(0,n)$-regular) if $\forall v \in S \exists \hbar \in S$ such that $v = v^m \hbar$ (resp. $v = \hbar v^n$).

Lemma 7. *The following assertions hold:*

(1) In $(m,0)$-regular S, $(\widehat{\mathscr{K}}, S) \in \mathscr{I}_{(m,0)}(U) \Rightarrow (\widehat{\mathscr{K}}, S) \in \mathscr{I}_R(U)$

(2) In $(0,n)$-regular S, $(\widehat{\mathscr{K}}, S) \in \mathscr{I}_{(0,n)}(U) \Rightarrow (\widehat{\mathscr{K}}, S) \in \mathscr{I}_L(U)$

Proof. Let $v, \hbar \in S$. Since S is $(m,0)$-regular, so $\exists \kappa \in S$ such that $v\hbar = v^m \kappa \hbar$. Therefore, we have

$$\widehat{\mathscr{K}}(v\hbar) = \widehat{\mathscr{K}}(v^m \kappa \hbar) = \widehat{\mathscr{K}}(v^m(\kappa\hbar)) \supseteq \widehat{\mathscr{K}}(v). \tag{25}$$

Hence, $(\widehat{\mathscr{K}}, S) \in \mathscr{I}_R(U)$. (2). Similarly, this can be proved. □

Lemma 8. *Let $(\varnothing \neq)\Omega \subseteq S$. Then, $\Omega \in \mathscr{M}_{(m,0)}$ (resp. $\Omega \in\in \mathscr{M}_{(0,n)}$) $\Leftrightarrow$ the $(\widehat{\chi_\Omega}, S) \in \mathscr{I}_{(m,0)}(U)$ (resp. $(\widehat{\chi_\Omega}, S) \in \mathscr{I}_{(0,n)}(U)$).*

Proof. $(\Rightarrow)$ Let $v_1, v_2, \ldots, v_m, \kappa \in S$. If $x_k \notin \Omega$, for some $k \in \{1, 2, \ldots, m\}$, then $\widehat{\chi_\Omega}(v_1 v_2, \ldots, v_m \kappa) \supseteq \widehat{\chi_\Omega}(v_1) \cap \widehat{\chi_\Omega}(v_2) \cap \cdots \cap \widehat{\chi_\Omega}(v_m)$. If $x_k \in \Omega$ for each $k \in \{1, 2, \ldots, m\}$, then $v_1 v_2, \ldots, v_m \kappa \in \Omega^m S \subseteq \Omega$. Therefore,

$$\widehat{\chi_\Omega}(v_1 v_2, \ldots, v_m c) = 1 \supseteq \widehat{\chi_\Omega}(v_1) \cap \widehat{\chi_\Omega}(v_2) \cap \cdots \cap \widehat{\chi_\Omega}(v_m). \tag{26}$$

Hence, $(\widehat{\chi_\Omega}, S) \in \mathscr{I}_{(m,0)}(U)$.

$(\Leftarrow)$ Let $v_1, v_2, \ldots, v_m \in \Omega$ and $\kappa \in S$. Then, $\widehat{\chi_\Omega}(v_1 v_2, \ldots, v_m c) \supseteq \widehat{\chi_\Omega}(v_1) \cap \widehat{\chi_\Omega}(v_2) \cap \cdots \cap \widehat{\chi_\Omega}(v_m) = 1$ implies $\widehat{\chi_\Omega}(v_1 v_2, \ldots, v_m c) = 1$. Therefore, $v_1 v_2, \ldots, v_m c \in \Omega$. Thus, $\Omega^m S \subseteq \Omega$, as required. □

Theorem 6. *Let $(\widehat{\mathscr{K}}, S)$ be any int-soft sub-semigroup over U. Then, $(\widehat{\mathscr{K}}, S) \in \mathscr{I}_{(m,0)}(U)$ (resp. $(\widehat{\mathscr{K}}, S) \in \mathscr{I}_{(0,n)}(U)$) $\Leftrightarrow (\widehat{\mathscr{K}}^m \circ \widehat{\chi_S}, S) \subseteq (\widehat{\mathscr{K}}, S)$ (resp. $(\widehat{\chi_S} \circ \widehat{\mathscr{K}}^n, S) \subseteq (\widehat{\mathscr{K}}, S)$).*

Proof. It is similar to the proof of Theorem 3. □

Lemma 9. *Let S be (m,n)-regular, $(\widehat{\mathscr{K}}, S) \in \mathscr{I}_{(m,0)}(U)$, and $(\widehat{\mathscr{F}}, S) \in \mathscr{I}_{(0,n)}(U)$. Then, $(\widehat{\mathscr{K}}, S) = (\widehat{\mathscr{K}} \circ \widehat{\mathscr{K}}, S)$ and $(\widehat{\mathscr{F}}, S) = (\widehat{\mathscr{F}} \circ \widehat{\mathscr{F}}, S)$.*

Proof. Let $(\widehat{\mathscr{K}}, S) \in \mathscr{I}_{(m,0)}(U)$. Then, $(\widehat{\mathscr{K}} \circ \widehat{\mathscr{K}}, S) \subseteq (\widehat{\mathscr{K}}, S)$. We have

$$\begin{aligned}
\widehat{\mathscr{K}}(x) &\subseteq \left(\widehat{\mathscr{K}}^m \circ \widehat{\chi_S} \circ \widehat{\mathscr{K}}^n\right)(x) = \left(\widehat{\mathscr{K}}^m \circ \widehat{\chi_S} \circ \widehat{\mathscr{K}}^{n-1} \circ \widehat{\mathscr{K}}\right)(x), \\
&\subseteq \left(\widehat{\mathscr{K}}^m \circ \widehat{\chi_S} \circ \widehat{\mathscr{K}}^{n-1} \circ \widehat{\mathscr{K}}^m \circ \widehat{\chi_S} \circ \widehat{\mathscr{K}}^n\right)(x) \\
&\subseteq \left(\widehat{\mathscr{K}}^m \circ \widehat{\chi_S} \circ \widehat{\mathscr{K}}^m \circ \widehat{\chi_S}\right)(x) \\
&\subseteq (\widehat{\mathscr{K}} \circ \widehat{\mathscr{K}})(x),
\end{aligned} \tag{27}$$

so we obtain $(\widehat{\mathscr{K}}, S) \subseteq (\widehat{\mathscr{K}} \circ \widehat{\mathscr{K}}, S)$. Hence, $(\widehat{\mathscr{K}}, S) = (\widehat{\mathscr{K}} \circ \widehat{\mathscr{K}}, S)$. □

Theorem 7. *In S, the following assertions are true:*

(1) S is $(m,0)$-regular $\Leftrightarrow (\widehat{\mathscr{K}}, S) \subseteq (\widehat{\mathscr{K}}^m \circ \widehat{\chi_S}, S)$, $\forall (\widehat{\mathscr{K}}, S) \in \mathscr{S}_S(U)$

(2) S is $(0,n)$-regular $\Leftrightarrow (\widehat{\mathscr{K}}, S) \subseteq (\widehat{\chi_S} \circ \widehat{\mathscr{K}}^n, S)$, $\forall (\widehat{\mathscr{K}}, S) \in \mathscr{S}_S(U)$

Proof. $(\Rightarrow)$ Let $v \in S$. Then, $\exists \hbar \in S$ such that $v = v^m \hbar$. Now, we have

$$\begin{aligned}
\left(\widehat{\mathscr{K}}^m \circ \widehat{\chi_S}\right)(v) &= \bigcup_{v = \kappa s} \left\{\left(\widehat{\mathscr{K}}^m\right)(\kappa) \cap \widehat{\chi_S}(s)\right\}, \\
&\supseteq \widehat{\mathscr{K}}^m(v^m) \cap \widehat{\chi_S}(\hbar) \\
&= \widehat{\mathscr{K}}^m(v^m) \\
&\supseteq \widehat{\mathscr{K}}(v).
\end{aligned} \tag{28}$$

Therefore, $(\widehat{\mathscr{K}}, S) \subseteq (\widehat{\mathscr{K}}^m \circ \widehat{\chi_S}, S)$.

$(\Leftarrow)$ Take any $v \in S$. Since $(\widehat{\chi_v}, S) \in \mathscr{S}_S(U)$, $(\widehat{\chi_v}, S) \subseteq (\widehat{\chi_v}^m \circ S, S)$. Therefore, $\widehat{\chi_v}(\hbar) \subseteq \widehat{\chi_v}^m \circ S(\hbar) = \widehat{\chi_{v^m S}}(\hbar)$. It follows that $v \in v^m S$ and so, S is $(m,0)$-regular. Similar to (1), (2) can be verified. □

Theorem 8. *The following assertions are true in S:*

(1) S is $(m,0)$-regular $\Leftrightarrow (\widehat{\mathscr{K}}, S) = (\widehat{\mathscr{K}}^m \circ \widehat{\chi_S}, S)$, $\forall (\widehat{\mathscr{K}}, S) \in \mathscr{I}_{(m,0)}(U)$

(2) S is $(0,n)$-regular $\Leftrightarrow (\widehat{\mathscr{K}}, S) = (\widehat{\chi_S} \circ \widehat{\mathscr{K}}^n, S)$, $\forall (\widehat{\mathscr{K}}, S) \in \mathscr{I}_{(0,n)}(U)$

Proof. (1) $(\Rightarrow)$ Suppose that S is $(m,0)$-regular and $(\widehat{\mathscr{K}}, S) \in \mathscr{I}_{(m,0)}(U)$. Then, by Theorems 7 and 6, we have $(\widehat{\mathscr{K}}, S) \subseteq (\widehat{\mathscr{K}}^m \circ \widehat{\chi_S}, S)$ and $(\widehat{\mathscr{K}}^m \circ \widehat{\chi_S}, S) \subseteq (\widehat{\mathscr{K}}, S)$. Hence, $(\widehat{\mathscr{K}}, S) = (\widehat{\mathscr{K}}^m \circ \widehat{\chi_S}, S)$.

$(\Leftarrow)$ Take $R \in\in \mathscr{M}_{(m,0)}$. By Lemma 8, $(\chi_R, S) \in \mathscr{I}_{(m,0)}(U)$. By hypothesis $(\chi_R, S) = (\chi_R^m \circ \widehat{\chi_S}, S)$. So, $\chi_R(\hbar) = \chi_R^m \circ \widehat{\chi_S}(\hbar) = \chi_{R^m S}(\hbar)$, and it follows that $R^m S = R$. Therefore, by Theorem 1 in [45], S is (m,n)-regular. Similar to (1), (2) can be verified. □

Theorem 9. *S is (m,n)-regular $\Leftrightarrow (\widehat{\mathscr{K}} \cap \widehat{\mathscr{F}}, S) = (\widehat{\mathscr{K}}^m \circ \widehat{\mathscr{F}}^n, S)$, $\forall (\widehat{\mathscr{K}}, S) \in \mathscr{I}_{(m,0)}(U)$, and $(\widehat{\mathscr{F}}, S) \in \mathscr{I}_{(0,n)}(U)$.*

Proof. $(\Rightarrow)$ Suppose that $(\widehat{\mathscr{K}}, S) \in \mathscr{I}_{(m,0)}(U)$ and $(\widehat{\mathscr{F}}, S) \in \mathscr{I}_{(0,n)}(U)$. As S is (m,n)-regular, we have

$$(\widehat{\mathcal{K}}\cap\widehat{\mathcal{F}},S)\subseteq\left(\left(\widehat{\mathcal{K}}\cap\widehat{\mathcal{F}}\right)^m\circ\widehat{\chi}_S\circ\left(\widehat{\mathcal{K}}\cap\widehat{\mathcal{F}}\right)^n,S\right)\subseteq\left(\widehat{\mathcal{K}}^m\circ\widehat{\chi}_S\circ\widehat{\mathcal{F}}^n,S\right). \tag{29}$$

By Theorem 8 and Lemma 9, we have $(\widehat{\chi}_S\circ\widehat{\mathcal{F}}^n,S)=(\widehat{\mathcal{F}},S)$ and $(\widehat{\mathcal{F}},S)=(\widehat{\mathcal{F}}^n,S)$. Therefore, $(\widehat{\mathcal{K}}\cap\widehat{\mathcal{F}},S)\subseteq(\widehat{\mathcal{K}}^m\circ\widehat{\mathcal{F}}^n,S)$. Also, $(\widehat{\mathcal{K}}^m\circ\widehat{\mathcal{F}}^n,S)\subseteq(\widehat{\mathcal{K}}\cap\widehat{\mathcal{F}},S)$. Therefore, $(\widehat{\mathcal{K}}\cap\widehat{\mathcal{F}},S)=(\widehat{\mathcal{K}}^m\circ\widehat{\mathcal{F}}^n,S)$.

($\Leftarrow$) Take $R\in\mathcal{M}_{(m,0)}$ and $L\in\mathcal{M}_{(0,n)}$. By Lemma 2, $(\widehat{\chi}_R,S)\in\mathcal{I}_{(m,0)}(U)$ and $(\widehat{\chi}_L,S)\in\mathcal{I}_{(0,n)}(U)$. By hypothesis, we have

$$\widehat{\chi}_{R\cap L}=\widehat{\chi}_R\wedge\widehat{\chi}_L=\widehat{\chi}_R^{\;m}\circ\widehat{\chi}_L^{\;n}=\widehat{\chi}_{R^mL^n}, \tag{30}$$

it follows that $R\cap L=R^mL^n$. Thus, by Theorem 12 in [44], S is (m,n)-regular. □

Corollary 1. *If S is (m,n)-regular, then $(\widehat{\mathcal{K}}\cap\widehat{\mathcal{F}},S)=(\widehat{\mathcal{K}}\circ\widehat{\mathcal{F}},S)$, $\forall(\widehat{\mathcal{K}},S)\in\mathcal{I}_{(m,0)}(U)$ and $(\widehat{\mathcal{F}},S)\in\mathcal{I}_{(0,n)}(U)$.*

Theorem 10. *S is (m,n)-regular $\Leftrightarrow(\widehat{\mathcal{K}}\cap\widehat{\mathcal{F}},S)=(\widehat{\mathcal{K}}^m\circ\widehat{\mathcal{F}}\cap\widehat{\mathcal{K}}\circ\widehat{\mathcal{F}}^n,S)\,\forall\,(\widehat{\mathcal{K}},S)\in\mathcal{I}_{(m,0)}(U)$ and $(\widehat{\mathcal{F}},S)\in\mathcal{I}_{(0,n)}(U)$.*

Proof. ($\Rightarrow$) Suppose that $(\widehat{\mathcal{K}},S)\in\mathcal{I}_{(m,0)}(U)$ and $(\widehat{\mathcal{F}},S)\in\mathcal{I}_{(0,n)}(U)$. As S is (m,n)-regular, we have

$$(\widehat{\mathcal{K}}\cap\widehat{\mathcal{F}},S)\subseteq\left(\left(\widehat{\mathcal{K}}\cap\widehat{\mathcal{F}}\right)^m\circ\widehat{\chi}_S\circ\left(\widehat{\mathcal{K}}\cap\widehat{\mathcal{F}}\right)^n,S\right)\subseteq\left(\widehat{\mathcal{K}}^m\circ\widehat{\chi}_S\circ\widehat{\mathcal{F}}^n,S\right)\subseteq\left(\widehat{\mathcal{K}}^m\circ\widehat{\mathcal{F}},S\right), \tag{31}$$

and so, $(\widehat{\mathcal{K}}\cap\widehat{\mathcal{F}},S)\subseteq(\widehat{\mathcal{K}}^m\circ\widehat{\mathcal{F}},S)$. Similarly, $(\widehat{\mathcal{K}}\cap\widehat{\mathcal{F}},S)\subseteq(\widehat{\mathcal{K}}\circ\widehat{\mathcal{F}}^n,S)$. Thus, $(\widehat{\mathcal{K}}\cap\widehat{\mathcal{F}},S)\subseteq(\widehat{\mathcal{K}}^m\circ\widehat{\mathcal{F}}\cap\widehat{\mathcal{K}}\circ\widehat{\mathcal{F}}^n,S)$. Since $(\widehat{\mathcal{K}},S)\in\mathcal{I}_{(m,0)}(U)$ and $(\widehat{\mathcal{F}},S)\in\mathcal{I}_{(0,n)}(U)$, the reverse inclusion holds. Hence, $(\widehat{\mathcal{K}}\cap\widehat{\mathcal{F}},S)=(\widehat{\mathcal{K}}^m\circ\widehat{\mathcal{F}}\cap\widehat{\mathcal{K}}\circ\widehat{\mathcal{F}}^n,S)$.

($\Leftarrow$) Take $R\in\mathcal{M}_{(m,0)}$ and $L\in\mathcal{M}_{(0,n)}$. By Lemma 8, $(\widehat{\chi}_R,S)\in\mathcal{I}_{(m,0)}(U)$ and $(\widehat{\chi}_L,S)\in\mathcal{I}_{(0,n)}(U)$. Observe that, by hypothesis, we have

$$\widehat{\chi}_{R\cap L}=\widehat{\chi}_R\cap\widehat{\chi}_L=\widehat{\chi}_R^{\;m}\circ\widehat{\chi}_L\cap\widehat{\chi}_R\circ\widehat{\chi}_L^{\;n}=\widehat{\chi}_{R^mL\cap RL^n}, \tag{32}$$

and it follows that $R\cap L=R^mL\cap RL^n$. Therefore, by Theorem 3 in [45], S is (m,n)-regular. □

Lemma 10. *For $(\widehat{\mathcal{K}},S)\in\mathcal{S}_S(U)$, $(\widehat{\mathcal{K}}\cup\widehat{\mathcal{K}}^m\circ\widehat{\chi}_S,S)\in\mathcal{I}_{(m,0)}(U)$ (resp. $(\widehat{\mathcal{K}}\cup\widehat{\chi}_S\circ\widehat{\mathcal{K}}^n,S)\in\mathcal{I}_{(0,n)}(U)$).*

Proof (straightforward). □

Lemma 11. *In (m,n)-regular semigroup S, for each $(\widehat{\mathcal{K}},S)\in\mathcal{I}_{(m,n)}(U)$, there exist $(\widehat{\mathcal{G}},S)\in\mathcal{I}_{(m,0)}(U)$ and $(\widehat{\mathcal{F}},S)\in\mathcal{I}_{(0,n)}(U)$ such that $(\widehat{\mathcal{K}},S)=(\widehat{\mathcal{G}}\circ\widehat{\mathcal{F}},S)$.*

Proof. Suppose that $(\widehat{\mathcal{K}},S)\in\mathcal{I}_{(m,n)}(U)$. Then, $(\widehat{\mathcal{K}}^m\circ\widehat{\chi}_S\circ\widehat{\mathcal{K}}^n,S)\subseteq(\widehat{\mathcal{K}},S)$. As S is (m,n)-regular, $(\widehat{\mathcal{K}},S)\subseteq(\widehat{\mathcal{K}}^m\circ\widehat{\chi}_S\circ\widehat{\mathcal{K}}^n,S)$. Therefore, $(\widehat{\mathcal{K}},S)=(\widehat{\mathcal{K}}^m\circ\widehat{\chi}_S\circ\widehat{\mathcal{K}}^n,S)$. Let $(\widehat{\mathcal{G}},S)=(\widehat{\mathcal{K}}\cup\widehat{\mathcal{K}}^m\circ\widehat{\chi}_S,S)$ and $(\widehat{\mathcal{F}},S)=(\widehat{\mathcal{K}}\cup\widehat{\chi}_S\circ\widehat{\mathcal{K}}^n,S)$. By Lemma 9, $(\widehat{\mathcal{G}},S)\in\mathcal{I}_{(m,0)}(U)$ and $(\widehat{\mathcal{F}},S)\in\mathcal{I}_{(0,n)}(U)$. Since S is (m,n)-regular, $(\widehat{\mathcal{G}},S)=(\widehat{\mathcal{K}}\cup\widehat{\mathcal{K}}^m\circ\widehat{\chi}_S,S)=(\widehat{\mathcal{K}}^m\circ\widehat{\chi}_S,S)$ and $(\widehat{\mathcal{F}},S)=(\widehat{\mathcal{K}}\cup\widehat{\chi}_S\circ\widehat{\mathcal{K}}^n,S)=(\widehat{\chi}_S\circ\widehat{\mathcal{K}}^n,S)$, so

$$(\widehat{\mathcal{G}}\circ\widehat{\mathcal{F}},S)=\left(\widehat{\mathcal{K}}^m\circ S\circ S\circ\widehat{\mathcal{K}}^n,S\right)=\left(\widehat{\mathcal{K}}^m\circ S\circ\widehat{\mathcal{K}}^n,S\right)=(\widehat{\mathcal{K}},S), \tag{33}$$

as required. □

Lemma 12. *In (m,n)-regular semigroup S, $\forall(\widehat{\mathcal{K}},S)\in\mathcal{I}_{(m,0)}(U)$ and $(\widehat{\mathcal{F}},S)\in\mathcal{S}_S(U)$, $(\widehat{\mathcal{K}}\circ\widehat{\mathcal{F}},S)\in\mathcal{I}_{(m,n)}(U)$.*

Proof. Let $(\widehat{\mathcal{K}},S)\in\mathcal{I}_{(m,0)}(U)$ and $(\widehat{\mathcal{F}},S)\in\mathcal{S}_S(U)$. Now,

$$\begin{aligned}
\left(\left(\widehat{\mathcal{K}}\circ\widehat{\mathcal{F}}\right)^m\circ\widehat{\chi}_S\circ\left(\widehat{\mathcal{K}}\circ\widehat{\mathcal{F}}\right)^n\right)(v)&=\left(\underbrace{(\widehat{\mathcal{K}}\circ\widehat{\mathcal{F}})\circ(\widehat{\mathcal{K}}\circ\widehat{\mathcal{F}})\circ\cdots\circ(\widehat{\mathcal{K}}\circ\widehat{\mathcal{F}})}_{m\text{-times}}\circ\widehat{\chi}_S\circ\underbrace{(\widehat{\mathcal{K}}\circ\widehat{\mathcal{F}})\circ(\widehat{\mathcal{K}}\circ\widehat{\mathcal{F}})\circ\cdots\circ(\widehat{\mathcal{K}}\circ\widehat{\mathcal{F}})}_{n\text{-times}}\right)(v),\\
&=\left((\widehat{\mathcal{K}}\circ\widehat{\mathcal{F}})\circ\underbrace{(\widehat{\mathcal{K}}\circ\widehat{\mathcal{F}})\circ(\widehat{\mathcal{K}}\circ\widehat{\mathcal{F}})\circ\cdots\circ(\widehat{\mathcal{K}}\circ\widehat{\mathcal{F}})}_{m-1\text{-times}}\circ\widehat{\chi}_S\circ\underbrace{(\widehat{\mathcal{K}}\circ\widehat{\mathcal{F}})\circ(\widehat{\mathcal{K}}\circ\widehat{\mathcal{F}})\circ\cdots\circ(\widehat{\mathcal{K}}\circ\widehat{\mathcal{F}})}_{n-1\text{-times}}\circ(\widehat{\mathcal{K}}\circ\widehat{\mathcal{F}})\right)(v)\\
&\subseteq\left((\widehat{\mathcal{K}}\circ\widehat{\mathcal{F}})\circ\widehat{\chi}_S\circ\widehat{\chi}_S\circ\widehat{\chi}_S\circ(\widehat{\mathcal{K}}\circ\widehat{\mathcal{F}})\right)(v)\subseteq\left(\widehat{\mathcal{K}}\circ\widehat{\chi}_S\circ\widehat{\mathcal{F}}\right)(v)\\
&\subseteq\left(\widehat{\mathcal{K}}^m\circ\widehat{\chi}_S\circ\widehat{\mathcal{F}}\right)(v)\quad\text{(by Lemma 9)}\\
&\subseteq(\widehat{\mathcal{K}}\circ\widehat{\mathcal{F}})(v).
\end{aligned} \tag{34}$$

Therefore, $(\widehat{\mathscr{K}}\circ\widehat{\mathscr{F}}, S) \in \mathscr{I}_{(m,n)}(U)$.

By Lemmas 11 and 12, we have the following. □

Theorem 11. *Let S be a (m,n)-regular and $(\widehat{\mathscr{K}}, S) \in \mathscr{S}_S(U)$. Then, $(\widehat{\mathscr{K}}, S) \in \mathscr{I}_{(m,n)}(U) \Leftrightarrow$ there exist $(\widehat{\mathscr{F}}, S) \in \mathscr{I}_{(m,0)}(U)$ and $(\widehat{\mathscr{F}}, S) \in \mathscr{I}_{(0,n)}(U)$ such that $(\widehat{\mathscr{K}}, S) = (\widehat{\mathscr{F}}\circ\widehat{\mathscr{F}}, S)$.*

Definition 5. An int-soft (m,n)-ideal $(\widehat{\mathscr{K}}, S)$ over U is called minimal if, for all int-soft (m,n)-ideal $(\widehat{\mathscr{K}}', S)$ over U, $(\widehat{\mathscr{K}}', S) \subseteq (\widehat{\mathscr{K}}, S)$ implies $(\widehat{\mathscr{K}}', S) = (\widehat{\mathscr{K}}, S)$.

Dually, a minimum int-soft $(m,0)$-ideal and minimal int-soft $(0,n)$-ideal over U can be described.

Theorem 12. *In (m,n)-regular semigroup S, a soft set $(\widehat{\mathscr{K}}, tS)$ over U is a minimal int-soft (m,n)-ideal over $U \Leftrightarrow$ there exist a minimal int-soft $(m,0)$-ideal $(\widehat{\mathscr{G}}, S)$ and a minimal int-soft $(0,n)$-ideal $(\widehat{\mathscr{F}}, S)$ over U such that $(\widehat{\mathscr{K}}, S) = (\widehat{\mathscr{G}}\circ\widehat{\mathscr{F}}, S)$.*

Proof. $(\Rightarrow)$ Let $(\widehat{\mathscr{K}}, S) \in \mathscr{I}_{(m,n)(U)}$ be minimal. By Lemma 11, $(\widehat{\mathscr{K}}, S) = (\widehat{\mathscr{K}} \cup \widehat{\mathscr{K}}^m \circ \widehat{\chi}_S, S) \circ (\widehat{\mathscr{K}} \cup \widehat{\chi}_S \circ \widehat{\mathscr{K}}^n, S)$. We show that $(\widehat{\mathscr{K}} \cup \widehat{\mathscr{K}}^m \circ \widehat{\chi}_S, S) \in \mathscr{I}_{(m,0)(U)}$ is minimal. To show this, let $(\widehat{\mathscr{K}}', S) \in \mathscr{I}_{(m,0)}(U)$ such that $(\widehat{\mathscr{K}}', S) \subseteq (\widehat{\mathscr{K}} \cup \widehat{\mathscr{K}}^m \circ \widehat{\chi}_S, S)$. Since S is (m,n)-regular, so, by Corollary 1, $(\widehat{\mathscr{K}} \cup \widehat{\mathscr{K}}^m \circ \widehat{\chi}_S, S) \cap (\widehat{\mathscr{K}} \cup \widehat{\chi}_S \circ \widehat{\mathscr{K}}^n, S) = (\widehat{\mathscr{K}} \cup \widehat{\mathscr{K}}^m \circ \widehat{\chi}_S, S) \circ (\widehat{\mathscr{K}} \cup \widehat{\chi}_S \circ \widehat{\mathscr{K}}^n, S)$. Again, by Corollary 1, $(\widehat{\mathscr{K}}', S) \circ (\widehat{\mathscr{K}} \cup \widehat{\chi}_S \circ \widehat{\mathscr{K}}^n, S) = (\widehat{\mathscr{K}}', S) \cap (\widehat{\mathscr{K}} \cup \widehat{\chi}_S \circ \widehat{\mathscr{K}}^n, S) \subseteq (\widehat{\mathscr{K}} \cup \widehat{\mathscr{K}}^m \circ \widehat{\chi}_S, S) \cap (\widehat{\mathscr{K}} \cup \widehat{\chi}_S \circ \widehat{\mathscr{K}}^n, S) = (\widehat{\mathscr{K}}, S)$. By Lemma 12, $(\widehat{\mathscr{K}}', S) \circ (\widehat{\mathscr{K}} \cup \widehat{\chi}_S \circ \widehat{\mathscr{K}}^n, S) \in \mathscr{I}_{(m,n)}(U)$. Since $(\widehat{\mathscr{K}}', S) \circ (\widehat{\mathscr{K}} \cup \widehat{\chi}_S \circ \widehat{\mathscr{K}}^n, S) \subseteq (\widehat{\mathscr{K}}, S)$, by minimality of the int-soft (m,n)-ideal $(\widehat{\mathscr{K}}, S)$ over U, we have $(\widehat{\mathscr{K}}', S) \circ (\widehat{\mathscr{K}} \cup \widehat{\chi}_S \circ \widehat{\mathscr{K}}^n, S) = (\widehat{\mathscr{K}}, S)$. Therefore, $(\widehat{\mathscr{K}} \cup \widehat{\mathscr{K}}^m \circ \widehat{\chi}_S, S) \cap (\widehat{\mathscr{K}} \cup \widehat{\chi}_S \circ \widehat{\mathscr{K}}^n, S) = (\widehat{\mathscr{K}}', S) \cap (\widehat{\mathscr{K}} \cup \widehat{\chi}_S \circ \widehat{\mathscr{K}}^n, S)$. As $(\widehat{\mathscr{K}}, S) \subseteq (\widehat{\mathscr{K}} \cup \widehat{\mathscr{K}}^m \circ \widehat{\chi}_S, S) \cap (\widehat{\mathscr{K}} \cup \widehat{\chi}_S \circ \widehat{\mathscr{K}}^n, S)$, we have $(\widehat{\mathscr{K}}, S) \subseteq (\widehat{\mathscr{K}}', S)$. So, $(\widehat{\mathscr{K}} \cup \widehat{\mathscr{K}}^m \circ \widehat{\chi}_S, S) \subseteq (\widehat{\mathscr{K}}', S)$. Hence, $(\widehat{\mathscr{K}}', S) = (\widehat{\mathscr{K}}, S)$. Thus, $(\widehat{\mathscr{K}} \cup \widehat{\mathscr{K}}^m \circ \widehat{\chi}_S, S) \in \mathscr{I}_{(m,0)(U)}$ is minimal. Similarly, $(\widehat{\mathscr{K}} \cup \widehat{\chi}_S \circ \widehat{\mathscr{K}}^n, S) \in \mathscr{I}_{(0,n)(U)}$ is minimal.

$(\Leftarrow)$ Assume that $(\widehat{\mathscr{K}}, S) = (\widehat{\mathscr{G}}\circ\widehat{\mathscr{F}}, S)$ for some minimal int-soft $(m,0)$-ideal $(\widehat{\mathscr{G}}, S)$ and minimal int-soft $(0,n)$-ideal $(\widehat{\mathscr{F}}, S)$ over U. By Lemma 11, $(\widehat{\mathscr{K}}, S) \in \mathscr{I}_{(m,n)}(U)$. To show that $(\widehat{\mathscr{K}}, S) \in \mathscr{I}_{(m,n)(U)}$ is minimal, let $(\widehat{\mathscr{W}}, S) \in \mathscr{I}_{(m,n)}(U)$ such that $(\widehat{\mathscr{W}}, S) \subseteq (\widehat{\mathscr{K}}, S)$. Then, $(\widehat{\mathscr{W}}^m \circ \widehat{\chi}_S, S) \subseteq (\widehat{\mathscr{K}}^m \circ \widehat{\chi}_S, S) \subseteq ((\widehat{\mathscr{G}}\circ\widehat{\mathscr{F}})^m \circ \widehat{\chi}_S, S) = ((\widehat{\mathscr{G}}\circ\widehat{\mathscr{F}}) \circ (\widehat{\mathscr{G}}\circ\widehat{\mathscr{F}}) \circ \cdots \circ (\widehat{\mathscr{G}}\circ\widehat{\mathscr{F}}) \circ \widehat{\chi}_S) \subseteq ((\widehat{\mathscr{G}}\circ\widehat{\mathscr{F}}) \circ (\widehat{\mathscr{G}}\circ\widehat{\mathscr{F}}) \circ \cdots \circ (\widehat{\mathscr{G}}\circ\widehat{\mathscr{F}}) \circ \widehat{\chi}_S \subseteq \widehat{\mathscr{G}} \circ \widehat{\chi}_S \subseteq (\widehat{\mathscr{G}}^m \circ \widehat{\chi}_S \circ \widehat{\mathscr{G}}^n) \circ \widehat{\chi}_S \subseteq \widehat{\mathscr{G}}^m \circ \widehat{\chi}_S \subseteq \widehat{\mathscr{G}}$.

As $(\widehat{\mathscr{W}}^m \circ \widehat{\chi}_S, S) \in \mathscr{I}_{(m,0)}(U)$ and $(\widehat{\mathscr{G}}, S) \in \mathscr{I}_{(m,0)(U)}$ is minimal, $(\widehat{\mathscr{W}}^m \circ \widehat{\chi}_S, S) = (\widehat{\mathscr{G}}, S)$. Similarly, $(\widehat{\chi}_S \circ \widehat{\mathscr{W}}^n, S) = (\widehat{\mathscr{F}}, S)$. Now, $(\widehat{\mathscr{K}}, S) = (\widehat{\mathscr{G}}\circ\widehat{\mathscr{F}}, S) = ((\widehat{\mathscr{W}}^m \circ \widehat{\chi}_S) \circ (\widehat{\chi}_S \circ \widehat{\mathscr{W}}^n), S) \subseteq (\widehat{\mathscr{W}}^m \circ \widehat{\chi}_S \circ \widehat{\mathscr{W}}^n, S) \subseteq (\widehat{\mathscr{W}}, S)$. Hence, $(\widehat{\mathscr{K}}, S) \in \mathscr{I}_{(m,n)(U)}$ is minimal. □

Corollary 2. *There is at least one minimal int-soft (m,n)-ideal over U in (m,n)-regular semigroup $S \Leftrightarrow S$ has at least one minimal int-soft $(m,0)$-ideal and one minimal int-soft $(0,n)$-ideal over U.*

5. Conclusion

The main purpose of this article is to present in semigroups the ideas of int-soft (m,n)-ideals, int-soft $(m,0)$-ideals, and int-soft $(0,n)$-ideals. If we take $m = 1 = n$ in the int-soft (m,n)-ideals, int-soft $(m,0)$-ideals, and int-soft $(0,n)$-ideals in particular, then we get the int-soft bi-ideals, int-soft right ideals, and int-soft left ideals. The ideas proposed in this paper can also be seen to be more general than int-soft bi-ideals, int-soft right ideals, and int-soft left ideals. Also, if we place $m = 1 = n$ in the results of this paper, then the results of [8] are deduced as corollaries, which is the main application of the results of this paper.

In the future work, one can further study these concepts to various algebraic structures such as semi-hypergroups, semi-hyperrings, rings, LA-semigroups, BL-algebras, MTL-algebras, R0-algebras, MV-algebras, EQ-algebras, and lattice implication algebras.

Data Availability

No data were used to support the study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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