Research Article

A Nonlinear Implicit Fractional Equation with Caputo Derivative

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In this paper, we study a nonlinear implicit differential equation with initial conditions. The considered problem involves the fractional Caputo derivatives under some conditions on the order. We prove an existence and uniqueness analytic result by application of Banach principle. Then, another result that deals with the existence of at least one solution is delivered and some sufficient conditions for this result are established by means of the fixed point theorem of Schaefer. At the end, we discuss two examples to illustrate the applicability of the main results.

1. Introduction

The theory of differential equations of fractional order and fractional calculus is very important since they can be used in analyzing and modeling real word phenomena. Recently, several researchers are interested in the important progress of differential equations of fractional order. For more information on these works and their applications, one can consult the references [1–9]. In particular, research on the existence of unique solutions for fractional differential equations is of big importance since it helps physicians to better understand the behaviour of real phenomena. See, for more details, the references [10–14].

The motivation for this work arises from both the development of the theory of fractional calculus itself and its wide applications to various fields of science, such as physics, chemistry, biology, electromagnetism of complex media, robotics, and economics.

Much attention has been paid to the existence and uniqueness of solutions of fractional dynamical systems [15–18] due to the fact that existence is the fundamental problem and a necessary condition for considering some other properties for fractional dynamical systems, such as controllability and stability. Chai [19] provided sufficient conditions for the existence of solutions to a class of anti-periodic boundary value problems for fractional differential equations, while Sheng and Jiang [20] considered a class of initial value problems for fractional differential systems. There are several operators studied in the field of fractional calculus, for example, see [21–26], but the difference in this work is that the operator considered is in the sense of Caputo derivative.

Motivated by the works of Benchohra et al. [27], we will establish in this paper existence and uniqueness results of the solutions of the fractional dynamical system with Caputo fractional derivative

$$\begin{cases} D^\alpha x(t) - AD^\beta x(t) = f(t, x(t), D^\beta x(t), D^\alpha x(t)), & t \in I = [0, 1], \\ x(0) = x_0, \\ x'(0) = x'_0, \\ x''(0) = x''_0, \\ x'''(0) = x'''_0, \end{cases} \quad (1)$$

where D^α is in the sense of Caputo, $f: I \times \mathbb{R}^n \times \mathbb{R}^n \longrightarrow \mathbb{R}^n$ is a given function, $x_0, x'_0, x''_0 \in \mathbb{R}^n$, A is an $n \times n$ matrix, and $1 < \beta < 2$, $3 < \alpha < 4$, with $\beta + 2 < \alpha$.

Rest of the paper is organised as follows: in Section 2, we recall some results and definitions which we use for the proof of our main results. In Section 3, we give and prove the main theorems of this paper, and we discuss some illustrative examples.

2. Preliminaries

In this section, we introduce some definitions, lemmas, and preliminaries facts which are used throughout this paper. See [7] for more information. Let $|\cdot|$ be a suitable norm in $\mathbb{R}^n$ and $\|\cdot\|$ be the matrix norm. We denote by $C(I, \mathbb{R}^n)$ the Banach space of continuous functions from I to $\mathbb{R}^n$ with the norm

$\|x\|_\infty = \sup\{|x|, x \in I\}$. We denote by $L^1(I, \mathbb{R}^n)$ the space of Lebesgue-integrable function $x: I \longrightarrow \mathbb{R}^n$ with the norm

$$\|x\|_{L^1} = \int_0^1 |x(t)| dt. \quad (2)$$

Let

$$X = \left\{ \begin{array}{l} x \in C(I, \mathbb{R}^n), \\ x'' \in C(I; \mathbb{R}^n) \end{array} \right\}, \quad (3)$$

with the norm

$$\|x\|_X = \|x\|_\infty + \|x''\|_\infty. \quad (4)$$

Definition 1. The Riemann–Liouville integral of order $\alpha > 0$ for a continuous function $\varphi \in L^1((0, 1], \mathbb{R})$ is given by

$$I^\alpha \varphi(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t - \tau)^{\alpha-1} \varphi(\tau) d\tau, \quad \forall t \in (0, 1], \quad (5)$$

with $\Gamma(\alpha) = \int_0^\infty e^{-u} u^{\alpha-1} du$.

Definition 2. If $\varphi \in C^n([0, 1], \mathbb{R})$ and $n - 1 < \alpha \leq n$, then the Caputo fractional derivative is given by

$$D^\alpha \varphi(t) = I^{n-\alpha} \frac{d^n}{dt^n} (\varphi(t)) = \frac{1}{\Gamma(n-\alpha)} \int_0^t (t-s)^{n-\alpha-1} \varphi^{(n)}(s) ds. \quad (6)$$

Lemma 1. Let $n \in \mathbb{N}^*$ and $n - 1 < \alpha < n$, then the general solution of $D^\alpha u(t) = 0$ is given by

$$u(t) = \sum_{i=0}^{n-1} c_i t^i, \quad (7)$$

such that $c_i \in \mathbb{R}$, $i = 0, 1, 2, \dots, n-1$.

Lemma 2. Taking $n \in \mathbb{N}^*$ and $n - 1 < \alpha < n$, we have

$$I^\alpha D^\alpha u(t) = u(t) + \sum_{k=0}^{n-1} \frac{u^{(k)}(0)}{k!} t^k, \quad (8)$$

with $t > 0$, $n - 1 < \alpha < n$.

Lemma 3. Let $1 < \beta < 2$ and $3 < \alpha < 4$. Then, it holds

$$I^\alpha D^\beta u(t) = I^{\alpha-\beta} u(t) - \frac{u(0)t^{\alpha-\beta}}{\Gamma(\alpha-\beta+1)} - \frac{u'(0)t^{\alpha-\beta+1}}{\Gamma(\alpha-\beta+2)}. \quad (9)$$

Proof. For this proof, we use the same method in [28]. We have

$$\begin{aligned} I^\alpha D^\beta u(t) &= \frac{1}{\Gamma(\alpha)\Gamma(2-\beta)} \int_0^t (t-\tau)^{\alpha-1} \int_0^t (\tau-s)^{1-\beta} u''(s) ds d\tau, \\ &= \frac{1}{\Gamma(\alpha)\Gamma(2-\beta)} \int_0^t u''(s) ds \int_s^t (t-\tau)^{\alpha-1} (\tau-s)^{1-\beta} d\tau. \end{aligned} \quad (10)$$

With the change of variable $\tau = s + (t-s)\eta$, we have

$$\int_s^t (t-\tau)^{\alpha-1} (\tau-s)^{1-\beta} d\tau = \frac{\Gamma(\alpha)\Gamma(2-\beta)}{\Gamma(\alpha-\beta+2)} (t-s)^{\alpha-\beta+1}. \quad (11)$$

Now, we get

$$I^\alpha D^\beta u(t) = \frac{1}{\Gamma(\alpha-\beta+2)} \int_0^t (t-s)^{\alpha-\beta+1} u''(s) ds = I^{\alpha-\beta} u(t) - \frac{u(0)t^{\alpha-\beta}}{\Gamma(\alpha-\beta+1)} - \frac{u'(0)t^{\alpha-\beta+1}}{\Gamma(\alpha-\beta+2)}. \quad (12)$$

Definition 3. Let X be a Banach space. Then, a map $T: X \rightarrow X$ is called a contraction mapping on X if there exists $q \in [0, 1)$ such that

$$\|T(x) - T(y)\| \leq q\|x - y\|, \quad (13)$$

for all $x, y \in X$.

Theorem 1 (Banach's fixed point theorem, see [29]). Let Ω be a nonempty closed subset of a Banach space X . Then, any contraction mapping T of Ω into itself has a unique fixed point.

Theorem 2 (Schaefer's fixed point theorem, see [29]). Let X be a Banach space, and let $N: X \rightarrow X$ be a completely continuous operator. If the set

$E = \{y \in X: y = \lambda Ny \text{ for some } \lambda \in (0, 1)\}$ is bounded, then N has fixed points.

3. Main Results

We begin this section by some results that help us for solving the problem considered in (1).

Lemma 4. For any $x \in X$ and $1 < \beta < 2$, we have

$$\|D^\beta x\|_\infty \leq \frac{1}{\Gamma(3-\beta)} \|x''\|_\infty \leq \frac{1}{\Gamma(3-\beta)} \|x''\|_X. \quad (14)$$

Proof. By the definition of the operator D^β , we have

$$|D^\beta x(t)|_\infty = \frac{1}{\Gamma(2-\beta)} \left| \int_0^t (t-s)^{1-\beta} x''(s) ds \right| \leq \|x''\|_\infty \frac{1}{\Gamma(2-\beta)} \int_0^1 (1-s)^{1-\beta} ds \leq \frac{1}{\Gamma(3-\beta)} \|x''\|_\infty. \quad (15)$$

□

Lemma 5. Let $1 < \beta < 2$, $3 < \alpha < 4$, and $G \in C(I, \mathbb{R}^n)$. Then, $I^\alpha D^\alpha x(t) - AD^\beta x(t) = G(t)$, $t \in I = [0, 1]$, has for solution the following function we can state that the problem,

$$\begin{cases} D^\alpha x(t) - AD^\beta x(t) = G(t), & t \in I = [0, 1], \\ x(0) = x_0, \\ x'(0) = x'_0, \\ x''(0) = x''_0, \\ x'''(0) = x'''_0, \end{cases} \quad (16)$$

$$\begin{aligned} x(t) = & x_0 + x'_0 t + \frac{1}{2} x''_0 t^2 + \frac{1}{6} x'''_0 t^3 - \frac{At^{\alpha-\beta}}{\Gamma(\alpha-\beta+1)} x_0 - \frac{At^{\alpha-\beta+1}}{\Gamma(\alpha-\beta+2)} x'_0 \\ & + \frac{A}{\Gamma(\alpha-\beta)} \int_0^t (t-s)^{\alpha-\beta-1} x(s) ds + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} G(s) ds. \end{aligned} \quad (17)$$

Proof. By applying I^α to both sides of equation (16), we have

$$I^\alpha D^\alpha x(t) - AI^\alpha D^\beta x(t) = I^\alpha G(t), \quad (18)$$

and using the property established in Lemmas 2 and 3, we find that

$$\begin{aligned} x(t) = & x(0) + x'(0)t + \frac{1}{2} x''(0)t^2 + \frac{1}{6} x'''(0)t^3 - \frac{At^{\alpha-\beta}}{\Gamma(\alpha-\beta+1)} x(0) - \frac{At^{\alpha-\beta+1}}{\Gamma(\alpha-\beta+2)} x'(0) \\ & + \frac{A}{\Gamma(\alpha-\beta)} \int_0^t (t-s)^{\alpha-\beta-1} x(s) ds + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} G(s) ds. \end{aligned} \quad (19)$$

Some of the initial conditions allow us to have the result.

Conversely, assume that $x(t)$ satisfy the equation (16), then we see easily the initial conditions.

We use the fact $D^\alpha I^\alpha G(t) = G(t)$ and $D^\alpha C = 0$, where C is a constant; we get

$$D^\alpha x(t) - AD^\beta x(t) = G(t), \quad t \in I = [0, 1]. \quad (20)$$

Let us now transform the above problem to a fixed point one. Consider the nonlinear operator $T: X \rightarrow X$ defined by

$$\begin{aligned} Tx(t) = & x_0 + x'_0 t + \frac{1}{2} x''_0 t^2 + \frac{1}{6} x'''_0 t^3 - \frac{At^{\alpha-\beta}}{\Gamma(\alpha-\beta+1)} x_0 - \frac{At^{\alpha-\beta+1}}{\Gamma(\alpha-\beta+2)} x'_0 \\ & + \frac{A}{\Gamma(\alpha-\beta)} \int_0^t (t-s)^{\alpha-\beta-1} x(s) ds + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(t, x(t), D^\beta x(t), D^\alpha x(t)) ds. \end{aligned} \quad (21)$$

To prove the main results, we need to work with the following hypotheses:

(H1) The function f defined on $I \times \mathbb{R}^{3n}$ is continuous.

(H2) There exist nonnegative constants c_1, c_2 , and $c_3 < 1$ such that, for any $t \in I$, $x_1, x_2, x_3, x_1^*, x_2^*, x_3^* \in \mathbb{R}^n$,

$$|f(t, x_1, x_2, x_3) - f(t, x_1^*, x_2^*, x_3^*)| \leq c_1 |x_1 - x_1^*| + c_2 |x_2 - x_2^*| + c_3 |x_3 - x_3^*|. \quad (22)$$

Also, we consider the quantities

$$\begin{aligned} D_1 &= \frac{\|A\|}{\Gamma(\alpha - \beta + 1)} + \frac{c_1\Gamma(3 - \beta) + c_3\|A\| + c_2}{(1 - c_3)\Gamma(\alpha + 1)\Gamma(3 - \beta)}, \\ D_2 &= \frac{\|A\|}{\Gamma(\alpha - \beta - 1)} + \frac{c_1\Gamma(3 - \beta) + c_3\|A\| + c_2}{(1 - c_3)\Gamma(\alpha - 1)\Gamma(3 - \beta)}. \end{aligned} \quad (23)$$

The first main result deals with the existence of a unique solution for (1). It is based on the application of Banach fixed point theorem for contraction mappings. $\square$

Theorem 3. *If the conditions (H1) and (H2) are satisfied and $D < 1$ ($D = D_1 + D_2$), then problem (1) has a unique solution on I .*

Proof. It is sufficient for us to prove that H is a contraction mapping.

Let $(x, y) \in X^2$. Then, we can write

$$|Tx(t) - Ty(t)| \leq \frac{\|A\|}{\Gamma(\alpha - \beta)} \int_0^t (t-s)^{\alpha-\beta-1} |x(s) - y(s)| ds + \frac{1}{\Gamma(\alpha)} \int_0^t (s-t)^{\alpha-1} |g(s) - h(s)| ds, \quad (24)$$

where $g, h \in C(I, \mathbb{R}^n)$ defined by
 $g(t) = f(t, x(t), D^\beta x(t), g(t) + AD^\beta x(t))$ and
 $h(t) = f(t, y(t), D^\beta y(t), h(t) + AD^\beta y(t)).$

From (H2) for each $t \in I$, we have

$$|g(t) - h(t)| \leq c_1|x(t) - y(t)| + c_2|D^\beta(x(t) - y(t))| + c_3|g(t) - h(t)| + c_3\|A\||x(t) - y(t)|, \quad (25)$$

and using Lemma 4, we have

Therefore, we have for each $t \in I$,

$$|g(t) - h(t)| \leq \frac{c_1\Gamma(3 - \beta) + c_3\|A\| + c_2}{(1 - c_3)\Gamma(3 - \beta)} \|x - y\|_X. \quad (26)$$

$$\|Tx - Ty\|_\infty \leq \left[\frac{\|A\|}{\Gamma(\alpha - \beta + 1)} + \frac{c_1\Gamma(3 - \beta) + c_3\|A\| + c_2}{(1 - c_3)\Gamma(\alpha + 1)\Gamma(3 - \beta)} \right] \|x - y\|_X \leq D_1\|x - y\|_X. \quad (27)$$

On the other hand, we have

$$\begin{aligned} (Tx)''(t) &= x_0'' + x_0''t - \frac{At^{\alpha-\beta-2}}{\Gamma(\alpha - \beta - 1)}x_0 - \frac{At^{\alpha-\beta-1}}{\Gamma(\alpha - \beta)}x_0' + \frac{A}{\Gamma(\alpha - \beta - 2)} \int_0^t (t-s)^{\alpha-\beta-3} x(s) ds \\ &\quad + \frac{1}{\Gamma(\alpha - 2)} \int_0^t (t-s)^{\alpha-3} f(t, x(t), D^\beta x(t), D^\alpha x(t)) ds, \end{aligned} \quad (28)$$

which is clear in $C(I, \mathbb{R}^n)$.

Then, with the same arguments as before, we have

$$\|(Tx)'' - (Ty)''\|_\infty \leq \left[\frac{\|A\|}{\Gamma(\alpha - \beta - 1)} + \frac{c_1\Gamma(3 - \beta) + c_3\|A\| + c_2}{(1 - c_3)\Gamma(\alpha - 1)\Gamma(3 - \beta)} \right] \|x - y\|_X \leq D_2 \|x - y\|_X. \quad (29)$$

Thus, we have

$$\|Tx - Ty\|_X \leq (D_1 + D_2) \|x - y\|_X. \quad (30)$$

Since $D < 1$, then the operator T is contraction. Hence, by Banach's contraction principle, T has a unique fixed point which is the unique solution of problem (1).

The following main result deals with the existence of at least one solution of the studied problem. $\square$

Theorem 4. *Under the hypotheses (H1) and (H2), problem (1) has at least one solution $u(t)$, $t \in I$.*

Proof. Let us prove the result by considering the following steps:

Continuous of T : if the proof is trivial, then it is omitted (we just apply the fact that f is continuous).

Uniform boundness of T : let us take $r > 0$ and consider the (bounded) ball $B_r = \{x \in X; \|x\|_X \leq r\}$. For $y \in B_r$, we can write

$$|Ty(t)| \leq |x_0| + \frac{1}{2}|x_0''| + \frac{1}{6}|x_0''| + \frac{\|A\||x_0|}{\Gamma(\alpha - \beta + 1)} + \frac{\|A\||x_0'|}{\Gamma(\alpha - \beta + 2)} + \frac{\|A\|}{\Gamma(\alpha - \beta - 1)} \|y\|_\infty + \frac{1}{\Gamma(\alpha + 1)} \|g\|_\infty. \quad (31)$$

With a simple calculus, we get

$$\|g\|_\infty \leq \frac{c_1\Gamma(3 - \beta) + c_3\|A\| + c_2}{(1 - c_3)\Gamma(3 - \beta)} \|y\|_X + m^*, \quad (32)$$

where $m^* = \sup_{t \in I} |f(t, 0, 0, 0)|$.

Then, we have

$$\|Ty\|_\infty \leq |x_0| + \frac{1}{2}|x_0''| + \frac{1}{6}|x_0''| + \frac{\|A\||x_0|}{\Gamma(\alpha - \beta + 1)} + \frac{\|A\||x_0'|}{\Gamma(\alpha - \beta + 2)} + \frac{m^*}{\Gamma(\alpha + 1)} + D_1 r < +\infty, \quad (33)$$

and also we have

$$\|(Ty)''\|_\infty \leq |x_0''| + |x_0''| + \frac{\|A\||x_0|}{\Gamma(\alpha - \beta - 1)} + \frac{\|A\||x_0'|}{\Gamma(\alpha - \beta)} + \frac{m^*}{\Gamma(\alpha - 1)} + D_2 r < +\infty. \quad (34)$$

The above two inequalities show that $\|Ty\|_X < +\infty$.

Consequently, T is uniformly bounded.

Equicontinuity of T : we prove that, for any bounded set B_r for instance, we obtain that $T(B_r)$ is an equicontinuous set of X .

Take $t_1, t_2 \in [0, 1]$, $t_1 < t_2$ and consider the above (bounded) ball B_r of X . So, by considering $y \in B_r$, we can state that

$$\begin{aligned} |Ty(t_2) - Ty(t_1)| &\leq |x_0'| |t_2 - t_1| + \frac{1}{2}|x_0''| |t_2^2 - t_1^2| + \frac{1}{6}|x_0''| |t_2^3 - t_1^3| + \frac{\|A\||x_0'|}{\Gamma(\alpha - \beta + 1)} |t_2^{\alpha - \beta} - t_1^{\alpha - \beta}| \\ &\quad + \frac{\|A\||x_0''|}{\Gamma(\alpha - \beta + 2)} |t_2^{\alpha - \beta + 1} - t_1^{\alpha - \beta + 1}| + \frac{\|A\|r}{\Gamma(\alpha - \beta + 1)} |t_2^{\alpha - \beta} - t_1^{\alpha - \beta}| + \frac{M}{\Gamma(\alpha + 1)} |t_2^\alpha - t_1^\alpha|, \end{aligned} \quad (35)$$

where $M = (c_1\Gamma(3-\beta) + c_3\|A\| + c_2) / (1 - c_3)\Gamma(3-\beta)r + m^*$.

As $t_2 \rightarrow t_1$, the right-hand side of the above inequality tends to zero, and we have also

$$\begin{aligned} |(Ty)''(t_2) - (Ty)''(t_1)| &\leq |x_0''| |t_2 - t_1| + \frac{\|A\||x_0'|}{\Gamma(\alpha - \beta - 1)} |t_2^{\alpha-\beta-2} - t_1^{\alpha-\beta-2}| + \frac{\|A\||x_0''|}{\Gamma(\alpha - \beta)} |t_2^{\alpha-\beta-1} - t_1^{\alpha-\beta-1}| \\ &\quad + \frac{\|A\|r}{\Gamma(\alpha - \beta - 1)} |t_2^{\alpha-\beta-2} - t_1^{\alpha-\beta-2}| + \frac{M}{\Gamma(\alpha - 1)} |t_2^{\alpha-2} - t_1^{\alpha-2}|. \end{aligned} \quad (36)$$

As $t_2 \rightarrow t_1$, the right-hand side of the above inequality tends to zero. From a consequence of the Ascoli-Arzelà's theorem, we conclude that T is completely continuous.

Let $y \in A_\gamma$. Then, we have $y = \gamma Ty$ for some $0 < \gamma < 1$. Hence, we can write

Boundness of A_γ : the set $A_\gamma = \{x \in X: x = \gamma Tx, \gamma \in]0, 1[\}$ is bounded.

$$\|y\|_\infty \leq \gamma \left(|x_0| + \frac{1}{2}|x_0'| + \frac{1}{6}|x_0''| + \frac{\|A\||x_0|}{\Gamma(\alpha - \beta + 1)} + \frac{\|A\||x_0'|}{\Gamma(\alpha - \beta + 2)} + \frac{m^*}{\Gamma(\alpha + 1)} + D_1 r \right), \quad (37)$$

$$\|(y)''\|_\infty \leq \gamma \left(|x_0''| + |x_0'| + \frac{\|A\||x_0|}{\Gamma(\alpha - \beta - 1)} + \frac{\|A\||x_0'|}{\Gamma(\alpha - \beta)} + \frac{m^*}{\Gamma(\alpha - 1)} + D_2 r \right). \quad (38)$$

From (37) and (38), we state that $\|y\|_X < \infty$. The set is thus bounded.

Example 1. Let us consider the following example:

Consequently, thanks to Schaefer fixed point theorem, we deduce that T has at least one fixed point. Thus, problem (1) has a solution. $\square$

$$\left\{ \begin{array}{l} D^\alpha x(t) - AD^\beta x(t) = f(t, x(t), D^\beta x(t), D^\alpha x(t)), \quad t \in I = [0, 1], \\ x(0) = \left(\frac{1}{2}, 0\right), \\ x'(0) = \left(0, \frac{1}{2}\right), \\ x''(0) = \left(\frac{1}{2}, 0\right), \\ x'''(0) = \left(0, \frac{1}{2}\right), \end{array} \right. \quad (39)$$

where

$$f: [0, 1] \times \mathbb{R}^2 \times \mathbb{R}^2 \times \mathbb{R}^2 \longrightarrow \mathbb{R}^2, \quad (40)$$

$$(t, u, v, w) \mapsto \left(\frac{1}{10e^{t+5}(1 + \|u\| + \|v\| + \|w\|)}, \frac{1}{10e^{t+7}(1 + \|u\| + \|v\| + \|w\|)} \right),$$

with $\|u\| = \max\{x_1, x_2\}$, $u = (x_1, x_2)$. We take $A = \begin{pmatrix} (1/20) & 0 \\ 0 & 0 \end{pmatrix}$, $\alpha = (15/4)$, and $\beta = (3/2)$.

For any $u, v, w, \bar{u}, \bar{v}, \bar{w} \in \mathbb{R}^2$ and $t \in [0, 1]$,

We can see clearly that the function f is continuous.

$$\left| \frac{1}{10e^{t+5}(1 + \|\bar{u}\| + \|\bar{v}\| + \|\bar{w}\|)} - \frac{1}{10e^{t+5}(1 + \|u\| + \|v\| + \|w\|)} \right| \leq \frac{1}{10e^5} (|u - \bar{u}| + |v - \bar{v}| + |w - \bar{w}|), \quad (41)$$

$$\left| \frac{1}{10e^{t+7}(1 + \|\bar{u}\| + \|\bar{v}\| + \|\bar{w}\|)} - \frac{1}{10e^{t+7}(1 + \|u\| + \|v\| + \|w\|)} \right| \leq \frac{1}{10e^7} (|u - \bar{u}| + |v - \bar{v}| + |w - \bar{w}|),$$

which give

$$|f(t, u, v, w) - f(t, \bar{u}, \bar{v}, \bar{w})| \leq c_1 |u - \bar{u}| + c_2 |v - \bar{v}| + c_3 |w - \bar{w}|, \quad (42)$$

where $c_1 = c_2 = c_3 = (1/10e^5)$.

Hence, the hypotheses (H1) and (H2) are satisfied.

With a simple computation, we get $D_1 = 0,0807$ and $D_2 = 0,24$, which imply $D < 1$.

Thus, all the assumptions from (H1)–(H3) are satisfied. From Theorem 3, we conclude that equation (1) has a unique solution.

4. Conclusion

In this work, we consider a nonlinear implicit fractional differential equation and we use the Caputo derivative operator. We prove two theorems and an example to illustrate our results. In the first theorem, we prove the existence and uniqueness of the solution and the second theorem deals with the existence of at least one solution. The methods used are the Banach's fixed point theorem and Schaefer's fixed point theorem. Here, two Caputo derivative operators of different fractional orders were used in the considered equation and it would be relevant to generalize this idea by considering several Caputo operators of different fractional orders.

Data Availability

No data were used to support this study.

Conflicts of Interest

The author declares no known conflicts of interest or personal relationships that could have appeared to influence the work reported in this paper.

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