Research Article

Multiple Solutions for a Nonlocal Elliptic Problem Involving $(p(x), q(x))$ -Biharmonic Operator

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In this paper, using the variational principle, the existence and multiplicity of solutions for $(p(x), q(x))$ -Kirchhoff type problem with Navier boundary conditions are proved. At the same time, the sufficient conditions for the multiplicity of solutions are obtained.

1. Introduction

In this paper, we will discuss the nonlocal elliptic problem involving $(p(x), q(x))$ -biharmonic operator:

$$\begin{cases} \Delta_{p(x)}^2 u(x) - M_1 \left(\int_{\Omega} \frac{|\nabla u(x)|^{p(x)}}{p(x)} dx \right) \Delta_{p(x)} u(x) + \rho_1(x) |u|^{p(x)-2} u(x) = \lambda F_u(x, u, v), & \text{in } \Omega, \\ \Delta_{q(x)}^2 v(x) - M_2 \left(\int_{\Omega} \frac{|\nabla v(x)|^{q(x)}}{q(x)} dx \right) \Delta_{q(x)} v(x) + \rho_2(x) |v|^{q(x)-2} v(x) = \lambda F_v(x, u, v), & \text{in } \Omega, \\ u = \Delta u = v = \Delta v = 0, & \text{on } \partial\Omega, \end{cases} \quad (1)$$

where $\Omega \subset \mathbb{R}^N$ ($N \geq 2$) is a smooth bounded region. $\lambda > 0$, $\Delta_{p(x)}^2 u(x) = \Delta(|\Delta u|^{p(x)-2} \Delta u)$. $\rho_1(x), \rho_2(x) \in L^\infty(\Omega)$, $p(x), q(x) \in C(\overline{\Omega})$, $(N/2) < p^- := \text{essinf}_{x \in \overline{\Omega}} p(x) \leq p^+ := \text{esssup}_{x \in \overline{\Omega}} p(x) < +\infty$, and $(N/2) < q^- := \text{essinf}_{x \in \overline{\Omega}} q(x) \leq q^+ := \text{esssup}_{x \in \overline{\Omega}} q(x) < +\infty$. Furthermore, the function $F: \overline{\Omega} \times \mathbb{R}^2 \rightarrow \mathbb{R}$ satisfies that $F(\cdot, s, t)$ is continuous in $\overline{\Omega}$, for all $(s, t) \in \mathbb{R}^2$, and $F(x, \cdot, \cdot)$ is C^1 in $\mathbb{R}^2$, for every $x \in \Omega$, and $\sup_{\{|s| \leq \tau, |t| \leq \tau\}} (|F_u(\cdot, s, t)| + |F_v(\cdot, s, t)|) \in L^1(\Omega)$, for all $\tau > 0$, where F_u and F_v are partial derivatives of u and v ,

respectively. Besides, the functions $M_i: [0, +\infty) \rightarrow \mathbb{R}$ are continuous and satisfy $m_0 \leq M_i(t) \leq m_1, i = 1, 2$, for every $t \geq 0$.

As we all know, the harmonic equation and biharmonic equation in partial differential equation are the most widely used equations in the field of theoretical research and engineering application. Among them, the focus and difficulty in science and engineering is to solve the boundary value problem of biharmonic equation.

Therefore, many scholars have carried out a lot of in-depth research on biharmonic problems, see [1–6]. In particular, in [3], when the nonlinear functions F and G satisfy certain conditions, Li and Tang studied the (p, q) -biharmonic problem,

$$\begin{cases} \Delta(|\Delta u|^{p-2}\Delta u) = \lambda F_u(x, u, v) + \mu G_u(x, u, v), & x \in \Omega, \\ \Delta(|\Delta v|^{q-2}\Delta v) = \lambda F_v(x, u, v) + \mu G_v(x, u, v), & x \in \Omega, \\ u = \Delta u = v = \Delta v = 0, & x \in \partial\Omega, \end{cases} \quad (2)$$

by applying the three critical point theorem by Ricceri [7]. Two years later, Masser et al. [8] supposed that the assumption in problem (2) μ was 0 and imposed appropriate conditions on F . Based on the critical point theorem introduced by Bonanno and Molica Bisci, infinitely many solutions were obtained. It well known that the variable exponent case possess more complicated properties than

the constant exponent case, and some methods used in the (p, q) -biharmonic case cannot be applied to the $(p(x), q(x))$ -biharmonic case. Therefore, Allaoui et al. [9] have made a great contribution to such problems, and they continued to extend (p, q) -biharmonic operator in [8] to $(p(x), q(x))$ -biharmonic case, on the basis of Ricceri's variational principle [10] and the basic theory of Sobolev space, and the following system is solved:

$$\begin{cases} \Delta(|\Delta u|^{p(x)-2}\Delta u) = \lambda F_u(x, u, v), & x \in \Omega, \\ \Delta(|\Delta v|^{q(x)-2}\Delta v) = \lambda F_v(x, u, v), & x \in \Omega, \\ u = \Delta u = v = \Delta v = 0, & x \in \partial\Omega. \end{cases} \quad (3)$$

With the deepening of the investigation, scholars are increasingly paying attention to the solution of nonlocal elliptic equation, see [11–15], for details.

In [16], Ferrara et al. researched the problem:

$$\begin{cases} \Delta(|\Delta u|^{p-2}\Delta u) - \left[M\left(\int_{\Omega} |\nabla u|^p dx\right) \right]^{p-1} \Delta_p u + \rho |u|^{p-2}u = \lambda f(x, u), & x \in \Omega, \\ u = \Delta u = 0, & x \in \partial\Omega. \end{cases} \quad (4)$$

If the nonlinear term satisfies the certain conditions, then we obtain that there are at least two nontrivial solutions to problem (4) by using the variational method and mountain pass lemma. In the same year, Hssini et al. [17] studied the same problem as in [16] according to Ricceri's variational principle and obtained multiple solutions of problem (4). In addition, in [18], Xiao and Miao extended problem (4) and continued to study the multiplicity of solutions of $(p_1, p_2, \dots, p_n)$ -Kirchhoff type problems. In [19], the author extended the conclusion of the problem (4) to the $p(x)$ -biharmonic operator.

So far, there are few results on the existence and multiplicity of solutions for $(p(x), q(x))$ -Kirchhoff type problems under Navier boundary condition. Therefore, inspired by the above research, in the present paper, our target is to show the existence and multiplicity of solutions of problem (1), according to the variational principle proposed by Ricceri and the basic results of Lebesgue or Sobolev spaces with a variable exponent.

This article consists of three sections. In Section 2, some basic conclusions and certain essential theorems or propositions of generalized Lebesgue–Sobolev spaces are obtained, which provides the significant framework for the research of variational problems and nonlocal elliptic problems with $(p(x), q(x))$ -biharmonic operator. In Section 3, we have the main results of this article and gain the corresponding proof mainly according to the variational principle by Ricceri. Besides, at the end of Section 3, we give an example involving $(p(x), q(x))$ -Kirchhoff type equations with $\Omega = [(-1, 1)]^2$, which is actually an application of Theorem 2, that is to say, this example explains our main results very well.

2. Preliminaries

Define the space as follows:

$$L^{p(x)}(\Omega) = \left\{ u: \Omega \longrightarrow \mathbb{R} \text{ measurable, } \int_{\Omega} |u(x)|^{p(x)} dx < \infty \right\}, \quad (5)$$

which has the norm

$$|u|_{p(x)} = \inf \left\{ \mu > 0: \int_{\Omega} \left| \frac{u(x)}{\mu} \right|^{p(x)} dx \leq 1 \right\}. \quad (6)$$

The Sobolev space with variable exponents is defined as

$$W^{k,p(x)}(\Omega) = \left\{ u \in L^{p(x)}(\Omega): D^{\sigma}u \in L^{p(x)}(\Omega), |\sigma| \leq k, k \in \mathbb{Z}_+ \right\}, \quad (7)$$

which has the norm

$$|u|_{k,p(x)} = \sum_{|\sigma| \leq k} |D^{\sigma}u|_{p(x)}, \quad (8)$$

where $\sigma = (\sigma_1, \sigma_2, \dots, \sigma_n)$ is a multi-index; moreover, $|\sigma| = \sum_{j=1}^n \sigma_j$ was established.

The closure of $C_0^{\infty}(\Omega)$ in $W^{k,p(x)}(\Omega)$ is the $W_0^{k,p(x)}(\Omega)$. Besides, from [20], we know that $L^{p(x)}(\Omega)$ and $W^{k,p(x)}(\Omega)$ are Banach space and satisfy separability, uniform convexity, and reflexivity.

Proposition 1 (see [20]). *Assume $(1/p_1(x)) + (1/p_2(x)) = 1$; then, $L^{p_1(x)}(\Omega)$ and $L^{p_2(x)}(\Omega)$ are conjugate spaces and satisfy the Holder inequality:*

$$\left| \int_{\Omega} u v dx \right| \leq \left(\frac{1}{p_1} + \frac{1}{p_2} \right) \|u\|_{p_1(x)} \|v\|_{p_2(x)}, \quad (9)$$

where $u \in L^{p_1(x)}(\Omega)$ and $v \in L^{p_2(x)}(\Omega)$.

Denote $X = (W_0^{1,p(x)}(\Omega) \cap W^{2,p(x)}(\Omega)) \times (W_0^{1,q(x)}(\Omega) \cap W^{2,q(x)}(\Omega))$, which is a separable and reflexive Banach space endowed with the norm:

$$\|(u, v)\| = \|u\|_{p(x)} + \|v\|_{q(x)}, \quad (10)$$

where

$$\begin{aligned} \|u\|_{p(x)} &= \inf \left\{ \omega > 0: \int_{\Omega} \left(\left| \frac{\Delta u(x)}{\omega} \right|^{p(x)} + \left| \frac{\nabla u(x)}{\omega} \right|^{p(x)} + \left| \frac{u(x)}{\omega} \right|^{p(x)} \right) dx \leq 1 \right\}, \\ \|v\|_{q(x)} &= \inf \left\{ \omega > 0: \int_{\Omega} \left(\left| \frac{\Delta v(x)}{\omega} \right|^{q(x)} + \left| \frac{\nabla v(x)}{\omega} \right|^{q(x)} + \left| \frac{v(x)}{\omega} \right|^{q(x)} \right) dx \leq 1 \right\}. \end{aligned} \quad (11)$$

From [21], we know that $|\cdot|_{2,p(\cdot)}$, $\|\cdot\|_{p(\cdot)}$, and $|\Delta \cdot|_{p(\cdot)}$ are equivalent norms of X .

Proposition 2 (see [9]). Since $p^-, q^- > (N/2)$ and $\Omega \subset \mathbb{R}^N$ is a bounded region, hence, the space X to $C(\overline{\Omega}) \times C(\overline{\Omega})$ is a compact embedding, and there exists a positive constant K such that

$$K = \max \left\{ \sup_{u \in (W^{2,p(x)}(\Omega) \cap W_0^{1,p(x)}(\Omega)) \setminus \{0\}} \frac{\max_{x \in \overline{\Omega}} |u(x)|}{\|u\|_{p(x)}}, \sup_{v \in (W^{2,q(x)}(\Omega) \cap W_0^{1,q(x)}(\Omega)) \setminus \{0\}} \frac{\max_{x \in \overline{\Omega}} |v(x)|}{\|v\|_{q(x)}} \right\} < +\infty. \quad (12)$$

Proposition 3 (see [9, 19]). Let $J(u) = \int_{\Omega} |\Delta u|^{p(x)} dx$, for $u \in W^{2,p(x)}(\Omega) \cap W_0^{1,p(x)}(\Omega)$; then, we deduce

$$(h1) \quad \|u\|_{p(x)} < 1 (= 1, > 1) \Leftrightarrow J(u) < 1 (= 1, > 1)$$

$$(h2) \quad \|u\|_{p(x)} \geq 1 \Rightarrow \|u\|_{p(x)}^{p^+} \leq J(u) \leq \|u\|_{p(x)}^{p^-}$$

$$(h3) \quad \|u\|_{p(x)} \leq 1 \Rightarrow \|u\|_{p(x)}^{p^+} \leq J(u) \leq \|u\|_{p(x)}^{p^-}$$

$$(h4) \quad \|u\|_{p(x)} \rightarrow 0 \Leftrightarrow J(u) \rightarrow 0$$

Definition 1. If the following equality

$$\begin{aligned} & \int_{\Omega} |\Delta u|^{p(x)-2} \Delta u \Delta \phi dx + \int_{\Omega} |\Delta v|^{q(x)-2} \Delta v \Delta \psi dx + M_1 \left(\int_{\Omega} \frac{|\nabla u|^{p(x)}}{p(x)} dx \right) \times \int_{\Omega} |\nabla u|^{p(x)-2} \nabla u \nabla \phi dx \\ & + M_2 \left(\int_{\Omega} \frac{|\nabla v|^{q(x)}}{q(x)} dx \right) \times \int_{\Omega} |\nabla v|^{q(x)-2} \nabla v \nabla \psi dx + \int_{\Omega} \rho_1(x) |u|^{p(x)-2} u \phi dx + \int_{\Omega} \rho_2(x) |v|^{q(x)-2} v \psi dx \\ & - \lambda \int_{\Omega} F_u(x, u, v) \phi dx - \lambda \int_{\Omega} F_v(x, u, v) \psi dx = 0, \end{aligned} \quad (13)$$

holds, for all $(\phi, \psi) \in X$, then $(u, v) \in X$ is a weak solution of problem (1).

Define the functional $I_{\lambda}: X \rightarrow \mathbb{R}$:

$$\begin{aligned} I_{\lambda}(u, v) &= \Phi(u, v) - \lambda \Psi(u, v), \\ \Phi(u, v) &= \int_{\Omega} \frac{|\Delta u|^{p(x)}}{p(x)} dx + \widehat{M}_1 \left(\int_{\Omega} \frac{|\nabla u|^{p(x)}}{p(x)} dx \right) + \int_{\Omega} \frac{\rho_1(x) |u|^{p(x)}}{p(x)} dx \\ & + \int_{\Omega} \frac{|\Delta v|^{q(x)}}{q(x)} dx + \widehat{M}_2 \left(\int_{\Omega} \frac{|\nabla v|^{q(x)}}{q(x)} dx \right) + \int_{\Omega} \frac{\rho_2(x) |v|^{q(x)}}{q(x)} dx, \\ \Psi(u, v) &= \int_{\Omega} F(x, u, v) dx, \end{aligned} \quad (14)$$

where $\widehat{M}_i(t) = \int_0^t M_i(s)ds, i = 1, 2$.

The functional $\Phi, \Psi: X \longrightarrow \mathbb{R}$ are well defined and Gateaux differentiable functions for $\forall (u, v) \in X$; we have

$$\begin{aligned} \langle \Phi'(u, v), (\phi, \psi) \rangle &= \int_{\Omega} |\Delta u|^{p(x)-2} \Delta u \Delta \phi dx + M_1 \left(\int_{\Omega} \frac{|\nabla u|^{p(x)}}{p(x)} dx \right) \times \int_{\Omega} |\nabla u|^{p(x)-2} \nabla u \nabla \phi dx \\ &\quad + \int_{\Omega} |\Delta v|^{q(x)-2} \Delta v \Delta \psi dx + M_2 \left(\int_{\Omega} \frac{|\nabla v|^{q(x)}}{q(x)} dx \right) \times \int_{\Omega} |\nabla v|^{q(x)-2} \nabla v \nabla \psi dx \\ &\quad + \int_{\Omega} \rho_1(x) |u|^{p(x)-2} u \phi dx + \int_{\Omega} \rho_2(x) |v|^{q(x)-2} v \psi dx, \\ \langle \Psi'(u, v), (\phi, \psi) \rangle &= \int_{\Omega} F_u(x, u, v) \phi dx + \int_{\Omega} F_v(x, u, v) \psi dx. \end{aligned} \quad (15)$$

Hence, according to Definition 1, we know that if $(u, v) \in X$ is the weak solution of problem (1), it is equivalent to that (u, v) is the critical point of I_{λ} . In addition, since the space X to $C(\overline{\Omega}) \times C(\overline{\Omega})$ is a compact embedding, so $\Phi, \Psi: X \longrightarrow \mathbb{R}$ are sequentially weakly lower semicontinuous, and it is clear that Φ is coercive.

Theorem 1 (see [10]). *Let X be a reflexive real Banach space. $\Phi, \Psi: X \longrightarrow \mathbb{R}$ are Gateaux differential functionals; Φ satisfies coercive and sequentially weakly lower semicontinuity and Ψ satisfies sequentially weakly upper semicontinuity. If $r > \inf_X \Phi$, denotes*

$$\begin{aligned} \varphi(r) &:= \inf_{u \in \Phi^{-1}((-\infty, r))} \frac{\sup_{v \in \Phi^{-1}((-\infty, r))} \Psi(v) - \Psi(u)}{r - \Phi(u)}, \\ \gamma &:= \lim_{r \rightarrow +\infty} \inf \varphi(r), \\ \delta &:= \lim_{r \rightarrow \left(\inf_X \Phi \right)^+} \inf \varphi(r), \end{aligned} \quad (16)$$

then one has

- (a) For every $r > \inf_X \Phi$ and every $\lambda \in (0, (1/\varphi(r)))$, the functional $I_{\lambda} = \Phi - \lambda \Psi$ has a global minimum in $\Phi^{-1}((-\infty, r))$, which is a critical point (local minimum) of I_{λ} in X .
- (b) If $\gamma < +\infty$, then $\forall \lambda \in (0, (1/\gamma))$; the following alternative holds: either
- (b1) I_{λ} has a global minimum or

(b2) I_{λ} has a series of critical points (local minimum) defined as $\{u_n\}$ satisfying $\lim_{n \rightarrow +\infty} \Phi(u_n) = +\infty$.

(c) If $\delta < +\infty$, then $\forall \lambda \in (0, (1/\delta))$; the following alternative holds: either

- (c1) Φ has a global minimum which is a local minimum of I_{λ} or
- (c2) I_{λ} has a series of pairwise distinct critical points (local minimum) which weakly converges to global minimum of Φ .

3. Main Results and Proof

Theorem 2. Suppose

- (A₁) $\forall (x, s, t) \in \Omega \times [0, +\infty)^2, F(x, s, t) \geq 0$.
- (A₂) $\exists x_0 \in \Omega$ and two real numbers $O_2 > O_1 > 0$ satisfy $R(x_0, O_2) \subseteq \Omega$. If we put

$$\alpha := \lim_{b \rightarrow +\infty} \inf \frac{\int_{\Omega} \sup_{|s|+|t|<b} F(x, s, t) dx}{b^{\min\{p^-, q^-\}}}, \quad (17)$$

$$\beta := \lim_{s, t \rightarrow +\infty} \sup \frac{\int_{R(x_0, O_1)} F(x, s, t) dx}{(s^{p^+}/p^-) + (t^{q^+}/q^-)},$$

one has

$$\alpha < L\beta, \quad (18)$$

where

$$\begin{aligned}
L &:= \min\{L_1, L_2\}, \\
L_1 &= \frac{1}{\left[K \left((p^+/M^-)^{(1/p^-)} + (q^+/M^-)^{(1/q^-)} \right) \right]^{\min\{p^-, q^-\}}} \cdot \frac{1}{M^+ (\theta_1 + \theta_2 + |\rho_1(x)|_\infty \theta_3)}, \\
L_2 &= \frac{1}{\left[K \left((p^+/M^-)^{(1/p^-)} + (q^+/M^-)^{(1/q^-)} \right) \right]^{\min\{p^-, q^-\}}} \cdot \frac{1}{M^+ (\theta_1 + \theta_2 + |\rho_2(x)|_\infty \theta_3)}, \\
\theta_1 &= \frac{2\pi^{(N/2)}}{N\Gamma(N/2)} \left(\frac{2N}{O_2^2 - O_1^2} \right)^{p^+} (O_2^N - O_1^N), \\
\theta_2 &= \frac{2\pi^{(N/2)}}{\Gamma(N/2)} \left(\frac{2}{O_2^2 - O_1^2} \right)^{p^+} \frac{O_2^{p^++N} - O_1^{p^++N}}{p^+ + N}, \\
\theta_3 &= \frac{2\pi^{(N/2)}}{\Gamma(N/2)} \left(\frac{O_1^N}{N} + \left(\frac{1}{O_2^2 - O_1^2} \right)^{p^+} \int_{O_1}^{O_2} (O_2^2 - r^2)^{p^+} r^{N-1} dr \right), \\
M^+ &= \max\{1, m_1\}, \\
M^- &= \min\{1, m_0\}.
\end{aligned} \tag{19}$$

Then, for every

$$\lambda \in \Lambda := \frac{1}{\left[K \left((p^+/M^-)^{(1/p^-)} + (q^+/M^-)^{(1/q^-)} \right) \right]^{\min\{p^-, q^-\}}} \left(\frac{1}{L\beta^2} \frac{1}{\alpha} \right). \tag{20}$$

Problem (1) has a sequence of solutions (u_n, u_n) such that $\lim_{n \rightarrow +\infty} \Phi(u_n, u_n) = +\infty$.

Proof. Since $\Phi(0, 0) = 0$ and $\Psi(0, 0) \geq 0$, if $r > 0$, then

$$\varphi(r) = \inf_{(w, z) \in \Phi^{-1}((-\infty, r))} \frac{\sup_{(u, v) \in \Phi^{-1}((-\infty, r))} \Psi(u, v) - \Psi(w, z)}{r - \Phi(w, z)} \leq \frac{\sup_{\Phi(u, v) < r} \int_{\Omega} F(x, u, v) dx}{r}. \tag{21}$$

From (A_2) , we know that there exists a sequence $\{\xi_n\}$ satisfying $\lim_{n \rightarrow +\infty} \xi_n = +\infty$ and

$$\lim_{n \rightarrow +\infty} \frac{\int_{\Omega} \sup_{|s|+|t| < \xi_n} F(x, s, t) dx}{\xi_n^{\min\{p^-, q^-\}}} = \alpha < +\infty. \tag{22}$$

Put

$$r_n = \frac{\xi_n^{\min\{p^-, q^-\}}}{\left[K \left((p^+/M^-)^{(1/p^-)} + (q^+/M^-)^{(1/q^-)} \right) \right]^{\min\{p^-, q^-\}}}, \tag{23}$$

for all $n \in \mathbb{N}$. When $(u, v) \in \Phi^{-1}((-\infty, r_n))$, we have $\Phi(u, v) < r_n$.

Since

$$\begin{aligned}
\Phi(u, v) &\geq \frac{1}{p^+} \left(\int_{\Omega} |\Delta u|^{p(x)} dx + m_0 \int_{\Omega} |\nabla u|^{p(x)} dx + \int_{\Omega} \rho_1(x) |u|^{p(x)} dx \right) \\
&\quad + \frac{1}{q^+} \left(\int_{\Omega} |\Delta v|^{q(x)} dx + m_0 \int_{\Omega} |\nabla v|^{q(x)} dx + \int_{\Omega} \rho_2(x) |v|^{q(x)} dx \right) \\
&\geq \frac{M^-}{p^+} \left[\int_{\Omega} (|\Delta u|^{p(x)} + |\nabla u|^{p(x)} + \rho_1(x) |u|^{p(x)}) dx \right] \\
&\quad + \frac{M^-}{q^+} \left[\int_{\Omega} (|\Delta v|^{q(x)} + |\nabla v|^{q(x)} + \rho_2(x) |v|^{q(x)}) dx \right],
\end{aligned} \tag{24}$$

according to Proposition 3, for $u \in X$, we infer that

$$\text{if } \|u\|_{p(x)} \geq 1, \|u\|_{p(x)}^{p^-} \leq \int_{\Omega} (|\Delta u|^{p(x)} + |\nabla u|^{p(x)} + \rho_1(x) |u|^{p(x)}) dx \leq \|u\|_{p(x)}^{p^+}, \tag{25}$$

$$\text{if } \|u\|_{p(x)} \leq 1, \|u\|_{p(x)}^{p^+} \leq \int_{\Omega} (|\Delta u|^{p(x)} + |\nabla u|^{p(x)} + \rho_1(x) |u|^{p(x)}) dx \leq \|u\|_{p(x)}^{p^-}. \tag{26}$$

Then,

$$\frac{M^-}{p^+} \min \left\{ \|u\|_{p(x)}^{p^-}, \|u\|_{p(x)}^{p^+} \right\} + \frac{M^-}{q^+} \min \left\{ \|v\|_{q(x)}^{q^-}, \|v\|_{q(x)}^{q^+} \right\} < r_n. \tag{27}$$

$$\|u\|_{p(x)} < \left(\frac{r_n p^+}{M^-} \right)^{(1/p^-)}, \tag{29}$$

$$\|v\|_{q(x)} < \left(\frac{r_n q^+}{M^-} \right)^{(1/q^-)},$$

Thus,

$$\begin{aligned}
\frac{M^-}{p^+} \min \left\{ \|u\|_{p(x)}^{p^-}, \|u\|_{p(x)}^{p^+} \right\} &< r_n, \\
\frac{M^-}{q^+} \min \left\{ \|v\|_{q(x)}^{q^-}, \|v\|_{q(x)}^{q^+} \right\} &< r_n.
\end{aligned} \tag{28}$$

from (12)

$$|u(x)| < K \left(\frac{r_n p^+}{M^-} \right)^{(1/p^-)}, \tag{30}$$

$$|v(x)| < K \left(\frac{r_n q^+}{M^-} \right)^{(1/q^-)},$$

If $\|u\|_{p(x)} \geq 1$, we obtain $(M^-/p^+) \|u\|_{p(x)}^{p^-} < r_n$; then, $\|u\|_{p(x)} < (r_n p^+/M^-)^{(1/p^-)}$; if $\|u\|_{p(x)} \leq 1$, we obtain $(M^-/p^+) \|u\|_{p(x)}^{p^+} < r_n$; then, $\|u\|_{p(x)} < (r_n p^+/M^-)^{(1/p^+)}$.

Hence, for n large enough ($r_n > M^-$),

thus

$$|u(x)| + |v(x)| < K \left[\left(\frac{p^+}{M^-} \right)^{(1/p^-)} + \left(\frac{q^+}{M^-} \right)^{(1/q^-)} \right] r_n^{(1/\min\{p^-, q^-\})} = \xi_n. \tag{31}$$

Then, the inclusion of sets is valid:

From (21)–(32), we have

$$\Phi^{-1}((-\infty, r_n)) \subseteq \{(u, v) \in X \mid |u(x)| + |v(x)| < \xi_n\}. \tag{32}$$

$$\varphi(r_n) \leq \frac{\sup_{\Phi(u,v) < r_n} \int_{\Omega} F(x, u, v) dx}{r_n} \leq \left[K \left(\left(\frac{p^+}{M^-} \right)^{(1/p^-)} + \left(\frac{q^+}{M^-} \right)^{(1/q^-)} \right) \right]^{\min\{p^-, q^-\}} \frac{\sup_{|s|+|t| < \xi_n} \int_{\Omega} F(x, s, t) dx}{\xi_n^{\min\{p^-, q^-\}}}. \tag{33}$$

Put

$$\gamma := \lim_{r \rightarrow +\infty} \inf \varphi(r). \tag{34}$$

From (22) and (33), we can deduce

$$\begin{aligned} \gamma &\leq \lim_{n \rightarrow +\infty} \inf \varphi(r_n) \leq \left[K \left(\left(\frac{p^+}{M^-} \right)^{(1/p^-)} + \left(\frac{q^+}{M^-} \right)^{(1/q^-)} \right) \right]^{\min\{p^-, q^-\}} \cdot \lim_{n \rightarrow +\infty} \frac{\int_{\Omega} \sup_{|s|+|t| < \xi_n} F(x, s, t) dx}{\xi_n^{\min\{p^-, q^-\}}} \\ &= \alpha \left[K \left(\left(\frac{p^+}{M^-} \right)^{(1/p^-)} + \left(\frac{q^+}{M^-} \right)^{(1/q^-)} \right) \right]^{\min\{p^-, q^-\}} < +\infty. \end{aligned} \quad (35)$$

Thus, $\Lambda \subset (0, (1/\gamma))$.

Next, it will prove when $\lambda \in \Lambda$; I_{λ} is unbounded from below. Indeed, on account of

$$\frac{1}{\lambda} < \left[K \left(\left(\frac{p^+}{M^-} \right)^{(1/p^-)} + \left(\frac{q^+}{M^-} \right)^{(1/q^-)} \right) \right]^{\min\{p^-, q^-\}} \cdot L\beta. \quad (36)$$

There exists a column of positive real numbers $\{\eta_n\}$ and $\theta > 0$ such that $\lim_{n \rightarrow +\infty} \eta_n = +\infty$ and

$$\frac{1}{\lambda} < \theta < L \left[K \left(\left(\frac{p^+}{M^-} \right)^{(1/p^-)} + \left(\frac{q^+}{M^-} \right)^{(1/q^-)} \right) \right]^{\min\{p^-, q^-\}} \cdot \frac{\int_{R(x_0, O_1)} F(x, \eta_n, \eta_n) dx}{(\eta_n^{p^+}/p^-) + (\eta_n^{q^+}/q^-)}. \quad (37)$$

Now, we define $u_n \in X$ as follows:

$$u_n(x) = \begin{cases} 0, & x \in \Omega \setminus R(x_0, O_2), \\ \eta_n, & x \in R(x_0, O_1), \\ \frac{\eta_n}{O_2^2 - O_1^2} (O_2^2 - |x - x_0|^2), & x \in R(x_0, O_2) \setminus R(x_0, O_1). \end{cases} \quad (38)$$

Then, $(u_n, u_n) \in X$ and

$$\begin{aligned} \int_{\Omega} |\Delta u_n|^{p(x)} dx &= \int_{R(x_0, O_2) \setminus R(x_0, O_1)} |\Delta u_n|^{p(x)} dx \leq \theta_1 \eta_n^{p^+}, \\ \int_{\Omega} |\nabla u_n|^{p(x)} dx &= \int_{R(x_0, O_2) \setminus R(x_0, O_1)} |\nabla u_n|^{p(x)} dx \leq \theta_2 \eta_n^{p^+}, \\ \int_{\Omega} |u_n|^{p(x)} dx &= \int_{R(x_0, O_1)} |\eta_n|^{p(x)} dx + \int_{R(x_0, O_2) \setminus R(x_0, O_1)} |u_n|^{p(x)} dx \leq \theta_3 \eta_n^{p^+}. \end{aligned} \quad (39)$$

Thus,

$$\begin{aligned}
\Phi(u_n, u_n) &\leq \frac{1}{p^-} \left(\int_{\Omega} |\Delta u_n|^{p(x)} dx + m_1 \int_{\Omega} |\nabla u_n|^{p(x)} dx + \int_{\Omega} \rho_1(x) |u_n|^{p(x)} dx \right) \\
&\quad + \frac{1}{q^-} \left(\int_{\Omega} |\Delta u_n|^{q(x)} dx + m_1 \int_{\Omega} |\nabla u_n|^{q(x)} dx + \int_{\Omega} \rho_2(x) |u_n|^{q(x)} dx \right) \\
&\leq \frac{M^+}{p^-} (\theta_1 + \theta_2 + |\rho_1(x)|_{\infty} \theta_3) \eta_n^{p^+} + \frac{M^+}{q^-} (\theta_1 + \theta_2 + |\rho_2(x)|_{\infty} \theta_3) \eta_n^{q^+} \\
&= \frac{1}{\left[K \left((p^+/M^-)^{(1/p^-)} + (q^+/M^-)^{(1/q^-)} \right) \right]^{\min\{p^-, q^-\}}} \cdot \left(\frac{\eta_n^{p^+}}{p^- L_1} + \frac{\eta_n^{q^+}}{q^- L_2} \right).
\end{aligned} \tag{40}$$

From (A_1) , we have

$$\Psi(u_n, u_n) = \int_{\Omega} F(x, u_n, u_n) dx \geq \int_{R(x_0, O_1)} F(x, \eta_n, \eta_n) dx. \tag{41}$$

When n is large enough, from (37), (40), and (41), we can obtain

$$\begin{aligned}
I_{\lambda}(u_n, u_n) &= \Phi(u_n, u_n) - \lambda \Psi(u_n, u_n) \\
&\leq \frac{1}{\left[K \left((p^+/M^-)^{(1/p^-)} + (q^+/M^-)^{(1/q^-)} \right) \right]^{\min\{p^-, q^-\}}} \cdot \left(\frac{\eta_n^{p^+}}{p^- L_1} + \frac{\eta_n^{q^+}}{q^- L_2} \right) - \lambda \int_{R(x_0, O_1)} F(x, \eta_n, \eta_n) dx \\
&\leq \frac{1}{L} \frac{1}{\left[K \left((p^+/M^-)^{(1/p^-)} + (q^+/M^-)^{(1/q^-)} \right) \right]^{\min\{p^-, q^-\}}} \cdot \left(\frac{\eta_n^{p^+}}{p^-} + \frac{\eta_n^{q^+}}{q^-} \right) - \lambda \int_{R(x_0, O_1)} F(x, \eta_n, \eta_n) dx \\
&\leq \frac{1 - \lambda \theta}{L} \cdot \left(\frac{\eta_n^{p^+}}{p^-} + \frac{\eta_n^{q^+}}{q^-} \right) \cdot \frac{1}{\left[K \left((p^+/M^-)^{(1/p^-)} + (q^+/M^-)^{(1/q^-)} \right) \right]^{\min\{p^-, q^-\}}},
\end{aligned} \tag{42}$$

so

$$I_{\lambda}(u_n, u_n) = -\infty, \quad \text{as } n \longrightarrow +\infty. \tag{43}$$

According to Theorem 1 (b), we know that Theorem 2 is established. $\square$

Theorem 3. Presume (A_1) holds and

$$(B_1), \quad \forall x \in \Omega, F(x, 0, 0) = 0$$

$$(B_2), \quad \exists x_0 \in \Omega, O_2 > O_1 > 0$$

$$\alpha^0 := \lim_{b \rightarrow 0^+} \inf \frac{\int_{\Omega} \sup_{|s|+|t|<b} F(x, s, t) dx}{b^{\max\{p^+, q^+\}}}, \tag{44}$$

$$\beta^0 := \lim_{s, t \rightarrow 0^+} \sup \frac{\int_{R(x_0, O_1)} F(x, s, t) dx}{(s^{p^-}/p^-) + (t^{q^-}/q^-)},$$

and one has

$$\alpha^0 < L^0 \beta^0, \tag{45}$$

where

$$\begin{aligned}
L^0 &:= \min\{L_3, L_4\}, \\
L_3 &= \frac{1}{\left[K \left((p^+/M^-)^{(1/p^+)} + (q^+/M^-)^{(1/q^+)} \right) \right]^{\max\{p^+, q^+\}}} \cdot \frac{1}{M^+ (\bar{\theta}_1 + \bar{\theta}_2 + |\rho_1(x)|_\infty \bar{\theta}_3)}, \\
L_4 &= \frac{1}{\left[K \left((p^+/M^-)^{(1/p^+)} + (q^+/M^-)^{(1/q^+)} \right) \right]^{\max\{p^+, q^+\}}} \cdot \frac{1}{M^+ (\bar{\theta}_1 + \bar{\theta}_2 + |\rho_2(x)|_\infty \bar{\theta}_3)}, \\
\bar{\theta}_1 &= \frac{2\pi^{(N/2)}}{N\Gamma(N/2)} \left(\frac{2N}{O_2^2 - O_1^2} \right)^{p^-} (O_2^N - O_1^N), \\
\bar{\theta}_2 &= \frac{2\pi^{(N/2)}}{\Gamma(N/2)} \left(\frac{2}{O_2^2 - O_1^2} \right)^{p^-} \frac{O_2^{p^-+N} - O_1^{p^-+N}}{p^- + N}, \\
\bar{\theta}_3 &= \frac{2\pi^{(N/2)}}{\Gamma(N/2)} \left(\frac{O_1^N}{N} + \left(\frac{1}{O_2^2 - O_1^2} \right)^{p^-} \int_{O_1}^{O_2} (O_2^2 - r^2)^{p^-} r^{N-1} dr \right), \\
M^+ &= \max\{1, m_1\}, \\
M^- &= \min\{1, m_0\}.
\end{aligned} \tag{46}$$

Then, for every

$$\lambda \in \bar{\Lambda} := \frac{1}{\left[K \left((p^+/M^-)^{(1/p^+)} + (q^+/M^-)^{(1/q^+)} \right) \right]^{\max\{p^+, q^+\}}} \left(\frac{1}{L^0 \beta^0}, \frac{1}{\alpha^0} \right), \tag{47}$$

problem (1) has infinity solutions converging to 0.

Proof. Obviously, $\min_X \Phi = \Phi(0, 0) = 0$.

There is a real sequence $\{\xi_n\}$ satisfying $\xi_n \rightarrow 0^+$ as $n \rightarrow +\infty$ and

$$\lim_{n \rightarrow +\infty} \frac{\int_{\Omega} \sup_{|s|+|t| < \xi_n} F(x, s, t) dx}{\xi_n^{\max\{p^+, q^+\}}} = \alpha^0 < +\infty. \tag{48}$$

Put $r_n = (\xi_n^{\max\{p^+, q^+\}} / [K((p^+/M^-)^{(1/p^+)} + (q^+/M^-)^{(1/q^+))])^{\max\{p^+, q^+\}}]$, for all $n \in \mathbb{N}$ and $\delta = \lim_{r \rightarrow 0^+} \inf \varphi(r)$; then, by assumption (B_2) , we have

$$\begin{aligned}
\delta &\leq \lim_{n \rightarrow +\infty} \inf \varphi(r_n) \leq \left[K \left(\left(\frac{p^+}{M^-} \right)^{(1/p^+)} + \left(\frac{q^+}{M^-} \right)^{(1/q^+)} \right) \right]^{\max\{p^+, q^+\}} \cdot \lim_{n \rightarrow +\infty} \frac{\int_{\Omega} \sup_{|s|+|t| < \xi_n} F(x, s, t) dx}{\xi_n^{\max\{p^+, q^+\}}} \\
&= \alpha^0 \left[K \left(\left(\frac{p^+}{M^-} \right)^{(1/p^+)} + \left(\frac{q^+}{M^-} \right)^{(1/q^+)} \right) \right]^{\max\{p^+, q^+\}} < +\infty.
\end{aligned} \tag{49}$$

Hence, $\bar{\Lambda} \subseteq (0, (1/\delta))$.

In the next step, we prove that I_λ does not have the local minimum at 0. For $\lambda \in \bar{\Lambda}$, we have

$$\frac{1}{\lambda} < \left[K \left(\left(\frac{p^+}{M^-} \right)^{(1/p^+)} + \left(\frac{q^+}{M^-} \right)^{(1/q^+)} \right) \right]^{\max\{p^+, q^+\}} \cdot L^0 \beta^0, \tag{50}$$

and there exists a column of positive real numbers $\{\eta_n\} \rightarrow 0^+$ as $n \rightarrow +\infty$ and $\theta > 0$ such that

$$\frac{1}{\lambda} < \theta < L^0 \left[K \left(\left(\frac{p^+}{M^-} \right)^{(1/p^+)} + \left(\frac{q^+}{M^-} \right)^{(1/q^+)} \right) \right]^{\max\{p^+, q^+\}} \cdot \frac{\int_{R(x_0, O_1)} F(x, \eta_n, \eta_n) dx}{\left(\eta_n^{p^-}/p^- \right) + \left(\eta_n^{q^-}/q^- \right)}. \quad (51)$$

Let $\{u_n(x)\}$ be defined by (38); then, we can obtain

$$\Phi(u_n, u_n) \leq \left(\frac{\eta_n^{p^-}}{p^- L_3} + \frac{\eta_n^{q^-}}{q^- L_4} \right) \frac{1}{\left[K \left((p^+/M^-)^{(1/p^+)} + (q^+/M^-)^{(1/q^+)} \right) \right]^{\max\{p^+, q^+\}}}. \quad (52)$$

Combining (41), (51), and (52), we have

$$\begin{aligned} I_\lambda(u_n, u_n) &= \Phi(u_n, u_n) - \lambda \Psi(u_n, u_n) \\ &\leq \frac{1}{L^0} \left(\frac{\eta_n^{p^-}}{p^-} + \frac{\eta_n^{q^-}}{q^-} \right) \frac{1}{\left[K \left((p^+/M^-)^{(1/p^+)} + (q^+/M^-)^{(1/q^+)} \right) \right]^{\max\{p^+, q^+\}}} - \lambda \int_{R(x_0, O_1)} F(x, \eta_n, \eta_n) dx \\ &\leq \frac{1 - \lambda \theta}{L^0} \cdot \left(\frac{\eta_n^{p^-}}{p^-} + \frac{\eta_n^{q^-}}{q^-} \right) \frac{1}{\left[K \left((p^+/M^-)^{(1/p^+)} + (q^+/M^-)^{(1/q^+)} \right) \right]^{\max\{p^+, q^+\}}} \\ &\leq 0 = I_\lambda(0, 0). \end{aligned} \quad (53)$$

Obviously, according to Theorem 1 (c), we have completed the proof of Theorem 3. Moreover, (u_n, u_n) is the solution satisfying the conditions and $\|(u_n, u_n)\| \rightarrow 0$. $\square$

Example 1. Let $\Omega = [(-1, 1)]^2$; for all $t > 0$, $M_1(t) = a + bt$ and $M_2(t) = c + dt$, where a, b, c , and d are positive

numbers and $m_0 = \min\{a, c\}$. On Ω , $p(x)$ and $q(x)$ are defined as

$$\begin{aligned} p(x) &= x^2 + y^2 + 2, \\ q(x) &= x^2 + y^2 + 3. \end{aligned} \quad (54)$$

Define the function $F: \mathbb{R}^2 \rightarrow \mathbb{R}$ by

$$F(s, t) = \begin{cases} (a_{n+1})^6 e^{-(1/(1-(s-a_{n+1})^2-(t-a_{n+1})^2))}, & \text{if } (s, t) \in \cup_{n \geq 1} R((a_{n+1}, a_{n+1}), 1), \\ 0, & \text{otherwise,} \end{cases} \quad (55)$$

where $a_1 = 3$ and $a_{n+1} = n(a_n)^3 + 3$ ($n \geq 1$) and $R((a_{n+1}, a_{n+1}), 1)$ is an open unit ball with its center point at (a_{n+1}, a_{n+1}) .

It is clear that $F \geq 0$ and $F \in C^1(\mathbb{R}^2)$, for all $n \in \mathbb{N}^+$. The maximum of F on $R((a_{n+1}, a_{n+1}), 1)$ is $F(a_{n+1}, a_{n+1}) = (a_{n+1})^6 e^{-1}$.

Hence,

$$\lim_{n \rightarrow +\infty} \sup \frac{F(a_{n+1}, a_{n+1})}{(a_{n+1}^4/2) + (a_{n+1}^5/3)} = +\infty. \quad (56)$$

Therefore,

$$\begin{aligned} \beta &:= \lim_{s, t \rightarrow +\infty} \sup \frac{\int_{R(x_0, O_1)} F(s, t) dx}{(s^4/2) + (t^5/3)} \\ &= |R(x_0, O_1)| \cdot \lim_{s, t \rightarrow +\infty} \frac{F(s, t)}{(s^4/2) + (t^5/3)} = +\infty, \end{aligned} \quad (57)$$

where $|R(x_0, O_1)|$ is the measure of $R(x_0, O_1)$. At the time, $\forall n \in \mathbb{N}^+$, and we deduce

$$\sup_{|s|+|t|<a_{n+1}-1} F(s, t) = (a_n)^6 e^{-1}, \quad (58)$$

that is to say

so

$$\lim_{n \rightarrow +\infty} \frac{\sup_{|s|+|t|<a_{n+1}-1} F(s, t)}{(a_{n+1} - 1)^2} = 0, \quad (59)$$

$$\lim_{b \rightarrow +\infty} \frac{\sup_{|s|+|t|<b} F(s, t)}{b^2} = 0. \quad (60)$$

Thus,

$$\alpha := \lim_{b \rightarrow +\infty} \inf \frac{\int_{\Omega} \sup_{|s|+|t|<b} F(s, t) dx}{b^2} = |\Omega| \lim_{b \rightarrow +\infty} \inf \frac{\sup_{|s|+|t|<b} F(s, t)}{b^2} = 0 < L\beta = +\infty, \quad (61)$$

$\forall \lambda > 0$; from Theorem 2, we can infer that the equations,

$$\begin{cases} \Delta(|\Delta u|^{x^2+y^2} \Delta u) - \left(a + b \int_{\Omega} \frac{|\nabla u(x)|^{x^2+y^2+2}}{x^2 + y^2 + 2} dx \right) \operatorname{div}(|\nabla u|^{x^2+y^2} \nabla u) + \rho_1(x) |u|^{x^2+y^2} u(x) = \lambda F_u(x, u, v), & \text{in } \Omega, \\ \Delta(|\Delta v|^{x^2+y^2+1} \Delta v) - \left(c + d \int_{\Omega} \frac{|\nabla v(x)|^{x^2+y^2+3}}{x^2 + y^2 + 3} dx \right) \operatorname{div}(|\nabla v|^{x^2+y^2+1} \nabla v) + \rho_2(x) |v|^{x^2+y^2+1} v(x) = \lambda F_v(x, u, v), & \text{in } \Omega, \\ u = \Delta u = v = \Delta v = 0, & \text{on } \partial\Omega, \end{cases} \quad (62)$$

have a sequence of unbounded weak solutions.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

Authors' Contributions

Each part of this paper is the result of the joint efforts of QZ and QM. They contributed equally to the final version of the paper. All the authors have read and approved the final manuscript.

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