Research Article

Existence and Uniqueness of Weak Solutions for Novel Anisotropic Nonlinear Diffusion Equations Related to Image Analysis

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Received 16 January 2021; Revised 16 February 2021; Accepted 17 February 2021; Published 8 March 2021

Academic Editor: Sun Young Cho

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This paper establishes the existence and uniqueness of weak solutions for the initial-boundary value problem of anisotropic nonlinear diffusion partial differential equations related to image processing and analysis. An implicit iterative method combined with a variational approach has been applied to construct approximate solutions for this problem. Then, under some a priori estimates and a monotonicity condition, the existence of unique weak solutions for this problem has been proven. This work has been complemented by a consistent and stable approximation scheme showing its great significance as an image restoration technique.

1. Introduction

In the last three decades, nonlinear diffusion equations have inspired numerous research studies in various application ranges. Perona and Malik [1] were the first to introduce such equation in image processing and analysis in the following manner:

$$\begin{cases} \frac{\partial u}{\partial t} - \nabla \cdot [c(|\nabla u|)\nabla u] = 0, & \text{in } \Omega \times (0, T], \\ \langle c(|\nabla u|)\nabla u, \mathbf{n} \rangle = 0, & \text{on } \partial\Omega \times (0, T], \\ u(x; 0) = u_0(x), & \text{in } \Omega, \end{cases} \quad (1)$$

where Ω is an image domain in $\mathbb{R}^2$ and c is a positive decreasing function defined on $\mathbb{R}_+$.

When it comes to processing a digital image, Perona and Malik chose the above model to preserve meaningful features such as edges while reducing irrelevant information such as noise in the homogeneous area. Nevertheless, this model, known as an isotropic nonlinear diffusion equation, handles

an image feature with the same amount of blurring in all its directions. For instance, this process cannot successfully eliminate noises at edges [2]. Accordingly, it might be wise to consider the orientation of essential features by using anisotropic diffusion. Weickert [2] introduced this property by defining an orientation descriptor using the structure tensor, which is convenient to identify features such as corners and T-junctions. Besides, digital images present some structural difficulties; that is, they are discrete in space and image intensity values. Accordingly, it would be of great interest to adapt the diffusion to digital images' structure by considering vertical, horizontal, and diagonal differential operators. Due to these reasons, we modeled and developed anisotropic nonlinear diffusion equations using a novel diffusion tensor.

Various tools can be used to examine the existence of solutions for nonlinear partial differential equations (PDEs), such as variational techniques, monotonicity method, fixed-point theorems, iterative methods, and truncation techniques; for more detailed information, we refer to [3–7] and the references therein. These PDEs have been motivated by various applications such as image restoration and reconstruction (see, for example, [3, 4, 8–11]). Moreover, the

image processing of the brain allows the localization of epileptogenic foci for the patient. A noninvasive method has been examined numerically as an inverse problem in [12].

Under some challenging conditions, the existence and uniqueness of weak solutions for the Perona and Malik model have been investigated in the bounded variation space $BV(\Omega)$ [3, 13]. In some other functional frameworks, Wang

and Zhou have thoroughly studied in [4] and proved the existence and uniqueness of weak solutions in the Orlicz space $L\log L(\Omega)$ using a new diffusion function $c(s) = ((s + 1)\log(s + 1))/(s(s + 1))$ for all $s \geq 0$.

In this paper, we suppose that Ω is an open-bounded domain of $\mathbb{R}^2$ with Lipschitz boundary $\partial\Omega$, and T is a positive number. We denote

$$\begin{cases} \partial_{x_1} u := u_{x_1} := \nabla u \cdot \mathbf{e}_1, \\ \partial_{x_2} u := u_{x_2} := \nabla u \cdot \mathbf{e}_2, \\ \partial_{x_{12}} u := u_{x_{12}} := \nabla u \cdot \frac{\mathbf{e}_1 + \mathbf{e}_2}{|\mathbf{e}_1 + \mathbf{e}_2|}, \\ \partial_{x_{-12}} u := u_{x_{-12}} := \nabla u \cdot \frac{-\mathbf{e}_1 + \mathbf{e}_2}{|-\mathbf{e}_1 + \mathbf{e}_2|}, \end{cases} \quad (2)$$

where (e_1, e_2) is the canonical basis of $\mathbb{R}^2$. We consider the following anisotropic nonlinear parabolic initial-boundary value problem:

$$\begin{cases} \frac{\partial u}{\partial t} - \nabla \cdot [\mathbf{D}_{\nabla u} \nabla u] = 0, & \text{in } \Omega \times (0, T], \\ \langle \mathbf{D}_{\nabla u} \nabla u, \mathbf{n} \rangle = 0, & \text{on } \partial\Omega \times (0, T], \\ u(x; 0) = u_0(x), & \text{in } \Omega, \end{cases} \quad (3)$$

where $\mathbf{D}_{\nabla u}$, the diffusion tensor, is a real symmetric positive definite matrix of $\mathbb{R}^{2 \times 2}$ defined as follows:

$$\mathbf{D}_{\nabla u} = \begin{pmatrix} g(|u_{x_1}|) + \frac{g(|u_{x_{12}}|) + g(|u_{x_{-12}}|)}{2} & \frac{g(|u_{x_{12}}|) - g(|u_{x_{-12}}|)}{2} \\ \frac{g(|u_{x_{12}}|) - g(|u_{x_{-12}}|)}{2} & g(|u_{x_2}|) + \frac{g(|u_{x_{12}}|) + g(|u_{x_{-12}}|)}{2} \end{pmatrix}, \quad (4)$$

and $g: \mathbb{R}_+ \rightarrow \mathbb{R}$ is a C^1 positive decreasing function. Then, we can define $\phi: \mathbb{R}_+ \rightarrow \mathbb{R}$ as a C^2 function such that

$$\phi(s) = \int_0^s r g(r) dr, \quad s \geq 0, \quad (5)$$

satisfying

$$\left\{ \begin{array}{l} \phi(0) = \phi'(0) = 0, \phi(s) > 0, \phi'(s) > 0, \quad \text{for } s \in \mathbb{R}_+^*, \\ \phi''(s) \geq 0, s\phi''(s) \leq \phi'(s), \quad \text{for } s \in \mathbb{R}_+, \\ 0 < \lim_{s \rightarrow \infty} \frac{\phi(s)}{s \log(s)} < \infty, 0 < \lim_{s \rightarrow \infty} \frac{\phi'(s)}{\log(s)} < \infty, \\ \lim_{s \rightarrow 0^+} \frac{\phi'(s)}{s} > 0, \lim_{s \rightarrow \infty} \frac{\phi'(s)}{s} = 0. \end{array} \right. \quad (6)$$

To construct an adaptive diffusion tensor, the function g is approximated numerically by a cubic Hermite spline [14] that interpolates numeric data specified at $0 = k_0 < k_1 < \dots < k_m$ with $m \in \mathbb{N}^*$:

$$g(s) = \begin{cases} p_{k_i} P_{1,k_i k_{i+1}}(s) + v_{k_i} P_{2,k_i k_{i+1}}(s) + p_{k_{i+1}} P_{1,k_{i+1} k_i}(s) + v_{k_{i+1}} P_{2,k_{i+1} k_i}(s), & s \in [k_i, k_{i+1}[\\ & i \in \{0, 1, \dots, m-1\}, \\ p_{k_m} g_{k_m,1}(s) + v_{k_m} g_{k_m,2}(s), & s \in [k_m, \infty[\end{cases} \quad (7)$$

where p_i and v_i are the coefficients used to define the position and the velocity vector at a specific point, k_i are the threshold parameters, $\{P_{j,cd}\}$ is the family of the basis functions composed of polynomials of degree 3 used on the interval $[c, d[$ such that

And we may consider

$$\left\{ \begin{array}{l} P_{1,cd}(s) = \frac{(s-d)^2(2s+d-3c)}{(d-c)^3}, \\ P_{2,cd}(s) = \frac{(s-d)^2(s-c)}{(d-c)^2}. \end{array} \right. \quad (8)$$

$$\left\{ \begin{array}{l} g_{k_m,1}(s) = \frac{k_m}{\log(k_m) + 2} \frac{2s(\log(s) + 1) - k_m \log(k_m)}{s^2}, \\ g_{k_m,2}(s) = \frac{k_m^2}{\log(k_m) + 2} \frac{s(\log(s) + 1) - k_m(\log(k_m) + 1)}{s^2}. \end{array} \right. \quad (9)$$

From the definition of ϕ , we can deduce

$$\phi(s) = \begin{cases} C_i + \sum_{j=0}^3 \frac{A_{k_i k_{i+1}, j}}{j+2} s^{j+2}, & s \in [k_i, k_{i+1}[, i \in \{0, 1, \dots, m-1\}, \\ A_{k_m, 2} s \log(s) + A_{k_m, 1} \log(s) + C_m, & s \in [k_m, \infty[, \end{cases} \quad (10)$$

where C_i and C_m are constants determined by the continuity of ϕ at each k_i . In this case, the values of the coefficients $A_{k_i k_{i+1}, j}$ are determined experimentally provided that ϕ satisfying the above conditions on $[0, k_m[$. Besides, we may introduce some sufficient conditions on k_m and A_{k_m} that guarantee the properties of ϕ on $[k_m, \infty[$:

$$\begin{cases} k_m \geq 1, \\ A_{k_m, 2} > 0, \\ A_{k_m, 1} < k_m A_{k_m, 2}, \\ A_{k_m, 1} \geq -\frac{k_m \log(k_m)}{2} A_{k_m, 2}. \end{cases} \quad (11)$$

Anisotropic diffusion model (3) allows strong directional smoothing within the areas where $|u_{x_1}|$, $|u_{x_2}|$, $|u_{x_{12}}|$, or $|u_{x_{-12}}|$ is small and prevents blurring boundaries, contours, or corners that separate neighboring areas, where one or a combination of these differential operators has significant value.

Moreover, the matrix $\mathbf{D}_{\nabla u}$ has two eigenvalues $\lambda_{+/-}$:

$$\begin{aligned} \lambda_{+/-} = & \frac{1}{2} \left(g(|u_{x_1}|) + g(|u_{x_2}|) + g(|u_{x_{12}}|) + g(|u_{x_{-12}}|) \right. \\ & \left. \pm \sqrt{\left(g(|u_{x_1}|) - g(|u_{x_2}|) \right)^2 + \left(g(|u_{x_{12}}|) - g(|u_{x_{-12}}|) \right)^2} \right), \end{aligned} \quad (12)$$

with $\theta_{+/-}$ are the corresponding eigenvectors. We can then expand the first equation of (3) into

$$\frac{\partial u}{\partial t} = \nabla \cdot [\lambda_+ \theta_+ \theta_+^T \nabla u] + \nabla \cdot [\lambda_- \theta_- \theta_-^T \nabla u]. \quad (13)$$

Accordingly, it is clear from the expression of $\lambda_{+/-}$ that $\lambda_+ \geq \lambda_- > 0$, which means that the diffusion towards θ_+ is privileged over θ_- . In fact, the difference

$(\lambda_+ - \lambda_-)^2 = (g(|u_{x_1}|) - g(|u_{x_2}|))^2 + (g(|u_{x_{12}}|) - g(|u_{x_{-12}}|))^2$ indicates the isotropic diffusion for zero value and anisotropic diffusion for positive values.

Henceforth, we will assume that the initial value satisfies

$$u_0 \in L^2(\Omega), \quad (14)$$

and we will introduce the following Orlicz space:

$$L \log L^{k_m}(\Omega) = \left\{ u: \Omega \longrightarrow \mathbb{R} \mid \int_{\Omega \cap \{|u| \geq k_m\}} |u| \log(|u|) dx < \infty \right\}. \quad (15)$$

Next, we define weak solutions for problem (3) on $Q_T := \Omega \times (0, T]$ with $T > 0$:

Definition 1. A function $u: \overline{Q}_T \longrightarrow \mathbb{R}$ is a weak solution for problem (3) if the following conditions are satisfied:

- (i) $u \in C([0, T]; L^2(\Omega)) \cap L^1(0, T; W^{1,1}(\Omega))$ with $\partial_{x_i} u \in L \log L^{k_m}(\Omega)$ for $i = 1, 2$.

- (ii) For any $\varphi \in C^1(\overline{Q}_T)$ with $\varphi(\cdot, T) = 0$, we have

$$-\int_{\Omega} u_0(x) \varphi(x, 0) dx + \int_0^T \int_{\Omega} [-u \varphi_t + \mathbf{D}_{\nabla u} \nabla u \cdot \nabla \varphi] dx dt = 0. \quad (16)$$

Now, we state our main theorem.

Theorem 1. Under assumption (14), there exists a unique weak solution for initial-boundary value problem (3).

Inspired by [4], this paper will investigate the existence and uniqueness of weak solutions for problem (3) according to the following steps:

- (i) First, we approximate nonlinear evolution problem (3) by nonlinear elliptic problems using an implicit iterative method (discretization in time-variable only), and then we prove the existence of a unique weak solution for each elliptic problem adopting a variational approach. These solutions constitute approximate solutions for problem (3).
- (ii) Next, we show the uniqueness of solutions for initial-boundary value problem (3) using the monotonicity of the vector field $\mathbf{D}_\nabla \nabla \cdot: u \in \mathbb{R}^2 \longrightarrow \mathbf{D}_\nabla u \nabla u \in \mathbb{R}^2$.
- (iii) Finally, passing to limits in some a priori energy estimates and using the monotonicity condition (17), we demonstrate the existence of weak solutions for problem (3).

2. Preliminaries

In this section, we state some useful lemmas that will be used later in the proofs.

Lemma 1. For all $a \geq 0$ and $b \geq 1$, we have $ab \leq a \exp(a) + b \log(b)$.

Proof. If $b \leq \exp(a)$, then $ab \leq a \exp(a) \leq a \exp(a) + b \log(b)$.

If $\exp(a) < b$, then $a < \log(b)$, which means $ab < b \log(b) < a \exp(a) + b \log(b)$. $\square$

Lemma 2. Suppose $\phi: \mathbb{R}_+ \longrightarrow \mathbb{R}_+$ is a C^2 convex function. Then, for all $\xi_0, \xi_1 \in \mathbb{R}^2$, we have

$$(\mathbf{D}_{\xi_1} \xi_1 - \mathbf{D}_{\xi_0} \xi_0) \cdot (\xi_1 - \xi_0) \geq 0. \quad (17)$$

Proof. For each $t \in [0, 1]$, we put $\xi_t = (1 - t)\xi_0 + t\xi_1$. Then, we have

$$\begin{aligned} \mathbf{D}_{\xi_1} \xi_1 - \mathbf{D}_{\xi_0} \xi_0 &= \int_0^1 d[(\xi_t \cdot \mathbf{e}_1)g(|\xi_t \cdot \mathbf{e}_1|)\mathbf{e}_1] + \int_0^1 d[(\xi_t \cdot \mathbf{e}_2)g(|\xi_t \cdot \mathbf{e}_2|)\mathbf{e}_2] \\ &\quad + \int_0^1 d[(\xi_t \cdot \mathbf{e}_{12})g(|\xi_t \cdot \mathbf{e}_{12}|)\mathbf{e}_{12}] + \int_0^1 d[(\xi_t \cdot \mathbf{e}_{-12})g(|\xi_t \cdot \mathbf{e}_{-12}|)\mathbf{e}_{-12}]. \end{aligned} \quad (18)$$

Since $\phi''(s) = g(s) + sg'(s)$, then we obtain

$$\begin{aligned} \mathbf{D}_{\xi_1} \xi_1 - \mathbf{D}_{\xi_0} \xi_0 &= \int_0^1 [\phi''(|\xi_t \cdot \mathbf{e}_1|)((\xi_1 - \xi_0) \cdot \mathbf{e}_1)\mathbf{e}_1] dt + \int_0^1 [\phi''(|\xi_t \cdot \mathbf{e}_2|)((\xi_1 - \xi_0) \cdot \mathbf{e}_2)\mathbf{e}_2] dt \\ &\quad + \int_0^1 [\phi''(|\xi_t \cdot \mathbf{e}_{12}|)((\xi_1 - \xi_0) \cdot \mathbf{e}_{12})\mathbf{e}_{12}] dt + \int_0^1 [\phi''(|\xi_t \cdot \mathbf{e}_{-12}|)((\xi_1 - \xi_0) \cdot \mathbf{e}_{-12})\mathbf{e}_{-12}] dt. \end{aligned} \quad (19)$$

We conclude then

$$\begin{aligned} (\mathbf{D}_{\xi_1} \xi_1 - \mathbf{D}_{\xi_0} \xi_0) \cdot (\xi_1 - \xi_0) &= \int_0^1 [\phi''(|\xi_t \cdot \mathbf{e}_1|)((\xi_1 - \xi_0) \cdot \mathbf{e}_1)^2] dt + \int_0^1 [\phi''(|\xi_t \cdot \mathbf{e}_2|)((\xi_1 - \xi_0) \cdot \mathbf{e}_2)^2] dt \\ &\quad + \int_0^1 [\phi''(|\xi_t \cdot \mathbf{e}_{12}|)((\xi_1 - \xi_0) \cdot \mathbf{e}_{12})^2] dt \\ &\quad + \int_0^1 [\phi''(|\xi_t \cdot \mathbf{e}_{-12}|)((\xi_1 - \xi_0) \cdot \mathbf{e}_{-12})^2] dt \geq 0, \end{aligned} \quad (20)$$

which completes the proof. $\square$

Lemma 3. Uniform integrability and weak convergence [15].

Assume $\Omega \subset \mathbb{R}^2$ is bounded, and let $\{u_i\}_{i=1}^\infty$ be a sequence of functions in $L^1(\Omega)$ satisfying

$$\sup_i \|u_i\|_{L^1(\Omega)} < \infty. \quad (21)$$

Suppose also

$$\lim_{l \rightarrow \infty} \sup_i \int_{\Omega \cap \{|u_i| \geq l\}} |u_i| dx = 0. \quad (22)$$

Then, there exist a subsequence $\{u_{i_j}\}_{j=1}^\infty$ and $\tilde{u} \in L^1(\Omega)$ such that

$$u_{i_j} \rightharpoonup \tilde{u}, \quad \text{weakly in } L^1(\Omega). \quad (23)$$

Lemma 4. Assume $\Omega \subset \mathbb{R}^2$ is bounded, and let $\{u_i\}_{i=1}^\infty$ be a sequence of functions in $L^1(\Omega)$ such that

$$\sup_i \int_{\Omega \cap \{|u_i| \geq k_m\}} |u_i| \log(|u_i|) dx < \infty. \quad (24)$$

Then, there exist a subsequence $\{u_{i_j}\}_{j=1}^\infty$ and $\tilde{u} \in L^1(\Omega)$ such that

$$u_{i_j} \rightharpoonup \tilde{u}, \quad \text{weakly in } L^1(\Omega), \quad (25)$$

with $\tilde{u} \in L \log L^{k_m}(\Omega)$.

Proof. Given $M > 0$, we may find an $l \geq k_m$ such that $Ms \leq s \log(s)$ for all $s \geq l$. Consequently,

$$\begin{aligned} \int_{\Omega} |u_i| dx &= \int_{\Omega \cap \{|u_i| < k_m\}} |u_i| dx + \int_{\Omega \cap \{|u_i| \geq k_m\}} |u_i| dx \\ &\leq k_m |\Omega| + \frac{1}{M} \int_{\Omega \cap \{|u_i| \geq k_m\}} |u_i| \log(|u_i|) dx, \end{aligned} \quad (26)$$

which implies that

$$\sup_i \int_{\Omega} |u_i| dx < \infty. \quad (27)$$

On the other hand, there exists a positive constant C such that

$$\begin{aligned} \int_{\Omega \cap \{|u_i| \geq l\}} |u_i| dx &\leq \frac{1}{M} \int_{\Omega \cap \{|u_i| \geq l\}} |u_i| \log(|u_i|) dx \\ &\leq \frac{1}{M} \int_{\Omega \cap \{|u_i| \geq k_m\}} |u_i| \log(|u_i|) dx \\ &\leq \frac{C}{M} = \varepsilon, \end{aligned} \quad (28)$$

which is true for all i and arbitrary $\varepsilon > 0$. It follows then that

$$\lim_{l \rightarrow \infty} \sup_i \int_{\Omega \cap \{|u_i| \geq l\}} |u_i| dx = 0. \quad (29)$$

Then, from Lemma 3, there exist a subsequence $\{u_{i_j}\}_{j=1}^\infty$ of $\{u_i\}_{i=1}^\infty$ and a function $\tilde{u} \in L^1(\Omega)$ such that

$$u_{i_j} \rightharpoonup \tilde{u}, \quad \text{weakly in } L^1(\Omega). \quad (30)$$

It remains to prove that $\tilde{u} \in L \log L^{k_m}(\Omega)$.

We know that the function $f(s) = s \log(s)$ for $s \geq 1$ is increasing and convex, and then the function $f(|s|)$ is also convex for all $s \geq 1$. Therefore, we obtain

$$f(|\tilde{u}|) \leq f(|u_{i_j}|) + f'(|\tilde{u}|)(\tilde{u} - u_{i_j}). \quad (31)$$

Integrating the above inequality over $\Omega_N \cap \{|u_{i_j}| \geq k_m\}$ with $\Omega_N = \Omega \cap \{k_m \leq |\tilde{u}| \leq N\}$, we have

$$\begin{aligned} \int_{\Omega_N} f(|\tilde{u}|) dx &\leq \int_{\Omega \cap \{|u_{i_j}| \geq k_m\}} f(|u_{i_j}|) dx + \int_{\Omega_N \cap \{|u_{i_j}| \geq k_m\}} f'(|\tilde{u}|)(\tilde{u} - u_{i_j}) dx \\ &\leq \int_{\Omega \cap \{|u_{i_j}| \geq k_m\}} f(|u_{i_j}|) dx + \int_{\Omega \cap \{|u_{i_j}| \geq k_m\}} f'(|\tilde{u}|) \chi_{\{k_m \leq |\tilde{u}| \leq N\}} (\tilde{u} - u_{i_j}) dx. \end{aligned} \quad (32)$$

Since $f'(|\tilde{u}|) \chi_{\{k_m \leq |\tilde{u}| \leq N\}} \in L^\infty(\Omega)$ and passing to limits as $j \rightarrow \infty$, we get

$$\int_{\Omega_N} f(|\tilde{u}|) dx \leq \liminf_{j \rightarrow \infty} \int_{\Omega \cap \{|u_{i_j}| \geq k_m\}} f(|u_{i_j}|) dx < \infty. \quad (33)$$

Then, passing to limits as $N \rightarrow \infty$, we deduce

$$\int_{\Omega \cap \{|\tilde{u}| \geq k_m\}} |\tilde{u}| \log(|\tilde{u}|) dx < \infty. \quad (34)$$

It follows then $\tilde{u} \in L \log L^{k_m}(\Omega)$. This finishes the proof. $\square$

3. Approximate Solutions

In this section, we will discretize the time-variable interval $[0, T]$ to get approximate solutions for problem (3). We denote $h = (T/N)$ with $N \in \mathbb{N}^*$, and we designate by u_n an approximate solution at time nh . We define gradually from $n = 1, 2, \dots, N$ the following elliptic problems:

$$\begin{cases} \frac{u_n - u_{n-1}}{h} - \nabla \cdot [\mathbf{D}_{\nabla u_n} \nabla u_n] = 0, & \text{in } \Omega, \\ \langle \mathbf{D}_{\nabla u_n} \nabla u_n, \mathbf{n} \rangle = 0, & \text{on } \partial\Omega. \end{cases} \quad (35)$$

To solve these equations step by step, we only need to prove the existence and uniqueness of weak solutions of the following elliptic problems:

$$\begin{cases} \frac{u - u_0}{h} - \nabla \cdot [\mathbf{D}_{\nabla u} \nabla u] = 0, & \text{in } \Omega, \\ \langle \mathbf{D}_{\nabla u} \nabla u, \mathbf{n} \rangle = 0, & \text{on } \partial\Omega, \end{cases} \quad (36)$$

where $h > 0$ and $u_0 \in L^2(\Omega)$.

Definition 2. A function $u \in L^2(\Omega) \cap W^{1,1}(\Omega)$ with $\partial_{x_i} u \in L \log L^{k_m}(\Omega)$ for $i = 1, 2$ is called a weak solution for problem (36); if for any $\varphi \in C^1(\overline{\Omega})$, we have

$$\int_{\Omega} \frac{u - u_0}{h} \varphi dx + \int_{\Omega} \mathbf{D}_{\nabla u} \nabla u \cdot \nabla \varphi dx = 0. \quad (37)$$

And when φ is a constant function, we obtain

$$\int_{\Omega} u dx = \int_{\Omega} u_0 dx. \quad (38)$$

In order to prove the existence and uniqueness of weak solutions for problem (36), we consider the variational problem

where

$$\min\{E(u) | u \in U\}, \quad (39)$$

$$U = \left\{ u \in L^2(\Omega) \cap W^{1,1}(\Omega) \mid \partial_{x_i} u \in L \log L^{k_m}(\Omega) \text{ with } i = 1, 2, \int_{\Omega} u dx = \int_{\Omega} u_0 dx \right\}, \quad (40)$$

and when $u \in U$, the functional E is defined as

$$E(u) = \int_{\Omega} \left[\phi(|u_{x_1}|) + \phi(|u_{x_2}|) + \phi(|u_{x_{12}}|) + \phi(|u_{x_{-12}}|) \right] + \frac{1}{2h} \int_{\Omega} (u - u_0)^2 dx. \quad (41)$$

It is easy to prove that (36) is the Euler–Lagrange equations of the functional E [16].

Theorem 2. *Problem (36) has a unique weak solution.*

Proof. Since

$$0 \leq \inf_{u \in U} E(u) \leq E(0) = \frac{1}{2h} \int_{\Omega} u_0^2 dx, \quad (42)$$

then we can construct a minimizing sequence $\{u_q\}_{q=1}^{\infty}$ in U such that $E(u_q) < E(0) + 1$ and

$$\lim_{q \rightarrow \infty} E(u_q) = \inf_{u \in U} E(u). \quad (43)$$

Besides,

$$\begin{aligned} \int_{\Omega} u_q^2 dx &= \int_{\Omega} (u_q - u_0 + u_0)^2 dx \\ &\leq 2 \int_{\Omega} (u_q - u_0)^2 + 2 \int_{\Omega} u_0^2 dx \\ &\leq 4h(E(u_q) + E(0)) \\ &\leq 4h(2E(0) + 1). \end{aligned} \quad (44)$$

It follows then

$$\sup_q \|u_q\|_{L^2(\Omega)} < \infty. \quad (45)$$

On the other hand, given $\varepsilon_0 > 0$, we may find $l_0 = k_m$ such that

$$\begin{aligned} \int_{\Omega \cap \{|\partial_{x_i} u_q| \geq k_m\}} |\partial_{x_i} u_q| \log(|\partial_{x_i} u_q|) dx &\leq C \int_{\Omega \cap \{|\partial_{x_i} u_q| \geq k_m\}} \phi(|\partial_{x_i} u_q|) dx \\ &\leq CE(u_q) < C(E(0) + 1) \end{aligned} \quad (46)$$

with $C = (\varepsilon_0 + (1/A_{k_m,2})) > 0$ and $i = 1, 2$. It follows then that for $i = 1, 2$,

$$\sup_q \int_{\Omega \cap \{|\partial_{x_i} u_q| \geq k_m\}} |\partial_{x_i} u_q| \log(|\partial_{x_i} u_q|) dx < \infty. \quad (47)$$

Therefore, thanks to Lemma 4 and the weak compactness of $L^2(\Omega)$, we can find a subsequence $\{u_{q_j}\}_{j=1}^{\infty}$ of $\{u_q\}_{q=1}^{\infty}$ and a function $u_1 \in L^2(\Omega) \cap W^{1,1}(\Omega)$ such that

$$u_{q_j} \rightharpoonup u_1, \quad \text{weakly in } L^2(\Omega), \quad (48)$$

and for $i = 1, 2$,

$$\begin{aligned} \partial_{x_i} u_{q_j} &\rightharpoonup \partial_{x_i} u_1, \quad \text{weakly in } L^1(\Omega), \\ \partial_{x_i} u_1 &\in L \log L^{k_m}(\Omega). \end{aligned} \quad (49)$$

Therefore, we have

$$\begin{aligned} \int_{\Omega} u_1 dx &= \lim_{j \rightarrow \infty} \int_{\Omega} u_{q_j} dx = \int_{\Omega} u_0 dx, \\ \int_{\Omega} (u_1 - u_0)^2 dx &\leq \liminf_{j \rightarrow \infty} \int_{\Omega} (u_{q_j} - u_0)^2 dx, \end{aligned} \quad (50)$$

and following the reasoning in the proof of Lemma 4, it is easy to show that for any $a \in \{x_1, x_2, x_{12}, x_{-12}\}$ and for a fixed $\varepsilon > 0$, there exists $l \geq k_m$ such that

$$\int_{\Omega \cap \{|\partial_a u_1| \geq l\}} |\partial_a u_1| \log(|\partial_a u_1|) dx \leq \liminf_{j \rightarrow \infty} \int_{\Omega \cap \{|\partial_a u_{q_j}| \geq l\}} |\partial_a u_{q_j}| \log(|\partial_a u_{q_j}|) dx, \quad (51)$$

$$\int_{\Omega \cap \{|\partial_a u_1| \geq l\}} \phi(|\partial_a u_1|) dx \leq \left(\varepsilon + A_{k_m, 2} \right) \left(\varepsilon + \frac{1}{A_{k_m, 2}} \right) \liminf_{j \rightarrow \infty} \int_{\Omega \cap \{|\partial_a u_{q_j}| \geq l\}} \phi(|\partial_a u_{q_j}|) dx. \quad (52)$$

Similarly, since ϕ is increasing and convex in $[0, l]$, then we can prove that

$$\int_{\Omega \cap \{|\partial_a u_1| < l\}} \phi(|\partial_a u_1|) dx \leq \liminf_{j \rightarrow \infty} \int_{\Omega \cap \{|\partial_a u_{q_j}| < l\}} \phi(|\partial_a u_{q_j}|) dx. \quad (53)$$

Therefore, we obtain from (52) and (53) that

$$\begin{aligned} \int_{\Omega} \phi(|\partial_a u_1|) dx &= \int_{\Omega \cap \{|\partial_a u_1| < l\}} \phi(|\partial_a u_1|) dx + \int_{\Omega \cap \{|\partial_a u_1| \geq l\}} \phi(|\partial_a u_1|) dx \\ &\leq \left(\varepsilon + A_{k_m, 2} \right) \left(\varepsilon + \frac{1}{A_{k_m, 2}} \right) \liminf_{j \rightarrow \infty} \int_{\Omega} \phi(|\partial_a u_{q_j}|) dx. \end{aligned} \quad (54)$$

Thus, by letting $\varepsilon \rightarrow 0$, we get

$$\int_{\Omega} \phi(|\partial_a u_1|) dx \leq \liminf_{j \rightarrow \infty} \int_{\Omega} \phi(|\partial_a u_{q_j}|) dx, \quad (55)$$

for any $a \in \{x_1, x_2, x_{12}, x_{-12}\}$. It follows then that

$$E(u_1) \leq \liminf_{j \rightarrow \infty} E(u_{q_j}) = \inf_{u \in U} E(u), \quad (56)$$

which signifies that $u_1 \in U$ is a minimizer of the energy functional $E(u)$, i.e.,

$$E(u_1) = \min_{u \in U} E(u). \quad (57)$$

Furthermore, for all $\varphi \in C^1(\overline{\Omega})$ and $t \in \mathbb{R}$, we have $u_1 + t(\varphi - \varphi_{\Omega}) \in U$ with $\varphi_{\Omega} = (1/|\Omega|) \int_{\Omega} \varphi dx$. Then, $\rho(0) \leq \rho(t)$ where

$$\rho(t) = E(u_1 + t(\varphi - \varphi_{\Omega})). \quad (58)$$

Hence, we have $\rho'(0) = 0$, which means

$$\int_{\Omega} \frac{u_1 - u_0}{h} (\varphi - \varphi_{\Omega}) dx + \int_{\Omega} \mathbf{D}_{\nabla u_1} \nabla u_1 \cdot \nabla \varphi dx = 0. \quad (59)$$

Because of (50), we get

$$\int_{\Omega} \frac{u_1 - u_0}{h} \varphi dx + \int_{\Omega} \mathbf{D}_{\nabla u_1} \nabla u_1 \cdot \nabla \varphi dx = 0. \quad (60)$$

We conclude then that u_1 is a weak solution for problem (36).

Now, assume that there is another weak solution $\hat{u}$ of (36). Then, for every $\varphi \in C^1(\overline{\Omega})$, we have

$$\int_{\Omega} \frac{\hat{u} - u_0}{h} \varphi dx + \int_{\Omega} \mathbf{D}_{\nabla \hat{u}} \nabla \hat{u} \cdot \nabla \varphi dx = 0, \quad (61)$$

which leads to

$$\int_{\Omega} \frac{\hat{u} - u_1}{h} \varphi dx + \int_{\Omega} [\mathbf{D}_{\nabla \hat{u}} \nabla \hat{u} - \mathbf{D}_{\nabla u_1} \nabla u_1] \cdot \nabla \varphi dx = 0. \quad (62)$$

Then, if we choose $\varphi = \hat{u} - u_1$ as a test function in (62), we get

$$\int_{\Omega} \frac{(\hat{u} - u_1)^2}{h} dx + \int_{\Omega} [\mathbf{D}_{\nabla \hat{u}} \nabla \hat{u} - \mathbf{D}_{\nabla u_1} \nabla u_1] \cdot (\nabla \hat{u} - \nabla u_1) dx = 0. \quad (63)$$

Thanks to Lemma 2, we deduce that

$$\int_{\Omega} \frac{(\hat{u} - u_1)^2}{h} dx = 0. \quad (64)$$

Therefore, $\hat{u} = u_1$ a.e. in Ω .

In conclusion, we have shown that there exists a unique weak solution $u_n \in U$ satisfying (35) for every $n \in \{1, 2, \dots, N\}$. Consequently, we define an approximate solution u_h for problem (3) as

$$u_h(x, t) = \begin{cases} u_0(x), & t = 0, \\ u_1(x), & t \in (0, h], \\ \dots\dots\dots, & \dots\dots\dots, \\ u_j(x), & t \in ((j-1)h, jh], \\ \dots\dots\dots, & \dots\dots\dots, \\ u_N(x), & t \in ((N-1)h, T], \end{cases} \quad (65)$$

for every $h = (T/N)$. $\square$

4. Existence and Uniqueness of Weak Solutions

Proof. of Theorem 1. In the beginning, we establish the uniqueness of solutions for problem (3). For this purpose, we suppose there exist two weak solutions u and v for problem (3). Then, we obtain the following:

$$\begin{cases} \frac{\partial(u-v)}{\partial t} - \nabla \cdot [\mathbf{D}_{\nabla u} \nabla u - \mathbf{D}_{\nabla v} \nabla v] = 0, & \text{in } Q_T, \\ \langle \mathbf{D}_{\nabla u} \nabla u - \mathbf{D}_{\nabla v} \nabla v, \mathbf{n} \rangle = 0, & \text{on } \partial\Omega \times (0, T], \\ (u-v)(x; 0) = 0, & \text{in } \Omega. \end{cases} \quad (66)$$

By multiplying the first equation of the above problem by $(u-v)$ and integrating over Ω and $[0, t]$, we get

$$\frac{1}{2} \int_{\Omega} (u-v)^2(t) dx + \int_0^t \int_{\Omega} [\mathbf{D}_{\nabla u} \nabla u - \mathbf{D}_{\nabla v} \nabla v] \cdot \nabla (u-v) dx d\tau = 0, \quad (67)$$

for every $t \in (0, T]$. Since the second term of the above equation is nonnegative (thanks to Lemma 2), it follows then $u = v$ a.e. in Q_T .

Let us now find our weak solution for problem (3). We intend to send h to zero and show that a subsequence of our solutions u_h of the approximate problems (35) converges to a weak solution for problem (3). To this end, we need to find some a priori estimates.

It follows from (35) that for every $\varphi \in C^1(\bar{\Omega})$,

$$\int_{\Omega} \frac{u_n - u_{n-1}}{h} \varphi dx + \int_{\Omega} \mathbf{D}_{\nabla u_n} \nabla u_n \cdot \nabla \varphi dx = 0. \quad (68)$$

Then, by taking u_n as a test function in (68) and using $u_n u_{n-1} \leq ((u_n^2 + u_{n-1}^2)/2)$, we get

$$\frac{1}{2} \int_{\Omega} u_n^2 dx + h \int_{\Omega} \mathbf{D}_{\nabla u_n} \nabla u_n \cdot \nabla u_n dx \leq \frac{1}{2} \int_{\Omega} u_{n-1}^2 dx. \quad (69)$$

For each $t \in (0, T]$, we can find $j \in \{1, \dots, N\}$ such that $t \in ((j-1)h, jh]$. Then, by adding all the inequalities (69) from $n = 1$ to $n = j$, we get

$$\sup_{0 \leq t \leq T} \int_{\Omega} u_h^2(x, t) dx + \int_0^T \int_{\Omega} \mathbf{D}_{\nabla u_h} \nabla u_h \cdot \nabla u_h dx d\tau \leq 2 \int_{\Omega} u_0^2 dx. \quad (73)$$

Recalling that $0 \leq \phi(s) \leq s\phi'(s)$ for all $s \geq 0$, then we can derive the following:

$$\begin{aligned} \mathbf{D}_{\nabla u_h} \nabla u_h \cdot \nabla u_h &= |\nabla u_h \cdot \mathbf{e}_1| \phi'(|\nabla u_h \cdot \mathbf{e}_1|) + |\nabla u_h \cdot \mathbf{e}_2| \phi'(|\nabla u_h \cdot \mathbf{e}_2|) \\ &\quad + |\nabla u_h \cdot \mathbf{e}_{12}| \phi'(|\nabla u_h \cdot \mathbf{e}_{12}|) + |\nabla u_h \cdot \mathbf{e}_{-12}| \phi'(|\nabla u_h \cdot \mathbf{e}_{-12}|) \\ &\geq \phi(|\nabla u_h \cdot \mathbf{e}_1|) + \phi(|\nabla u_h \cdot \mathbf{e}_2|) + \phi(|\nabla u_h \cdot \mathbf{e}_{12}|) + \phi(|\nabla u_h \cdot \mathbf{e}_{-12}|). \end{aligned} \quad (74)$$

$$\frac{1}{2} \int_{\Omega} u_j^2 dx + h \sum_{n=1}^j \int_{\Omega} \mathbf{D}_{\nabla u_n} \nabla u_n \cdot \nabla u_n dx \leq \frac{1}{2} \int_{\Omega} u_0^2 dx. \quad (70)$$

Then, by definition of u_h , we obtain for $t \in ((j-1)h, jh]$ that

$$\frac{1}{2} \int_{\Omega} u_h^2(x, t) dx + \int_0^{jh} \int_{\Omega} \mathbf{D}_{\nabla u_h} \nabla u_h \cdot \nabla u_h dx d\tau \leq \frac{1}{2} \int_{\Omega} u_0^2 dx. \quad (71)$$

Since $\mathbf{D}_{\nabla u_h}$ is a symmetric positive definite matrix, we have also

$$\frac{1}{2} \int_{\Omega} u_h^2(x, t) dx + \int_0^t \int_{\Omega} \mathbf{D}_{\nabla u_h} \nabla u_h \cdot \nabla u_h dx d\tau \leq \frac{1}{2} \int_{\Omega} u_0^2 dx. \quad (72)$$

Therefore, after taking the supremum over $[0, T]$, we deduce that

Besides, as in (46), for $|\partial_{x_1} u_h|, |\partial_{x_2} u_h| \geq k_m$, we may find a positive constant C such that

$$\begin{aligned} |\partial_{x_1} u_h| \log(|\partial_{x_1} u_h|) + |\partial_{x_2} u_h| \log(|\partial_{x_2} u_h|) &\leq C(\phi(|\nabla u_h \cdot \mathbf{e}_1|) + \phi(|\nabla u_h \cdot \mathbf{e}_2|)) \\ &\leq C \mathbf{D}_{\nabla u_h} \nabla u_h \cdot \nabla u_h. \end{aligned} \quad (75)$$

Thus, we conclude

$$\begin{cases} \sup_{0 \leq t \leq T} \int_{\Omega} u_h^2(x, t) dx < \infty, \\ \int_0^T \int_{\Omega \cap \{|\partial_{x_1} u_h| \geq k_m\}} |\partial_{x_1} u_h| \log(|\partial_{x_1} u_h|) dx d\tau < \infty, \\ \int_0^T \int_{\Omega \cap \{|\partial_{x_2} u_h| \geq k_m\}} |\partial_{x_2} u_h| \log(|\partial_{x_2} u_h|) dx d\tau < \infty. \end{cases} \quad (76)$$

By Lemma 4, we can find a subsequence of $\{u_h\}$ (for simplicity, we also denote it by u_h) such that [17]

$$\begin{aligned} u_h &\rightharpoonup u, \quad \text{weakly-}^* \text{ in } L^\infty(0, T; L^2(\Omega)), \\ u_h &\rightharpoonup u, \quad \text{weakly in } L^1(0, T; W^{1,1}(\Omega)), \end{aligned} \quad (77)$$

with

$$\begin{cases} \sup_{0 \leq t \leq T} \int_{\Omega} u^2(x, t) dx < \infty, \\ \int_0^T \int_{\Omega \cap \{|\partial_{x_1} u| \geq k_m\}} |\partial_{x_1} u| \log(|\partial_{x_1} u|) dx d\tau < \infty, \\ \int_0^T \int_{\Omega \cap \{|\partial_{x_2} u| \geq k_m\}} |\partial_{x_2} u| \log(|\partial_{x_2} u|) dx d\tau < \infty. \end{cases} \quad (78)$$

So, it remains to prove that u is just a weak solution for problem (3). Let us now denote $\xi_h = \mathbf{D}_{\nabla u_h} \nabla u_h$. We will show that ξ_h is bounded in $[L^2(Q_T)]^2$, so we may find a

subsequence of ξ_h that converges weakly in $[L^2(Q_T)]^2$ to a particular vector-valued function. Then, we will prove that this vector-valued function is equal almost everywhere to $\mathbf{D}_{\nabla u} \nabla u$ in Q_T through monotonicity condition (17).

From the expression of $\mathbf{D}_{\nabla u_h}$, we can derive the following:

$$\begin{aligned} |\xi_h| &= \left| \frac{\partial_{x_1} u_h}{|\partial_{x_1} u_h|} \phi'(|\partial_{x_1} u_h|) \mathbf{e}_1 + \frac{\partial_{x_2} u_h}{|\partial_{x_2} u_h|} \phi'(|\partial_{x_2} u_h|) \mathbf{e}_2 \right. \\ &\quad \left. + \frac{\partial_{x_{12}} u_h}{|\partial_{x_{12}} u_h|} \phi'(|\partial_{x_{12}} u_h|) \mathbf{e}_{12} + \frac{\partial_{x_{-12}} u_h}{|\partial_{x_{-12}} u_h|} \phi'(|\partial_{x_{-12}} u_h|) \mathbf{e}_{-12} \right| \\ &\leq 4\phi'(|\partial_{x_1} u_h| + |\partial_{x_2} u_h|). \end{aligned} \quad (79)$$

Given $\varepsilon_1, \varepsilon_2 > 0$, we may find $l_1 = l_2 = k_m$ such that

$$\begin{aligned} \phi'(s) &\leq M \log(s), \\ s &\leq \varepsilon_2 s \log(s), \end{aligned} \quad (80)$$

for all $s \geq k_m$ with $M = (\varepsilon_1 + A_{k_m, 2})$. Thus, we can distinguish two cases:

- (i) If $|\partial_{x_1} u_h| + |\partial_{x_2} u_h| < k_m$ then $|\xi_h|^2 \leq (4\phi'(k_m))^2$.
- (ii) If $|\partial_{x_1} u_h| + |\partial_{x_2} u_h| \geq k_m$ then

$$\begin{aligned} |\xi_h| &\leq 4M \log(|\partial_{x_1} u_h| + |\partial_{x_2} u_h|), \\ |\xi_h|^2 &\leq (4M)^2 (1 + \varepsilon_2 \log(2)) (|\partial_{x_1} u_h| \log(|\partial_{x_1} u_h|) + |\partial_{x_2} u_h| \log(|\partial_{x_2} u_h|)), \end{aligned} \quad (81)$$

$$|\xi_h| \exp\left(\frac{|\xi_h|}{4M}\right) \leq 4M (1 + \varepsilon_2 \log(2)) (|\partial_{x_1} u_h| \log(|\partial_{x_1} u_h|) + |\partial_{x_2} u_h| \log(|\partial_{x_2} u_h|)).$$

Then, $\{\xi_h\}$ is bounded in $[L^2(Q_T)]^2$, which means that we can find a subsequence of $\{\xi_h\}$ (denote it also by $\{\xi_h\}$) and a function $\xi \in [L^2(Q_T)]^2$ such that

$$\xi_h \rightharpoonup \xi, \quad \text{weakly in } [L^2(Q_T)]^2. \quad (82)$$

Since $s \mapsto s \exp(s)$ ($s \geq 0$) is increasing and convex, then as in the proof of Lemma 4, we deduce that

$$\begin{aligned} \int_0^T \int_{\Omega \cap \{|\partial_{x_1} u_h| + |\partial_{x_2} u_h| \geq k_m\}} |\xi| \exp\left(\frac{|\xi|}{4M}\right) dx dt &\leq \\ \liminf_{h \rightarrow 0} \int_0^T \int_{\Omega \cap \{|\partial_{x_1} u_h| + |\partial_{x_2} u_h| \geq k_m\}} |\xi_h| \exp\left(\frac{|\xi_h|}{4M}\right) dx dt &< \infty. \end{aligned} \quad (83)$$

Then, by using Lemma 1, we get

$$\begin{aligned}
\int_0^T \int_{\Omega} |\xi \cdot \nabla u| dx dt &\leq \int_0^T \int_{\Omega} |\xi| |\nabla u| dx dt \\
&\leq \int_0^T \int_{\Omega} |\xi| (|\partial_{x_1} u| + |\partial_{x_2} u|) dx dt \\
&\leq k_m \int_0^T \int_{\Omega \cap \{|\partial_{x_1} u| + |\partial_{x_2} u| < k_m\}} |\xi| dx dt \\
&\quad + \int_0^T \int_{\Omega \cap \{|\partial_{x_1} u| + |\partial_{x_2} u| \geq k_m\}} |\xi| \exp\left(\frac{|\xi|}{4M}\right) dx dt \\
&\quad + 4M(1 + \varepsilon_2) \left[\int_0^T \int_{\Omega \cap \{|\partial_{x_1} u| \geq k_m\}} |\partial_{x_1} u| \log(|\partial_{x_1} u|) dx dt \right. \\
&\quad \left. + \int_0^T \int_{\Omega \cap \{|\partial_{x_2} u| \geq k_m\}} |\partial_{x_2} u| \log(|\partial_{x_2} u|) dx dt \right] < \infty.
\end{aligned} \tag{84}$$

It follows then $\xi \cdot \nabla u \in L^1(Q_T)$. Next, we will show that $\xi = \mathbf{D}_{\nabla u} \nabla u$ a.e. in Q_T .

For each $\varphi \in C^1(Q_T)$ with $\varphi(\cdot, T) = 0$, we take $\varphi(x, nh)$ as a test function in (35):

$$\int_{\Omega} \frac{u_n(x) - u_{n-1}(x)}{h} \varphi(x, nh) dx + \int_{\Omega} \mathbf{D}_{\nabla u_n} \nabla u_n \cdot \nabla \varphi(x, nh) dx = 0, \tag{85}$$

$$\begin{aligned}
& -\frac{1}{h} \int_{\Omega} u_0(x) \varphi(x, 0) dx + \sum_{n=0}^{N-1} \int_{\Omega} u_n(x) \frac{\varphi(x, nh) - \varphi(x, (n+1)h)}{h} dx \\
& + \sum_{n=1}^N \int_{\Omega} \mathbf{D}_{\nabla u_n} \nabla u_n \cdot \nabla \varphi(x, nh) dx = 0.
\end{aligned} \tag{86}$$

From the definition of u_h (65), we have

with $n \in \{1, 2, \dots, N\}$. By summing n from 1 to N , we obtain

$$\begin{aligned}
\sum_{n=0}^{N-1} \int_{\Omega} u_n(x) \frac{\varphi(x, nh) - \varphi(x, (n+1)h)}{h} dx &= - \sum_{n=0}^{N-1} \int_{nh}^{(n+1)h} \int_{\Omega} u_h(x, t) \frac{\varphi_t(x, t)}{h} dx dt \\
&= -\frac{1}{h} \int_0^T \int_{\Omega} u_h(x, t) \varphi_t(x, t) dx dt.
\end{aligned} \tag{87}$$

Therefore,

$$\begin{aligned}
& - \int_{\Omega} u_0(x) \varphi(x, 0) dx - \int_0^T \int_{\Omega} u_h(x, t) \varphi_t(x, t) dx dt + \int_0^T \int_{\Omega} \mathbf{D}_{\nabla u_h} \nabla u_h \cdot \nabla \varphi dx dt \\
& + \sum_{n=1}^N \int_{(n-1)h}^{nh} \int_{\Omega} \mathbf{D}_{\nabla u_n} \nabla u_n \cdot [\nabla \varphi(x, nh) - \nabla \varphi(x, t)] dx dt = 0.
\end{aligned} \tag{88}$$

Letting h tend to zero, we get

$$\int_{\Omega} u_0(x) \varphi(x, 0) dx + \int_0^T \int_{\Omega} u \varphi_t dx dt = \int_0^T \int_{\Omega} \xi \cdot \nabla \varphi dx dt. \tag{89}$$

On the other hand, we let $v \in L^1(Q_T)$ with

$$\int_0^T \int_{\Omega \cap \{|\partial_{x_i} v| \geq k_m\}} |\partial_{x_i} v| \log(|\partial_{x_i} v|) dx dt < \infty, \quad (90)$$

for $i = 1, 2$. We sum up inequalities (69):

$$\frac{1}{2} \int_{\Omega} u_h^2(T) dx + \int_0^T \int_{\Omega} \mathbf{D}_{\nabla u_h} \nabla u_h \cdot \nabla u_h dx dt \leq \frac{1}{2} \int_{\Omega} u_0^2 dx. \quad (91)$$

We have from Lemma 2 that

$$\int_0^T \int_{\Omega} (\mathbf{D}_{\nabla u_h} \nabla u_h - \mathbf{D}_{\nabla v} \nabla v) \cdot (\nabla u_h - \nabla v) dx dt \geq 0. \quad (92)$$

Then, we obtain

$$\begin{aligned} & \frac{1}{2} \int_{\Omega} u_h^2(T) dx + \int_0^T \int_{\Omega} \mathbf{D}_{\nabla u_h} \nabla u_h \cdot \nabla v dx dt + \int_0^T \int_{\Omega} \mathbf{D}_{\nabla v} \nabla v \cdot \nabla u_h dx dt \\ & - \int_0^T \int_{\Omega} \mathbf{D}_{\nabla v} \nabla v \cdot \nabla v dx dt \leq \frac{1}{2} \int_{\Omega} u_0^2 dx. \end{aligned} \quad (93)$$

Letting $h \rightarrow 0$ and noting that

$$\int_{\Omega} u^2(T) dx \leq \liminf_{h \rightarrow 0} \int_{\Omega} u_h^2(T) dx, \quad (94)$$

we obtain

$$\begin{aligned} & \frac{1}{2} \int_{\Omega} u^2(T) dx + \int_0^T \int_{\Omega} \xi \cdot \nabla v dx dt + \int_0^T \int_{\Omega} \mathbf{D}_{\nabla v} \nabla v \cdot \nabla u dx dt \\ & - \int_0^T \int_{\Omega} \mathbf{D}_{\nabla v} \nabla v \cdot \nabla v dx dt \leq \frac{1}{2} \int_{\Omega} u_0^2 dx. \end{aligned} \quad (95)$$

By using $\varphi = u$ in (89), we get

$$\frac{1}{2} \int_{\Omega} u^2(T) dx + \frac{1}{2} \int_{\Omega} u_0^2 dx = \int_0^T \int_{\Omega} \xi \cdot \nabla u dx dt. \quad (96)$$

Combining (95) with (96), we have

$$\int_0^T \int_{\Omega} (\xi - \mathbf{D}_{\nabla v} \nabla v) \cdot (\nabla v - \nabla u) dx dt \leq - \int_{\Omega} u^2(T) dx. \quad (97)$$

Now, setting $v = u + \lambda w$ for any $\lambda > 0$, $w \in W^{1,2}(Q_T)$, we derive from the above inequality that

$$\int_0^T \int_{\Omega} (\xi - \mathbf{D}_{\nabla(u+\lambda w)} \nabla(u+\lambda w)) \cdot \nabla w dx dt \leq 0. \quad (98)$$

By letting $\lambda \rightarrow 0$ and using Lebesgue's dominated convergence theorem, we obtain

$$\int_0^T \int_{\Omega} (\xi - \mathbf{D}_{\nabla u} \nabla u) \cdot \psi dx dt = 0, \quad (99)$$

for every $\psi \in [L^2(\Omega)]^2$. It follows then

$$\xi = \mathbf{D}_{\nabla u} \nabla u, \quad \text{a.e. in } Q_T. \quad (100)$$

Therefore, we conclude from (89) that

$$- \int_{\Omega} u_0(x) \varphi(x, 0) dx + \int_0^T \int_{\Omega} [-u \varphi_t + \mathbf{D}_{\nabla u} \nabla u \cdot \nabla \varphi] dx dt = 0, \quad (101)$$

for any $\varphi \in C^1(\overline{Q_T})$ with $\varphi(\cdot, T) = 0$. Finally, we need to prove that $u \in C([0, T], L^2(\Omega))$. If we choose $\varphi \in C_0^\infty(Q_T)$ in (89), we obtain

$$\int_0^T \int_{\Omega} u \varphi_t dx dt = \int_0^T \int_{\Omega} \xi \cdot \nabla \varphi dx dt. \quad (102)$$

Since $\xi \in [L^2(Q_T)]^2$, we conclude that $u_t \in L^1(0, T; H^{-1}(\Omega))$ where $H^{-1}(\Omega)$ is the dual space of $W_0^{1,2}(\Omega)$. Since

$$\begin{aligned} u &= \int_0^t u_\tau d\tau + u_0, \\ u_0 &\in L^2(\Omega) \hookrightarrow H^{-1}(\Omega). \end{aligned} \quad (103)$$

It follows then that $u \in C(0, T; H^{-1}(\Omega))$. Besides, for every $h > 0$, let $v_h(x, t) = u(x, t + h)$ be the weak solution for problem (3) satisfying $v_h(x; 0) = u(x, h)$. Then, $w_h(x, t) = u(x, t + h) - u(x, t)$ satisfies

$$\begin{cases} \frac{\partial w_h}{\partial t} - \nabla \cdot [\mathbf{D}_{\nabla v_h} \nabla v_h - \mathbf{D}_{\nabla u} \nabla u] = 0, & \text{in } \Omega \times (0, T], \\ \langle \mathbf{D}_{\nabla v_h} \nabla v_h - \mathbf{D}_{\nabla u} \nabla u, \mathbf{n} \rangle = 0, & \text{on } \partial\Omega \times (0, T], \\ w_h(x; 0) = u(x, h) - u_0(x), & \text{in } \Omega. \end{cases} \quad (104)$$

For each $t_0 \in [0, T]$, we may choose w_h as a test function in the first equation for problem (104) over $[0, t_0]$:

$$\frac{1}{2} \int_{\Omega} w_h^2(x, t_0) dx + \int_0^{t_0} \int_{\Omega} (\mathbf{D}_{\nabla v_h} \nabla v_h - \mathbf{D}_{\nabla u} \nabla u) \cdot (\nabla v_h - \nabla u) dx dt \leq \frac{1}{2} \int_{\Omega} w_h^2(x, 0) dx. \quad (105)$$

Because of Lemma 2, we deduce

$$\int_{\Omega} |u(x, t_0 + h) - u(x, t_0)|^2 dx \leq \int_{\Omega} |u(x, h) - u_0(x)|^2 dx. \quad (106)$$

Now, in order to prove that $u \in C([0, T], L^2(\Omega))$, we need to prove

$$\limsup_{h \rightarrow 0^+} \int_{\Omega} |u(x, h) - u_0(x)|^2 dx = 0. \quad (107)$$

We suppose that (107) is not true. Then, there exist a positive number δ and a sequence $\{h_i\}$ with $h_i \rightarrow 0$ as $i \rightarrow \infty$ such that

$$\lim_{h_i \rightarrow 0^+} \int_{\Omega} |u(x, h_i) - u_0(x)|^2 dx \geq \delta. \quad (108)$$

From estimate (72), we have

$$\int_{\Omega} |u(x, h_i)|^2 dx \leq \int_{\Omega} |u_0(x)|^2 dx. \quad (109)$$

Then, from (108), we get

$$\liminf_{h_i \rightarrow 0^+} \left(\int_{\Omega} |u_0(x)|^2 dx - \int_{\Omega} u_0(x) u(x, h_i) dx \right) \geq \frac{\delta}{2}. \quad (110)$$

From (109), we conclude that $\{u(x, h_i)\}$ is a bounded sequence in $L^2(\Omega)$. Then, we may find a subsequence (denote it also by $\{u(x, h_i)\}$) such that there exists a function $\tilde{u}_0 \in L^2(\Omega)$ such that

$$u(x, h_i) \rightharpoonup \tilde{u}_0, \quad \text{weakly in } L^2(\Omega). \quad (111)$$

Since $u \in C(0, T; H^{-1}(\Omega))$, it follows that

$$u(x, h_i) \rightarrow u_0, \quad \text{weakly in } H^{-1}(\Omega). \quad (112)$$

Therefore, we must have $\tilde{u}_0 = u_0$, and since $u \in C(0, T; H^{-1}(\Omega))$, it follows that

$$u(x, h_i) \rightarrow u_0, \quad \text{weakly in } L^2(\Omega), \quad (113)$$

which is contradictory with (110). Therefore, we conclude that (107) is true and $u \in C([0, T], L^2(\Omega))$. This completes the proof of Theorem 1. $\square$

5. Numerical Implementation and Experimental Results

5.1. Consistent and Stable Symmetric Finite Difference Approximation. In this section, we provide a consistent and stable discretization scheme using symmetric finite difference approximation: at time $t_n = n\delta_t$, $n \geq 0$, and the mesh points $x_i = i\delta$, $y_j = j\delta$ ($0 \leq i \leq N+1$ and $0 \leq j \leq M+1$), and we denote by $u_{i,j}^n$ the finite difference approximation of $u(x_i, y_j; t_n)$. The time-space derivatives are discretized in the following manner:

$$\begin{aligned}
u_{x_1}(x_i, y_j; t_n) &= \frac{u(x_{i+(1/2)}, y_j; t_n) - u(x_{i-(1/2)}, y_j; t_n)}{\delta} + \mathcal{O}(\delta^2), \\
u_{x_2}(x_i, y_j; t_n) &= \frac{u(x_i, y_{j+(1/2)}; t_n) - u(x_i, y_{j-(1/2)}; t_n)}{\delta} + \mathcal{O}(\delta^2), \\
u_{x_{12}}(x_i, y_j; t_n) &= \frac{u(x_{i+(1/2)}, y_{j+(1/2)}; t_n) - u(x_{i-(1/2)}, y_{j-(1/2)}; t_n)}{\sqrt{2}\delta} + \mathcal{O}(\delta^2), \\
u_{x_{-12}}(x_i, y_j; t_n) &= \frac{u(x_{i-(1/2)}, y_{j+(1/2)}; t_n) - u(x_{i+(1/2)}, y_{j-(1/2)}; t_n)}{\sqrt{2}\delta} + \mathcal{O}(\delta^2), \\
u_t(x_i, y_j; t_n) &= \frac{u(x_i, y_j; t_{n+1}) - u(x_i, y_j; t_n)}{\delta_t} + \mathcal{O}(\delta_t).
\end{aligned} \tag{114}$$

By assume $\delta = 1$ and denote

$$\left\{ \begin{aligned} g_{N_{i,j}}^n &= g(|\Delta_N u_{i,j}^n|), \\ g_{E_{i,j}}^n &= g(|\Delta_E u_{i,j}^n|), \\ g_{S_{i,j}}^n &= g(|\Delta_S u_{i,j}^n|), \\ g_{W_{i,j}}^n &= g(|\Delta_W u_{i,j}^n|), \\ g_{NE_{i,j}}^n &= g\left(|\frac{\Delta_{NE} u_{i,j}^n}{\sqrt{2}}|\right), \\ g_{SE_{i,j}}^n &= g\left(|\frac{\Delta_{SE} u_{i,j}^n}{\sqrt{2}}|\right), \\ g_{SW_{i,j}}^n &= g\left(|\frac{\Delta_{SW} u_{i,j}^n}{\sqrt{2}}|\right), \\ g_{NW_{i,j}}^n &= g\left(|\frac{\Delta_{NW} u_{i,j}^n}{\sqrt{2}}|\right), \end{aligned} \right. \quad \text{with} \quad \left\{ \begin{aligned} \Delta_N u_{i,j}^n &= u_{i,j+1}^n - u_{i,j}^n, \\ \Delta_E u_{i,j}^n &= u_{i+1,j}^n - u_{i,j}^n, \\ \Delta_S u_{i,j}^n &= u_{i,j-1}^n - u_{i,j}^n, \\ \Delta_W u_{i,j}^n &= u_{i-1,j}^n - u_{i,j}^n, \\ \Delta_{NE} u_{i,j}^n &= u_{i+1,j+1}^n - u_{i,j}^n, \\ \Delta_{SE} u_{i,j}^n &= u_{i+1,j-1}^n - u_{i,j}^n, \\ \Delta_{SW} u_{i,j}^n &= u_{i-1,j-1}^n - u_{i,j}^n, \\ \Delta_{NW} u_{i,j}^n &= u_{i-1,j+1}^n - u_{i,j}^n. \end{aligned} \right. \tag{115}$$

Then, we may approximate problem (3) using the above scheme to obtain the following nonlinear diffusion filter:

$$u_{i,j}^{n+1} = u_{i,j}^n + \delta_t \left[g_N \Delta_N u + g_E \Delta_E u + g_S \Delta_S u + g_W \Delta_W u + \frac{g_{NE} \Delta_{NE} u + g_{SE} \Delta_{SE} u + g_{SW} \Delta_{SW} u + g_{NW} \Delta_{NW} u}{2} \right]_{i,j}^n, \tag{116}$$

FIGURE 1: Brain MRI scans: Patient30: sagittal T1 of a 30-year-old female patient [20]. Patient50: coronal T2 of a 50-year-old female patient [21]. Patient55: axial T2 of a 55-year-old patient [22]. (a) Patient30, (b) Patient50, and (c) Patient55.

for $1 \leq i \leq N$, $1 \leq j \leq 1 \leq i \leq M$, and $n \geq 0$, with the initial condition $u_{i,j}^0$ and the discrete Neumann boundary condition:

$$\begin{cases} u_{0,j}^n = u_{1,j}^n, u_{N+1,j}^n = u_{N,j}^n, & \text{for } 1 \leq j \leq M, \\ u_{i,0}^n = u_{i,1}^n, u_{i,M+1}^n = u_{i,M}^n, & \text{for } 1 \leq i \leq N, \\ u_{0,0}^n = u_{1,1}^n, u_{N+1,0}^n = u_{N,1}^n, \\ u_{0,M+1}^n = u_{1,M}^n, u_{N+1,M+1}^n = u_{N,M}^n. \end{cases} \quad (117)$$

A unique sequence $(u^n)_{n \in \mathbb{N}}$ is produced when using filter (116) on a particular initial image u^0 [2]. Besides, due to the continuity of the function g , the sequence u^n depends continuously on u^0 for every finite n . Furthermore, equation (116) satisfies the following maximum-minimum principle, which describes a stability condition for the discrete scheme.

Theorem 3. *Discrete extremum principle [1, 2].*

For an iteration step satisfying

$$0 < \delta_t < \frac{1}{6g(0)}, \quad (118)$$

scheme (116) satisfies

$$g(s) = \begin{cases} p_0 P_{1,0k}(s) + v_0 P_{2,0k}(s) + p_k P_{1,k0}(s) + v_k P_{2,k0}(s), & s \in [0, k[, \\ p_k g_{k,1}(s) + v_k g_{k,2}(s), & s \in [k, \infty[. \end{cases} \quad (120)$$

(ii) The Wang and Zhou diffusion function (WZ) [4]:

$$g(s) = \frac{1}{s+1} + \frac{\log(s+1)}{s}. \quad (121)$$

Additionally, we will consider real test images Figure 1 and evaluate our model's performance on these images, which will be corrupted with different levels of Gaussian white noises with zero mean and variance σ^2 .

Table 1 shows the quantitative results on real images, corrupted with various Gaussian noises, filtered by discrete

$$\min_{i,j} u_{i,j}^0 \leq u_{i,j}^n \leq \max_{i,j} u_{i,j}^0, \quad (119)$$

for all $1 \leq i \leq N$, $1 \leq j \leq M$, and $n \in \mathbb{N}$.

5.2. Experimental Results. This section will show the performance of proposed diffusion filter (116) in the image denoising process, under the boundary and initial conditions (117) while respecting the requirements concerning ϕ (Section 1), and δ_t (118). We will use the Peak Signal-to-Noise Ratio (PSNR that is a positive value) [18] and the Structural SIMilarity Index (SSIM that lies in $(0, 1)$) [19] to evaluate the quality of the restored images. The best results for the denoising process are equivalent to the higher value of these metrics.

For comparative purposes, we will examine the proposed diffusion function with another one that has the same properties using the same filter (116). Therefore, we will use the following diffusion functions:

(i) The proposed diffusion function ($m = 1$ for instance):

model (116) using proposed diffusion function (120) and the one proposed by WZ (121). These results are obtained using the optimal parameters determined experimentally, as in Table 2 for each diffusion function.

It can be seen from Table 1 and Figure 2 that the proposed model shows remarkable results against the WZ model. From a visual comparison, Figure 2 shows that the restored images using the proposed diffusion function have considerable noise removal and preserve the image essential features better than the restored images by the WZ diffusion function. Besides, compared with the WZ diffusion function,

TABLE 1: PSNR and SSIM values of the images in Figure 1 affected by different values of Gaussian noise σ^2 and their corresponding iteration number for both functions.

	σ^2	Noisy		WZ [4]		Iter	Proposed		Iter
		PSNR	SSIM	PSNR	SSIM		PSNR	SSIM	
Patient30	0.005	23.4311	0.3426	33.3757	0.9114	30	33.8333	0.9372	13
	0.010	20.6839	0.2432	31.3367	0.8766	41	31.5737	0.9145	21
	0.015	19.1190	0.1960	30.1565	0.8521	48	30.2008	0.8980	24
	0.020	17.9763	0.1660	29.0905	0.8298	56	29.0007	0.8825	26
	0.100	12.0756	0.0596	22.6821	0.6535	117	21.8953	0.7389	56
Patient50	0.005	23.6686	0.4424	31.1185	0.8708	24	31.2934	0.8769	11
	0.010	20.7561	0.3278	29.0893	0.8186	35	29.1527	0.8258	19
	0.015	19.1313	0.2707	27.9137	0.7820	41	27.8981	0.7920	26
	0.020	17.9838	0.2334	27.0011	0.7508	47	26.9714	0.7625	27
	0.100	11.9985	0.0911	21.4469	0.5585	93	20.9674	0.5932	62
Patient55	0.005	24.0179	0.3867	31.3190	0.9021	26	31.4668	0.9258	19
	0.010	21.2303	0.2892	29.1310	0.8600	36	29.0997	0.8887	28
	0.015	19.5990	0.2403	27.6640	0.8221	44	27.5635	0.8561	35
	0.020	18.4292	0.2096	26.7717	0.7938	49	26.5468	0.8305	42
	0.100	12.2234	0.0882	20.6948	0.5642	98	20.1755	0.6237	99

TABLE 2: The best possible parameters for different diffusion functions.

	σ^2	WZ [4]			Proposed			
		δ_t	δ_t	k	p_0	p_k	v_0	v_k
Patient30	0.005	0.08331	0.14701	4.61411	1.13191	0.66151	-0.00011	-0.10441
	0.010	0.08331	0.14991	5.00191	1.10891	0.45921	-0.00011	-0.04351
	0.015	0.08331	0.15051	5.20221	1.10601	0.45671	-0.00021	-0.03851
	0.020	0.08331	0.14701	5.86411	1.13281	0.46991	-0.00021	-0.03961
	0.100	0.08331	0.15001	5.08081	1.10111	0.45791	-0.00091	-0.03951
Patient50	0.005	0.08331	0.14941	3.18011	1.09241	0.56271	-0.00021	-0.01151
	0.010	0.08321	0.16231	1.89931	1.00171	0.56991	-0.00001	-0.00201
	0.015	0.08331	0.16591	2.06951	0.99011	0.56981	-0.00011	-0.11471
	0.020	0.08331	0.16101	1.79891	1.00001	0.56981	-0.00011	-0.01051
	0.100	0.08331	0.15301	3.87001	0.98701	0.55511	-0.00031	-0.13411
Patient55	0.005	0.08321	0.14681	3.50081	1.12891	0.58971	-0.00021	-0.16571
	0.010	0.08331	0.14701	3.31431	1.12621	0.58281	-0.00131	-0.17311
	0.015	0.08331	0.15141	3.09991	1.09021	0.58881	-0.01401	-0.18991
	0.020	0.08331	0.14881	3.39691	1.08721	0.53891	-0.00011	-0.15801
	0.100	0.08331	0.14991	2.70091	1.08901	0.56861	-0.00021	-0.20951

FIGURE 2: Continued.

FIGURE 2: Visual comparison on different real images corrupted by Gaussian noise with $\sigma^2 = 0.1$ and restored images using both functions. (a) Original-Patient30. (b) Original-Patient50. (c) Original-Patient55. (d) Noisy-Patient30. (e) Noisy-Patient50. (f) Noisy-Patient55. (g) WZ-Patient30. (h) WZ-Patient50. (i) WZ-Patient55. (j) Proposed-Patient30. (k) Proposed-Patient50. (l) Proposed-Patient55.

the results from Table 1 prove that the suggested approach has higher values in SSIM, whereas the WZ model shows significant results in PSNR while σ^2 -value increases.

6. Conclusion

This paper principally investigates the class of anisotropic diffusion partial differential equations related to image processing and analysis. The existence and uniqueness of weak solutions for this problem have been proven under sufficient conditions satisfied by ϕ . A consistent and stable numerical approximation has been applied, and a discrete nonlinear filter has been tested and revealed its efficiency in the image restoration field.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that there are no conflicts of interest regarding the publication of this paper.

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