

Research Article

Dynamic Analysis and Hopf Bifurcation of a Lengyel–Epstein System with Two Delays

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In this paper, a Lengyel–Epstein model with two delays is proposed and considered. By choosing the different delay as a parameter, the stability and Hopf bifurcation of the system under different situations are investigated in detail by using the linear stability method. Furthermore, the sufficient conditions for the stability of the equilibrium and the Hopf conditions are obtained. In addition, the explicit formula determining the direction of Hopf bifurcation and the stability of bifurcating periodic solutions are obtained with the normal form theory and the center manifold theorem to delay differential equations. Some numerical examples and simulation results are also conducted at the end of this paper to validate the developed theories.

1. Introduction

It is of great significance to study the dynamical behaviors of chemical reaction models to understand the reaction mechanism and evolution law of the reaction process of reactants. Experiments show that the description of the internal reaction mechanism of the chemical reaction system by delay reaction-diffusion equation is more realistic. Therefore, many scholars have conducted in-depth studies on the equation and obtained a large number of practical conclusions. In 1990, De Kepper et al. discovered Turing patterns in the CIMA reaction experiment, which also verified Turing's theoretical work over 40 years ago. In 1991 and 1992, Lengyel and Epstein in [1, 2] proposed the following mathematical model to depict the experimental process of the CIMA:

$$\begin{cases} \frac{\partial u}{\partial t} = \Delta u + a - u - \frac{4uv}{1+u^2}, & t > 0, x \in \Omega, \\ \frac{\partial v}{\partial t} = \sigma \left[c\Delta v + b \left(u - \frac{uv}{1+u^2} \right) \right], & t > 0, x \in \Omega, \\ \frac{\partial u}{\partial n} = \frac{\partial v}{\partial n} = 0, & t > 0, x \in \partial\Omega, \\ u(x, 0) = u_0(x) \geq 0, v(x, 0) = v_0(x) \geq 0, & x \in \Omega, \end{cases} \quad (1)$$

where $u(x, t)$ and $v(x, t)$ are the concentrations of two chemical reactants, respectively, a and b are positive constants representing the quantities related to the feed concentrations, the ratio of diffusion coefficients and the rescaling parameter are denoted as c and $\sigma > 0$, respectively, Ω is a bounded open domain in \mathbb{R}^n with smooth boundary $\partial\Omega$, and Δ is the Laplace operator.

System (1) has been extensively studied by many scholars (see [3–7]). In [5], the researchers mainly analyzed some basic properties of system (1). It can be found through the study that when the parameter Ω (size of the reactor), $d = c/b$ (effective diffusion rate), and a (initial concentration of the reactants) are relatively small, the system has no nonconstant steady state. In addition, when the initial concentration of the reactant a is within an appropriate range, the system has nonconstant steady states for a large diffusion rate, $d = c/b$. In 2005, on the basis of [3], Jang et al. studied the global bifurcation problem of the nonconstant positive steady-state solutions in one-dimensional case and considered its limiting behavior by using the shadow system method. In [6], Yi et al. discussed the conditions of homogeneous equilibrium solution and periodic solution and gave the detailed Hopf bifurcation analysis for the ODE and PDE models. In 2009, Yi et al. [7] proved the global asymptotical behavior of the constant positive equilibrium

solution and convergence of all solutions of the system by constructing the Lyapunov function. In [4], Jin et al. proved the bifurcations of steady-state solutions and spatially nonhomogeneous periodic solutions of (1).

Obviously, in the initial study of the model, most scholars did not consider the influence of time-delay factors. It is well known that the ordinary differential equation reflects that the development of things only depends on the current state, while the delay differential equation is used to describe the development system which depends on both the current state and the past state. In practical problems, if the delay factor is not considered, the properties and states of the system may change, even lead to wrong conclusions. Usually, a long delay will destroy the stability of the equilibrium point of the system. Therefore, it is more realistic to consider the delay factor in the system (see [8–15]). Based on this fact, Celik et al. [16] considered a coupled delayed-Lengyel–Epstein model as follows:

$$\begin{cases} \dot{u}(t) = a - u(t) - \frac{4u(t)v(t-\tau)}{1+u^2(t)}, & t > 0, \\ \dot{v}(t) = \sigma b \left(u(t) - \frac{u(t)v(t-\tau)}{1+u^2(t)} \right), & t > 0. \end{cases} \quad (2)$$

By choosing τ as the varying parameter and analyzing the related characteristic equations, Celik et al. [16] investigated the stability of the constant equilibrium point and the existence of Hopf bifurcation of system (2) and determined the necessary conditions for the parameters.

In [17, 18], Merdan and Kayan proposed the following delayed models:

$$\begin{cases} \dot{u}(t) = a - u(t) - \frac{4u(t-\tau)v(t)}{1+u^2(t)}, & t > 0, \\ \dot{v}(t) = \sigma b \left(u(t) - \frac{u(t-\tau)v(t)}{1+u^2(t)} \right), & t > 0, \end{cases} \quad (3)$$

and

$$\begin{cases} \dot{u}(t) = a - u(t-\tau) - \frac{4u(t-\tau)v(t)}{1+u^2(t-\tau)}, & t > 0, \\ \dot{v}(t) = \sigma b \left(u(t-\tau) - \frac{u(t-\tau)v(t)}{1+u^2(t-\tau)} \right), & t > 0. \end{cases} \quad (4)$$

They investigated the existence of Hopf bifurcation of the systems and the properties of Hopf bifurcation.

Recently, Zhang and He [19] assumed that the delay only occurred in the self-decomposition of the activator and further studied the delayed differential equation model as the following form:

$$\begin{cases} \dot{u}(t) = a - u(t-\tau) - \frac{4u(t)v(t)}{1+u^2(t)}, & t > 0, \\ \dot{v}(t) = \sigma b \left(u(t) - \frac{u(t)v(t)}{1+u^2(t)} \right), & t > 0. \end{cases} \quad (5)$$

They analyzed the influence of the change of delay τ on the dynamical behaviors and found that the equilibrium of system (5) would eventually become unstable after passing through several stable switches and Hopf bifurcations at some certain critical values of τ .

For a long time, most scholars are more concerned about biological mathematics. Therefore, in a period of time, biological mathematics has made a rapid development, and some practical conclusions have been obtained. However, there are many chemical reaction functional differential equations in nature, so it is of great practical significance to study the existence, stability, and Hopf bifurcation of periodic solutions of these functional differential equations in the field of chemistry. Of course, researchers did not realize the influence of time delay on the properties of periodic solutions of differential equations in the early stages. With the maturity of the biological mathematics theory, researchers began to realize the importance of time delay in chemical reaction systems. Apparently, these models (2)–(5) discussed above are based on the hypothesis that the delay effect is a single activator or inhibitor alone. In the field of chemistry, scholars are concerned about the range of time delay, when the system produces Hopf bifurcation and the equilibrium point is stable, which means that the parameters of the model should be selected in an appropriate range to save energy to the greatest extent and control environmental pollution for the model of the system. Many models represent the dynamic systems of chemical reactions through ordinary differential equations with no delay or differential equations with the same delay. However, the effects of time delay in the activator and inhibitor are completely different. Based on the fact, a more reasonable mathematical model to describe CIMA reaction should be depicted by the Lengyel–Epstein model with two delays as follows:

$$\begin{cases} \dot{u}(t) = a - u(t-\tau_1) - \frac{4u(t-\tau_1)v(t-\tau_2)}{1+u^2(t-\tau_1)}, & t > 0, \\ \dot{v}(t) = \sigma b \left(u(t-\tau_2) - \frac{u(t-\tau_2)v(t-\tau_1)}{1+u^2(t-\tau_2)} \right), & t > 0, \end{cases} \quad (6)$$

where $\tau_i \geq 0$ ($i = 1, 2$) are the time lags of the activator and the inhibitor.

We know that the most widely studied delay differential equations are the cases with a single delay or multiple identical delays. When the delay is taken as a parameter, the unique positive equilibrium of the system often loses stability at the critical value after Hopf bifurcation. However,

there are multiple stable intervals and unstable intervals in the system when it has many different time delays because of their different critical values. We pay more attention to the changes happened to the dynamic properties of the system in this case. If we fix one of the time delays to its stable interval and take the other as a dynamic parameter, we can study the rich dynamic properties of the system. Since the time lags between catalysts and inhibitors are different in actual chemical reactions, it would be interesting and realistic to consider the two different delays in the system. To the best knowledge of the authors, the Lengyel–Epstein model with two different delays is seldom studied by scholars. By choosing different delays as the bifurcation parameter, the main purpose of this article is to analyze the effect of the two different delays $\tau_i \geq 0$ ($i = 1, 2$) on the dynamical behaviors of system (6).

This article is organized as follows. In Section 2, we investigate the stability of the positive equilibrium and the existence of local Hopf bifurcation of system (6) in three cases according to different symbols of τ_i . In Section 3, by choosing τ_1 as a parameter and fixing τ_2 in its stable interval, the direction and stability of bifurcating periodic solutions are obtained by using the normal form method and the center manifold reduction for delayed differential equations developed by Hassard et al. [20]. In Section 4, some numerical examples and simulations results are included to illustrate the validity of the main results.

2. Stability and Hopf Bifurcation

In this section, we use the method in [21] to discuss the system’s positivity and uniform boundedness.

Lemma 1. *Let $u(0) > 0$ and $v(0) > 0$. Then, the solutions $u(t)$ and $v(t)$ of system (6) are nonnegative, for all $t > 0$.*

Proof. From the first equation, we have

$$\frac{du}{dt} \geq -u - \frac{4uv}{1+u^2} = -u \left(1 + \frac{4v}{1+u^2} \right) \geq -u(1+4v), \quad (7)$$

that is,

$$u(t) \geq u(0)e^{-\int_0^t (1+4v)d\omega} > 0. \quad (8)$$

Therefore, one can show that $u(t) > 0$, for all $t > 0$.

Similarly, we can apply the same method to prove $v(t) > 0$, for all $t > 0$. In fact, from the second equation, we can obtain

$$\frac{dv}{dt} \geq -\sigma b \frac{uv}{1+u^2}, \quad (9)$$

that is,

$$v(t) \geq v(0)e^{-\int_0^t (\sigma bu/(1+u^2))d\omega} > 0. \quad (10)$$

Therefore, one can show that $u(t) > 0, v(t) > 0$, for all $t > 0$.

For the boundedness of all solutions of system (6), according to [5], we have the following result. \square

Lemma 2. *Let $u(0) > 0$ and $v(0) > 0$. Then, all solutions of system (6) are uniformly bounded, that is,*

$$\begin{aligned} \limsup_{t \rightarrow +\infty} u(t) &\leq a, \\ \limsup_{t \rightarrow +\infty} v(t) &\leq 1 + a^2. \end{aligned} \quad (11)$$

Now, we will analyze the stability of the positive equilibrium point and the existence of local Hopf bifurcation of system (6) in detail in three cases $\tau_1 = \tau_2 = 0, \tau_1 = 0$ and $\tau_2 > 0$, and $\tau_1 > 0$ and $\tau_2 > 0$.

Let $\hat{\alpha} = (a/5)$; it is easy to get that system (6) has a unique positive equilibrium point $E^*(u^*, v^*) = (\hat{\alpha}, 1 + \hat{\alpha}^2)$. Then, the linearized system of (6) is given as follows:

$$\begin{cases} \dot{u}(t) = P_1 u(t - \tau_1) - Q_1 v(t - \tau_2), \\ \dot{v}(t) = P_2 u(t - \tau_1) - Q_2 v(t - \tau_2), \end{cases} \quad (12)$$

where

$$\begin{aligned} P_1 &= \frac{3\hat{\alpha}^2 - 5}{1 + \hat{\alpha}^2}, \\ P_2 &= \frac{2\sigma b \hat{\alpha}^2}{1 + \hat{\alpha}^2}, \\ Q_1 &= -\frac{4\hat{\alpha}}{1 + \hat{\alpha}^2}, \\ Q_2 &= -\frac{\sigma b \hat{\alpha}}{1 + \hat{\alpha}^2}. \end{aligned} \quad (13)$$

The associated characteristic equation of the linear system (12) is given by the following form:

$$\begin{vmatrix} \lambda - P_1 e^{-\lambda \tau_1} & -Q_1 e^{-\lambda \tau_2} \\ -P_2 e^{-\lambda \tau_2} & \lambda - Q_2 e^{-\lambda \tau_1} \end{vmatrix} = 0. \quad (14)$$

That is,

$$\lambda^2 - (P_1 + Q_2)\lambda e^{-\lambda \tau_1} + P_1 Q_2 e^{-2\lambda \tau_1} - P_2 Q_1 e^{-2\lambda \tau_2} = 0. \quad (15)$$

Case 1. $\tau_1 = \tau_2 = 0$.

Then, the characteristic equation (15) can be simplified to

$$\lambda^2 - (P_1 + Q_2)\lambda + P_1 Q_2 - P_2 Q_1 = 0. \quad (16)$$

By the Routh–Hurwitz criterion, we can see that all roots of (16) have always negative real parts if the condition,

$$\begin{aligned} (H1): P_1 + Q_2 &< 0, \\ P_1 Q_2 - P_2 Q_1 &> 0, \end{aligned} \quad (17)$$

holds.

Hence, the positive equilibrium of system (6) is locally asymptotically stable if (H1) holds.

Case 2. $\tau_1 = 0, \tau_2 > 0$.

Then, the associated characteristic equation (15) can be transformed to

$$\lambda^2 - (P_1 + Q_2)\lambda + P_1Q_2 - P_2Q_1e^{-2\lambda\tau_2} = 0. \quad (18)$$

Let $\lambda = i\omega$ ($\omega > 0$) be a root of the characteristic equation (18); then, we have

$$\begin{aligned} -\omega^2 - i(P_1 + Q_2)\omega + P_1Q_2 - P_2Q_1(\cos 2\omega\tau_2 \\ - i \sin 2\omega\tau_2) = 0. \end{aligned} \quad (19)$$

Separating the real and imaginary parts of equation (19), we obtain

$$\begin{cases} \omega^2 - P_1Q_2 = -P_2Q_1 \cos 2\omega\tau_2, \\ (P_1 + Q_2)\omega = P_2Q_1 \sin 2\omega\tau_2. \end{cases} \quad (20)$$

It follows from (20) that

$$\omega^4 + (P_1^2 + Q_2^2)\omega^2 + P_1^2Q_2^2 - P_2^2Q_1^2 = 0. \quad (21)$$

Let $z = \omega^2$. Then, (21) becomes

$$z^2 + (P_1^2 + Q_2^2)z + P_1^2Q_2^2 - P_2^2Q_1^2 = 0. \quad (22)$$

Notice that $P_1^2 + Q_2^2 > 0$ and

$$\begin{aligned} \Delta &= (P_1^2 + Q_2^2)^2 - 4(P_1^2Q_2^2 - P_2^2Q_1^2) \\ &= (P_1^2 - Q_2^2)^2 + 4P_2^2Q_1^2 > 0. \end{aligned} \quad (23)$$

Hence, one can see that if the condition

$$(H2): (P_1Q_2)^2 - (P_2Q_1)^2 > 0, \quad (24)$$

holds, then equation (22) has no positive solution. Therefore, all solutions of equation (18) have negative real parts when $\tau_2 > 0$, and condition (H2) holds.

Theorem 1. Assume that $\tau_1 = 0$ and (H2) holds; then, the positive equilibrium E^* of system (6) is asymptotically stable, for all $\tau_2 > 0$.

Now, we suppose that the following condition is given:

$$(H3): (P_1Q_2)^2 - (P_2Q_1)^2 < 0. \quad (25)$$

Then, one can obtain that z_0 is a unique positive root of equation (22) and $\pm i\omega_0$ is a pair of purely imaginary roots of the characteristic equation (18). Here,

$$\omega_0 = \sqrt{z_0} = \sqrt{\frac{1}{2} \left[-(P_1^2 + Q_2^2) + \sqrt{(P_1^2 - Q_2^2)^2 + 4P_2^2Q_1^2} \right]}. \quad (26)$$

By substituting ω_0 into (20), one can solve a series of τ_2 as

$$\tau_{2j} = \frac{1}{2\omega_0} \arccos \frac{\omega_0^2 - P_1Q_2}{-P_2Q_1} + \frac{2j\pi}{\omega_0}, \quad (j = 0, 1, 2, \dots). \quad (27)$$

Let $\lambda(\tau_2) = \alpha(\tau_2) + i\omega(\tau_2)$ be a root of (18) near $\tau_2 = \tau_{2j}$ such that $\alpha(\tau_{2j}) = 0$ and $\omega(\tau_{2j}) = \omega_0$. Differentiating both sides of equation (18) with respect to τ_2 , the following can be performed:

$$2\lambda \frac{d\lambda}{d\tau_2} - (P_1 + Q_2) \frac{d\lambda}{d\tau_2} - P_2Q_1e^{-2\lambda\tau_2} \left(-2\tau_2 \frac{d\lambda}{d\tau_2} - 2\lambda \right) = 0. \quad (28)$$

It follows that

$$\begin{aligned} \left[\frac{d\lambda}{d\tau_2} \right]^{-1} &= \frac{2\lambda - (P_1 + Q_2) + 2\tau_2 P_2Q_1 e^{-2\lambda\tau_2}}{-2\lambda P_2Q_1 e^{-2\lambda\tau_2}} \\ &= \frac{[2\lambda - (P_1 + Q_2)]e^{2\lambda\tau_2}}{-2\lambda P_2Q_1} - \frac{\tau_2}{\lambda}. \end{aligned} \quad (29)$$

Furthermore, we can know that

$$\begin{aligned} \left[\frac{d(\operatorname{Re} \lambda(\tau_2))}{d\tau_2} \right]_{\tau_2=\tau_{2j}}^{-1} &= \operatorname{Re} \left\{ \frac{[2\lambda - (P_1 + Q_2)]e^{2\lambda\tau_2}}{-2\lambda P_2Q_1} \right\} \Big|_{\tau_2=\tau_{2j}} \\ &= \frac{2\omega_0^2 + P_1^2 + Q_2^2}{2(P_2Q_1)^2} > 0. \end{aligned} \quad (30)$$

According to the above discussion and Corollary 2.4 of [3], we can present the following results.

Theorem 2. Assume that $\tau_1 = 0$ and (H3) hold, τ_{2j} ($j = 0, 1, 2, \dots$) can be defined by (27), and ω_0 can be given by (26). Then,

- (i) The positive equilibrium E^* of system (6) is asymptotically stable, for all $\tau_2 \in [0, \tau_{20})$.
- (ii) The positive equilibrium E^* of system (6) is unstable for $\tau_2 \in (\tau_{20}, +\infty)$
- (iii) System (6) undergoes Hopf bifurcation at the positive equilibrium E^* for $\tau_2 = \tau_{20}$

Case 3. $\tau_1 > 0, \tau_2 > 0$.

In this case, we consider the characteristic equation (15) by choosing τ_1 as a parameter and fixing τ_2 in its stable interval $[0, \tau_{20})$. Without loss of generality, we analyze system (6) under condition (H3).

Multiplying both sides of equation (15) by $e^{\lambda\tau_1}$, then one can get the following equation:

$$\lambda^2 e^{\lambda\tau_1} - (P_1 + Q_2)\lambda + P_1Q_2 e^{-\lambda\tau_1} - P_2Q_1 e^{-2\lambda\tau_2} e^{-\lambda\tau_1} = 0. \quad (31)$$

Let $\lambda = i\omega$ ($\omega > 0$) be a root of the characteristic equation (15); then, equation (31) becomes the following form:

$$\begin{aligned}
 & -\omega^2(\cos \omega\tau_1 + i \sin \omega\tau_1) - i(P_1 + Q_2)\omega \\
 & + P_1Q_2(\cos \omega\tau_1 - i \sin \omega\tau_1) \\
 & - P_2Q_1(\cos 2\omega\tau_2 - i \sin 2\omega\tau_2)(\cos \omega\tau_1 + i \sin \omega\tau_1) = 0.
 \end{aligned}
 \tag{32}$$

Separating the real and imaginary parts of equation (32), we obtain

$$\begin{cases} (-\omega^2 + P_1Q_2 - P_2Q_1 \cos 2\omega\tau_2)\cos \omega\tau_1 - (P_2Q_1 \sin 2\omega\tau_2)\sin \omega\tau_1 = 0, \\ (-\omega^2 - P_1Q_2 - P_2Q_1 \cos 2\omega\tau_2)\sin \omega\tau_1 + (P_2Q_1 \sin 2\omega\tau_2)\cos \omega\tau_1 = (P_1 + Q_2)\omega. \end{cases}
 \tag{33}$$

It follows from (33) that

$$\begin{cases} \sin \omega\tau_1 = \frac{(P_1 + Q_2)\omega(-\omega^2 + P_1Q_2 - P_2Q_1 \cos 2\omega\tau_2)}{(-\omega^2 - P_2Q_1 \cos 2\omega\tau_2)^2 - (P_1Q_2)^2 + (P_2Q_1 \sin 2\omega\tau_2)^2}, \\ \cos \omega\tau_1 = \frac{(P_1 + Q_2)\omega P_2Q_1 \sin 2\omega\tau_2}{(-\omega^2 - P_2Q_1 \cos 2\omega\tau_2)^2 - (P_1Q_2)^2 + (P_2Q_1 \sin 2\omega\tau_2)^2}. \end{cases}
 \tag{34}$$

According to (34), we can obtain

$$\omega^8 + l_1\omega^6 + l_2\omega^4 + l_3\omega^2 + l_4 = 0, \tag{35}$$

where

$$\begin{aligned}
 l_1 &= 4P_2Q_1 \cos 2\omega\tau_2 - (P_1 + Q_2)^2, \\
 l_2 &= 4(P_2Q_1 \cos 2\omega\tau_2)^2 + 2((P_2Q_1)^2 - (P_1Q_2)^2) \\
 &\quad + 2(P_1 + Q_2)^2 P_2Q_1 \cos 2\omega\tau_2, \\
 l_3 &= 4P_2Q_1((P_2Q_1)^2 - (P_1Q_2)^2)\cos 2\omega\tau_2 \\
 &\quad - (P_1 + Q_2)^2((P_1Q_2)^2 + (P_2Q_1)^2) \\
 &\quad + 2(P_1 + Q_2)^2 P_1Q_2P_2Q_1 \cos 2\omega\tau_2, \\
 l_4 &= [(P_2Q_1)^2 - (P_1Q_2)^2]^2.
 \end{aligned}
 \tag{36}$$

Now, we give the following assumption for equation (35).

(H4): assume that equation (35) has at least one positive root.

If condition (H4) holds, then equation (35) has a positive root ω_* such that equation (15) has a pair of purely imaginary roots $\pm i\omega_*$. Thus, we have

$$\tau_{1j} = \frac{1}{\omega_*} \arccos \left\{ \frac{(P_1 + Q_2)\omega_*^2 P_2Q_1 \sin 2\omega_*\tau_2}{[-\omega_*^2 - P_2Q_1 \cos 2\omega_*\tau_2]^2 - (P_1Q_2)^2 + (P_2Q_1 \sin 2\omega_*\tau_2)^2} \right\} + \frac{2j\pi}{\omega_*}, (j \in \mathbb{N}_0). \tag{37}$$

Differentiating both sides of equation (15) with respect to τ_1 , the following equation can be easily obtained:

$$\begin{aligned}
 & 2\lambda \frac{d\lambda}{d\tau_1} - (P_1 + Q_2) \left[\frac{d\lambda}{d\tau_1} e^{-\lambda\tau_1} + \lambda e^{-\lambda\tau_1} \left(-\frac{d\lambda}{d\tau_1} \tau_1 - \lambda \right) \right] \\
 & + P_1 Q_2 e^{-2\lambda\tau_1} \left(-2 \frac{d\lambda}{d\tau_1} \tau_1 - 2\lambda \right) \\
 & - P_2 Q_1 e^{-2\lambda\tau_2} \left(-2\tau_2 \frac{d\lambda}{d\tau_1} \right) = 0.
 \end{aligned}
 \tag{38}$$

It follows that

$$\begin{aligned}
 \left[\frac{d\lambda}{d\tau_1} \right]^{-1} &= \frac{2\lambda - (P_1 + Q_2)e^{-\lambda\tau_1} + (P_1 + Q_2)\lambda\tau_1 e^{-\lambda\tau_1} - 2P_1 Q_2 \tau_1 e^{-2\lambda\tau_1} + 2P_2 Q_1 \tau_2 e^{-2\lambda\tau_2}}{-(P_1 + Q_2)\lambda^2 e^{-\lambda\tau_1} - 2\lambda P_1 Q_2 e^{-2\lambda\tau_1}} \\
 &= \frac{2\lambda - (P_1 + Q_2)e^{-\lambda\tau_1} + 2P_2 Q_1 \tau_2 e^{-2\lambda\tau_2}}{-(P_1 + Q_2)\lambda^2 e^{-\lambda\tau_1} + 2\lambda P_1 Q_2 e^{-2\lambda\tau_1}} - \frac{\tau_1}{\lambda} \\
 &= \frac{2\lambda e^{\lambda\tau_1} - (P_1 + Q_2) + 2P_2 Q_1 \tau_2 e^{-2\lambda\tau_2} e^{\lambda\tau_1}}{-(P_1 + Q_2)\lambda^2 + 2\lambda P_1 Q_2 e^{-\lambda\tau_1}} - \frac{\tau_1}{\lambda}.
 \end{aligned}
 \tag{39}$$

Let $\lambda = i\omega_*$, and we can know that

$$\left[\frac{d(\operatorname{Re}\lambda(\tau_1))}{d\tau_1} \right]_{\tau_1=\tau_{1j}}^{-1} = \frac{M_1(\omega_*)N_1(\omega_*) + M_2(\omega_*)N_2(\omega_*)}{M_1^2(\omega_*) + M_2^2(\omega_*)},
 \tag{40}$$

where

$$\begin{aligned}
 M_1(\omega_*) &= (P_1 + Q_2)\omega_*^2 + 2P_1 Q_2 \omega_* \sin \omega_* \tau_1, \\
 M_2(\omega_*) &= 2P_1 Q_2 \omega_* \cos \omega_* \tau_1, \\
 N_1(\omega_*) &= 2P_2 Q_1 \tau_2 \cos 2\omega_* \tau_2 \cos \omega_* \tau_1 \\
 &\quad - (2\omega_* - 2P_2 Q_1 \tau_2 \sin 2\omega_* \tau_2) \sin \omega_* \tau_1 \\
 &\quad - (P_1 + Q_2), \\
 N_2(\omega_*) &= 2P_2 Q_1 \tau_2 \cos 2\omega_* \tau_2 \sin \omega_* \tau_1 \\
 &\quad + (2\omega_* - 2P_2 Q_1 \tau_2 \sin 2\omega_* \tau_2) \cos \omega_* \tau_1.
 \end{aligned}
 \tag{41}$$

Next, we assume that the following condition holds:

$$(H5): M_1(\omega_*)N_1(\omega_*) + M_2(\omega_*)N_2(\omega_*) \neq 0.
 \tag{42}$$

Therefore, from the above discussions and Hopf bifurcation Theorem 2, the following results can be directly deduced.

Theorem 3. Assume that (H3), (H4), and (H5) hold, $\tau_2 \in [0, \tau_{20})$ and τ_{1j} ($j \in \mathbb{N}_0$) can be defined by (37), and ω_* can be given by (35). Then,

- (i) The positive equilibrium E^* of system (6) is asymptotically stable for all $\tau_1 \in [0, \tau_{10})$
- (ii) The positive equilibrium E^* of system (6) is unstable for $\tau_1 \in (\tau_{10}, +\infty)$

(iii) System (6) undergoes Hopf bifurcation at the positive equilibrium E^* for $\tau_1 = \tau_{10}$

3. Direction of Hopf Bifurcation and Stability of Bifurcating Periodic Solution

In the previous section, the stability of the positive equilibrium E^* and the existence of Hopf bifurcation of system (6) by regarding different delays τ_i ($i = 1, 2$) as the bifurcation parameter were discussed. In this section, by applying the normal form theory and the center manifold theorem of Hassard et al. [20], we derive the explicit formulas, determining the direction of Hopf bifurcation and the stability of bifurcation periodic solutions. We always assume that system (6) undergoes Hopf bifurcation at the positive equilibrium E^* for $\tau_1 = \tau_{10}$, and the corresponding purely imaginary roots of the characteristic equation at E^* are denoted by $\pm i\omega_*$.

Let us assume that $\tau_2^* < \tau_{10}$, where $\tau_2 \in (0, \tau_{20})$ and τ_{20} is defined by (27). For the convenience of discussion, let $\bar{u}(t) = u(t) - u^*$, $\bar{v}(t) = v(t) - v^*$, $t = \tau\bar{t}$, and $\tau_1 = \tau_{10} + \mu, \mu \in \mathbb{R}$; then, $\mu = 0$ is the Hopf bifurcation value of (6). We drop the bars of these notations to simplify the next discussion, and system (6) can be concisely regarded as the abstract FDE in $C = C([-1, 0], \mathbb{R}^2)$:

$$\dot{u}(t) = L_\mu(u_t) + F(\mu, u_t),
 \tag{43}$$

where $u(t) = (u_1(t), u_2(t))^T$, $u_t(\theta) = u(t + \theta)$, $L_\mu: C \rightarrow \mathbb{R}^2$, and $F(\mu, \cdot): \mathbb{R} \times C \rightarrow \mathbb{R}^2$ are, respectively, the linear operator and the nonlinear operator with the following forms:

$$L_\mu(\phi) = (\tau_{10} + \mu) \left[A \begin{pmatrix} \phi_1\left(\frac{\tau_2^*}{\tau_{10}}\right) \\ \phi_2\left(\frac{\tau_2^*}{\tau_{10}}\right) \end{pmatrix} + B \begin{pmatrix} \phi_1(-1) \\ \phi_2(-1) \end{pmatrix} \right],$$

$$F(\mu, \phi) = (\tau_{10} + \mu)(f_1, f_2)^T, \tag{44}$$

where $\phi(\theta) = (\phi_1(\theta), \phi_2(\theta))^T \in C$, $A = \begin{pmatrix} 0 & Q_1 \\ P_2 & 0 \end{pmatrix}$,

$$B = \begin{pmatrix} P_1 & 0 \\ 0 & Q_2 \end{pmatrix}, \text{ and}$$

$$f_1 = \frac{4\alpha(3 - \alpha^2)}{(1 + \alpha^2)^2} \phi_1^2(-1) + \frac{-4(1 - \alpha^2)}{(1 + \alpha^2)^2} \phi_1(-1)\phi_2$$

$$\cdot \left(\frac{\tau_2^*}{\tau_{10}}\right) + \dots,$$

$$f_2 = \frac{\sigma b \alpha(3 - \alpha^2)}{(1 + \alpha^2)^2} \phi_1^2\left(\frac{\tau_2^*}{\tau_{10}}\right)$$

$$+ \frac{-\sigma b(1 - \alpha^2)}{(1 + \alpha^2)^2} \phi_1\left(\frac{\tau_2^*}{\tau_{10}}\right) \phi_2(-1) + \dots. \tag{45}$$

According to Riesz representation theorem, there exists a matrix function $\eta(\theta, \mu), \theta \in [-1, 0]$ with bounded variation components such that

$$L_\mu(\phi) = \int_{-1}^0 d\eta(\theta, \mu) \phi(\theta), \phi(\theta) \in C([-1, 0], \mathbb{R}^2). \tag{46}$$

In fact, we can take

$$\eta(\theta, \mu) = \begin{cases} (\tau_{10} + \mu)(A + B), & \theta = 0, \\ (\tau_{10} + \mu)(A + B), & \theta \in \left[-\frac{\tau_2^*}{\tau_{10}}, 0\right], \\ (\tau_{10} + \mu)B, & \theta \in \left(-1, -\frac{\tau_2^*}{\tau_{10}}\right), \\ 0, & \theta = -1. \end{cases} \tag{47}$$

In the following discussion, we define the linear differential operators $A(\mu)$ and $R(\mu)$ by

$$A(\mu)\phi = \begin{cases} \frac{d\phi(\theta)}{d\theta}, & \theta \in [-1, 0), \\ \int_{-1}^0 d\eta(s, \mu)\phi(s), & \theta = 0, \end{cases} \tag{48}$$

$$R(\mu)\phi = \begin{cases} 0, & \theta \in [-1, 0), \\ F(\mu, \phi), & \theta = 0. \end{cases}$$

Then, system (43) can be further represented as

$$\dot{u}(t) = A(\mu)u_t + R(\mu)u_t, \tag{49}$$

where $u_t(\theta) = u(t + \theta), \theta \in [-1, 0]$. For $\psi \in C^1([0, 1], (\mathbb{R}^2)^*)$, the linear differential operator A^* is defined by

$$A^*\psi(s) = \begin{cases} -\frac{d\psi(s)}{ds}, & s \in (0, 1], \\ \int_{-1}^0 d\eta^T(t, 0)\psi(-t), & s = 0. \end{cases} \tag{50}$$

Furthermore, for $\phi \in C^1([-1, 0], (\mathbb{R}^2)^*)$ and $\psi \in C^1([0, 1], (\mathbb{R}^2)^*)$, we give the definition of the bilinear inner product:

$$\langle \psi(s), \phi(\theta) \rangle = \bar{\psi}(0)\phi(0) - \int_{-1}^0 \int_{\xi=0}^\theta \bar{\psi}(\xi - \theta) d\eta(\theta)\phi(\xi) d\xi, \tag{51}$$

where $\eta(\theta) = \eta(\theta, 0)$. Obviously, $A(0)$ and A^* are adjoint operators. It is easy to see that they are eigenvalues of the linear operator $A(0)$. It follows that they are also eigenvalues of the linear operator A^* .

Assume that $q(\theta) = (1, \alpha)^T e^{i\omega_* \tau_{10} \theta}$ is the eigenvector of the operator $A(0)$ corresponding to the eigenvalues $i\omega_* \tau_{10}$; then, $A(0)q(\theta) = i\omega_* \tau_{10} q(\theta)$. From the definitions of $A, L_\mu \phi$, and $\eta(\theta, \mu)$, we can obtain

$$\begin{pmatrix} i\omega_* - P_1 e^{-i\omega_* \tau_{10}} & -Q_1 e^{-i\omega_* \tau_2^*} \\ -P_2 e^{-i\omega_* \tau_2^*} & i\omega_* - Q_2 e^{-i\omega_* \tau_{10}} \end{pmatrix} \begin{pmatrix} 1 \\ \alpha \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}. \tag{52}$$

By simple calculation, we have

$$\alpha = \frac{i\omega_* - P_1 e^{-i\omega_* \tau_{10}}}{Q_1 e^{-i\omega_* \tau_2^*}}. \tag{53}$$

By analogy, we can get the eigenvector $q^*(s) = D(1, \alpha^*) e^{i\omega_* \tau_{10} s}$ of the operator A^* corresponding to the eigenvalues $-i\omega_* \tau_{10}$, where

$$\alpha^* = \frac{i\omega_* + P_1 e^{-i\omega_* \tau_{10}}}{P_2 e^{-i\omega_* \tau_2^*}}. \tag{54}$$

Now, we evaluate the value of D such that $\langle q^*(s), q(\theta) \rangle = 1$. From the bilinear inner product of (51), it follows that

$$\begin{aligned}
 \langle q^*(s), q(\theta) \rangle &= \bar{q}^*(0)q(0) - \int_{-1}^0 \int_{\xi=0}^{\theta} \bar{q}^*(\xi - \theta) d\eta(\theta) q(\xi) d\xi \\
 &= \bar{D}(1, \bar{\alpha}^*)(1, \alpha)^T - \int_{-1}^0 \int_{\xi=0}^{\theta} \bar{D}(1, \bar{\alpha}^*) e^{-i\omega_* \tau_{10}(\xi - \theta)} d\eta(\theta) (1, \alpha)^T e^{i\omega_* \tau_{10} \xi} d\xi \\
 &= \bar{D}(1, \bar{\alpha}^*)(1, \alpha)^T - \bar{D}(1, \bar{\alpha}^*) \int_{-1}^0 \int_{\xi=0}^{\theta} d\eta(\theta) (1, \alpha)^T e^{i\omega_* \tau_{10} \theta} d\xi \\
 &= \bar{D}(1, \bar{\alpha}^*)(1, \alpha)^T - \bar{D}(1, \bar{\alpha}^*) \int_{-1}^0 \theta e^{i\omega_* \tau_{10} \theta} d\eta(\theta) (1, \alpha)^T \\
 &= \bar{D}(1, \bar{\alpha}^*)(1, \alpha)^T - \bar{D}(1, \bar{\alpha}^*) \left[\tau_{10} A \phi \left(-\frac{\tau_2^*}{\tau_{10}} \right) + \tau_{10} B \phi(-1) \right] (1, \alpha)^T \\
 &= \bar{D}(1, \bar{\alpha}^*)(1, \alpha)^T - \bar{D}(1, \bar{\alpha}^*) \left[\tau_{10} \begin{pmatrix} 0 & Q_1 \\ P_2 & 0 \end{pmatrix} \begin{pmatrix} -\tau_2^* \\ \tau_{10} \end{pmatrix} e^{-i\omega_* \tau_2^*} - \tau_{10} \begin{pmatrix} P_1 & 0 \\ 0 & Q_2 \end{pmatrix} e^{-i\omega_* \tau_{10}} \right] (1, \alpha)^T \\
 &= \bar{D}(1 + \alpha \bar{\alpha}^*) + \bar{D} \tau_{10} \left[P_1 e^{-i\omega_* \tau_{10}} + P_2 \bar{\alpha}^* \frac{\tau_2^*}{\tau_{10}} e^{-i\omega_* \tau_2^*} + \alpha Q_1 \frac{\tau_2^*}{\tau_{10}} e^{-i\omega_* \tau_2^*} + Q_2 \alpha \bar{\alpha}^* e^{-i\omega_* \tau_{10}} \right] \\
 &= \bar{D}(1 + \alpha \bar{\alpha}^*) + \bar{D} \left[P_1 \tau_{10} e^{-i\omega_* \tau_{10}} + P_2 \bar{\alpha}^* \tau_2^* e^{-i\omega_* \tau_2^*} + \alpha Q_1 \tau_2^* e^{-i\omega_* \tau_2^*} + Q_2 \tau_{10} \alpha \bar{\alpha}^* e^{-i\omega_* \tau_{10}} \right].
 \end{aligned} \tag{55}$$

Thus, we have

$$D = \frac{1}{1 + \bar{\alpha} \alpha^* + P_1 \tau_{10} e^{i\omega_* \tau_{10}} + P_2 \alpha^* \tau_2^* e^{i\omega_* \tau_2^*} + \alpha Q_1 \tau_2^* e^{i\omega_* \tau_2^*} + Q_2 \tau_{10} \bar{\alpha} \alpha^* e^{i\omega_* \tau_{10}}}. \tag{56}$$

In addition, from $\langle \psi, A\phi \rangle = \langle A^* \psi, \phi \rangle$ and $A\bar{q}(\theta) = -i\omega^* \bar{q}(\theta)$, we can obtain

$$\begin{aligned}
 -i\omega^* \langle q^*, \bar{q} \rangle &= \langle q^*, A\bar{q} \rangle = \langle A^* q^*, \bar{q} \rangle \\
 &= \langle -i\omega^* q^*, \bar{q} \rangle = i\omega^* \langle q^*, \bar{q} \rangle.
 \end{aligned} \tag{57}$$

Hence, $\langle q^*(\theta), \bar{q}(\theta) \rangle = 0$.

In the next, we calculate the coordinates describing the center manifold C_0 at $\mu = 0$ by using the method of Hassard et al. [20]. Let u_t be the solution of equation (43) and $z(t) = \langle q^*, u_t \rangle$; then, from (49) and (51), we have

$$\begin{aligned}
 \dot{z}(t) &= \langle q^*, \dot{u}_t \rangle = \langle q^*, A(0)u_t + R(0)u_t \rangle \\
 &= \langle q^*, A(0)u_t \rangle + \langle q^*, R(0)u_t \rangle \\
 &= \langle A^*(0)q^*, u_t \rangle + \bar{q}^*(0)F(0, u_t) \\
 &= i\omega_* \tau_{10} z + g(z, \bar{z}),
 \end{aligned} \tag{58}$$

where

$$\begin{aligned}
 g(z, \bar{z}) &= \bar{q}^*(0)F(0, u_t) = g_{20}(\theta) \frac{z^2}{2} + g_{11}(\theta) z \bar{z} \\
 &+ g_{02}(\theta) \frac{\bar{z}^2}{2} + g_{21}(\theta) \frac{z^2 \bar{z}}{2} + \dots
 \end{aligned} \tag{59}$$

Assume

$$\begin{aligned}
 W(t, \theta) &= u_t(\theta) - z(t)q(\theta) - \bar{z}(t)\bar{q}(\theta) \\
 &= u_t(\theta) - 2\text{Re}\{z(t)q(\theta)\}.
 \end{aligned} \tag{60}$$

Then, we have

$$\begin{aligned}
 W(t, \theta) &= W(z(t), \bar{z}(t), \theta) = W_{20} \frac{z^2}{2} + W_{11} z \bar{z} \\
 &+ W_{02} \frac{\bar{z}^2}{2} + W_{30} \frac{z^3}{6} + \dots,
 \end{aligned} \tag{61}$$

where z and \bar{z} refer to local coordinates on the center manifold C_0 in the directions of q^* and \bar{q}^* . As W is real, if u_t

is real, we only discuss real solutions. From (60), it follows that

Therefore,

$$\begin{aligned}
 u_t(\theta) &= (u_{1t}(\theta), u_{2t}(\theta))^T \\
 &= W(t, \theta) + 2\operatorname{Re}\{z(t)q(\theta)\} \\
 &= W(t, \theta) + z(t)q(\theta) + \bar{z}(t)\bar{q}(\theta) \\
 &= (1, \alpha)^T e^{i\omega_* \tau_{10} \theta} z + (1, \bar{\alpha})^T e^{-i\omega_* \tau_{10} \theta} \bar{z} \\
 &\quad + W_{20} \frac{z^2}{2} + W_{11} z\bar{z} + W_{02} \frac{\bar{z}^2}{2} + \dots.
 \end{aligned} \tag{62}$$

$$\begin{aligned}
 u_{1t}(-1) &= ze^{-i\omega_* \tau_{10}} + \bar{z}e^{i\omega_* \tau_{10}} + W_{20}^{(1)}(-1) \frac{z^2}{2} + W_{11}^{(1)}(-1) z\bar{z} + W_{02}^{(1)}(-1) \frac{\bar{z}^2}{2} + \dots, \\
 u_{2t}(-1) &= z\alpha e^{-i\omega_* \tau_{10}} + \bar{z}\bar{\alpha} e^{i\omega_* \tau_{10}} + W_{20}^{(2)}(-1) \frac{z^2}{2} + W_{11}^{(2)}(-1) z\bar{z} + W_{02}^{(2)}(-1) \frac{\bar{z}^2}{2} + \dots, \\
 u_{1t}\left(-\frac{\tau_2^*}{\tau_{10}}\right) &= ze^{-i\omega_* \tau_2^*} + \bar{z}e^{i\omega_* \tau_2^*} + W_{20}^{(1)}\left(-\frac{\tau_2^*}{\tau_{10}}\right) \frac{z^2}{2} + W_{11}^{(1)}\left(-\frac{\tau_2^*}{\tau_{10}}\right) z\bar{z} + W_{02}^{(1)}\left(-\frac{\tau_2^*}{\tau_{10}}\right) \frac{\bar{z}^2}{2} + \dots, \\
 u_{2t}\left(-\frac{\tau_2^*}{\tau_{10}}\right) &= z\alpha e^{-i\omega_* \tau_2^*} + \bar{z}\bar{\alpha} e^{i\omega_* \tau_2^*} + W_{20}^{(2)}\left(-\frac{\tau_2^*}{\tau_{10}}\right) \frac{z^2}{2} + W_{11}^{(2)}\left(-\frac{\tau_2^*}{\tau_{10}}\right) z\bar{z} + W_{02}^{(2)}\left(-\frac{\tau_2^*}{\tau_{10}}\right) \frac{\bar{z}^2}{2} + \dots.
 \end{aligned} \tag{63}$$

Furthermore, we have

$$\begin{aligned}
 f_1 &= \frac{4\alpha(3-\alpha^2)}{(1+\alpha^2)^2} \phi_1^2(-1) + \frac{-4(1-\alpha^2)}{(1+\alpha^2)^2} \phi_1(-1)\phi_2\left(-\frac{\tau_2^*}{\tau_{10}}\right) + \dots, \\
 f_2 &= \frac{\sigma b\alpha(3-\alpha^2)}{(1+\alpha^2)^2} \phi_1^2\left(-\frac{\tau_2^*}{\tau_{10}}\right) + \frac{-\sigma b(1-\alpha^2)}{(1+\alpha^2)^2} \phi_1\left(-\frac{\tau_2^*}{\tau_{10}}\right)\phi_2(-1) + \dots,
 \end{aligned}$$

$$g(z, \bar{z}) = \bar{q}^*(0)F(0, u_t) = \bar{D}(1, \bar{\alpha}^*)F(0, u_t)$$

$$= \bar{D}(1, \bar{\alpha}^*)\tau_{10} \begin{pmatrix} \alpha_1 u_{1t}^2(-1) + \alpha_2 u_{1t}(-1)u_{2t}\left(-\frac{\tau_2^*}{\tau_{10}}\right) \\ \beta_1 u_{1t}^2\left(-\frac{\tau_2^*}{\tau_{10}}\right) + \beta_2 u_{1t}\left(-\frac{\tau_2^*}{\tau_{10}}\right)u_{2t}(-1) \end{pmatrix}$$

$$\begin{aligned}
 &= \overline{D}\tau_{10} \left(\begin{aligned}
 &\alpha_1 \left(ze^{-i\omega_*\tau_{10}} + \overline{z}e^{i\omega_*\tau_{10}} + W_{20}^{(1)}(-1)\frac{z^2}{2} + W_{11}^{(1)}(-1)z\overline{z} + W_{02}^{(1)}(-1)\frac{\overline{z}^2}{2} + \dots \right)^2 \\
 &+ \alpha_2 \left(ze^{-i\omega_*\tau_{10}} + \overline{z}e^{i\omega_*\tau_{10}} + W_{20}^{(1)}(-1)\frac{z^2}{2} + W_{11}^{(1)}(-1)z\overline{z} + W_{02}^{(1)}(-1)\frac{\overline{z}^2}{2} + \dots \right) \\
 &\cdot \left(z\alpha e^{-i\omega_*\tau_2^*} + \overline{z}\overline{\alpha}e^{i\omega_*\tau_2^*} + W_{20}^{(2)}\left(-\frac{\tau_2^*}{\tau_{10}}\right)\frac{z^2}{2} + W_{11}^{(2)}\left(-\frac{\tau_2^*}{\tau_{10}}\right)z\overline{z} + W_{02}^{(2)}\left(-\frac{\tau_2^*}{\tau_{10}}\right)\frac{\overline{z}^2}{2} + \dots \right) \\
 &+ \overline{\alpha}^*\beta_1 \left(ze^{-i\omega_*\tau_2^*} + \overline{z}e^{i\omega_*\tau_2^*} + W_{20}^{(1)}\left(-\frac{\tau_2^*}{\tau_{10}}\right)\frac{z^2}{2} + W_{11}^{(1)}\left(-\frac{\tau_2^*}{\tau_{10}}\right)z\overline{z} + W_{02}^{(1)}\left(-\frac{\tau_2^*}{\tau_{10}}\right)\frac{\overline{z}^2}{2} + \dots \right)^2 \\
 &+ \overline{\alpha}^*\beta_2 \left(ze^{-i\omega_*\tau_2^*} + \overline{z}e^{i\omega_*\tau_2^*} + W_{20}^{(1)}\left(-\frac{\tau_2^*}{\tau_{10}}\right)\frac{z^2}{2} + W_{11}^{(1)}\left(-\frac{\tau_2^*}{\tau_{10}}\right)z\overline{z} + W_{02}^{(1)}\left(-\frac{\tau_2^*}{\tau_{10}}\right)\frac{\overline{z}^2}{2} + \dots \right) \\
 &\cdot \left(z\alpha e^{-i\omega_*\tau_{10}} + \overline{z}\overline{\alpha}e^{i\omega_*\tau_{10}} + W_{20}^{(2)}(-1)\frac{z^2}{2} + W_{11}^{(2)}(-1)z\overline{z} + W_{02}^{(2)}(-1)\frac{\overline{z}^2}{2} + \dots \right)
 \end{aligned} \right) \tag{64}
 \end{aligned}$$

$$\begin{aligned}
 &= \overline{D}\tau_{10} \left\{ \left[\alpha_1 e^{-2i\omega_*\tau_{10}} + \alpha_2 \alpha e^{-i\omega_*\tau_{10}} e^{-i\omega_*\tau_2^*} + \overline{\alpha}^*\beta_1 e^{-2i\omega_*\tau_2^*} + \overline{\alpha}^*\beta_2 \alpha e^{-i\omega_*\tau_{10}} e^{-i\omega_*\tau_2^*} \right] z^2 \right. \\
 &+ \left[2\alpha_1 + \alpha_2 \overline{\alpha} e^{-i\omega_*\tau_{10}} e^{i\omega_*\tau_2^*} + \alpha_2 \alpha e^{i\omega_*\tau_{10}} e^{-i\omega_*\tau_2^*} + 2\overline{\alpha}^*\beta_1 + \overline{\alpha}^*\beta_2 (\overline{\alpha} e^{i\omega_*\tau_{10}} e^{-i\omega_*\tau_2^*} + \alpha e^{-i\omega_*\tau_{10}} e^{i\omega_*\tau_2^*}) \right] z\overline{z} \\
 &+ \left[\alpha_1 e^{2i\omega_*\tau_{10}} + \alpha_2 \overline{\alpha} e^{i\omega_*\tau_{10}} e^{i\omega_*\tau_2^*} + \overline{\alpha}^*\beta_1 e^{2i\omega_*\tau_2^*} + \overline{\alpha}^*\beta_2 \overline{\alpha} e^{i\omega_*\tau_{10}} e^{i\omega_*\tau_2^*} \right] \overline{z}^2 \\
 &+ \left[\alpha_1 (2W_{11}^{(1)}(-1) + W_{20}^{(1)}(-1)) e^{-i\omega_*\tau_{10}} + \alpha_2 \left(W_{11}^{(2)}\left(-\frac{\tau_2^*}{\tau_{10}}\right) e^{-i\omega_*\tau_{10}} + \frac{1}{2}W_{20}^{(2)}\left(-\frac{\tau_2^*}{\tau_{10}}\right) \overline{\alpha} e^{i\omega_*\tau_{10}} \right. \right. \\
 &+ \left. \left. \frac{1}{2}W_{20}^{(1)}(-1) \overline{\alpha} e^{i\omega_*\tau_2^*} + W_{11}^{(1)}(-1) \alpha e^{-i\omega_*\tau_2^*} \right) + \overline{\alpha}^*\beta_1 \left(2W_{11}^{(1)}\left(-\frac{\tau_2^*}{\tau_{10}}\right) e^{-i\omega_*\tau_2^*} + W_{20}^{(1)}\left(-\frac{\tau_2^*}{\tau_{10}}\right) e^{i\omega_*\tau_2^*} \right) \right. \\
 &+ \left. \left. \overline{\alpha}^*\beta_2 \left(W_{11}^{(2)}(-1) e^{-i\omega_*\tau_2^*} + \frac{1}{2}W_{20}^{(2)}(-1) e^{i\omega_*\tau_2^*} + \frac{1}{2}W_{20}^{(1)}\left(-\frac{\tau_2^*}{\tau_{10}}\right) \overline{\alpha} e^{i\omega_*\tau_{10}} + W_{11}^{(1)}\left(-\frac{\tau_2^*}{\tau_{10}}\right) \alpha e^{-i\omega_*\tau_{10}} \right) \right] \right\} z^2 \overline{z} + \dots,
 \end{aligned}$$

where

$$\begin{aligned}
 \alpha_1 &= \frac{4\alpha(3 - \alpha^2)}{(1 + \alpha^2)^2}, \\
 \alpha_2 &= \frac{-4(1 - \alpha^2)}{(1 + \alpha^2)^2}, \\
 \beta_1 &= \frac{\sigma b \alpha (3 - \alpha^2)}{(1 + \alpha^2)^2}, \\
 \beta_2 &= \frac{-\sigma b (1 - \alpha^2)}{(1 + \alpha^2)^2}.
 \end{aligned} \tag{65}$$

According to the above analysis, we can obtain

$$\begin{aligned}
 g_{20} &= 2\bar{D}\tau_{10} \left[\alpha_1 e^{-2i\omega_*\tau_{10}} + \alpha_2 \alpha e^{-i\omega_*\tau_{10}} e^{-i\omega_*\tau_2^*} + \bar{\alpha}^* \beta_1 e^{-2i\omega_*\tau_2^*} + \bar{\alpha}^* \beta_2 \alpha e^{-i\omega_*\tau_{10}} e^{-i\omega_*\tau_2^*} \right], \\
 g_{11} &= 2\bar{D}\tau_{10} \left[\alpha_1 + \alpha_2 \operatorname{Re} \left\{ \alpha e^{i\omega_*\tau_{10}} e^{-i\omega_*\tau_2^*} \right\} + \bar{\alpha}^* \beta_1 + \bar{\alpha}^* \beta_2 \operatorname{Re} \left\{ \alpha e^{-i\omega_*\tau_{10}} e^{i\omega_*\tau_2^*} \right\} \right]; \\
 g_{02} &= 2\bar{D}\tau_{10} \left[\alpha_1 e^{2i\omega_*\tau_{10}} + \alpha_2 \bar{\alpha} e^{i\omega_*\tau_{10}} e^{i\omega_*\tau_2^*} + \bar{\alpha}^* \beta_1 e^{2i\omega_*\tau_2^*} + \bar{\alpha}^* \beta_2 \bar{\alpha} e^{i\omega_*\tau_{10}} e^{i\omega_*\tau_2^*} \right], \\
 g_{21} &= 2\bar{D}\tau_{10} \left[\alpha_1 \left(2W_{11}^{(1)}(-1) + W_{20}^{(1)}(-1) \right) e^{-i\omega_*\tau_{10}} + \alpha_2 \left(W_{11}^{(2)} \left(-\frac{\tau_2^*}{\tau_{10}} \right) e^{-i\omega_*\tau_{10}} + \frac{1}{2} W_{20}^{(2)} \left(-\frac{\tau_2^*}{\tau_{10}} \right) \bar{\alpha} e^{i\omega_*\tau_{10}} \right. \right. \\
 &\quad \left. \left. + \frac{1}{2} W_{20}^{(1)}(-1) \bar{\alpha} e^{i\omega_*\tau_2^*} + W_{11}^{(1)}(-1) \alpha e^{-i\omega_*\tau_2^*} \right) + \bar{\alpha}^* \beta_1 \left(2W_{11}^{(1)} \left(-\frac{\tau_2^*}{\tau_{10}} \right) e^{-i\omega_*\tau_2^*} + W_{20}^{(1)} \left(-\frac{\tau_2^*}{\tau_{10}} \right) e^{i\omega_*\tau_2^*} \right) \right. \\
 &\quad \left. + \bar{\alpha}^* \beta_2 \left(W_{11}^{(2)}(-1) e^{-i\omega_*\tau_2^*} + \frac{1}{2} W_{20}^{(2)}(-1) e^{i\omega_*\tau_2^*} + \frac{1}{2} W_{20}^{(1)} \left(-\frac{\tau_2^*}{\tau_{10}} \right) \bar{\alpha} e^{i\omega_*\tau_{10}} + W_{11}^{(1)} \left(-\frac{\tau_2^*}{\tau_{10}} \right) \alpha e^{-i\omega_*\tau_{10}} \right) \right].
 \end{aligned} \tag{66}$$

After g_{20} , g_{21} , and g_{02} are obtained, $W_{20}(\theta)$ and $W_{11}(\theta)$ in g_{21} can be determined by using method in [22–24]. From (49) and (60), we can obtain

$$\begin{aligned}
 \dot{W} = \dot{u}_t - \dot{z}q - \dot{\bar{z}}\bar{q} &= \begin{cases} AW - 2\operatorname{Re}\{\bar{q}^*(0)F(0, u_t)q(\theta)\} & \theta \in [-1, 0) \\ AW - 2\operatorname{Re}\{\bar{q}^*(0)F(0, u_t)q(0)\} + F(0, u_t) & \theta = 0 \end{cases} \\
 &\triangleq AW + H(z, \bar{z}, \theta),
 \end{aligned} \tag{67}$$

where

$$H(z, \bar{z}, \theta) = H_{20}(\theta) \frac{z^2}{2} + H_{11}(\theta) z\bar{z} + H_{02}(\theta) \frac{\bar{z}^2}{2} + \dots \tag{68}$$

Thus, we have

$$\begin{aligned}
 AW(t, \theta) - \dot{W} &= -H(z, \bar{z}, \theta) = -H_{20}(\theta) \frac{z^2}{2} \\
 &\quad - H_{11}(\theta) z\bar{z} - H_{02}(\theta) \frac{\bar{z}^2}{2} - \dots.
 \end{aligned} \tag{69}$$

From (61), one can obtain

$$\begin{aligned}
 AW(t, \theta) &= AW_{20}(\theta) \frac{z^2}{2} + AW_{11}(\theta) z\bar{z} + AW_{02}(\theta) \frac{\bar{z}^2}{2} \\
 &\quad + AW_{30}(\theta) \frac{z^3}{6} + \dots, \\
 \dot{W} &= W_z \dot{z} + W_{\bar{z}} \dot{\bar{z}} = W_{20}(\theta) z\dot{z} + W_{11}(\theta) (\dot{z}\bar{z} + z\dot{\bar{z}}) \\
 &\quad + \dots = 2i\omega_*\tau_{10} W_{20}(\theta) \frac{z^2}{2} + \dots.
 \end{aligned} \tag{70}$$

Therefore, we can further obtain

$$\begin{aligned}
 (A - 2i\omega_*\tau_{10})W_{20}(\theta) &= -H_{20}(\theta), \\
 AW_{11}(\theta) &= -H_{11}(\theta).
 \end{aligned} \tag{71}$$

For $\theta \in [-1, 0)$, we have

$$\begin{aligned}
 H(z, \bar{z}, \theta) &= -\bar{q}^*(0)F(0, u_t)q(\theta) - q^*(0)\bar{F}(0, u_t)\bar{q}(\theta) \\
 &= -g(z, \bar{z})q(\theta) - \bar{g}(z, \bar{z})\bar{q}(\theta).
 \end{aligned} \tag{72}$$

A comparison of coefficients for (72) and (68) leads to

$$H_{20}(\theta) = -(g_{20}q(\theta) + \bar{g}_{02}\bar{q}(\theta)), \tag{73}$$

$$H_{11}(\theta) = -(g_{11}q(\theta) + \bar{g}_{11}\bar{q}(\theta)). \tag{74}$$

By (71) and (73) and the definition of A , one can obtain

$$\dot{W}_{20}(\theta) = 2i\omega_*\tau_{10}W_{20}(\theta) + g_{20}q(\theta) + \bar{g}_{02}\bar{q}(\theta). \tag{75}$$

Notice that $q(\theta) = q(0)e^{i\omega_*\tau_{10}\theta} = (1, \alpha)^T e^{i\omega_*\tau_{10}\theta}$. By computing (75), we have

$$\begin{aligned}
 W_{20}(\theta) &= \frac{ig_{20}}{\omega_*\tau_{10}} q(0)e^{i\omega_*\tau_{10}\theta} + \frac{i\bar{g}_{02}}{3\omega_*\tau_{10}} \bar{q}(0)e^{-i\omega_*\tau_{10}\theta} \\
 &\quad + E_1 e^{2i\omega_*\tau_{10}\theta},
 \end{aligned} \tag{76}$$

where $E_1 = (E_1^{(1)}, E_1^{(2)}) \in \mathbb{R}^2$ is also a two-dimensional constant vector.

Similarly, by (71) and (74) and the definition of A , one can obtain

$$\dot{W}_{11}(\theta) = g_{11}q(\theta) + \bar{g}_{11}\bar{q}(\theta), \tag{77}$$

$$W_{11}(\theta) = -\frac{ig_{11}}{\omega_*\tau_{10}}q(0)e^{i\omega_*\tau_{10}\theta} + \frac{i\bar{g}_{11}}{\omega_*\tau_{10}}\bar{q}(0)e^{-i\omega_*\tau_{10}\theta} + E_2, \tag{78}$$

where $E_2 = (E_2^{(1)}, E_2^{(2)}) \in \mathbb{R}^2$ is a two-dimensional constant vector.

Next, we need to compute E_1 and E_2 in (76) and (78). By the definition of A and (71), we obtain that

$$\int_{-1}^0 d\eta(\theta)W_{20}(\theta) = 2i\omega_*\tau_{10}W_{20}(0) - H_{20}(0), \tag{79}$$

$$\int_{-1}^0 d\eta(\theta)W_{11}(\theta) = -H_{11}(0). \tag{80}$$

Thus, we have

$$H_{20}(0) = -g_{20}q(0) - \bar{g}_{02}\bar{q}(0) + 2\tau_{10} \begin{pmatrix} \alpha_1 e^{-2i\omega_*\tau_{10}} + \alpha_2 \alpha e^{-i\omega_*\tau_{10}} e^{-i\omega_*\tau_2^*} \\ \beta_1 e^{-2i\omega_*\tau_2^*} + \beta_2 \alpha e^{-i\omega_*\tau_{10}} e^{-i\omega_*\tau_2^*} \end{pmatrix}, \tag{81}$$

$$H_{11}(0) = -g_{11}q(0) - \bar{g}_{11}\bar{q}(0) + 2\tau_{10} \begin{pmatrix} \alpha_1 + \alpha_2 \operatorname{Re}\{\alpha e^{i\omega_*\tau_{10}} e^{-i\omega_*\tau_2^*}\} \\ \beta_1 + \beta_2 \operatorname{Re}\{\alpha e^{-i\omega_*\tau_{10}} e^{i\omega_*\tau_2^*}\} \end{pmatrix}. \tag{82}$$

Noting that

$$\left(-i\omega_*\tau_{10}I - \int_{-1}^0 e^{-i\omega_*\tau_{10}\theta} d\eta(\theta)\right)\bar{q}(0) = 0. \tag{83}$$

Substituting (76) and (81) into equation (80) yields

$$\begin{aligned} &\left(2i\omega_*\tau_{10}I - \int_{-1}^0 e^{2i\omega_*\tau_{10}\theta} d\eta(\theta)\right)E_1 \\ &= 2\tau_{10} \begin{pmatrix} \alpha_1 e^{-2i\omega_*\tau_{10}} + \alpha_2 \alpha e^{-i\omega_*\tau_{10}} e^{-i\omega_*\tau_2^*} \\ \beta_1 e^{-2i\omega_*\tau_2^*} + \beta_2 \alpha e^{-i\omega_*\tau_{10}} e^{-i\omega_*\tau_2^*} \end{pmatrix}. \end{aligned} \tag{84}$$

That is,

$$\begin{aligned} &\begin{pmatrix} 2i\omega_* - P_1 e^{-2i\omega_*\tau_{10}} & -Q_1 e^{-2i\omega_*\tau_2^*} \\ -P_2 e^{-2i\omega_*\tau_2^*} & 2i\omega_* - Q_2 e^{-2i\omega_*\tau_{10}} \end{pmatrix} E_1 \\ &= 2 \begin{pmatrix} \alpha_1 e^{-2i\omega_*\tau_{10}} + \alpha_2 \alpha e^{-i\omega_*\tau_{10}} e^{-i\omega_*\tau_2^*} \\ \beta_1 e^{-2i\omega_*\tau_2^*} + \beta_2 \alpha e^{-i\omega_*\tau_{10}} e^{-i\omega_*\tau_2^*} \end{pmatrix}. \end{aligned} \tag{85}$$

It follows that

$$\begin{aligned} E_1^{(1)} &= \frac{2}{\Delta_1} \begin{vmatrix} \alpha_1 e^{-2i\omega_*\tau_{10}} + \alpha_2 \alpha e^{-i\omega_*\tau_{10}} e^{-i\omega_*\tau_2^*} & -Q_1 e^{-2i\omega_*\tau_2^*} \\ \beta_1 e^{-2i\omega_*\tau_2^*} + \beta_2 \alpha e^{-i\omega_*\tau_{10}} e^{-i\omega_*\tau_2^*} & 2i\omega_* - Q_2 e^{-2i\omega_*\tau_{10}} \end{vmatrix}, \\ E_1^{(2)} &= \frac{2}{\Delta_1} \begin{vmatrix} 2i\omega_* - P_1 e^{-2i\omega_*\tau_{10}} & \alpha_1 e^{-2i\omega_*\tau_{10}} + \alpha_2 \alpha e^{-i\omega_*\tau_{10}} e^{-i\omega_*\tau_2^*} \\ -P_2 e^{-2i\omega_*\tau_2^*} & \beta_1 e^{-2i\omega_*\tau_2^*} + \beta_2 \alpha e^{-i\omega_*\tau_{10}} e^{-i\omega_*\tau_2^*} \end{vmatrix}, \end{aligned} \tag{86}$$

where

$$\Delta_1 = \begin{vmatrix} 2i\omega_* - P_1 e^{-2i\omega_*\tau_{10}} & -Q_1 e^{-2i\omega_*\tau_2^*} \\ -P_2 e^{-2i\omega_*\tau_2^*} & 2i\omega_* - Q_2 e^{-2i\omega_*\tau_{10}} \end{vmatrix}. \tag{87}$$

Similarly, substituting (78) and (82) into (80) yields

$$\int_{-1}^0 d\eta(\theta)E_2 = -2\tau_{10} \begin{pmatrix} \alpha_1 + \alpha_2 \operatorname{Re}\{\alpha e^{i\omega_*\tau_{10}} e^{-i\omega_*\tau_2^*}\} \\ \beta_1 + \beta_2 \operatorname{Re}\{\alpha e^{-i\omega_*\tau_{10}} e^{i\omega_*\tau_2^*}\} \end{pmatrix}. \tag{88}$$

That is,

$$\begin{pmatrix} -P_1 & -Q_1 \\ -P_2 & -Q_2 \end{pmatrix} E_2 = 2 \begin{pmatrix} \alpha_1 + \alpha_2 \operatorname{Re}\{\alpha e^{i\omega_*\tau_{10}} e^{-i\omega_*\tau_2^*}\} \\ \beta_1 + \beta_2 \operatorname{Re}\{\alpha e^{-i\omega_*\tau_{10}} e^{i\omega_*\tau_2^*}\} \end{pmatrix}. \tag{89}$$

It follows that

$$\begin{aligned} E_2^{(1)} &= \frac{2}{\Delta_2} \begin{vmatrix} \alpha_1 + \alpha_2 \operatorname{Re}\{\alpha e^{i\omega_*\tau_{10}} e^{-i\omega_*\tau_2^*}\} & -Q_1 \\ \beta_1 + \beta_2 \operatorname{Re}\{\alpha e^{-i\omega_*\tau_{10}} e^{i\omega_*\tau_2^*}\} & -Q_2 \end{vmatrix}, \\ E_2^{(2)} &= \frac{2}{\Delta_2} \begin{vmatrix} -P_1 & \alpha_1 + \alpha_2 \operatorname{Re}\{\alpha e^{i\omega_*\tau_{10}} e^{-i\omega_*\tau_2^*}\} \\ -P_2 & \beta_1 + \beta_2 \operatorname{Re}\{\alpha e^{-i\omega_*\tau_{10}} e^{i\omega_*\tau_2^*}\} \end{vmatrix}, \end{aligned} \tag{90}$$

where

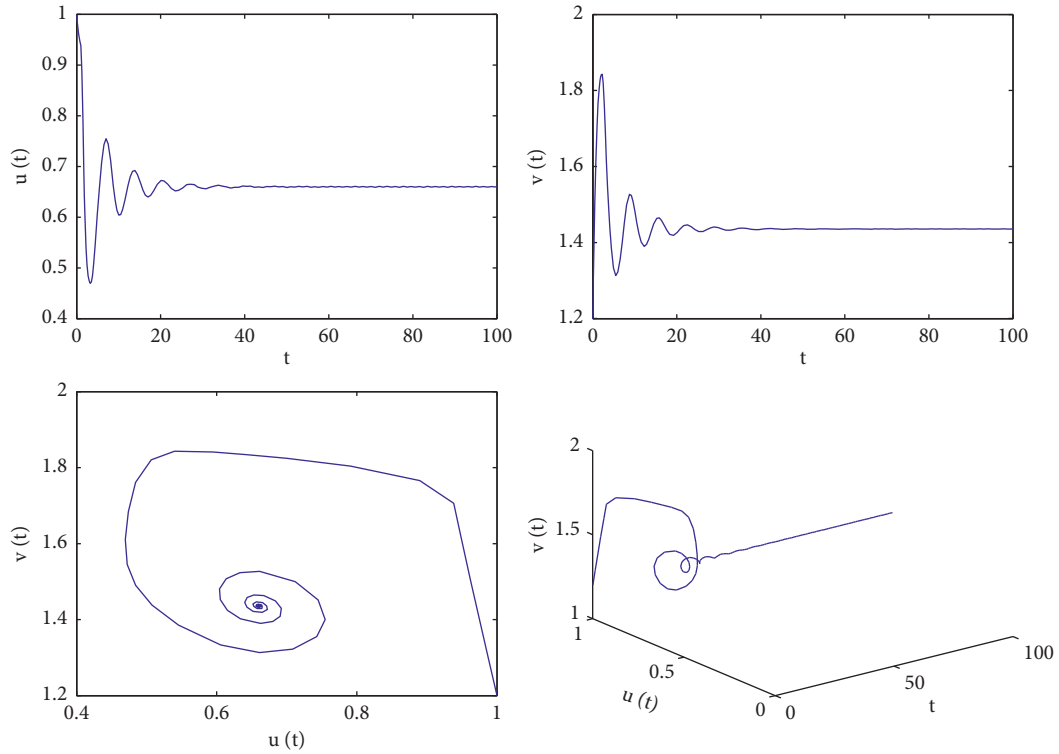


FIGURE 1: The trajectories graphs and phase graphs of system (93) with $\tau_1 = 0$ and $\tau_2 = 1$ at the initial value $(1, 1.2)$.

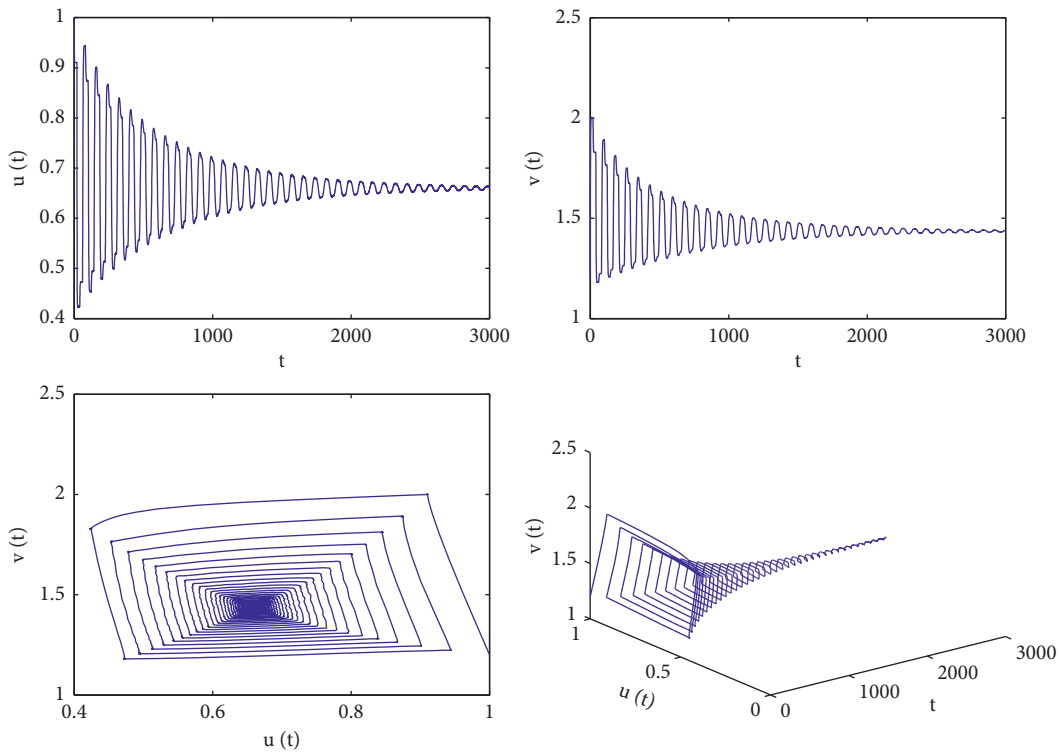


FIGURE 2: The trajectories graphs and phase graphs of system (93) with $\tau_1 = 0$ and $\tau_2 = 20$ at the initial value $(1, 1.2)$.

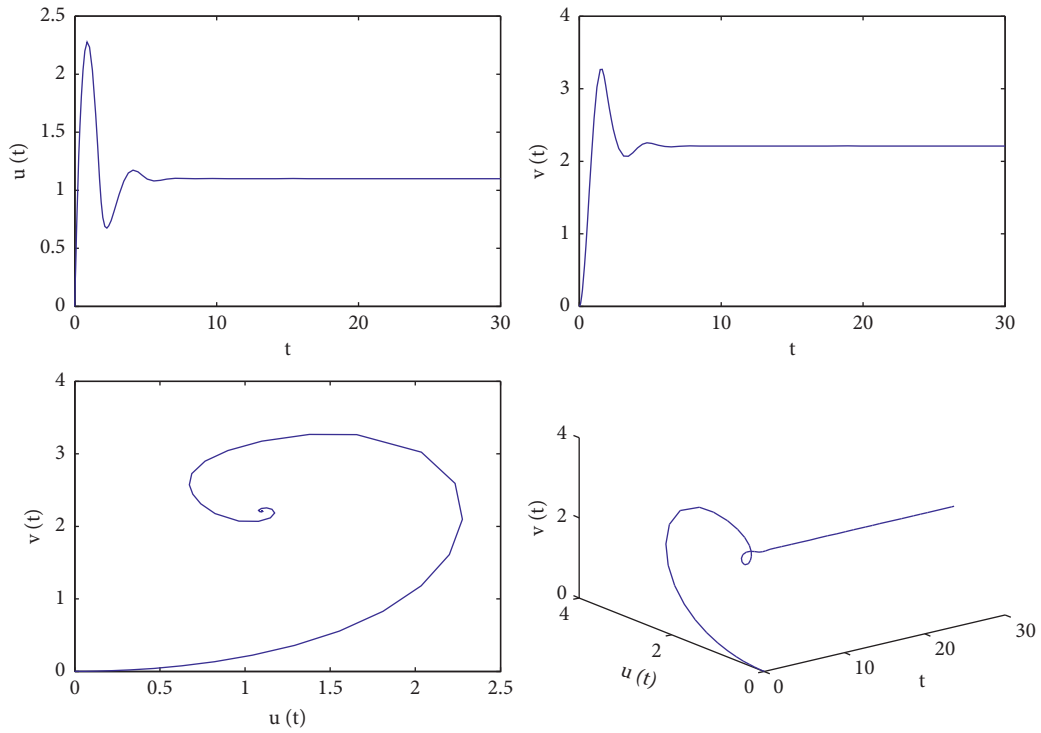


FIGURE 3: The trajectories graphs and phase graphs of system (94) with $\tau_1 = \tau_2 = 0$ at the initial value $(0, 0)$.

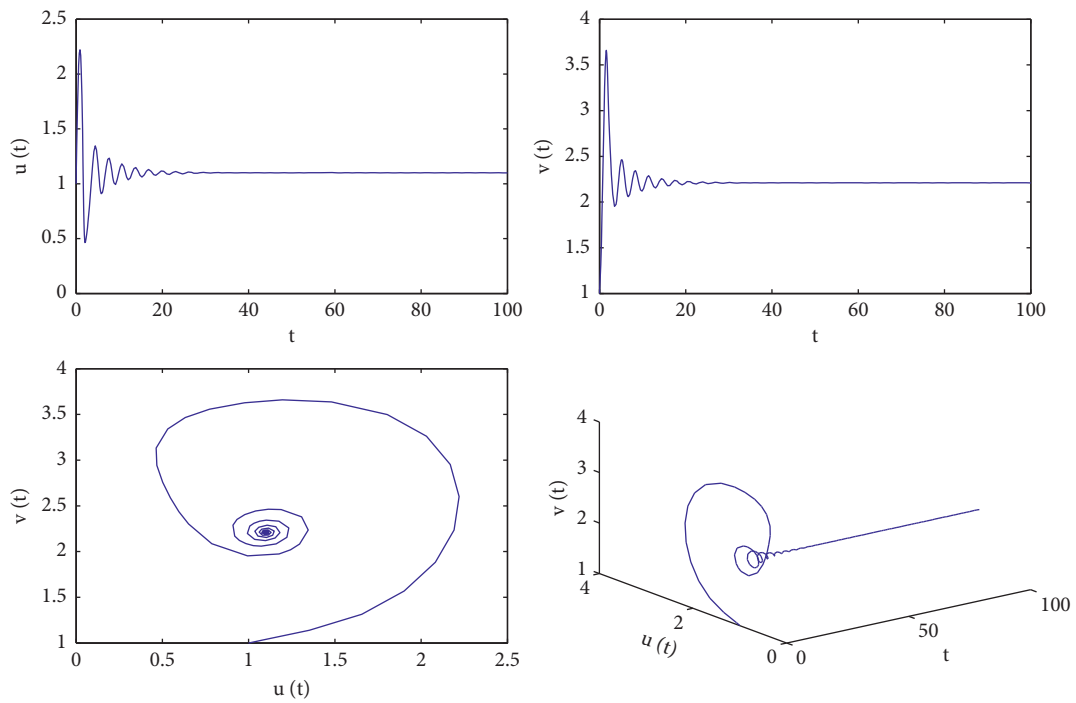


FIGURE 4: The trajectories graphs and phase graphs of system (94) with $\tau_1 = 0, \tau_2 = 0.15 < \tau_{20} = 0.2061$ at the initial value $(1, 1)$.

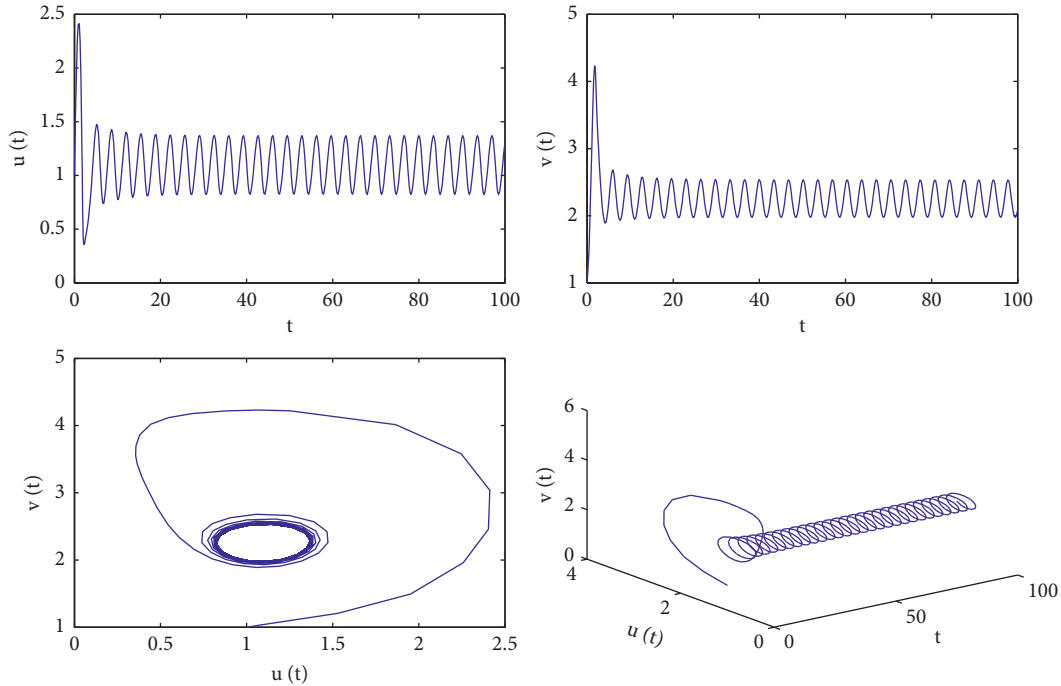


FIGURE 5: The trajectories graphs and phase graphs of system (94) with $\tau_1 = 0, \tau_2 = 0.23 > \tau_{20} = 0.2061$ at the initial value $(1, 1)$.

$$\Delta_2 = \begin{vmatrix} P_1 & Q_1 \\ P_2 & Q_2 \end{vmatrix}. \tag{91}$$

Thus, from (76), (78), (86), and (90), we can determine g_{21} and compute the following values:

$$\left\{ \begin{array}{l} c_1(0) = \frac{i}{2\omega_*\tau_{10}} \left(g_{20}g_{11} - 2|g_{11}|^2 - \frac{1}{3}|g_{02}|^2 \right) + \frac{1}{2}g_{21}, \\ \mu_2 = -\frac{\text{Re}\{c_1(0)\}}{\text{Re}\{\lambda'(\tau_{10})\}}, \\ \beta_2 = 2\text{Re}\{c_1(0)\}, \\ T_2 = -\frac{1}{\omega_*\tau_{10}} (\text{Im}\{c_1(0)\} + \mu_2\text{Im}\{\lambda'(\tau_{10})\}). \end{array} \right. \tag{92}$$

Now, the main results in this section are given.

Theorem 4. *If $\mu_2 > 0$ (resp. $\mu_2 < 0$), then the periodic solution is supercritical (resp. subcritical); if $\beta_2 < 0$ (resp. $\beta_2 > 0$), then the bifurcating periodic solutions are orbitally asymptotically stable with an asymptotical phase (resp. unstable); if $T_2 > 0$ (resp. $T_2 < 0$), then the period of the bifurcating periodic solutions increases (resp. decreases).*

4. Examples and Numerical Simulations

In this section, by applying the Matlab software package, we shall give some examples and numerical simulations to verify the results developed in the previous sections.

Example 1. Let $a = 3.3$ and $\sigma b = 2$; then, system (6) becomes

$$\begin{cases} \dot{u}(t) = 3.3 - u(t - \tau_1) - \frac{4u(t - \tau_1)v(t - \tau_2)}{1 + u^2(t - \tau_1)}, \\ \dot{v}(t) = 2 \left(u(t - \tau_2) - \frac{u(t - \tau_2)v(t - \tau_1)}{1 + u^2(t - \tau_2)} \right). \end{cases} \tag{93}$$

By calculating, system (93) has a unique positive equilibrium $(u^*, v^*) = (0.66, 1.4356)$. When $\tau_1 = 0$, it is easily obtained that the condition (H2): $(P_1Q_2)^2 - (P_2Q_1)^2 = 0.6136 > 0$ is satisfied. By Theorem 1, the positive equilibrium E^* of (93) is asymptotically stable for any $\tau_2 > 0$, as shown in Figures 1 and 2.

Example 2. Let $a = 5.5$ and $\sigma b = 2$; we have the following system:

$$\begin{cases} \dot{u}(t) = 5.5 - u(t - \tau_1) - \frac{4u(t - \tau_1)v(t - \tau_2)}{1 + u^2(t - \tau_1)}, \\ \dot{v}(t) = 2 \left(u(t - \tau_2) - \frac{u(t - \tau_2)v(t - \tau_1)}{1 + u^2(t - \tau_2)} \right). \end{cases} \tag{94}$$

By calculation with Matlab, system (94) has a unique positive equilibrium $(u^*, v^*) = (1.1, 2.21)$. When $\tau_1 = \tau_2 = 0$, we obtain $P_1 + Q_2 = -1.6154 < 0$ and $P_1Q_2 - P_2Q_1 = 4.9774 > 0$. That is, condition (H1) holds. Thus, the positive equilibrium E^* of (94) is asymptotically stable, as depicted in Figure 3. When $\tau_1 = 0, \tau_2 > 0$, we obtain that condition (H3): $(P_1Q_2)^2 - (P_2Q_1)^2 = -18.6311 < 0$ holds and $\omega_0 = 1.9192, \tau_{20} = 0.2061, [d(\text{Re}\lambda(\tau_2))]/$

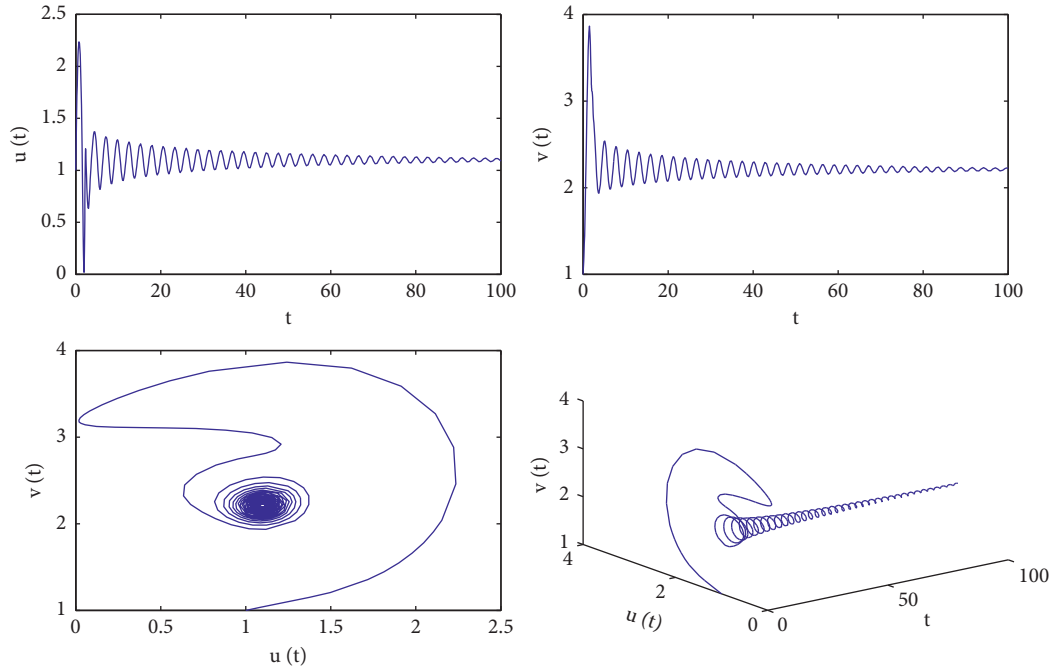


FIGURE 6: The trajectories graphs and phase graphs of system (94) with $\tau_1 = 0.2 < \tau_{10} = 0.2107$, $\tau_2 = 0.15 \in [0, 0.2061]$ at the initial value $(1, 1)$.

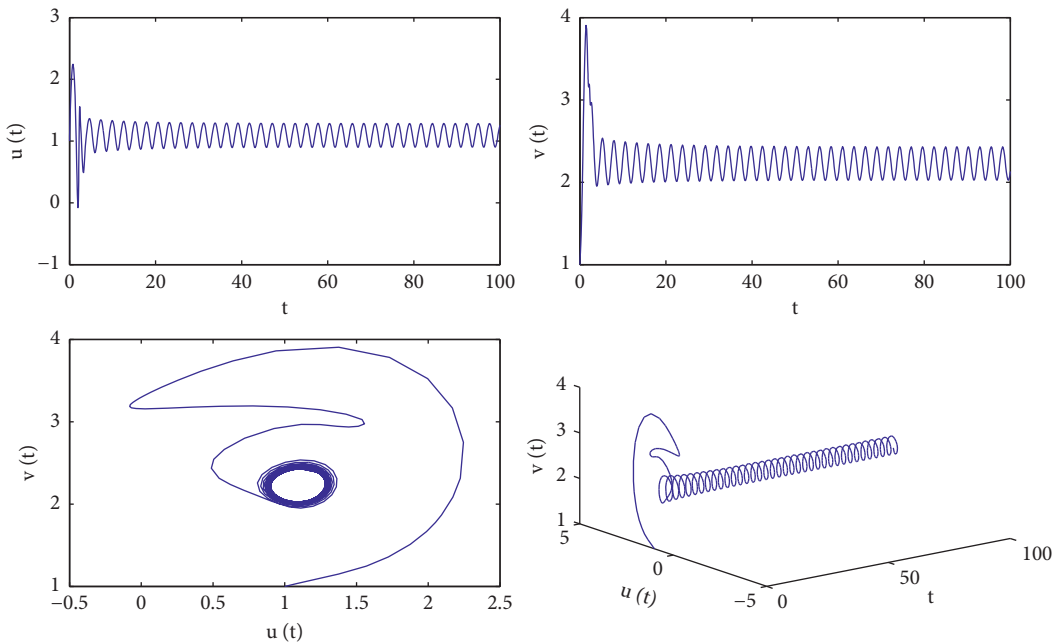


FIGURE 7: The trajectories graphs and phase graphs of system (94) with $\tau_1 = 0.23 > \tau_{10} = 0.2107$, $\tau_2 = 0.15 \in [0, 0.2061]$ at the initial value $(1, 1)$.

$d\tau_2|_{\lambda=i\omega_*, \tau_2=\tau_{20}}^{-1} = 0.2299 > 0$. It is known from Theorem 2, the positive equilibrium E^* of system (94) is asymptotically stable when $\tau_2 \in [0, \tau_{20})$. However, τ_2 passes through the critical value τ_{20} , the positive equilibrium E^* loses its stability, and a Hopf bifurcation occurs, as shown in Figures 4 and 5.

Let $\tau_2 = 0.15 \in [0, 0.2061]$, and choose $\tau_1 > 0$ as a parameter. The numerical results show that $\omega_* = 2.3361$ and $\tau_{10} = 0.2107$ and the conditions (H3): $(P_1Q_2)^2 - (P_2Q_1)^2 = -18.6311 < 0$, (H4): $\omega_* = 2.3361 > 0$, and (H5): $M_1(\omega_*)N_1(\omega_*) - M_2(\omega_*)N_2(\omega_*) = 0.4023 > 0$ are satisfied. By Theorem 3, the positive equilibrium E^* of

system (94) is asymptotically stable. And, the transversality condition $[d(\operatorname{Re}\lambda(\tau_1))/d\tau_1]_{\lambda=iw^*, \tau_1=\tau_{10}}^{-1} = 0.0065 > 0$ is also satisfied. Therefore, when τ_1 passes through the critical value τ_{10} , the positive equilibrium E^* loses its stability and a family of periodic solutions bifurcates from the positive equilibrium E^* , as shown in Figures 6 and 7. From formula (92) in Section 3, it follows that $c_{10} = -1.1543 + 7.05i$, $\mu_2 = 1.7611$, $\beta_2 = -2.3087$, and $T_2 = -22.7964$. Since $\mu_2 > 0$ and $\beta_2 < 0$, the direction of Hopf bifurcation of system (94) at $\tau_{10} = 0.2107$ is supercritical, and these bifurcating periodic solutions are stable, as shown in Figure 7.

5. Conclusions

Researchers often apply differential equations with parametric-to-dynamic system modeling because the solution of the equations depends on the behaviors of the parameters, which can effectively help researchers understand the system. In the past few decades, the dynamic behavior of the time-delay model has been widely concerned in many fields (chemistry, economics, medicine, ecology, etc.), which is inseparable from its biological and physical significance. The research on the dynamics of delay differential equations has always been the focus of attention in various fields. Of course, it is especially widely used in chemistry. The main reason is that the chemical reaction of the system with a particular input is often not immediate but delayed, and the system has different time delays for a chemical reaction with different inputs. Therefore, it is necessary to analyze the dynamical behaviors of the system with multiple delays.

In this article, we investigated the dynamical behaviors of the Lengyel–Epstein model with two delays which are different from the one studied in [5, 12, 14, 16]. By choosing the delay τ_i ($i = 1, 2$) as the bifurcation parameter, the stability of the positive equilibrium and the existence of Hopf bifurcation of the system are discussed in detail in three different cases. When $\tau_1 = 0$ and $\tau_2 > 0$ and condition (H2) holds, the positive equilibrium E^* of system (6) is asymptotically stable for any $\tau_2 > 0$. For the same τ_i , when condition (H3) holds, the positive equilibrium of system (6) is asymptotically stable for $\tau_2 \in [0, \tau_{20})$, and with the increase of τ_2 , the positive equilibrium loses its stability and a sequence of Hopf bifurcations occur. More generally, we further analyze the case, where τ_2 is fixed to the stable interval and τ_1 is taken as the bifurcation parameter. When $\tau_1 > 0$ and $\tau_2 > 0$ and conditions (H3) – (H5) hold, the positive equilibrium of system (6) is asymptotically stable for $\tau_1 \in [0, \tau_{10})$. Therefore, under certain conditions, if the value of the time-delay factor is lower than the critical value, the system is in an ideal stable state, which is convenient for the effective control of the chemical reaction in the system (See Figure 6). With the increase of τ_1 , the positive equilibrium loses its stability and a sequence of Hopf bifurcations occurs. In this case, the phenomenon of small amplitude periodic solution is generated near the equilibrium point of the system, which is not conducive to the effective control of the reaction in the system (see Figure 7). In addition, by applying the normal form and the center manifold theorem, the explicit formula determining the direction of Hopf bifurcation and the

stability of the bifurcating periodic solutions are obtained. Moreover, some numerical examples also support our theoretical results. The above results indicate that the stability of chemical reactions can be controlled by setting parameters, which plays a positive role in the safety and effectiveness of chemical experiments. Secondly, it is worth noting that there are many factors that affect the dynamic properties of the chemical reaction system. In this paper, we only consider the influence of delay parameters on the system, and there are many works worth study, such as considering the influence of the concentration of reactants in the chemical reaction process, diffusion, and other factors on the dynamic properties of the chemical reaction system and analyzing the global Hopf bifurcation of the model. Furthermore, due to the discrete nature of chemical reactions, the study of discrete systems is meaningful and interesting, and there are opportunities to further consider the dynamic properties of discrete systems with multiple delays in the future.

Data Availability

The data used to support the findings of this study are available from the corresponding author upon request.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

Acknowledgments

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