Research Article

Embedding of Besov Spaces and the Volterra Integral Operator

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The boundedness and compactness of the inclusion mapping from Besov spaces to tent spaces are studied in this paper. Meanwhile, the boundedness, compactness, and essential norm of the Volterra integral operator T_g from Besov spaces to a class of general function spaces are also investigated.

1. Introduction

Let $\mathcal{D}$ be the unit disk in the complex plane $\mathbb{C}$ and $H(\mathcal{D})$ be the class of functions analytic in $\mathcal{D}$. For $1 < p < \infty$, the Besov space, denoted by B_p , is the space of all functions $f \in H(\mathcal{D})$ satisfying

$$\|f\|_{B_p}^p = |f(0)|^p + \int_{\mathcal{D}} |f'(z)|^p (1 - |z|^2)^{p-2} dA(z) < \infty. \quad (1)$$

Let $0 < p < \infty$, $-2 < q < \infty$, and $0 \leq s < \infty$. The space $F(p, q, s)$ is the space consisting of all $f \in H(\mathcal{D})$ such that

$$\|f\|_{F(p, q, s)}^p = |f(0)|^p + \sup_{a \in \mathcal{D}} \int_{\mathcal{D}} |f'(z)|^p (1 - |z|^2)^q (1 - |\sigma_a(z)|^2)^s dA(z) < \infty, \quad (2)$$

where $\sigma_a(z) = ((a - z)/(1 - \bar{a}z))$. This space was first introduced by Zhao in [1]. $F(2, 0, s)$ is the Q_s space (see [2]). $F(2, 0, 1)$ is the BMOA space. $F(p, \alpha, 0)$ is called the Dirichlet-type space, denoted by $\mathcal{D}_\alpha^p$. In particular, $F(p, p - 2, 0)$ is the Besov space B_p . $F(p, p, 0)$ is just the classical Bergman space A^p . When $s > 1$, from [1], we see that $F(p, p - 2, s)$ is equivalent to the Bloch space, denoted by $\mathcal{B}$, which consists of all $f \in H(\mathcal{D})$ such that

$$\|f\|_{\mathcal{B}} = |f(0)| + \sup_{z \in \mathcal{D}} (1 - |z|^2) |f'(z)| < \infty. \quad (3)$$

For $1 < q < \infty$ and $0 < s, t < \infty$, let $\mathcal{LF}(q, q - 2, s, t)$ denote the space of all $f \in H(\mathcal{D})$ such that

$$\|f\|_{\mathcal{LF}(q, q-2, s, t)}^q = \sup_{a \in \mathcal{D}} \frac{1}{(\log(2/(1 - |a|^2)))^t} \int_{\mathcal{D}} |f'(z)|^q (1 - |z|^2)^{q-2} (1 - |\sigma_a(z)|^2)^s dA(z) < \infty. \quad (4)$$

The norm for $f \in \mathcal{LF}(q, q - 2, s, t)$ is given by $\|f\|_{\mathcal{LF}(q, q-2, s, t)} = |f(0)| + \|f\|_{\text{Log}}$. The space $\mathcal{LF}(q, q - 2, s, t)$ is a Banach space when $q > 1$.

The Volterra integral operator T_g was introduced by Pommerenke in [3]. Here,

$$T_g f(z) = \int_0^z f(w) g'(w) dw, \quad f \in H(\mathcal{D}). \quad (5)$$

In [3], Pommerenke showed that T_g is bounded on H^2 if and only if $g \in \text{BMOA}$. Aleman and Siskakis showed that T_g is bounded on H^p ($p \geq 1$) if and only if $g \in \text{BMOA}$ in [4]. In [5], Aleman and Siskakis proved that $T_g: A^p \rightarrow A^p$ is bounded (compact) if and only if $g \in \mathcal{B}$ ($g \in \mathcal{B}_0$). Recently, the operator T_g has been receiving much attention. See [4–18] and the references therein for more study of the operator T_g .

For any arc $I \subseteq \partial\mathbb{D}$, the boundary of $\mathcal{D}$, let $|I| = (1/2\pi) \int_I |d\zeta|$ denote the normalized length of I and $S(I)$ be the Carleson box defined by

$$S(I) = \left\{ z \in \mathcal{D} : 1 - |I| \leq |z| < 1, \frac{z}{|z|} \in I \right\}. \quad (6)$$

Let $0 < s < \infty$ and μ be a positive Borel measure on $\mathcal{D}$. We say that μ is a α -Carleson measure if

$$\|\mu\|_{\text{CM}_\alpha} = \sup_{I \subset \partial\mathcal{D}} \frac{\mu(S(I))}{|I|^\alpha} < \infty. \quad (7)$$

When $\alpha = 1$, it gives the classical Carleson measure. μ is said to be a vanishing α -Carleson measure if $\lim_{|I| \rightarrow 0} (\mu(S(I))/|I|^\alpha) = 0$. The Carleson measure is very useful in the theory of function spaces and operator theory. The famous embedding theorem says that the inclusion mapping $I_d: H^2 \rightarrow L^2(d\mu)$ is bounded if and only if μ is a Carleson measure (see [19]). See [7, 20] for the study of the inclusion mapping $I_d: B_p \rightarrow L^p(d\mu)$.

Let $0 < s, q < \infty, 0 \leq t < \infty$, and μ be a positive Borel measure on $\mathbb{D}$. Let $\mathcal{T}_{s,t}^q(\mu)$ denote the space of all μ -measurable functions f such that (see, e.g., [21])

$$\sup_{I \subset \partial\mathcal{D}} \frac{1}{|I|^s (\log(2/|I|))^t} \int_{S(I)} |f(z)|^q d\mu(z) < \infty. \quad (8)$$

The tent space $\mathcal{T}_{s,t}^q(\mu)$ was introduced by Liu et al. in [21]. When $t = 0$, $\mathcal{T}_{s,t}^q(\mu)$ will be denoted by $\mathcal{T}_s^q(\mu)$ for the simplicity. In [21], Liu et al. studied the embedding of some Möbius invariant spaces, such as the Bloch space and the Q_p space, into $\mathcal{T}_{s,t}^q$.

In [12], Pau and Zhao showed that the inclusion mapping $I_d: F(p, p-2, s) \rightarrow \mathcal{T}_s^p(\mu)$ is bounded if and only if μ is a p -logarithmic s -Carleson measure. In [9], Li et al. proved that the inclusion mapping $I_d: \mathcal{D}_{p-1}^p \rightarrow \mathcal{T}_s^p(\mu)$ is bounded if and only if μ is a $(s+1)$ -Carleson measure. In [14], Qian and Li proved that the inclusion mapping $I_d: B_p \rightarrow \mathcal{T}_s^q(\mu)$ is bounded (resp. compact) if and only if μ is a $q(1 - (1/p))$ -logarithmic s -Carleson measure (resp. vanishing $q(1 - (1/p))$ -logarithmic s -Carleson measure) under the assumption that $1 < p < q < \infty$ and $0 < s < \infty$.

Motivated by [14, 21], in this paper, we study the boundedness and compactness of the inclusion mapping $I_d: B_p \rightarrow \mathcal{T}_{s,q(1-(1/p))}^q(\mu)$. More precisely, we show that $I_d: B_p \rightarrow \mathcal{T}_{s,q(1-(1/p))}^q(\mu)$ is bounded (resp. compact) if and only if μ is an s -Carleson measure (resp. vanishing s -Carleson measure) under the assumption that $1 < p < q < \infty$ and $0 < s < \infty$. As an application, we study the boundedness of the operator $T_g: B_p \rightarrow \mathcal{LF}(q, q-2, s, q - (q/p))$. Moreover, the compactness and essential norm of the operator $T_g: B_p \rightarrow \mathcal{LF}(q, q-2, s, q - (q/p))$ are also investigated.

In this paper, the symbol $f \approx g$ means that $f \lesssim g \lesssim f$. We say that $f \lesssim g$ if there exists a constant C such that $f \leq Cg$.

2. Embedding the Besov Space B_p into $\mathcal{T}_{s,t}^q(\mu)$

We need the following equivalent description of α -Carleson measure (see Lemma 2.2 in [12]).

Lemma 1. Let $0 < \alpha, t < \infty$ and μ be a positive Borel measure on $\mathbb{D}$. Then, μ is an α -Carleson measure if and only if

$$\sup_{a \in \mathbb{D}} \int_{\mathbb{D}} \frac{(1 - |a|^2)^t}{|1 - \bar{a}z|^{\alpha+t}} d\mu(z) < \infty. \quad (9)$$

Moreover,

$$\|\mu\|_{\text{CM}_\alpha} \approx \sup_{a \in \mathbb{D}} \int_{\mathbb{D}} \frac{(1 - |a|^2)^t}{|1 - \bar{a}z|^{\alpha+t}} d\mu(z). \quad (10)$$

Using Lemma 3.10 in [22], we can easily obtain the following result.

Lemma 2. Let $1 < p < \infty$ and $w \in \mathcal{D}$. Set

$$f_w(z) = \left(\frac{1}{\log(2/1 - |w|^2)} \right)^{1/p} \log \frac{2}{1 - \bar{w}z}, \quad z \in \mathcal{D}. \quad (11)$$

Then, $f_w \in B_p$.

Theorem 1. Let $1 < p < q < \infty, 0 < s < \infty$, and μ be a positive Borel measure on $\mathcal{D}$. Then, the inclusion mapping $I_d: B_p \rightarrow \mathcal{T}_{s,q(1-(1/p))}^q(\mu)$ is bounded if and only if μ is an s -Carleson measure.

Proof. First, we assume that $I_d: B_p \rightarrow \mathcal{T}_{s,q(1-(1/p))}^q(\mu)$ is bounded. For any given arc $I \subseteq \partial\mathcal{D}$, set $a = (1 - |I|)\eta$, and η is the center point of I . It is easy to see that

$$|1 - \bar{a}z| \approx 1 - |a|^2 \approx |I|, \quad z \in S(I). \quad (12)$$

Let

$$f_a(z) = \left(\frac{1}{\log(2/1 - |a|^2)} \right)^{1/p} \log \frac{2}{1 - \bar{a}z}. \quad (13)$$

By Lemma 2, we see that $f_a \in B_p$. From the boundedness of $I_d: B_p \rightarrow \mathcal{T}_{s,q(1-(1/p))}^q(\mu)$, we have

$$\begin{aligned} \|f_a\|_{\mathcal{T}_s^q(\mu)}^q &= \sup_{I \subset \partial\mathcal{D}} \frac{1}{|I|^s (\log(2/|I|))^{q(1-(1/p))}} \\ &\quad \cdot \int_{S(I)} |f_a(z)|^q d\mu(z) < \infty. \end{aligned} \quad (14)$$

By the fact that $|f_a(z)| \approx (\log(2/|I|))^{1-(1/p)}$ when $z \in S(I)$, we get

$$\sup_{I \subset \partial\mathcal{D}} \frac{\mu(S(I))}{|I|^s} < \infty. \quad (15)$$

Hence, μ is an s -Carleson measure.

Conversely, assume that μ is an s -Carleson measure. Let $f \in B_p$. For any given arc $I \subseteq \partial\mathcal{D}$, set $w = (1 - |I|)\eta$, and η is the center point of I . Then,

$$\begin{aligned} \frac{1}{|I|^s (\log(2/|I|))^{q(1-(1/p))}} \int_{S(I)} |f(z)|^q d\mu(z) &\leq \frac{1}{|I|^s (\log(2/|I|))^{q(1-(1/p))}} \int_{S(I)} |f(z) - f(w)|^q d\mu(z) \\ &+ \frac{1}{|I|^s (\log(2/|I|))^{q(1-(1/p))}} \int_{S(I)} |f(w)|^q d\mu(z) \\ &= A + B, \end{aligned} \quad (16)$$

where

$$\begin{aligned} A &= \frac{1}{|I|^s (\log(2/|I|))^{q(1-(1/p))}} \int_{S(I)} |f(z) - f(w)|^q d\mu(z), \\ B &= \frac{1}{|I|^s (\log(2/|I|))^{q(1-(1/p))}} \int_{S(I)} |f(w)|^q d\mu(z). \end{aligned} \quad (17)$$

Since

$$|f(w)| \leq \left(\log \frac{2}{1-|w|^2} \right)^{1-(1/p)} \|f\|_{B_p} \leq \left(\log \frac{2}{|I|} \right)^{1-(1/p)} \|f\|_{B_p}, \quad (18)$$

we get

$$B \leq \frac{\mu(S(I))}{|I|^s} \|f\|_{B_p}^q \leq \|f\|_{B_p}^q. \quad (19)$$

Now, we turn to estimate A . By Theorem 1 in [7], we see that μ is an s -Carleson measure if and only if $I_d: D_{p-2+(ps/q)}^p \rightarrow L^q(\mu)$ is bounded. Note that

$$f \in B_p \subseteq D_{p-2+(ps/q)}^p. \quad (20)$$

Then,

$$\begin{aligned} A &\leq (1 - |w|^2)^s \int_{S(I)} \left| \frac{f(z) - f(w)}{(1 - \bar{w}z)^{(2s/q)}} \right|^q d\mu(z) \\ &\leq \left((1 - |z|^2)^{(ps/q)} \int_{\mathcal{D}} \left| \left(\frac{f(z) - f(w)}{(1 - \bar{w}z)^{(2s/q)}} \right)' \right|^p \right. \\ &\quad \left. \cdot (1 - |z|^2)^{p-2+(ps/q)} dA(z) \right)^{(q/p)}. \end{aligned} \quad (21)$$

Since

$$\left(\frac{f(z) - f(w)}{(1 - \bar{w}z)^{(2s/q)}} \right)' = \frac{f'(z)(1 - \bar{w}z)^{(2s/q)} + \bar{w}(2s/q)(f(z) - f(w))(1 - \bar{w}z)^{(2s/q)-1}}{(1 - \bar{w}z)^{(4s/q)}}, \quad (22)$$

we deduce that $A \leq (W_1 + W_2)^{(q/p)}$, where

$$\begin{aligned} W_1 &= (1 - |w|^2)^{(ps/q)} \int_{\mathcal{D}} \frac{|f'(z)|^p}{|1 - \bar{w}z|^{(2ps/q)}} (1 - |z|^2)^{p-2+(ps/q)} dA(z), \\ W_2 &= (1 - |w|^2)^{(ps/q)} \int_{\mathcal{D}} \frac{|f(z) - f(w)|^p}{|1 - \bar{w}z|^{(2ps/q)+p}} (1 - |z|^2)^{p-2+(ps/q)} dA(z). \end{aligned} \quad (23)$$

Since

$$\frac{(1 - |w|^2)(1 - |z|^2)}{|1 - \bar{w}z|^2} = (1 - |\sigma_w(z)|^2) \leq 1, \quad (24)$$

we get that

$$W_1 \leq \|f\|_{B_p}^p. \quad (25)$$

Making the change of variable $\eta = \sigma_w(z)$ and combining with Proposition 4.2 in [22], we have

$$\begin{aligned}
W_2 &= (1 - |w|^2)^{(ps/q)} \int_{\mathcal{D}} \frac{|(f^\circ \sigma_w)(\eta) - (f^\circ \sigma_w)(0)|^p}{|1 - \bar{w}\sigma_w(\eta)|^{(2ps/q)+p}} (1 - |\sigma_w(\eta)|^2)^{p-2+(ps/q)} \times \frac{(1 - |w|^2)^2}{|1 - \bar{w}\eta|^4} dA(\eta) \\
&= \int_{\mathcal{D}} |(f^\circ \sigma_w)(\eta) - (f^\circ \sigma_w)(0)|^p \frac{(1 - |\eta|^2)^{p-2+(ps/q)}}{|1 - \bar{w}\eta|^p} dA(\eta) \\
&\leq \int_{\mathcal{D}} |(f^\circ \sigma_w)'(\eta)|^p \frac{(1 - |\eta|^2)^{2p-2+(ps/q)}}{|1 - \bar{w}\eta|^p} dA(\eta) \\
&\leq \int_{\mathcal{D}} |f'(\sigma_w(\eta))|^p (1 - |\sigma_w(\eta)|^2)^p \frac{(1 - |\eta|^2)^{p-2+(ps/q)}}{|1 - \bar{w}\eta|^p} dA(\eta) \\
&\leq \int_{\mathcal{D}} |f'(z)|^p (1 - |z|^2)^p \frac{(1 - |\sigma_w(z)|^2)^{p-2+(ps/q)}}{|1 - \bar{w}\sigma_w(z)|^p} \frac{(1 - |w|^2)^2}{|1 - \bar{w}z|^4} dA(z) \\
&\leq \int_{\mathcal{D}} |f'(z)|^p \frac{(1 - |w|^2)^{(ps/q)} (1 - |z|^2)^{2p-2+(ps/q)}}{|1 - \bar{w}z|^{p+(2ps/q)}} dA(z) \\
&\leq \|f\|_{B_p}^p.
\end{aligned} \tag{26}$$

Therefore,

$$\sup_{I \subset \partial \mathbb{D}} \frac{1}{|I|^s (\log(2/|I|))^{q(1-(1/p))}} \int_{S(I)} |f(z)|^q d\mu(z) \leq \|f\|_{B_p}^q, \tag{27}$$

which implies the desired result. The proof is completed.

We say that the inclusion mapping $I_d: B_p \longrightarrow \mathcal{T}_{s,q(1-(1/p))}^q(\mu)$ is compact if

$$\lim_{n \rightarrow \infty} \frac{1}{|I|^s (\log(2/|I|))^{q(1-(1/p))}} \int_{S(I)} |f_n(z)|^q d\mu(z) = 0 \tag{28}$$

whenever $I \subset \partial \mathbb{D}$ and $\{f_n\}$ is a bounded sequence in B_p that converges to 0 uniformly on compact subsets of $\mathbb{D}$. $\square$

Theorem 2. Let $1 < p < q < \infty$ and $0 < s < \infty$. Let μ be a nonnegative Borel measure on $\mathbb{D}$ such that point evaluation is a bounded functional on $\mathcal{T}_{s,q(1-(1/p))}^q(\mu)$. Then, the inclusion mapping $I_d: B_p \longrightarrow \mathcal{T}_{s,q(1-(1/p))}^q(\mu)$ is compact if and only if μ is a vanishing s -Carleson measure.

Proof. First, we assume that $I_d: B_p \longrightarrow \mathcal{T}_{s,q(1-(1/p))}^q(\mu)$ is compact. Let $\{I_k\}$ be a sequence arc with $\lim_{k \rightarrow \infty} |I_k| = 0$. Set $a_k = (1 - |I_k|)\eta_k$, where η_k is the midpoint of arc I_k . Take

$$f_k(z) = \left(\frac{1}{\log(2/1 - |a_k|^2)} \right)^{1/p} \log \frac{2}{1 - \bar{a}_k z}. \tag{29}$$

We see that $f_k \in B_p$, and $\{f_k\}$ converges to 0 uniformly on compact subsets of $\mathcal{D}$ when $k \longrightarrow \infty$. Then, we get

$$\frac{\mu(S(I_k))}{|I_k|^s} \leq \frac{1}{|I_k|^s (\log(2/|I_k|))^{q(1-(1/p))}} \int_{S(I_k)} |f_k(z)|^q d\mu(z) \longrightarrow 0, \tag{30}$$

as $k \longrightarrow \infty$, which implies that μ is a vanishing s -Carleson measure.

Conversely, assume that μ is a vanishing s -Carleson measure. From [12], we see that

$$\|\mu - \mu_r\|_{\text{CM}_s} \longrightarrow 0, \quad r \longrightarrow 1. \tag{31}$$

Here, $\mu_r(z) = \mu(z)$ for $|z| < r$ and $\mu_r(z) = 0$ for $r \leq |z| < 1$. Let $\|f_k\|_{B_p} \leq 1$ and $\{f_k\}$ converge to 0 uniformly on compact subsets of $\mathcal{D}$. Then,

$$\begin{aligned}
&\frac{1}{|I|^s (\log(2/|I|))^{q(1-(1/p))}} \int_{S(I)} |f_k(z)|^q d\mu(z) \\
&\leq \frac{1}{|I|^s (\log(2/|I|))^{q(1-(1/p))}} \int_{S(I)} |f_k(z)|^q d\mu_r(z) \\
&\quad + \frac{1}{|I|^s (\log(2/|I|))^{q(1-(1/p))}} \int_{S(I)} |f_k(z)|^q d(\mu - \mu_r)(z) \\
&\leq \frac{1}{|I|^s (\log(2/|I|))^{q(1-(1/p))}} \int_{S(I)} |f_k(z)|^q d\mu_r(z) + \|\mu - \mu_r\|_{\text{CM}_s} \|f_k\|_{B_p}^q \\
&\leq \frac{1}{|I|^s (\log(2/|I|))^{q(1-(1/p))}} \int_{S(I)} |f_k(z)|^q d\mu_r(z) + \|\mu - \mu_r\|_{\text{CM}_s}.
\end{aligned} \tag{32}$$

Letting $k \longrightarrow \infty$ and then $r \longrightarrow 1$, we have

$$\lim_{k \rightarrow \infty} \|f_k\|_{\mathcal{T}_{s,q(1-(1/p))}^q(\mu)} = 0. \tag{33}$$

Therefore, $I_d: B_p \longrightarrow \mathcal{T}_{s,q(1-(1/p))}^q(\mu)$ is compact. The proof is completed. $\square$

3. Volterra Integral Operator $T_g: B_p \longrightarrow \mathcal{LF}(q, q-2, s, q-(q/p))$

Lemma 3. Let $1 < q < \infty$ and $0 < s, t < \infty$. Then, $f \in \mathcal{LF}(q, q-2, s, t)$ if and only if

$$\sup_{I \subset \partial \mathcal{D}} \frac{1}{|I|^s (\log(2/|I|))^t} \int_{S(I)} |f'(z)|^q (1-|z|^2)^{q-2+s} dA(z) < \infty. \quad (34)$$

Proof. The proof is similar to that of Proposition 1 in [15]. Thus, we omit the details of the proof. $\square$

Theorem 3. Let $1 < p < q < \infty$ and $0 < s < \infty$. Then, $T_g: B_p \longrightarrow \mathcal{LF}(q, q-2, s, q-(q/p))$ is bounded if and only if $g \in F(q, q-2, s)$.

Proof. Assume that $T_g: B_p \longrightarrow \mathcal{LF}(q, q-2, s, q-(q/p))$ is bounded. For any fixed arc $I \subset \partial \mathcal{D}$, let $e^{i\theta}$ denote the center of I and $a = (1-|I|)e^{i\theta}$. Set

$$f_a(z) = \left(\frac{1}{\log(2/1-|a|^2)} \right)^{1/p} \log \frac{2}{1-\bar{a}z}. \quad (35)$$

By Lemma 2, we have $f_a \in B_p$ for $1 < p < \infty$. In addition, it is easy to see that

$$|1-\bar{a}z| \approx 1-|a| = |I|, \quad (36)$$

$$|f_a(z)| \approx \left(\log \frac{2}{|I|} \right)^{(1-1/p)},$$

when $z \in S(I)$. By the boundedness of $T_g: B_p \longrightarrow \mathcal{LF}(q, q-2, s, q-(q/p))$, we get

$$\begin{aligned} & \infty > \|T_g f_a\|_{\mathcal{LF}(q, q-2, s, q-(q/p))}^q \\ & \geq \frac{1}{|I|^s (\log(2/|I|))^{q(1-1/p)}} \int_{S(I)} |f_a(z)|^q |g'(z)|^q \\ & \quad (1-|z|^2)^{q-2+s} dA(z) \\ & \approx \frac{1}{|I|^s} \int_{S(I)} |g'(z)|^q (1-|z|^2)^{q-2+s} dA(z), \end{aligned} \quad (37)$$

which implies that $g \in F(q, q-2, s)$ by [1].

Conversely, suppose that $g \in F(q, q-2, s)$. By [1], we see that $d\mu_g(z) = |g'(z)|^q (1-|z|^2)^{q-2+s} dA(z)$ is an s -Carleson measure. Let $f \in B_p$. By Theorem 1, we see that $I_d: B_p \longrightarrow \mathcal{T}_{s,q(1-(1/p))}^q(\mu_g)$ is bounded, i.e.,

$$\begin{aligned} & \sup_{I \subset \partial \mathcal{D}} \frac{1}{|I|^s (\log(2/|I|))^{q(1-1/p)}} \int_{S(I)} |(T_g f)'(z)|^q (1-|z|^2)^{q-2+s} dA(z) \\ & = \sup_{I \subset \partial \mathcal{D}} \frac{1}{|I|^s (\log(2/|I|))^{q(1-1/p)}} \int_{S(I)} |f(z)|^q |g'(z)|^q (1-|z|^2)^{q-2+s} dA(z) \\ & = \sup_{I \subset \partial \mathcal{D}} \frac{1}{|I|^s (\log(2/|I|))^{q(1-1/p)}} \int_{S(I)} |f(z)|^q d\mu_g(z) \leq \|f\|_{B_p}^q < \infty. \end{aligned} \quad (38)$$

By Lemma 3, we get that

$$\sup_{a \in \mathcal{D}} \frac{1}{(\log(2/1-|a|^2))^{q(1-(1/p))}} \int_{\mathcal{D}} |(T_g f)'(z)|^q (1-|z|^2)^{q-2} (1-|\sigma_a(z)|^2)^s dA(z) < \infty. \quad (39)$$

Therefore, $T_g: B_p \longrightarrow \mathcal{LF}(q, q-2, s, q-(q/p))$ is bounded.

Next, we give an estimation for the essential norm of T_g . First, we recall some definitions. The essential norm of $T: X \longrightarrow Y$, denoted by $\|T\|_{e, X \longrightarrow Y}$, is defined by

$$\|T\|_{e, X \longrightarrow Y} = \inf_K \{\|T - K\|_{X \longrightarrow Y} : K \text{ is compact from } X \text{ to } Y\}. \quad (40)$$

Here, X and Y are Banach spaces, and $T: X \longrightarrow Y$ is a bounded linear operator. It is easy to see that $T: X \longrightarrow Y$ is compact if and only if $\|T\|_{e, X \longrightarrow Y} = 0$. Let A be a closed subspace of X . Given $f \in X$, the distance from f to A , denoted by $\text{dist}_X(f, A)$, is defined by

$$\text{dist}_X(f, A) = \inf_{g \in A} \|f - g\|_X. \quad (41)$$

$\square$

Lemma 4. (see [14]). Let $1 < q < \infty$ and $0 < s < \infty$. If $g \in F(q, q-2, s)$, then

$$\begin{aligned} \text{dist}_{F(q, q-2, s)}(g, F_0(q, q-2, s)) &\approx \limsup_{r \rightarrow 1^-} \|g - g_r\|_{F(q, q-2, s)} \\ &\approx \limsup_{|a| \rightarrow 1} \int_{\mathcal{D}} |g'(z)|^q (1 - |z|^2)^{q-2} (1 - |\sigma_a(z)|^2)^s dA(z). \end{aligned} \quad (42)$$

Here, $g_r(z) = g(rz)$, $0 < r < 1$, $z \in \mathcal{D}$.

Lemma 5. Let $1 < p < q < \infty$ and $0 < s < \infty$. If $0 < r < 1$ and $g \in F(q, q-2, s)$, then $T_g: B_p \rightarrow \mathcal{LF}(q, q-2, s, q - (q/p))$ is compact.

Proof. Given $\{f_n\} \subset B_p$ such that $\{f_n\}$ converges to zero uniformly on any compact subset of $\mathcal{D}$ and $\sup_n \|f_n\|_{B_p} \leq 1$. Since $g \in F(q, q-2, s) \subset \mathcal{B}$, we get that

$$|g'_r(z)| \leq \frac{\|g\|_{F(q, q-2, s)}}{1 - r^2}. \quad (43)$$

Hence,

$$\begin{aligned} &\|T_g f_n\|_{\mathcal{LF}(q, q-2, s, q - (q/p))}^q \\ &= \sup_{a \in \mathcal{D}} \frac{1}{(\log(2/1 - |a|^2))^{q(1-1/p)}} \int_{\mathcal{D}} |f_n(z)|^q |g'_r(z)|^q (1 - |z|^2)^{q-2} (1 - |\sigma_a(z)|^2)^s dA(z) \\ &\leq \frac{\|g\|_{\mathcal{B}}^q}{(1 - r^2)^q} \sup_{a \in \mathcal{D}} \int_{\mathcal{D}} |f_n(z)|^q (1 - |z|^2)^{q-2} (1 - |\sigma_a(z)|^2)^s dA(z) \\ &\leq \frac{\|g\|_{F(q, q-2, s)}^q \|f_n\|_{B_p}^{q-p}}{(1 - r^2)^q} \sup_{a \in \mathcal{D}} \int_{\mathcal{D}} |f_n(z)|^p (1 - |z|^2)^{p-2} (1 - |\sigma_a(z)|^2)^s dA(z) \\ &\leq \frac{\|g\|_{F(q, q-2, s)}^q \|f_n\|_{B_p}^{q-p}}{(1 - r^2)^q} \int_{\mathcal{D}} |f_n(z)|^p (1 - |z|^2)^{p-2} dA(z) \\ &\leq \frac{\|g\|_{F(q, q-2, s)}^q \|f_n\|_{B_p}^{q-p}}{(1 - r^2)^q} \int_{\mathcal{D}} |f'_n(z)|^p (1 - |z|^2)^{p-2} dA(z) \\ &\leq \frac{\|g\|_{F(q, q-2, s)}^q \|f_n\|_{B_p}^q}{(1 - r^2)^q}. \end{aligned} \quad (44)$$

By the dominated convergence theorem, we get the desired result. The proof is completed. $\square$

Lemma 6. (see [23]). Let X and Y be two Banach spaces of analytic functions on $\mathcal{D}$. Suppose that

- (1) The point evaluation functionals on Y are continuous
- (2) The closed unit ball of X is a compact subset of X in the topology of uniform convergence on compact sets
- (3) $T: X \rightarrow Y$ is continuous when X and Y are given the topology of uniform convergence on compact sets

Then, T is a compact operator if and only if for any bounded sequence $\{f_n\}$ in X such that $\{f_n\}$ converges to zero uniformly on every compact set of $\mathcal{D}$, the sequence $\{Tf_n\}$ converges to zero in the norm of Y .

Theorem 4. Let $1 < p < q < \infty$ and $0 < s < \infty$. If $T_g: B_p \rightarrow \mathcal{LF}(q, q-2, s, q - (q/p))$ is bounded, then

$$\|T_g\|_{e, B_p} \rightarrow \mathcal{LF}(q, q-2, s, q - (q/p)) \approx \text{dist}_{F(q, q-2, s)}(g, F_0(q, q-2, s)). \quad (45)$$

Proof. Let $\{I_n\} \subset \partial\mathcal{D}$ and $|I_n| \rightarrow 0$ as $n \rightarrow \infty$. Suppose $e^{i\theta_n}$ is the center of I_n and $w_n = (1 - |I_n|)e^{i\theta_n}$. For each n , let

$$f_{w_n}(z) = \left(\frac{1}{\log(2/1 - |w_n|^2)} \right)^{1/p} \log \frac{2}{1 - w_n z}. \quad (46)$$

Then, $|f_{w_n}(z)| \approx (\log(2/|I_n|))^{1-1/p}$ when $z \in S(I_n)$, and $\{f_{w_n}\}$ is bounded in B_p . Furthermore, $\{f_{w_n}\}$ converges to zero uniformly on every compact subset of $\mathcal{D}$. Given a compact operator $K: B_p \rightarrow \mathcal{LF}(q, q-2, s, q - (q/p))$, by Lemma 6, we have

$$\lim_{n \rightarrow \infty} \|K f_{w_n}\|_{\mathcal{LF}(q, q-2, s, q - (q/p))} = 0. \quad (47)$$

So,

$$\begin{aligned}
\|T_g - K\| &> \limsup_{n \rightarrow \infty} \|(T_g - K)f_{w_n}\|_{\mathcal{LF}(q, q-2, s, q-(q/p))} \\
&> \limsup_{n \rightarrow \infty} \left(\|T_g f_{w_n}\|_{\mathcal{LF}(q, q-2, s, q-(q/p))} - \|K f_{w_n}\|_{\mathcal{LF}(q, q-2, s, q-(q/p))} \right) \\
&= \limsup_{n \rightarrow \infty} \|T_g f_{w_n}\|_{\mathcal{LF}(q, q-2, s, q-(q/p))} \\
&\geq \limsup_{n \rightarrow \infty} \left(\frac{1}{(\log(2/1 - |w_n|^2))^{q(1-1/p)}} \int_{\mathcal{D}} |f_{w_n}(z)|^q |g'(z)|^q (1 - |z|^2)^{q-2} (1 - |\sigma_{w_n}(z)|^2)^s dA(z) \right)^{(1/q)} \\
&\geq \limsup_{n \rightarrow \infty} \left(\frac{1}{(\log(2/1 - |w_n|^2))^{q(1-1/p)}} \int_{S(I_n)} |f_{w_n}(z)|^q |g'(z)|^q (1 - |z|^2)^{q-2} (1 - |\sigma_{w_n}(z)|^2)^s dA(z) \right)^{(1/q)} \\
&> \limsup_{n \rightarrow \infty} \left(\frac{1}{|I_n|^s} \int_{S(I_n)} |g'(z)|^q (1 - |z|^2)^{q-2+s} dA(z) \right)^{(1/q)},
\end{aligned} \tag{48}$$

which implies that

$$\|T_g\|_{e, B_p \rightarrow \mathcal{LF}(q, q-2, s, q-(q/p))} > \limsup_{n \rightarrow \infty} \left(\int_{\mathcal{D}} |g'(z)|^q (1 - |z|^2)^{q-2} (1 - |\sigma_{w_n}(z)|^2)^s dA(z) \right)^{(1/q)}. \tag{49}$$

It follows from Lemma 4 that

$$\|T_g\|_{e, B_p \rightarrow \mathcal{LF}(q, q-2, s, q-(q/p))} > \text{dist}_{F(q, q-2, s)}(g, F_0(q, q-2, s)). \tag{50}$$

On the contrary, by Lemma 5, $T_{g_r}: B_p \rightarrow \mathcal{LF}(q, q-2, s, q-(q/p))$ is compact. Then,

$$\begin{aligned}
\|T_g\|_{e, B_p \rightarrow \mathcal{LF}(q, q-2, s, q-(q/p))} &\leq \|T_g - T_{g_r}\| = \|T_{g-g_r}\| \\
&\approx \|g - g_r\|_{F(q, q-2, s)}.
\end{aligned} \tag{51}$$

Using Lemma 4 again, we have

$$\begin{aligned}
\|T_g\|_{e, B_p \rightarrow \mathcal{LF}(q, q-2, s, q-(q/p))} &\leq \limsup_{r \rightarrow 1^-} \|g - g_r\|_{F(q, q-2, s)} \\
&\approx \text{dist}_{F(q, q-2, s)}(g, F_0(q, q-2, s)).
\end{aligned} \tag{52}$$

The proof is completed.

The following result can be deduced by Theorem 4 directly. $\square$

Corollary 1. Let $1 < p < q < \infty$ and $0 < s < \infty$. If $T_g: B_p \rightarrow \mathcal{LF}(q, q-2, s, q-(q/p))$ is bounded, then $T_g: B_p \rightarrow \mathcal{LF}(q, q-2, s, q-(q/p))$ is compact if and only if

$$g \in F_0(q, q-2, s). \tag{53}$$

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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