Hindawi
Journal of Mathematics
Volume 2023, Article ID 9810931, 1 page
https://doi.org/10.1155/2023/9810931

Retraction

Retracted: Hamacher Interactive Hybrid Weighted Averaging Operators under Fermatean Fuzzy Numbers

Journal of Mathematics

Received 10 October 2023; Accepted 10 October 2023; Published 11 October 2023

This article has been retracted by Hindawi following an investigation undertaken by the publisher [1]. This investigation has uncovered evidence of one or more of the following indicators of systematic manipulation of the publication process:

(1) Discrepancies in scope
(2) Discrepancies in the description of the research reported
(3) Discrepancies between the availability of data and the research described
(4) Inappropriate citations
(5) Incoherent, meaningless and/or irrelevant content included in the article
(6) Peer-review manipulation

The presence of these indicators undermines our confidence in the integrity of the article's content and we cannot, therefore, vouch for its reliability. Please note that this notice is intended solely to alert readers that the content of this article is unreliable. We have not investigated whether authors were aware of or involved in the systematic manipulation of the publication process.

Wiley and Hindawi regrets that the usual quality checks did not identify these issues before publication and have since put additional measures in place to safeguard research integrity.

We wish to credit our own Research Integrity and Research Publishing teams and anonymous and named external researchers and research integrity experts for contributing to this investigation.

The corresponding author, as the representative of all authors, has been given the opportunity to register their agreement or disagreement to this retraction. We have kept a record of any response received.

References

[1] G. Shahzadi, G. Muhiuddin, M. Arif Butt, and A. Ashraf, "Hamacher Interactive Hybrid Weighted Averaging Operators under Fermatean Fuzzy Numbers," *Journal of Mathematics*, vol. 2021, Article ID 5556017, 17 pages, 2021.

Hindawi
Journal of Mathematics
Volume 2021, Article ID 5556017, 17 pages
https://doi.org/10.1155/2021/5556017

Research Article

Hamacher Interactive Hybrid Weighted Averaging Operators under Fermatean Fuzzy Numbers

Gulfam Shahzadi,[1] G. Muhiuddin ,[2] Muhammad Arif Butt,[3] and Ather Ashraf[3]

[1]*Department of Mathematics, University of the Punjab New Campus, Lahore, Pakistan*
[2]*Department of Mathematics, University of Tabuk, P. O. Box 741, Tabuk 71491, Saudi Arabia*
[3]*Punjab University College of Information Technology, University of the Punjab Old Campus, Lahore-54000, Pakistan*

Correspondence should be addressed to G. Muhiuddin; chishtygm@gmail.com

Received 9 February 2021; Revised 24 February 2021; Accepted 10 March 2021; Published 24 March 2021

Academic Editor: Sami Ullah Khan

A Fermatean fuzzy set is a more powerful tool to deal with uncertainties in the given information as compared to intuitionistic fuzzy set and Pythagorean fuzzy set and has energetic applications in decision-making. Aggregation operators are very helpful for assessing the given alternatives in the decision-making process, and their purpose is to integrate all the given individual evaluation values into a unified form. In this research article, some new aggregation operators are proposed under the Fermatean fuzzy set environment. Some deficiencies of the existing operators are discussed, and then, new operational law, by considering the interaction between the membership degree and nonmembership degree, is discussed to reduce the drawbacks of existing theories. Based on Hamacher's norm operations, new averaging operators, namely, Fermatean fuzzy Hamacher interactive weighted averaging, Fermatean fuzzy Hamacher interactive ordered weighted averaging, and Fermatean fuzzy Hamacher interactive hybrid weighted averaging operators, are introduced. Some interesting properties related to these operators are also presented. To get the optimal alternative, a multiattribute group decision-making method has been given under proposed operators. Furthermore, we have explicated the comparison analysis between the proposed and existing theories for the exactness and validity of the proposed work.

1. Introduction

The process of multiattribute group decision-making (MAGDM) yields the best alternative when the list of all possible alternatives has been compiled according to some certain attributes. Previously, the data about alternatives corresponding to attributes and their weights were given in crisp values. However, nowadays, uncertainties play an important part in the decision-making (DM) approach. Each alternative is allotted a preference to some certain degree to deal with the complicated system. However, information regarding real-world system is indefinite and fuzzy with a lot of ambiguities. Such type of conditions is appropriately explained by fuzzy set (FS) [1] and intuitionistic fuzzy set (IFS) [2] rather than crisp values. IFS is a more efficient tool to deal with vague information because it has both the membership degree (MD) and nonmembership degree (NMD), but there are some drawbacks. The sum of MD and NMD is constrained to unit interval in IFS's model. Pythagorean fuzzy set (PFS) was introduced by Yager [3] to tackle vague decisions more effectively. However, this model also has some restrictions; if MD of an element is 0.8 and NMD is 0.76, then $0.8^2 + 0.76^2 > 1$. Therefore, Yager [4] narrated the theory of q-rung orthopair fuzzy set (q-ROFS) with condition $0 \leq \varrho^q + \sigma^q \leq 1$. The basic notions about Fermatean fuzzy set (FFS) were studied by Senapati and Yager [5].

The idea of aggregation operators (AOs) performs a crucial role in getting an optimal solution when there are a lot of choices for one given problem. The idea of aggregation of infinite sequences was presented by Mesiar and Pap [6]. Xu [7] gave the theory of intuitionistic fuzzy (IF) AOs.

Zhao et al. [8] developed the theory of generalized AOs for IFS. The Einstein hybrid AOs under IF environment were studied by Zhao and Wei [9]. The concept of IF AOs using Einstein operations were discussed by Wang and Liu [10]. Garg [11] combined the theories of IFS and interactive averaging AOs. Garg et al. [12] gave the idea of Choquet integral aggregation operators for interval-valued IFS. Garg [13] introduced IF Hamacher AOs with entropy weight. Alcantud et al. [14] elaborated the idea of aggregation of infinite chains of IFS. Wu and Wei [15] gave the theory of Pythagorean fuzzy (PF) Hamacher AOs. Wei [16] proposed the PF interaction AOs. Shahzadi and Akram [17] combined the concept of PF numbers and Yager operators. The theory of novel interactive hybrid weighted AOs with PF environment was studied by Li et al. [18]. The idea of q-ROF power Maclaurin symmetric mean operators was narrated by Liu et al. [19]. q-rung orthopair fuzzy (q-ROF) weighted AOs were expressed by Liu and Wang [20]. The exponential aggregation operators for q-ROFS were defined by Peng et al. [21]. Some confidence levels about q-ROF AOs were studied by Joshi and Gegov [22]. The hybrid DM model under q-ROF Yager AOs was developed by Akram and Shahzadi [23]. Akram et al. [24] presented the Einstein geometric operators for q-ROF information. Akram et al. [25] gave the protraction of Einstein operators under q-ROF environment. Darko and Liang [26] examined q-ROF Hamacher AOs and their application in MAGDM with modified EDAS method. Senapati and Yager [27] elaborated the theory of Fermatean fuzzy (FF) averaging/geometric operators. Senapati and Yager [28] studied subtraction, division, and Fermatean arithmetic mean operations over FFS. Many new operations for FFS were defined by Senapati and Yager [28]. Garg et al. [29] developed the theory for the choice of a most suitable laboratory for COVID-19 test under FF environment. The effectiveness of a sanitizer in COVID-19 was discussed by Akram et al. [30]. For more knowledge and applications, the readers are suggested to study [31–44].

1.1. Motivations of Proposed Work

(i) The proposed operators have the ability to deal with the interaction between the MD and NMD.
(ii) The proposed theory shows that the change in MD will affect the NMD.
(iii) The developed operators show that there will be nonzero NMD of the whole aggregated FF numbers (FFNs) even if at least one of them is zero. Therefore, the others grades of nonmembership function of FFNs perform a significant role in the aggregation process (AP).

1.2. Contributions of Proposed Work

(i) Some novel operators such as Fermatean fuzzy Hamacher interactive weighted averaging (FFHIWA), Fermatean fuzzy Hamacher interactive ordered weighted averaging (FFHIOWA), and Fermatean fuzzy Hamacher interactive hybrid weighted averaging (FFHIHWA) operators are explored here.
(ii) Some special cases of these operators along with their attractive properties are discussed, which reduce the shortcomings of the existing operators.
(iii) Some basic steps for MAGDM under proposed operators are explained with the help of a numerical example.
(iv) The comparison analysis with other developed approaches shows the validity of proposed theory.

1.3. Framework and Organization of the Paper. The remaining paper is arranged as follows: Section 2 recalls some elementary definitions. Section 3 defines the hybrid structure of Hamacher, interactive operators, and FFNs such as FFHIWA operator along with some fundamental properties. In Section 4, we elaborate the idea of FFHIOWA operator with some attractive properties. Section 5 presents the notion of FFHIHWA operator. Section 6 discusses an algorithm to deal with MAGDM along with a numerical example. Section 7 gives a comparison analysis with FF Einstein weighted averaging (FFEWA) operator for the validity and importance of proposed theory. In Section 8, we have summarized the results.

2. Preliminaries

In this section, we recall some basic definitions.

Definition 1 (see [5]). A FFS $\mathfrak{P}$ on nonempty set $\mathfrak{D}$ is given by

$$\mathfrak{P} = \left\{\left\langle \mathfrak{r}, \varrho_{\mathfrak{P}}(\mathfrak{r}), \sigma_{\mathfrak{P}}(\mathfrak{r})\right\rangle\right\}, \tag{1}$$

where $\varrho_{\mathfrak{P}}\colon \mathfrak{D} \longrightarrow [0,1]$, $\sigma_{\mathfrak{P}}\colon \mathfrak{D} \longrightarrow [0,1]$, and $\varpi_{\mathfrak{P}}(\mathfrak{r}) = \sqrt[3]{1 - (\varrho_{\mathfrak{P}}(\mathfrak{r}))^3 - (\sigma_{\mathfrak{P}}(\mathfrak{r}))^3}$ indicate MD, NMD, and indeterminacy degree (InD), respectively.

Definition 2 (see [5]). For FFN $\mathfrak{P} = (\varrho_{\mathfrak{P}}, \sigma_{\mathfrak{P}})$, the score function and accuracy function are given as

$$\begin{aligned} S(\mathfrak{P}) &= \varrho_{\mathfrak{P}}^3 - \sigma_{\mathfrak{P}}^3,\ S(\mathfrak{P}) \in [-1,1],\\ \mathscr{A}(\mathfrak{P}) &= \varrho_{\mathfrak{P}}^3 + \sigma_{\mathfrak{P}}^3,\ \mathscr{A}(\mathfrak{P}) \in [0,1]. \end{aligned} \tag{2}$$

Definition 3 (see [5]). Consider two FFNs $\mathfrak{P}_1 = \left\langle \varrho_{\mathfrak{P}_1}, \sigma_{\mathfrak{P}_1}\right\rangle$ and $\mathfrak{P}_2 = \left\langle \varrho_{\mathfrak{P}_2}, \sigma_{\mathfrak{P}_2}\right\rangle$. Then, the following holds:

(1) If $S(\mathfrak{P}_1) < S(\mathfrak{P}_2)$, then $\mathfrak{P}_1 \prec \mathfrak{P}_2$.
(2) If $S(\mathfrak{P}_1) > S(\mathfrak{P}_2)$, then $\mathfrak{P}_1 \succ \mathfrak{P}_2$.
(3) If $S(\mathfrak{P}_1) = S(\mathfrak{P}_2)$, then
 (a) If $\mathscr{A}(\mathfrak{P}_1) < \mathscr{A}(\mathfrak{P}_2)$, then $\mathfrak{P}_1 \prec \mathfrak{P}_2$.
 (b) If $\mathscr{A}(\mathfrak{P}_1) > \mathscr{A}(\mathfrak{P}_2)$, then $\mathfrak{P}_1 \succ \mathfrak{P}_2$.
 (c) If $\mathscr{A}(\mathfrak{P}_1) = \mathscr{A}(\mathfrak{P}_2)$, then $\mathfrak{P}_1 \sim \mathfrak{P}_2$.

$$T(\mathfrak{r},\mathfrak{s}) = \frac{\mathfrak{r}\mathfrak{s}}{\delta + (1-\delta)(\mathfrak{r}+\mathfrak{s}-\mathfrak{r}\mathfrak{s})},$$
$$T^{*}(\mathfrak{r},\mathfrak{s}) = \frac{\mathfrak{r}+\mathfrak{s}-\mathfrak{r}\mathfrak{s}-(1-\delta)\mathfrak{r}\mathfrak{s}}{1-(1-\delta)\mathfrak{r}\mathfrak{s}}. \quad (3)$$

(i) For $\delta = 1$, these operations become algebraic t-norm and t-conorm $T(\mathfrak{r},\mathfrak{s})) = \mathfrak{r}\mathfrak{s}$ and $T^{*}(\mathfrak{r},\mathfrak{s}) = \mathfrak{r} + \mathfrak{s} - \mathfrak{r}\mathfrak{s}$.

(ii) For $\delta = 2$, these operations become Einstein t-norm and t-conorm $T(\mathfrak{r},\mathfrak{s}) = \mathfrak{r}\mathfrak{s}/1 + (1-\mathfrak{r})(1-\mathfrak{s})$ and $T^{*}(\mathfrak{r},\mathfrak{s}) = \mathfrak{r} + \mathfrak{s}/1 + \mathfrak{r}\mathfrak{s}$.

3. Fermatean Fuzzy Hamacher Interactive Average Operators

Definition 4. Let $\mathfrak{P}_1 = \langle \varrho_1, \sigma_1 \rangle$, $\mathfrak{P}_2 = \langle \varrho_2, \sigma_2 \rangle$, and $\mathfrak{P} = \langle \varrho, \sigma \rangle$ be three FFNs and $\beta > 0$. Then, some arithmetic operations between them by using Hamacher norms are as follows:

(i) $\mathfrak{P}_1 \oplus \mathfrak{P}_2 = \left\langle \sqrt[3]{\prod_{\mathfrak{i}=1}^{2}(1+(\delta-1)\varrho_{\mathfrak{i}}^{3}) - \prod_{\mathfrak{i}=1}^{2}(1-\varrho_{\mathfrak{i}}^{3}) / \prod_{\mathfrak{i}=1}^{2}(1+(\delta-1)\varrho_{\mathfrak{i}}^{3}) + (\delta-1)\prod_{\mathfrak{i}=1}^{2}(1-\varrho_{\mathfrak{i}}^{3})}, \sqrt[3]{\delta\left\{\prod_{\mathfrak{i}=1}^{2}(1-\varrho_{\mathfrak{i}}^{3}) - \prod_{\mathfrak{i}=1}^{2}(1-\varrho_{\mathfrak{i}}^{3}-\sigma_{\mathfrak{i}}^{3})\right\} / \prod_{\mathfrak{i}=1}^{2}(1+(\delta-1)\varrho_{\mathfrak{i}}^{3}) + (\delta-1)\prod_{\mathfrak{i}=1}^{2}(1-\varrho_{\mathfrak{i}}^{3})} \right\rangle$,

(ii) $\beta.\mathfrak{P} = \left\langle \sqrt[3]{(1+(\delta-1)\varrho^{3})^{\beta} - (1-\varrho^{3})^{\beta} / (1+(\delta-1)\varrho^{3})^{\beta} + (\delta-1)(1-\varrho^{3})^{\beta}}, \sqrt[3]{\delta\left\{(1-\varrho^{3}) - (1-\varrho^{3}-\sigma^{3})^{\beta}\right\} / (1+(\delta-1)\varrho^{3})^{\beta} + (\delta-1)(1-\varrho^{3})^{\beta}} \right\rangle$.

3.1. Weighted Average Aggregation Operators. Let $\mathfrak{P}_{\mathfrak{i}} = (\varrho_{\mathfrak{i}}, \sigma_{\mathfrak{i}})(\mathfrak{i} = 1, 2, \ldots, \mathfrak{y})$ be a collection of FFNs and $\kappa = (\kappa_1, \kappa_2, \ldots, \kappa_{\mathfrak{y}})^{T}$ be its weight vector (WV) such that $\kappa_{\mathfrak{i}} > 0$ and $\sum_{\mathfrak{i}=1}^{\mathfrak{y}} \kappa_{\mathfrak{i}} = 1$, then FFHIWA: $\Omega^{\mathfrak{y}} \longrightarrow \Omega$ is defined as

$$\mathrm{FFHIWA}\left(\mathfrak{P}_1, \mathfrak{P}_2, \ldots, \mathfrak{P}_{\mathfrak{y}}\right) = \kappa_1\mathfrak{P}_1 \oplus \kappa_2\mathfrak{P}_2 \oplus \cdots \kappa_{\mathfrak{y}}\mathfrak{P}_{\mathfrak{y}}. \quad (4)$$

Theorem 1. *Let $\mathfrak{P}_{\mathfrak{i}} = (\varrho_{\mathfrak{i}}, \sigma_{\mathfrak{i}})$ be a collection of FFNs, then*

$$\mathrm{FFHIWA}\left(\mathfrak{P}_1, \mathfrak{P}_2, \ldots, \mathfrak{P}_{\mathfrak{y}}\right) = \left\langle \sqrt{\prod_{\mathfrak{i}=1}^{\mathfrak{y}}\left(1+(\delta-1)\varrho_{\mathfrak{i}}^{3}\right)^{\kappa_{\mathfrak{i}}} - \prod_{\mathfrak{i}=1}^{\mathfrak{y}}\left(1-\varrho_{\mathfrak{i}}^{3}\right)^{\kappa_{\mathfrak{i}}} / \prod_{\mathfrak{i}=1}^{\mathfrak{y}}\left(1+(\delta-1)\varrho_{\mathfrak{i}}^{3}\right)^{\kappa_{\mathfrak{i}}} + (\delta-1)\prod_{\mathfrak{i}=1}^{\mathfrak{y}}\left(1-\varrho_{\mathfrak{i}}^{3}\right)^{\kappa_{\mathfrak{i}}} 3}, \sqrt{\delta\left\{\prod_{\mathfrak{i}=1}^{\mathfrak{y}}\left(1-\varrho_{\mathfrak{i}}^{3}\right)^{\kappa_{\mathfrak{i}}} - \prod_{\mathfrak{i}=1}^{\mathfrak{y}}\left(1-\varrho_{\mathfrak{i}}^{3}-\sigma_{\mathfrak{i}}^{3}\right)^{\kappa_{\mathfrak{i}}}\right\} / \prod_{\mathfrak{i}=1}^{\mathfrak{y}}\left(1+(\delta-1)\varrho_{\mathfrak{i}}^{3}\right)^{\kappa_{\mathfrak{i}}} + (\delta-1)\prod_{\mathfrak{i}=1}^{\mathfrak{y}}\left(1-\varrho_{\mathfrak{i}}^{3}\right)^{\kappa_{\mathfrak{i}}} 3} \right\rangle. \quad (5)$$

Proof. For $\mathfrak{y} = 1, \kappa = \kappa_1 = 1$,

$$\begin{aligned}\mathrm{FFHIWA}(\mathfrak{P}_1) &= \kappa_1\mathfrak{P}_1 \\ &= \mathfrak{P}_1 \\ &= (\varrho_1, \sigma_1) \\ &= \left\langle \sqrt[3]{\frac{\left(1+(\delta-1)\varrho_1^{3}\right) - \left(1-\varrho_1^{3}\right)}{\left(1+(\delta-1)\varrho_1^{3}\right) + (\delta-1)\left(1-\varrho_1^{3}\right)}}, \sqrt[3]{\frac{\delta\left\{\left(1-\varrho_1^{3}\right) - \left(1-\varrho_1^{3}-\sigma_1^{3}\right)\right\}}{\left(1+(\delta-1)\varrho_1^{3}\right)(\delta-1)\left(1-\varrho_1^{3}\right)}} \right\rangle.\end{aligned} \quad (6)$$

Thus, the result holds for $\mathfrak{y} = 1$. Suppose that result holds for $\mathfrak{y} = \mathfrak{p}$, i.e.,

$$\mathrm{FFHIWA}\left(\mathfrak{P}_1, \mathfrak{P}_2, \ldots, \mathfrak{P}_{\mathfrak{p}}\right) = \left\langle \sqrt[3]{\frac{\prod_{\mathfrak{i}=1}^{\mathfrak{p}}\left(1+(\delta-1)\varrho_{\mathfrak{i}}^{3}\right)^{\kappa_{\mathfrak{i}}} - \prod_{\mathfrak{i}=1}^{\mathfrak{p}}\left(1-\varrho_{\mathfrak{i}}^{3}\right)^{\kappa_{\mathfrak{i}}}}{\prod_{\mathfrak{i}=1}^{\mathfrak{p}} 1+(\delta-1)\varrho_{\mathfrak{i}}^{3} + (\delta-1)\prod_{\mathfrak{i}=1}^{\mathfrak{p}}\left(1-\varrho_{\mathfrak{i}}^{3}\right)^{\kappa_{\mathfrak{i}}}}}, \cdot \sqrt[3]{\frac{\delta\left\{\prod_{\mathfrak{i}=1}^{\mathfrak{p}}\left(1-\varrho_{\mathfrak{i}}^{3}\right)^{\kappa_{\mathfrak{i}}} - \prod_{\mathfrak{i}=1}^{\mathfrak{p}}\left(1-\varrho_{\mathfrak{i}}^{3}-\sigma_{\mathfrak{i}}^{3}\right)^{\kappa_{\mathfrak{i}}}\right\}}{\prod_{\mathfrak{i}=1}^{\mathfrak{p}} 1+(\delta-1)\varrho_{\mathfrak{i}}^{3} + (\delta-1)\prod_{\mathfrak{i}=1}^{\mathfrak{p}}\left(1-\varrho_{\mathfrak{i}}^{3}\right)^{\kappa_{\mathfrak{i}}}}} \right\rangle. \quad (7)$$

Now, for $\mathfrak{y} = \mathfrak{p} + 1$,

$$\begin{aligned}
\mathrm{FFHIWA}\left(\mathfrak{P}_1, \mathfrak{P}_2, \ldots, \mathfrak{P}_{\mathfrak{p}+1}\right) &= \mathop{\oplus}_{\mathfrak{i}=1}^{\mathfrak{p}+1} \kappa_{\mathfrak{i}} \mathfrak{P}_{\mathfrak{i}} \\
&= \left\langle \sqrt[3]{\frac{\prod_{\mathfrak{i}=1}^{\mathfrak{p}}\left(1+(\delta-1)\varrho_{\mathfrak{i}}^3\right)^{\kappa_{\mathfrak{i}}} - \prod_{\mathfrak{i}=1}^{\mathfrak{p}}\left(1-\varrho_{\mathfrak{i}}^3\right)^{\kappa_{\mathfrak{i}}}}{\prod_{\mathfrak{i}=1}^{\mathfrak{p}}\left(1+(\delta-1)\varrho_{\mathfrak{i}}^3\right)^{\kappa_{\mathfrak{i}}} + (\delta-1)\prod_{\mathfrak{i}=1}^{\mathfrak{p}}\left(1-\varrho_{\mathfrak{i}}^3\right)^{\kappa_{\mathfrak{i}}}}}, \sqrt[3]{\frac{\delta\left\{\prod_{\mathfrak{i}=1}^{\mathfrak{p}}\left(1-\varrho_{\mathfrak{i}}^3\right)^{\kappa_{\mathfrak{i}}} - \prod_{\mathfrak{i}=1}^{\mathfrak{p}}\left(1-\varrho_{\mathfrak{i}}^3-\sigma_{\mathfrak{i}}^3\right)^{\kappa_{\mathfrak{i}}}\right\}}{\prod_{\mathfrak{i}=1}^{\mathfrak{p}}\left(1+(\delta-1)\varrho_{\mathfrak{i}}^3\right)^{\kappa_{\mathfrak{i}}} + (\delta-1)\prod_{\mathfrak{i}=1}^{\mathfrak{p}}\left(1-\varrho_{\mathfrak{i}}^3\right)^{\kappa_{\mathfrak{i}}}}} \right\rangle \\
&\oplus \left\langle \sqrt[3]{\frac{\left(1+(\delta-1)\varrho_{\mathfrak{p}+1}^3\right)^{\kappa_{\mathfrak{p}+1}} - \left(1-\varrho_{\mathfrak{p}+1}^3\right)^{\kappa_{\mathfrak{p}+1}}}{\left(1+(\delta-1)\varrho_{\mathfrak{p}+1}^3\right) + (\delta-1)\left(1-\varrho_{\mathfrak{p}+1}^3\right)^{\kappa_{\mathfrak{p}+1}}}}, \sqrt[3]{\frac{\delta\left\{\left(1-\varrho_{\mathfrak{p}+1}^3\right)^{\kappa_{\mathfrak{p}+1}} - \left(1-\varrho_{\mathfrak{p}+1}^3-\sigma_{\mathfrak{p}+1}^3\right)^{\kappa_{\mathfrak{p}+1}}\right\}}{\left(1+(\delta-1)\varrho_{\mathfrak{p}+1}^3\right)^{\kappa_{\mathfrak{p}+1}} + (\delta-1)\left(1-\varrho_{\mathfrak{p}+1}^3\right)^{\kappa_{\mathfrak{p}+1}}}} \right\rangle \\
&= \left\langle \sqrt[3]{\frac{\prod_{\mathfrak{i}=1}^{\mathfrak{p}}\left(1+(\delta-1)\varrho_{\mathfrak{i}}^3\right)^{\kappa_{\mathfrak{i}}} - \prod_{\mathfrak{i}=1}^{\mathfrak{p}}\left(1-\varrho_{\mathfrak{i}}^3\right)^{\kappa_{\mathfrak{i}}}}{\prod_{\mathfrak{i}=1}^{\mathfrak{p}}\left(1+(\delta-1)\varrho_{\mathfrak{i}}^3\right)^{\kappa_{\mathfrak{i}}} + (\delta-1)\prod_{\mathfrak{i}=1}^{\mathfrak{p}}\left(1-\varrho_{\mathfrak{i}}^3\right)^{\kappa_{\mathfrak{i}}}}}, \sqrt[3]{\frac{\delta\left\{\prod_{\mathfrak{i}=1}^{\mathfrak{p}}\left(1-\varrho_{\mathfrak{i}}^3\right)^{\kappa_{\mathfrak{i}}} - \prod_{\mathfrak{i}=1}^{\mathfrak{p}}\left(1-\varrho_{\mathfrak{i}}^3-\sigma_{\mathfrak{i}}^3\right)^{\kappa_{\mathfrak{i}}}\right\}}{\prod_{\mathfrak{i}=1}^{\mathfrak{p}}\left(1+(\delta-1)\varrho_{\mathfrak{i}}^3\right)^{\kappa_{\mathfrak{i}}} + (\delta-1)\prod_{\mathfrak{i}=1}^{\mathfrak{p}}\left(1-\varrho_{\mathfrak{i}}^3\right)^{\kappa_{\mathfrak{i}}}}} \right\rangle.
\end{aligned} \tag{8}$$

$\Rightarrow$ Result holds $\forall\, \mathfrak{y}$. □

Remark 1. We elaborate two cases of the FFHIWA operator:

(i) For $\delta = 1$, FFHIWA operator becomes FF interactive weighted averaging (FFIWA) operator:

$$\mathrm{FFIWA}\left(\mathfrak{P}_1, \mathfrak{P}_2, \ldots, \mathfrak{P}_{\mathfrak{y}}\right) = \left\langle \sqrt[3]{1 - \prod_{\mathfrak{i}=1}^{\mathfrak{y}}\left(1-\varrho_{\mathfrak{i}}^3\right)^{\kappa_{\mathfrak{i}}}}, \sqrt[3]{\prod_{\mathfrak{i}=1}^{\mathfrak{y}}\left(1-\varrho_{\mathfrak{i}}^3\right)^{\kappa_{\mathfrak{i}}} - \prod_{\mathfrak{i}=1}^{\mathfrak{y}}\left(1-\varrho_{\mathfrak{i}}^3-\sigma_{\mathfrak{i}}^3\right)^{\kappa_{\mathfrak{i}}}} \right\rangle. \tag{9}$$

(ii) For $\delta = 2$, FFHIWA operator becomes FF Einstein interactive weighted averaging (FFEIWA) operator:

$$\begin{aligned}
\mathrm{FFEIWA}\left(\mathfrak{P}_1, \mathfrak{P}_2, \ldots, \mathfrak{P}_{\mathfrak{y}}\right) &= \left\langle \sqrt[3]{\frac{\prod_{\mathfrak{i}=1}^{\mathfrak{y}}\left(1+\varrho_{\mathfrak{i}}^3\right)^{\kappa_{\mathfrak{i}}} - \prod_{\mathfrak{i}=1}^{\mathfrak{y}}\left(1+\varrho_{\mathfrak{i}}^3\right)^{\kappa_{\mathfrak{i}}}}{\prod_{\mathfrak{i}=1}^{\mathfrak{y}}\left(1+\varrho_{\mathfrak{i}}^3\right)^{\kappa_{\mathfrak{i}}} + \prod_{\mathfrak{i}=1}^{\mathfrak{y}}\left(1+\varrho_{\mathfrak{i}}^3\right)^{\kappa_{\mathfrak{i}}}}}, \right. \\
&\left. \cdot \sqrt[3]{\frac{2\left\{\prod_{\mathfrak{i}=1}^{\mathfrak{y}}\left(1+\varrho_{\mathfrak{i}}^3\right)^{\kappa_{\mathfrak{i}}} - \prod_{\mathfrak{i}=1}^{\mathfrak{y}}\left(1-\varrho_{\mathfrak{i}}^3-\sigma_{\mathfrak{i}}^3\right)^{\kappa_{\mathfrak{i}}}\right\}}{\prod_{\mathfrak{i}=1}^{\mathfrak{y}}\left(1+\varrho_{\mathfrak{i}}^3\right)^{\kappa_{\mathfrak{i}}} + \prod_{\mathfrak{i}=1}^{\mathfrak{y}}\left(1+\varrho_{\mathfrak{i}}^3\right)^{\kappa_{\mathfrak{i}}}}} \right\rangle.
\end{aligned} \tag{10}$$

Theorem 2. *Let $\mathfrak{P}_{\mathfrak{i}} = (\varrho_{\mathfrak{i}}, \sigma_{\mathfrak{i}})$ be FFNs, then the accumulated value by using FFHIWA operator is a FFN, i.e.,*

$$\mathrm{FFHIWA}\left(\mathfrak{P}_1, \mathfrak{P}_2, \ldots, \mathfrak{P}_{\mathfrak{y}}\right) \in \mathrm{FFN}. \tag{11}$$

Proof. As $\mathfrak{P}_{\mathfrak{i}}'s$ are FFNs, $0 \le \varrho_{\mathfrak{i}}, \sigma_{\mathfrak{i}} \le 1$ and $0 \le \varrho_{\mathfrak{i}}^3 + \sigma_{\mathfrak{i}}^3 \le 1$. Therefore,

$$\begin{aligned}
\frac{\prod_{\mathfrak{i}=1}^{\mathfrak{y}}\left(1+(\delta-1)\varrho_{\mathfrak{i}}^3\right)^{\kappa_{\mathfrak{i}}} - \prod_{\mathfrak{i}=1}^{\mathfrak{y}}\left(1-\varrho_{\mathfrak{i}}^3\right)^{\kappa_{\mathfrak{i}}}}{\prod_{\mathfrak{i}=1}^{\mathfrak{y}}\left(1+(\delta-1)\varrho_{\mathfrak{i}}^3\right)^{\kappa_{\mathfrak{i}}} + (\delta-1)\prod_{\mathfrak{i}=1}^{\mathfrak{y}}\left(1-\varrho_{\mathfrak{i}}^3\right)^{\kappa_{\mathfrak{i}}}} &= 1 - \frac{\delta\prod_{\mathfrak{i}=1}^{\mathfrak{y}}\left(1-\varrho_{\mathfrak{i}}^3\right)^{\kappa_{\mathfrak{i}}}}{\prod_{\mathfrak{i}=1}^{\mathfrak{y}}\left(1+(\delta-1)\varrho_{\mathfrak{i}}^3\right)^{\kappa_{\mathfrak{i}}} + (\delta-1)\prod_{\mathfrak{i}=1}^{\mathfrak{y}}\left(1-\varrho_{\mathfrak{i}}^3\right)^{\kappa_{\mathfrak{i}}}} \\
&\le 1 - \prod_{\mathfrak{i}=1}^{\mathfrak{y}}\left(1-\varrho_{\mathfrak{i}}^3\right)^{\kappa_{\mathfrak{i}}} \le 1.
\end{aligned} \tag{12}$$

Also, $(1+(\delta-1)\varrho_{\mathfrak{i}}^3)\geq (1-\varrho_{\mathfrak{i}}^3)\Rightarrow\prod_{\mathfrak{i}=1}^{\mathfrak{y}}(1+(\delta-1)\varrho_{\mathfrak{i}}^3)-\prod_{\mathfrak{i}=1}^{\mathfrak{y}}(1-\varrho_{\mathfrak{i}}^3)\geq 0$. Therefore,

$$\frac{\prod_{\mathfrak{i}=1}^{\mathfrak{y}}\left(1+(\delta-1)\varrho_{\mathfrak{i}}^3\right)^{\kappa_{\mathfrak{i}}}-\prod_{\mathfrak{i}=1}^{\mathfrak{y}}\left(1-\varrho_{\mathfrak{i}}^3\right)^{\kappa_{\mathfrak{i}}}}{\prod_{\mathfrak{i}=1}^{\mathfrak{y}}\left(1+(\delta-1)\varrho_{\mathfrak{i}}^3\right)^{\kappa_{\mathfrak{i}}}+(\delta-1)\prod_{\mathfrak{i}=1}^{\mathfrak{y}}\left(1-\varrho_{\mathfrak{i}}^3\right)^{\kappa_{\mathfrak{i}}}}\geq 0,$$
$$\Rightarrow\sqrt[3]{\frac{\prod_{\mathfrak{i}=1}^{\mathfrak{y}}\left(1+(\delta-1)\varrho_{\mathfrak{i}}^3\right)^{\kappa_{\mathfrak{i}}}-\prod_{\mathfrak{i}=1}^{\mathfrak{y}}\left(1-\varrho_{\mathfrak{i}}^3\right)^{\kappa_{\mathfrak{i}}}}{\prod_{\mathfrak{i}=1}^{\mathfrak{y}}\left(1+(\delta-1)\varrho_{\mathfrak{i}}^3\right)^{\kappa_{\mathfrak{i}}}+(\delta-1)\prod_{\mathfrak{i}=1}^{\mathfrak{y}}\left(1-\varrho_{\mathfrak{i}}^3\right)^{\kappa_{\mathfrak{i}}}}}\geq 0. \tag{13}$$

Thus,
Moreover,

$$\frac{\delta\left\{\prod_{\mathfrak{i}=1}^{\mathfrak{y}}\left(1-\varrho_{\mathfrak{i}}^3\right)^{\kappa_{\mathfrak{i}}}-\prod_{\mathfrak{i}=1}^{\mathfrak{y}}\left(1-\varrho_{\mathfrak{i}}^3-\sigma_{\mathfrak{i}}^3\right)^{\kappa_{\mathfrak{i}}}\right\}}{\prod_{\mathfrak{i}=1}^{\mathfrak{y}}\left(1+(\delta-1)\varrho_{\mathfrak{i}}^3\right)^{\kappa_{\mathfrak{i}}}+(\delta-1)\prod_{\mathfrak{i}=1}^{\mathfrak{y}}\left(1-\varrho_{\mathfrak{i}}^3\right)^{\kappa_{\mathfrak{i}}}}$$
$$\leq\frac{\delta\left\{\prod_{\mathfrak{i}=1}^{\mathfrak{y}}\left(1-\varrho_{\mathfrak{i}}^3\right)^{\kappa_{\mathfrak{i}}}\right\}}{\prod_{\mathfrak{i}=1}^{\mathfrak{y}}\left(1+(\delta-1)\varrho_{\mathfrak{i}}^3\right)^{\kappa_{\mathfrak{i}}}+(\delta-1)\prod_{\mathfrak{i}=1}^{\mathfrak{y}}\left(1-\varrho_{\mathfrak{i}}^3\right)^{\kappa_{\mathfrak{i}}}} \tag{14}$$
$$\leq\leq\prod_{\mathfrak{i}=1}^{\mathfrak{y}}\left(1-\varrho_{\mathfrak{i}}^3\right)^{\kappa_{\mathfrak{i}}}\leq 1.$$

Also,

$$\prod_{\mathfrak{i}=1}^{\mathfrak{y}}\left(1-\varrho_{\mathfrak{i}}^3\right)^{\kappa_{\mathfrak{i}}}-\prod_{\mathfrak{i}=1}^{\mathfrak{y}}\left(1-\varrho_{\mathfrak{i}}^3-\sigma_{\mathfrak{i}}^3\right)^{\kappa_{\mathfrak{i}}}\geq 0,$$
$$\frac{\delta\left\{\prod_{\mathfrak{i}=1}^{\mathfrak{y}}\left(1-\varrho_{\mathfrak{i}}^3\right)^{\kappa_{\mathfrak{i}}}-\prod_{\mathfrak{i}=1}^{\mathfrak{y}}\left(1-\varrho_{\mathfrak{i}}^3-\sigma_{\mathfrak{i}}^3\right)^{\kappa_{\mathfrak{i}}}\right\}}{\prod_{\mathfrak{i}=1}^{\mathfrak{y}}\left(1+(\delta-1)\varrho_{\mathfrak{i}}^3\right)^{\kappa_{\mathfrak{i}}}+(\delta-1)\prod_{\mathfrak{i}=1}^{\mathfrak{y}}\left(1-\varrho_{\mathfrak{i}}^3\right)^{\kappa_{\mathfrak{i}}}}\geq 0,$$
$$\sqrt[3]{\frac{\delta\left\{\prod_{\mathfrak{i}=1}^{\mathfrak{y}}\left(1-\varrho_{\mathfrak{i}}^3\right)^{\kappa_{\mathfrak{i}}}-\prod_{\mathfrak{i}=1}^{\mathfrak{y}}\left(1-\varrho_{\mathfrak{i}}^3-\sigma_{\mathfrak{i}}^3\right)^{\kappa_{\mathfrak{i}}}\right\}}{\prod_{\mathfrak{i}=1}^{\mathfrak{y}}\left(1+(\delta-1)\varrho_{\mathfrak{i}}^3\right)^{\kappa_{\mathfrak{i}}}+(\delta-1)\prod_{\mathfrak{i}=1}^{\mathfrak{y}}\left(1-\varrho_{\mathfrak{i}}^3\right)^{\kappa_{\mathfrak{i}}}}}\geq 0. \tag{15}$$

Thus, □

Property 1. (idempotency). If $\mathfrak{P}_{\mathfrak{i}}=\mathfrak{P}_o=(\varrho_o,\sigma_o)$, $\forall\,\mathfrak{i}$, then

$$\text{FFHIWA}\left(\mathfrak{P}_1,\mathfrak{P}_2,\ldots,\mathfrak{P}_{\mathfrak{y}}\right)=\mathfrak{P}_o. \tag{16}$$

Proof. Since $\mathfrak{P}_{\mathfrak{i}}=\mathfrak{P}_o=(\varrho_o,\sigma_o)\,(\forall\,\mathfrak{i}=1,2.\ldots,\mathfrak{y})$ and $\sum_{\mathfrak{i}=1}^{\mathfrak{y}}\kappa_{\mathfrak{i}}=1$, by Theorem 1,

$$\begin{aligned}
\text{FFHIWA}\left(\mathfrak{P}_1,\mathfrak{P}_2,\ldots,\mathfrak{P}_{\mathfrak{y}}\right)&=\left\langle\sqrt[3]{\frac{\prod_{\mathfrak{i}=1}^{\mathfrak{y}}\left(1+(\delta-1)\varrho_o^3\right)^{\kappa_{\mathfrak{i}}}-\prod_{\mathfrak{i}=1}^{\mathfrak{y}}\left(1-\varrho_o^3\right)^{\kappa_{\mathfrak{i}}}}{\prod_{\mathfrak{i}=1}^{\mathfrak{y}}\left(1+(\delta-1)\varrho_o^3\right)^{\kappa_{\mathfrak{i}}}+(\delta-1)\prod_{\mathfrak{i}=1}^{\mathfrak{y}}\left(1-\varrho_o^3\right)^{\kappa_{\mathfrak{i}}}}},\sqrt[3]{\frac{\delta\left\{\prod_{\mathfrak{i}=1}^{\mathfrak{y}}\left(1-\varrho_o^3\right)^{\kappa_{\mathfrak{i}}}-\prod_{\mathfrak{i}=1}^{\mathfrak{y}}\left(1-\varrho_o^3-\sigma_o^3\right)^{\kappa_{\mathfrak{i}}}\right\}}{\prod_{\mathfrak{i}=1}^{\mathfrak{y}}\left(1+(\delta-1)\varrho_o^3\right)^{\kappa_{\mathfrak{i}}}+(\delta-1)\prod_{\mathfrak{i}=1}^{\mathfrak{y}}\left(1-\varrho_o^3\right)^{\kappa_{\mathfrak{i}}}}}\right\rangle\\
&=\left\langle\sqrt[3]{\frac{\left(1+(\delta-1)\varrho_o^3\right)^{\sum_{\mathfrak{i}=1}^{\mathfrak{y}}\kappa_{\mathfrak{i}}}-\left(1-\varrho_o^3\right)^{\sum_{\mathfrak{i}=1}^{\mathfrak{y}}\kappa_{\mathfrak{i}}}}{\left(1+(\delta-1)\varrho_o^3\right)^{\sum_{\mathfrak{i}=1}^{\mathfrak{y}}\kappa_{\mathfrak{i}}}+(\delta-1)\left(1-\varrho_o^3\right)^{\sum_{\mathfrak{i}=1}^{\mathfrak{y}}\kappa_{\mathfrak{i}}}}},\sqrt[3]{\frac{\delta\left\{\left(1-\varrho_o^3\right)^{\sum_{\mathfrak{i}=1}^{\mathfrak{y}}\kappa_{\mathfrak{i}}}-\left(1-\varrho_o^3-\sigma_o^3\right)^{\sum_{\mathfrak{i}=1}^{\mathfrak{y}}\kappa_{\mathfrak{i}}}\right\}}{\left(1+(\delta-1)\varrho_o^3\right)^{\sum_{\mathfrak{i}=1}^{\mathfrak{y}}\kappa_{\mathfrak{i}}}+(\delta-1)\left(1-\varrho_o^3\right)^{\sum_{\mathfrak{i}=1}^{\mathfrak{y}}\kappa_{\mathfrak{i}}}}}\right\rangle\\
&=\left\langle\sqrt[3]{\frac{\left(1+(\delta-1)\varrho_o^3\right)-\left(1-\varrho_o^3\right)}{\left(1+(\delta-1)\varrho_o^3\right)+(\delta-1)\left(1-\varrho_o^3\right)}},\sqrt[3]{\frac{\delta\left\{\left(1-\varrho_o^3\right)-\left(1-\varrho_o^3-\sigma_o^3\right)\right\}}{\left(1+(\delta-1)\varrho_o^3\right)+(\delta-1)\left(1-\varrho_o^3\right)}}\right\rangle\\
&=(\varrho_o,\sigma_o).
\end{aligned} \tag{17}$$

□

Property 2. (boundedness). Let $\mathfrak{P}^-=(\min_{\mathfrak{i}}(\varrho_{\mathfrak{i}}),\max_{\mathfrak{i}}(\sigma_{\mathfrak{i}}))$ and $\mathfrak{P}^+=(\min_{\mathfrak{i}}(\varrho_{\mathfrak{i}}),\max_{\mathfrak{i}}(\sigma_{\mathfrak{i}}))$, then

$$\mathfrak{P}^-\leq\text{FFHIWA}\left(\mathfrak{P}_1,\mathfrak{P}_2,\ldots,\mathfrak{P}_{\mathfrak{y}}\right)\leq\mathfrak{P}^+. \tag{18}$$

Proof. Let $f(\mathfrak{r})=1-\mathfrak{r}/1+(\delta-1)\mathfrak{r}$, $\mathfrak{r}\in[0,1]$, then $f'(\mathfrak{r})=-\delta/(1+(\delta-1)\mathfrak{r})^2<0$, so $f(\mathfrak{r})$ is a decreasing function (DF). As $\varrho_{\mathfrak{i},\min}^3\leq\varrho_{\mathfrak{i}}^3\leq\varrho_{\mathfrak{i},\max}^3$, $\forall\,\mathfrak{i}=1,2,\ldots,\mathfrak{y}$, then $f(\varrho_{\mathfrak{i},\max}^3)\leq f(\varrho_{\mathfrak{i}}^3)\leq f(\varrho_{\mathfrak{i},\min}^3)$, $\forall\,\mathfrak{i}$; that is, $1-\varrho_{\mathfrak{i},\max}^3/1+(\delta-1)\varrho_{\mathfrak{i},\max}^3\leq 1-\varrho_{\mathfrak{i}}^3/1+(\delta-1)\varrho_{\mathfrak{i}}^3\leq 1-\varrho_{\mathfrak{i},\min}^3/1+(\delta-1)\varrho_{\mathfrak{i},\min}^3$, $\forall\,\mathfrak{i}$. Let $\kappa_{\mathfrak{i}}\in[0,1]$ and $\sum_{\mathfrak{i}=1}^{\mathfrak{y}}\kappa_{\mathfrak{i}}=1$, we have

$$\left(\frac{1-\varrho_{\mathfrak{i},\max}^3}{1+(\delta-1)\varrho_{\mathfrak{i},\max}^3}\right)^{\kappa_{\mathfrak{i}}}\leq\left(\frac{1-\varrho_{\mathfrak{i}}^3}{1+(\delta-1)\varrho_{\mathfrak{i}}^3}\right)^{\kappa_{\mathfrak{i}}}\leq\left(\frac{1-\varrho_{\mathfrak{i},\min}^3}{1+(\delta-1)\varrho_{\mathfrak{i},\min}^3}\right)^{\kappa_{\mathfrak{i}}}$$
$$\prod_{\mathfrak{i}=1}^{\mathfrak{y}}\left(\frac{1-\varrho_{\mathfrak{i},\max}^3}{1+(\delta-1)\varrho_{\mathfrak{i},\max}^3}\right)^{\kappa_{\mathfrak{i}}}\leq\prod_{\mathfrak{i}=1}^{\mathfrak{y}}\left(\frac{1-\varrho_{\mathfrak{i}}^3}{1+(\delta-1)\varrho_{\mathfrak{i}}^3}\right)^{\kappa_{\mathfrak{i}}}\leq\prod_{\mathfrak{i}=1}^{\mathfrak{y}}\left(\frac{1-\varrho_{\mathfrak{i},\min}^3}{1+(\delta-1)\varrho_{\mathfrak{i},\min}^3}\right)^{\kappa_{\mathfrak{i}}}$$

$$\Leftrightarrow\left(\frac{1-\varrho_{\mathfrak{i},\max}^{3}}{1+(\delta-1)\varrho_{\mathfrak{i},\max}^{3}}\right)^{\sum_{\mathfrak{i}=1}^{\mathfrak{y}}\kappa_{\mathfrak{i}}}\leq\prod_{\mathfrak{i}=1}^{\mathfrak{y}}\left(\frac{1-\varrho_{\mathfrak{i}}^{3}}{1+(\delta-1)\varrho_{\mathfrak{i}}^{3}}\right)^{\kappa_{\mathfrak{i}}}\leq\left(\frac{1-\varrho_{\mathfrak{i},\min}^{3}}{1+(\delta-1)\varrho_{\mathfrak{i},\min}^{3}}\right)^{\sum_{\mathfrak{i}=1}^{\mathfrak{y}}\varpi_{\mathfrak{i}}}$$

$$\Leftrightarrow\left(\frac{1-\varrho_{\mathfrak{i},\max}^{3}}{1+(\delta-1)\varrho_{\mathfrak{i},\max}^{3}}\right)\leq\prod_{\mathfrak{i}=1}^{\mathfrak{y}}\left(\frac{1-\varrho_{\mathfrak{i}}^{3}}{1+(\delta-1)\varrho_{\text{math fraki}}^{3}}\right)^{\kappa_{\mathfrak{i}}}\leq\left(\frac{1-\varrho_{\mathfrak{i},\min}^{3}}{1+(\delta-1)\varrho_{\mathfrak{i},\min}^{3}}\right)$$

$$\Leftrightarrow(\delta-1)\left(\frac{1-\varrho_{\mathfrak{i},\max}^{3}}{1+(\delta-1)\varrho_{\mathfrak{i},\max}^{3}}\right)\leq(\delta-1)\prod_{\mathfrak{i}=1}^{\mathfrak{y}}\left(\frac{1-\varrho_{\mathfrak{i}}^{3}}{1+(\delta-1)\varrho_{\mathfrak{i}}^{3}}\right)^{\varpi_{\mathfrak{i}}}\leq(\delta-1)\left(\frac{1-\varrho_{\mathfrak{i},\min}^{3}}{1+(\delta-1)\varrho_{\mathfrak{i},\min}^{3}}\right)$$

$$\Leftrightarrow\left(\frac{\delta}{1+(\delta-1)\varrho_{\mathfrak{i},\max}^{3}}\right)\leq 1+(\delta-1)\prod_{\mathfrak{i}=1}^{\mathfrak{y}}\left(\frac{1-\varrho_{\mathfrak{i}}^{3}}{1+(\delta-1)\varrho_{\mathfrak{i}}^{3}}\right)^{\kappa_{\mathfrak{i}}}\leq\left(\frac{\delta}{1+(\delta-1)\varrho_{\mathfrak{i},\min}^{3}}\right)$$

$$\Leftrightarrow\left(\frac{1+(\delta-1)\varrho_{\mathfrak{i},\min}^{3}}{\delta}\right)\leq\frac{1}{1+(\delta-1)\prod_{\mathfrak{i}=1}^{\mathfrak{y}}\left(1-\varrho_{\mathfrak{i}}^{3}/1+(\delta-1)\varrho_{\mathfrak{i}}^{3}\right)^{\kappa_{\mathfrak{i}}}}\leq\left(\frac{1+(\delta-1)\varrho_{\mathfrak{i},\max}^{3}}{\delta}\right)$$

$$\Leftrightarrow\left(1+(\delta-1)\varrho_{\mathfrak{i},\min}^{3}\right)\leq\frac{\delta}{1+(\delta-1)\prod_{\mathfrak{i}=1}^{\mathfrak{y}}\left(1-\varrho_{\mathfrak{i}}^{3}/1+(\delta-1)\varrho_{\mathfrak{i}}^{3}\right)^{\kappa_{\mathfrak{i}}}}\leq\left(1+(\delta-1)\varrho_{\mathfrak{i},\max}^{3}\right)$$

$$\Leftrightarrow(\delta-1)\varrho_{\mathfrak{i},\min}^{3}\leq\frac{\delta}{1+(\delta-1)\prod_{\mathfrak{i}=1}^{\mathfrak{y}}\left(1-\varrho_{\mathfrak{i}}^{3}/1+(\delta-1)\varrho_{\mathfrak{i}}^{3}\right)^{\kappa_{\mathfrak{i}}}}-1\leq(\delta-1)\varrho_{\mathfrak{i},\max}^{3}$$

$$\Leftrightarrow\varrho_{\mathfrak{i},\min}^{3}\leq\frac{\prod_{\mathfrak{i}=1}^{\mathfrak{y}}\left(1+(\delta-1)\varrho_{\mathfrak{i}}^{3}\right)^{\kappa_{\mathfrak{i}}}-\prod_{\mathfrak{i}=1}^{\mathfrak{y}}\left(1-\varrho_{\mathfrak{i}}^{3}\right)^{\kappa_{\mathfrak{i}}}}{\prod_{\mathfrak{i}=1}^{\mathfrak{y}}\left(1+(\delta-1)\varrho_{\mathfrak{i}}^{3}\right)^{\varpi_{\mathfrak{i}}}+(\delta-1)\prod_{\mathfrak{i}=1}^{\mathfrak{y}}\left(1-\varrho_{\mathfrak{i}}^{3}\right)^{\varpi_{\mathfrak{i}}}}\leq\varrho_{\mathfrak{i},\max}^{3}. \tag{19}$$

Thus,

$$\varrho_{\mathfrak{i},\min}\leq\sqrt[3]{\frac{\prod_{\mathfrak{i}=1}^{\mathfrak{y}}\left(1+(\delta-1)\varrho_{\mathfrak{i}}^{3}\right)^{\kappa_{\mathfrak{i}}}-\prod_{\mathfrak{i}=1}^{\mathfrak{y}}\prod\left(1-\varrho_{\mathfrak{i}}^{3}\right)^{\kappa_{\mathfrak{i}}}}{\prod_{\mathfrak{i}=1}^{\mathfrak{y}}\left(1+(\delta-1)\varrho_{\mathfrak{i}}^{3}\right)^{\kappa_{\mathfrak{i}}}+(\delta-1)\prod_{\mathfrak{i}=1}^{\mathfrak{y}}\left(1-\varrho_{\mathfrak{i}}^{3}\right)^{\varpi_{\mathfrak{i}}}}}\leq\varrho_{\mathfrak{i},\max}. \tag{20}$$

Consider $g(\mathfrak{s})=\delta-(\delta-1)\mathfrak{s}/(\delta-1)\mathfrak{s}$, $\mathfrak{s}\in(0,1]$, then $g'(\mathfrak{s})=-\delta/(\delta-1)\mathfrak{s}^{2}$; i.e., $g(\mathfrak{s})$ is a DF on $(0,1]$. Since $1-\varrho_{\mathfrak{i},\max}^{3}\leq 1-\varrho_{\mathfrak{i}}^{3}\leq 1-\varrho_{\mathfrak{i},\min}^{3}$, $\forall\,\mathfrak{i}$, then $g(1-\varrho_{i,\max}^{3})\leq g(1-\varrho_{\mathfrak{i}}^{3})\leq g(1-\varrho_{\mathfrak{i},\max}^{3})$, $\forall\,\mathfrak{i}$, that is, $\delta-(\delta-1)(1-\varrho_{\mathfrak{i},\min}^{3})/(\delta-1)(1-\varrho_{\mathfrak{i},\min}^{3})\leq\delta-(\delta-1)(1-\varrho_{\mathfrak{i}}^{3})/(\delta-1)(1-\varrho_{\mathfrak{i}}^{3})\leq\delta-(\delta-1)(1-\varrho_{\mathfrak{i},\max}^{3})/(\delta-1)(1-\varrho_{\mathfrak{i},\max}^{3})$. Then,

$$\left(\frac{\delta-(\delta-1)\left(1-\varrho_{\mathfrak{i},\min}^{3}\right)}{(\delta-1)\left(1-\varrho_{\mathfrak{i},\min}^{3}\right)}\right)^{\kappa_{\mathfrak{i}}}\leq\left(\frac{\delta-(\delta-1)(1-\varrho_{\mathfrak{i}}^{3})}{(\delta-1)\left(1-\varrho_{\mathfrak{i}}^{3}\right)}\right)^{\kappa_{\mathfrak{i}}}\leq\left(\frac{\delta-(\delta-1)\left(1-\varrho_{\mathfrak{i},\max}^{3}\right)}{(\delta-1)\left(1-\varrho_{\mathfrak{i},\max}^{3}\right)}\right)^{\kappa_{\mathfrak{i}}}$$

$$\prod_{\mathfrak{i}=1}^{\mathfrak{y}}\left(\frac{\delta-(\delta-1)\left(1-\varrho_{\mathfrak{i},\min}^{3}\right)}{(\delta-1)\left(1-\varrho_{\mathfrak{i},\min}^{3}\right)}\right)^{\kappa_{\mathfrak{i}}}\leq\prod_{\mathfrak{i}=1}^{\mathfrak{y}}\left(\frac{\delta-(\delta-1)(1-\varrho_{\mathfrak{i}}^{3})}{(\delta-1)\left(1-\varrho_{\mathfrak{i}}^{3}\right)}\right)^{\kappa_{\mathfrak{i}}}\leq\prod_{\mathfrak{i}=1}^{\mathfrak{y}}\left(\frac{\delta-(\delta-1)\left(1-\varrho_{\mathfrak{i},\max}^{3}\right)}{(\delta-1)\left(1-\varrho_{\mathfrak{i},\max}^{3}\right)}\right)^{\kappa_{\mathfrak{i}}}$$

$$\Rightarrow\left(\frac{\delta-(\delta-1)\left(1-\varrho_{\mathfrak{i},\min}^{3}\right)}{(\delta-1)\left(1-\varrho_{\mathfrak{i},\min}^{3}\right)}\right)^{\sum_{\mathfrak{i}=1}^{\mathfrak{y}}\kappa_{\mathfrak{i}}}\leq\prod_{\mathfrak{i}=1}^{\mathfrak{y}}\left(\frac{\delta-(\delta-1)(1-\varrho_{\mathfrak{i}}^{3})}{(\delta-1)\left(1-\varrho_{\mathfrak{i}}^{3}\right)}\right)^{\kappa_{\mathfrak{i}}}\leq\left(\frac{\delta-(\delta-1)\left(1-\varrho_{\mathfrak{i},\max}^{3}\right)}{(\delta-1)\left(1-\varrho_{\mathfrak{i},\max}^{3}\right)}\right)^{\sum_{\mathfrak{i}=1}^{\mathfrak{y}}\kappa_{\mathfrak{i}}}$$

$$\Rightarrow\left(\frac{\delta-(\delta-1)\left(1-\varrho_{\mathfrak{i},\min}^{3}\right)}{(\delta-1)\left(1-\varrho_{\mathfrak{i},\min}^{3}\right)}\right)\leq\prod_{\mathfrak{i}=1}^{\mathfrak{y}}\left(\frac{\delta-(\delta-1)(1-\varrho_{\mathfrak{i}}^{3})}{(\delta-1)\left(1-\varrho_{\mathfrak{i}}^{3}\right)}\right)^{\kappa_{\mathfrak{i}}}\leq\left(\frac{\delta-(\delta-1)\left(1-\varrho_{\mathfrak{i},\max}^{3}\right)}{(\delta-1)\left(1-\varrho_{\mathfrak{i},\max}^{3}\right)}\right)$$

$$\Rightarrow\left(\frac{\delta}{(\delta-1)\left(1-\varrho_{\mathfrak{i},\min}^{3}\right)}\right)\leq\prod_{\mathfrak{i}=1}^{\mathfrak{y}}\left(\frac{\delta-(\delta-1)\left(1-\varrho_{\mathfrak{i}}^{3}\right)}{(\delta-1)\left(1-\varrho_{\mathfrak{i}}^{3}\right)}\right)^{\kappa_{\mathfrak{i}}}+1\leq\left(\frac{\delta}{(\delta-1)\left(1-\varrho_{\mathfrak{i},\max}^{3}\right)}\right)$$

$$\Rightarrow\left(\frac{(\delta-1)\left(1-\varrho_{\mathfrak{i},\max}^{3}\right)}{\delta}\right)\leq\frac{1}{\prod_{\mathfrak{i}=1}^{\mathfrak{y}}\left(\delta-(\delta-1)\left(1-\varrho_{\mathfrak{i}}^{3}\right)/(\delta-1)\left(1-\varrho_{\mathfrak{i}}^{3}\right)\right)^{\kappa_{\mathfrak{i}}}+1}\leq\left(\frac{(\delta-1)\left(1-\varrho_{\mathfrak{i},\min}^{3}\right)}{\delta}\right)$$

$$\Rightarrow\left(1-\varrho_{\mathfrak{i},\max}^{3}\right)\leq\frac{\delta}{(\delta-1)\prod_{\mathfrak{i}=1}^{\mathfrak{y}}\left(\delta-(\delta-1)\left(1-\varrho_{\mathfrak{i}}^{3}\right)/(\delta-1)\left(1-\varrho_{\mathfrak{i}}^{3}\right)\right)^{\kappa_{\mathfrak{i}}}+(\delta-1)}\leq\left(1-\varrho_{\mathfrak{i},\min}^{3}\right)$$

$$\Rightarrow\left(1-\varrho_{\mathfrak{i},\max}^{3}\right)\leq\frac{\delta}{(\delta-1)\prod_{\mathfrak{i}=1}^{\mathfrak{y}}\left(\delta-(\delta-1)\sigma_{\mathfrak{i}}^{3}/(\delta-1)\sigma_{\mathfrak{i}}^{3}\right)^{\kappa_{\mathfrak{i}}}+(\delta-1)}\leq\left(1-\varrho_{\mathfrak{i},\min}^{3}\right). \tag{21}$$

Let FFHIWA $(\mathfrak{P}_1,\mathfrak{P}_2,\ldots,\mathfrak{P}_{\mathfrak{y}})=\mathfrak{P}=\langle\varrho_{\mathfrak{P}},\sigma_{\mathfrak{P}}\rangle$, then from inequalities (20) and (21), $\varrho_{\min}\leq\varrho_{\mathfrak{P}}\leq\varrho_{\max}$, $\sigma_{\max}\leq\sigma_{\mathfrak{P}}\leq\sigma_{\min}$, where $\varrho_{\min}=\min_i\{\varrho_{\mathfrak{i}}\}$, $\varrho_{\max}=\max_{\mathfrak{i}}\{\varrho_{\mathfrak{i}}\}$, $\sigma_{\min}=\min_{\mathfrak{i}}\{\sigma_{\mathfrak{i}}\}$, and $\sigma_{\max}=\max_{\mathfrak{i}}\{\sigma_{\mathfrak{i}}\}$. So $S(\mathfrak{P})=\varrho_{\mathfrak{P}}^3-\sigma_{\mathfrak{P}}^3\leq\varrho_{\max}^3-\sigma_{\max}^3=S(\mathfrak{P}^+)$ and $S(\mathfrak{P})=\varrho_{\mathfrak{P}}^3-\sigma_{\mathfrak{P}}^3\geq\varrho_{\min}^3-\sigma_{\min}^3=S(\mathfrak{P}^-)$. As $S(\mathfrak{P})<S(\mathfrak{P}^+)$ and $S(\mathfrak{P})>S(\mathfrak{P}^-)$,

$$\mathfrak{P}^-\leq\text{FFHIWA}\left(\mathfrak{P}_1,\mathfrak{P}_2,\ldots,\mathfrak{P}_{\mathfrak{y}}\right)\leq\mathfrak{P}^+. \tag{22}$$

□

Property 3. (monotonicity). When $\mathfrak{P}_{\mathfrak{i}}\leq\mathscr{T}_{\mathfrak{i}}$, $\forall\,\mathfrak{i}$, then

$$\text{FFHIWA}\left(\mathfrak{P}_1,\mathfrak{P}_2,\ldots,\mathfrak{P}_{\mathfrak{y}}\right)\leq\text{FFHIWA}\left(\mathscr{T}_1,\mathscr{T}_2,\ldots,\mathscr{T}_{\mathfrak{y}}\right). \tag{23}$$

Proof. It is similar to above. □

Property 4. (shift invariance). If $\mathscr{T}=(\varrho_{\mathscr{T}},\sigma_{\mathscr{T}})$ is another FFN, then

$$\begin{aligned}&\text{FFHIWA}\left(\mathfrak{P}_1\oplus\mathscr{T},\mathfrak{P}_2\oplus\mathscr{T},\ldots,\mathfrak{P}_{\mathfrak{y}}\oplus\mathscr{T}\right)\\&=\text{FFHIWA}\left(\mathfrak{P}_1,\mathfrak{P}_2,\ldots,\mathfrak{P}_{\mathfrak{y}}\right)\oplus\mathscr{T}.\end{aligned} \tag{24}$$

Proof. As $\mathfrak{P}_{\mathfrak{i}},\mathscr{T}\in$ FFNs,

$$\begin{aligned}\mathfrak{P}_{\mathfrak{i}}\oplus\mathscr{T}=\Bigg\langle&\sqrt[3]{\frac{\left(1+(\delta-1)\varrho_{\mathfrak{i}}^{3}\right)\left(1+(\delta-1)\varrho_{\mathscr{T}}^{3}\right)-\left(1-\varrho_{\mathfrak{i}}^{3}\right)\left(1-\varrho_{\mathscr{T}}^{3}\right)}{\left(1+(\delta-1)\varrho_{\mathfrak{i}}^{3}\right)\left(1+(\delta-1)\varrho_{\mathscr{T}}^{3}\right)+(\delta-1)\left(1-\varrho_{\mathfrak{i}}^{3}\right)\left(1-\varrho_{\mathscr{T}}^{3}\right)}},\\&\cdot\sqrt[3]{\frac{\delta\left\{\left(1-\varrho_{\mathfrak{i}}^{3}\right)\left(1-\varrho_{\mathscr{T}}^{3}\right)-\left(1-\varrho_{\mathfrak{i}}^{3}-\sigma_{\mathfrak{i}}^{3}\right)\left(1-\varrho_{\mathscr{T}}^{3}-\sigma_{\mathscr{T}}^{3}\right)\right\}}{\left(1+(\delta-1)\varrho_{\mathfrak{i}}^{3}\right)\left(1+(\delta-1)\varrho_{\mathscr{T}}^{3}\right)+(\delta-1)\left(1-\varrho_{\mathfrak{i}}^{3}\right)\left(1-\varrho_{\mathscr{T}}^{3}\right)}}\Bigg\rangle\end{aligned} \tag{25}$$

Therefore,

$$\begin{aligned}\text{FFHIWA}\left(\mathfrak{P}_1\oplus\mathscr{T},\mathfrak{P}_2\oplus\mathscr{T},\ldots,\mathfrak{P}_{\mathfrak{y}}\oplus\mathscr{T}\right)=\Bigg\langle&\sqrt[3]{\frac{\prod_{\mathfrak{i}=1}^{\mathfrak{y}}\left(\left(1+(\delta-1)\varrho_{\mathfrak{i}}^{3}\right)\left(1+(\delta-1)\varrho_{\mathscr{T}}^{3}\right)\right)^{\kappa_{\mathfrak{i}}}-\prod_{\mathfrak{i}=1}^{\mathfrak{y}}\left(\left(1-\varrho_{\mathfrak{i}}^{3}\right)\left(1-\varrho_{\mathscr{T}}^{3}\right)\right)^{\kappa_{\mathfrak{i}}}}{\prod_{\mathfrak{i}=1}^{\mathfrak{y}}\left(\left(1+(\delta-1)\varrho_{\mathfrak{i}}^{3}\right)\left(1+(\delta-1)\varrho_{\mathscr{T}}^{3}\right)\right)^{\kappa_{\mathfrak{i}}}+(\delta-1)\prod_{\mathfrak{i}=1}^{\mathfrak{y}}\left(\left(1-\varrho_{\mathfrak{i}}^{3}\right)\left(1-\varrho_{\mathscr{T}}^{3}\right)\right)^{\kappa_{\mathfrak{i}}}}},\\&\cdot\sqrt[3]{\frac{\delta\left\{\prod_{\mathfrak{i}=1}^{\mathfrak{y}}\left(\left(1-\varrho_{\mathfrak{i}}^{3}\right)\left(1-\varrho_{\mathscr{T}}^{3}\right)\right)^{\kappa_{\mathfrak{i}}}-\prod_{\mathfrak{i}=1}^{\mathfrak{y}}\left(\left(1-\varrho_{\mathfrak{i}}^{3}-\sigma_{\mathfrak{i}}^{3}\right)\left(1-\varrho_{\mathscr{T}}^{3}-\sigma_{\mathscr{T}}^{3}\right)\right)^{\kappa_{\mathfrak{i}}}\right\}}{\prod_{\mathfrak{i}=1}^{\mathfrak{y}}\left(\left(1+(\delta-1)\varrho_{\mathfrak{i}}^{3}\right)\left(1+(\delta-1)\varrho_{\mathscr{T}}^{3}\right)\right)^{\kappa_{\mathfrak{i}}}+(\delta-1)\prod_{\mathfrak{i}=1}^{\mathfrak{y}}\left(\left(1-\varrho_{\mathfrak{i}}^{3}\right)\left(1-\varrho_{\mathscr{T}}^{3}\right)\right)^{\kappa_{\mathfrak{i}}}}}\Bigg\rangle\\=\Bigg\langle&\sqrt[3]{\frac{\prod_{\mathfrak{i}=1}^{\mathfrak{y}}\left(1+(\delta-1)\varrho_{\mathfrak{i}}^{3}\right)^{\kappa_{\mathfrak{i}}}\left(1+(\delta-1)\varrho_{\mathscr{T}}^{3}\right)^{\kappa_{\mathfrak{i}}}-\prod_{\mathfrak{i}=1}^{\mathfrak{y}}\left(1-\varrho_{\mathfrak{i}}^{3}\right)^{\kappa_{\mathfrak{i}}}\left(1-\varrho_{\mathscr{T}}^{3}\right)^{\kappa_{\mathfrak{i}}}}{\prod_{\mathfrak{i}=1}^{\mathfrak{y}}\left(1+(\delta-1)\varrho_{\mathfrak{i}}^{3}\right)^{\kappa_{\mathfrak{i}}}\left(1+(\delta-1)\varrho_{\mathscr{T}}^{3}\right)^{\kappa_{\mathfrak{i}}}+(\delta-1)\prod_{\mathfrak{i}=1}^{\mathfrak{y}}\left(1-\varrho_{\mathfrak{i}}^{3}\right)^{\kappa_{\mathfrak{i}}}\left(1-\varrho_{\mathscr{T}}^{3}\right)^{\kappa_{\mathfrak{i}}}}},\end{aligned}$$

$$
\begin{aligned}
&\cdot\sqrt[3]{\frac{\delta\left\{\prod_{i=1}^{\mathfrak{y}}\left(1-\varrho_i^3\right)^{\kappa_i}\left(1-\varrho_{\mathscr{T}}^3\right)^{\kappa_i}-\prod_{i=1}^{\mathfrak{y}}\left(1-\varrho_i^3-\sigma_i^3\right)^{\kappa_i}\left(1-\varrho_{\mathscr{T}}^3-\sigma_{\mathscr{T}}^3\right)^{\kappa_i}\right\}}{\prod_{i=1}^{\mathfrak{y}}\left(1+(\delta-1)\varrho_i^3\right)^{\kappa_i}\left(1+(\delta-1)\varrho_{\mathscr{T}}^3\right)^{\kappa_i}+(\delta-1)\prod_{i=1}^{\mathfrak{y}}\left(1-\varrho_i^3\right)^{\kappa_i}\left(1-\varrho_{\mathscr{T}}^3\right)^{\kappa_i}}}\Bigg\rangle \\
&=\Bigg\langle\sqrt[3]{\frac{\left\{\prod_{i=1}^{\mathfrak{y}}\left(1+(\delta-1)\varrho_i^3\right)^{\kappa_i}\right\}\left(1+(\delta-1)\varrho_{\mathscr{T}}^3\right)-\left\{\prod_{i=1}^{\mathfrak{y}}\left(1-\varrho_i^3\right)^{\kappa_i}\right\}\left(1-\varrho_{\mathscr{T}}^3\right)}{\left\{\prod_{i=1}^{\mathfrak{y}}\left(1+(\delta-1)\varrho_i^3\right)^{\kappa_i}\right\}\left(1+(\delta-1)\varrho_{\mathscr{T}}^3\right)+(\delta-1)\left\{\prod_{i=1}^{\mathfrak{y}}\left(1-\varrho_i^3\right)^{\kappa_i}\right\}\left(1-\varrho_{\mathscr{T}}^3\right)}}, \\
&\cdot\sqrt[3]{\frac{\delta\left(\left\{\prod_{i=1}^{\mathfrak{y}}\left(1-\varrho_i^3\right)^{\kappa_i}\right\}\left(1-\varrho_{\mathscr{T}}^3\right)-\left\{\prod_{i=1}^{\mathfrak{y}}\left(1-\varrho_i^3-\sigma_i^3\right)^{\kappa_i}\right\}\left(1-\varrho_{\mathscr{T}}^3-\sigma_{\mathscr{T}}^3\right)\right)}{\left\{\prod_{i=1}^{\mathfrak{y}}\left(1+(\delta-1)\varrho_i^3\right)^{\kappa_i}\right\}\left(1+(\delta-1)\varrho_{\mathscr{T}}^3\right)+(\delta-1)\left\{\prod_{i=1}^{\mathfrak{y}}\left(1-\varrho_i^3\right)^{\kappa_i}\right\}\left(1-\varrho_{\mathscr{T}}^3\right)}}\Bigg\rangle \\
&=\mathrm{FFHIWA}\left(\mathfrak{P}_1,\mathfrak{P}_2,\ldots,\mathfrak{P}_{\mathfrak{y}}\right)\oplus\mathscr{T}.
\end{aligned}
\tag{26}
$$

□

Property 5. (homogeneity). Let $\beta>0$, then

$$
\begin{aligned}
&\mathrm{FFHIWA}\left(\beta\mathfrak{P}_1,\beta\mathfrak{P}_2,\ldots,\beta\mathfrak{P}_{\mathfrak{y}}\right) \\
&\quad=\beta\mathrm{FFHIWA}\left(\mathfrak{P}_1,\mathfrak{P}_2,\ldots,\mathfrak{P}_{\mathfrak{y}}\right).
\end{aligned}
\tag{27}
$$

Proof. Since $\mathfrak{P}_i=(\varrho_i,\sigma_i)$ are FFNs, for $\beta>0$,

$$
\beta\mathfrak{P}_i=\Bigg\langle\sqrt[3]{\frac{\left(1+(\delta-1)\varrho^3\right)^{\beta}-\left(1-\varrho^3\right)^{\beta}}{\left(1+(\delta-1)\varrho^3\right)^{\beta}+(\delta-1)\left(1-\varrho^3\right)^{\beta}}},\sqrt[3]{\frac{\delta\left\{\left(1-\varrho^3\right)^{\beta}-\left(1-\varrho^3-\sigma^3\right)^{\beta}\right\}}{\left(1+(\delta-1)\varrho^3\right)^{\beta}+(\delta-1)\left(1-\varrho^3\right)^{\beta}}}\Bigg\rangle.
\tag{28}
$$

Therefore,

$$
\begin{aligned}
\mathrm{FFHIWA}\left(\beta\mathfrak{P}_1,\beta\mathfrak{P}_2,\ldots,\beta\mathfrak{P}_{\mathfrak{y}}\right)&=\Bigg\langle\sqrt[3]{\frac{\prod_{i=1}^{\mathfrak{y}}\left(\left(1+(\delta-1)\varrho^3\right)^{\beta}\right)^{\kappa_i}-\prod_{i=1}^{\mathfrak{y}}\left(\left(1-\varrho^3\right)^{\beta}\right)^{\kappa_i}}{\prod_{i=1}^{\mathfrak{y}}\left(\left(1+(\delta-1)\varrho^3\right)^{\beta}\right)^{\kappa_i}+(\delta-1)\prod_{i=1}^{\mathfrak{y}}\left(\left(1-\varrho^3\right)^{\beta}\right)^{\kappa_i}}}, \\
&\cdot\sqrt[3]{\frac{\delta\left\{\prod_{i=1}^{\mathfrak{y}}\left(\left(1-\varrho^3\right)^{\beta}\right)^{\kappa_i}-\prod_{i=1}^{\mathfrak{y}}\left(\left(1-\varrho^3-\sigma^3\right)^{\beta}\right)^{\kappa_i}\right\}}{\prod_{i=1}^{\mathfrak{y}}\left(\left(1+(\delta-1)\varrho^3\right)^{\beta}\right)^{\kappa_i}+(\delta-1)\prod_{i=1}^{\mathfrak{y}}\left(\left(1-\varrho^3\right)^{\beta}\right)^{\kappa_i}}}\Bigg\rangle \\
&=\Bigg\langle\sqrt[3]{\frac{\left(\prod_{i=1}^{\mathfrak{y}}\left(1+(\delta-1)\varrho^3\right)^{\kappa_i}\right)^{\beta}-\left(\prod_{i=1}^{\mathfrak{y}}\left(1-\varrho^3\right)^{\kappa_i}\right)^{\beta}}{\left(\prod_{i=1}^{\mathfrak{y}}\left(1+(\delta-1)\varrho^3\right)^{\kappa_i}\right)^{\beta}+(\delta-1)\left(\left(\prod_{i=1}^{\mathfrak{y}}\left(1-\varrho^3\right)^{\kappa_i}\right)\right)^{\beta}}}, \\
&\cdot\sqrt[3]{\frac{\delta\left\{\left(\prod_{i=1}^{\mathfrak{y}}\left(1-\varrho^3\right)^{\kappa_i}\right)^{\beta}-\left(\prod_{i=1}^{\mathfrak{y}}\left(1+(\delta-1)\varrho^3\right)^{\kappa_i}\right)^{\beta}\right\}}{\left(\prod_{i=1}^{\mathfrak{y}}\left(1+(\delta-1)\varrho^3\right)^{\kappa_i}\right)^{\beta}+(\delta-1)\left(\prod_{i=1}^{\mathfrak{y}}\left(1-\varrho^3\right)^{\kappa_i}\right)^{\beta}}}\Bigg\rangle \\
&=\Bigg\langle\beta\sqrt[3]{\frac{\prod_{i=1}^{\mathfrak{y}}\left(1+(\delta-1)\varrho_i^3\right)^{\kappa_i}-\prod_{i=1}^{\mathfrak{y}}\left(1-\varrho_i^3\right)^{\kappa_i}}{\prod_{i=1}^{\mathfrak{y}}\left(1+(\delta-1)\varrho_i^3\right)^{\kappa_i}+(\delta-1)\prod_{i=1}^{\mathfrak{y}}\left(1-\varrho_i^3\right)^{\kappa_i}}}, \\
&\cdot\sqrt[3]{\frac{\delta\left\{\prod_{i=1}^{\mathfrak{y}}\left(1-\varrho_i^3\right)^{\kappa_i}-\prod_{i=1}^{\mathfrak{y}}\left(1-\varrho_i^3-\sigma_i^3\right)^{\kappa_i}\right\}}{\prod_{i=1}^{\mathfrak{y}}\left(1+(\delta-1)\varrho_i^3\right)^{\kappa_i}+(\delta-1)\prod_{i=1}^{\mathfrak{y}}\left(1-\varrho_i^3\right)^{\kappa_i}}}\Bigg\rangle \\
&=\beta\mathrm{FFHIWA}\left(\mathfrak{P}_1,\mathfrak{P}_2,\ldots,\mathfrak{P}_{\mathfrak{y}}\right).
\end{aligned}
\tag{29}
$$

□

Property 6. Let $\mathfrak{P}_{i} = (\varrho_{\mathfrak{P}_i}, \sigma_{\mathfrak{P}_i})$ and $\mathscr{T}_{i} = (\varrho_{\mathscr{T}_i}, \sigma_{\mathscr{T}_i})$ be two collections of FFNs, then

$$\begin{aligned}&\mathrm{FFHIWA}\left(\mathfrak{P}_1 \oplus \mathscr{T}_1, \mathfrak{P}_2 \oplus \mathscr{T}_2, \ldots, \mathfrak{P}_{\mathfrak{y}} \oplus \mathscr{T}_{\mathfrak{y}}\right)\\ &\quad = \mathrm{FFHIWA}\left(\mathfrak{P}_1, \mathfrak{P}_2, \ldots, \mathfrak{P}_{\mathfrak{y}}\right) \oplus \mathrm{FFHIWA}\left(\mathscr{T}_1, \mathscr{T}_2, \ldots, \mathscr{T}_{\mathfrak{y}}\right).\end{aligned} \tag{30}$$

Proof. As $\mathfrak{P}_{i} = (\varrho_{\mathfrak{P}_i}, \sigma_{\mathfrak{P}_i})$ and $\mathscr{T}_{i} = (\varrho_{\mathscr{T}_i}, \sigma_{\mathscr{T}_i})$ are two collections of FFNs, then

$$\begin{aligned}\mathfrak{P}_i \oplus \mathscr{T}_i = \Bigg\langle &\sqrt[3]{\frac{\left(1+(\delta-1)\varrho_{\mathfrak{P}_i}^3\right)\left(1+(\delta-1)\varrho_{\mathscr{T}_i}^3\right) - \left(1-\varrho_{\mathfrak{P}_i}^3\right)\left(1-\varrho_{\mathscr{T}_i}^3\right)}{\left(1+(\delta-1)\varrho_{\mathfrak{P}_i}^3\right)\left(1+(\delta-1)\varrho_{\mathscr{T}_i}^3\right) + (\delta-1)\left(1-\varrho_{\mathfrak{P}_i}^3\right)\left(1-\varrho_{\mathscr{T}_i}^3\right)}},\\ &\cdot\sqrt[3]{\frac{\delta\left\{\left(1-\varrho_{\mathfrak{P}_i}^3\right)\left(1-\varrho_{\mathscr{T}_i}^3\right) - \left(1-\varrho_{\mathfrak{P}_i}^3-\sigma_{\mathfrak{P}_i}^3\right)\left(1-\varrho_{\mathscr{T}_i}^3-\sigma_{\mathscr{T}_i}^3\right)\right\}}{\left(1+(\delta-1)\varrho_{\mathfrak{P}_i}^3\right)\left(1+(\delta-1)\varrho_{\mathscr{T}_i}^3\right) + (\delta-1)\left(1-\varrho_{\mathfrak{P}_i}^3\right)\left(1-\varrho_{\mathscr{T}_i}^3\right)}}\Bigg\rangle\end{aligned} \tag{31}$$

Therefore,

$$\begin{aligned}&\mathrm{FFHIWA}\left(\mathfrak{P}_1 \oplus \mathscr{T}_1, \mathfrak{P}_2 \oplus \mathscr{T}_2, \ldots, \mathfrak{P}_{\mathfrak{y}} \oplus \mathscr{T}_{\mathfrak{y}}\right)\\
&= \Bigg\langle \sqrt[3]{\frac{\prod_{i=1}^{\mathfrak{y}}\left(\left(1+(\delta-1)\varrho_{\mathfrak{P}_i}^3\right)\left(1+(\delta-1)\varrho_{\mathscr{T}_i}^3\right)\right)^{\kappa_i} - \prod_{i=1}^{\mathfrak{y}}\left(\left(1-\varrho_{\mathfrak{P}_i}^3\right)\left(1-\varrho_{\mathscr{T}_i}^3\right)\right)^{\kappa_i}}{\prod_{i=1}^{\mathfrak{y}}\left(\left(1+(\delta-1)\varrho_{\mathfrak{P}_i}^3\right)\left(1+(\delta-1)\varrho_{\mathscr{T}_i}^3\right)\right)^{\kappa_i} + (\delta-1)\prod_{i=1}^{\mathfrak{y}}\left(\left(1-\varrho_{\mathfrak{P}_i}^3\right)\left(1-\varrho_{\mathscr{T}_i}^3\right)\right)^{\kappa_i}}},\\
&\quad\cdot\sqrt[3]{\frac{\delta\left\{\prod_{i=1}^{\mathfrak{y}}\left(\left(1-\varrho_{\mathfrak{P}_i}^3\right)\left(1-\varrho_{\mathscr{T}_i}^3\right)\right)^{\kappa_i} - \prod_{i=1}^{\mathfrak{y}}\left(\left(1-\varrho_{\mathfrak{P}_i}^3-\sigma_{\mathfrak{P}_i}^3\right)\left(1-\varrho_{\mathscr{T}_i}^3-\sigma_{\mathscr{T}_i}^3\right)\right)^{\kappa_i}\right\}}{\prod_{i=1}^{\mathfrak{y}}\left(\left(1+(\delta-1)\varrho_{\mathfrak{P}_i}^3\right)\left(1+(\delta-1)\varrho_{\mathscr{T}_i}^3\right)\right)^{\kappa_i} + (\delta-1)\prod_{i=1}^{\mathfrak{y}}\left(\left(1-\varrho_{\mathfrak{P}_i}^3\right)\left(1-\varrho_{\mathscr{T}_i}^3\right)\right)^{\kappa_i}}}\Bigg\rangle\\
&= \Bigg\langle \sqrt[3]{\frac{\prod_{i=1}^{\mathfrak{y}}\left(1+(\delta-1)\varrho_{\mathfrak{P}_i}^3\right)^{\kappa_i}\prod_{i=1}^{\mathfrak{y}}\left(1+(\delta-1)\varrho_{\mathscr{T}_i}^3\right)^{\kappa_i} - \prod_{i=1}^{\mathfrak{y}}\left(1-\varrho_{\mathfrak{P}_i}^3\right)^{\kappa_i}\prod_{i=1}^{\mathfrak{y}}\left(1-\varrho_{\mathscr{T}_i}^3\right)^{\kappa_i}}{\prod_{i=1}^{\mathfrak{y}}\left(1+(\delta-1)\varrho_{\mathfrak{P}_i}^3\right)^{\kappa_i}\prod_{i=1}^{\mathfrak{y}}\left(1+(\delta-1)\varrho_{\mathscr{T}_i}^3\right)^{\kappa_i} + (\delta-1)\prod_{i=1}^{\mathfrak{y}}\left(1-\varrho_{\mathfrak{P}_i}^3\right)^{\kappa_i}\prod_{i=1}^{\mathfrak{y}}\left(1-\varrho_{\mathscr{T}_i}^3\right)^{\kappa_i}}},\\
&\quad\cdot\sqrt[3]{\frac{\delta\left\{\prod_{i=1}^{\mathfrak{y}}\left(1-\varrho_{\mathfrak{P}_i}^3\right)^{\kappa_i}\prod_{i=1}^{\mathfrak{y}}\left(1-\varrho_{\mathscr{T}_i}^3\right)^{\kappa_i} - \prod_{i=1}^{\mathfrak{y}}\left(1-\varrho_{\mathfrak{P}_i}^3-\sigma_{\mathfrak{P}_i}^3\right)^{\kappa_i}\prod_{i=1}^{\mathfrak{y}}\left(1-\varrho_{\mathscr{T}_i}^3-\sigma_{\mathscr{T}_i}^3\right)^{\kappa_i}\right\}}{\prod_{i=1}^{\mathfrak{y}}\left(1+(\delta-1)\varrho_{\mathfrak{P}_i}^3\right)^{\kappa_i}\prod_{i=1}^{\mathfrak{y}}\left(1+(\delta-1)\varrho_{\mathscr{T}_i}^3\right)^{\kappa_i} + (\delta-1)\prod_{i=1}^{\mathfrak{y}}\left(1-\varrho_{\mathfrak{P}_i}^3\right)^{\kappa_i}\prod_{i=1}^{\mathfrak{y}}\left(1-\varrho_{\mathscr{T}_i}^3\right)^{\kappa_i}}}\Bigg\rangle\\
&= \Bigg\langle \sqrt[3]{\frac{\prod_{i=1}^{\mathfrak{y}}\left(1+(\delta-1)\varrho_{\mathfrak{P}_i}^3\right)^{\kappa_i} - \prod_{i=1}^{\mathfrak{y}}\left(1-\varrho_{\mathfrak{P}_i}^3\right)^{\kappa_i}}{\prod_{i=1}^{\mathfrak{y}}\left(1+(\delta-1)\varrho_{\mathfrak{P}_i}^3\right)^{\kappa_i} + (\delta-1)\prod_{i=1}^{\mathfrak{y}}\left(1-\varrho_{\mathfrak{P}_i}^3\right)^{\kappa_i}}}, \sqrt[3]{\frac{\delta\left\{\prod_{i=1}^{\mathfrak{y}}\left(1-\varrho_{\mathfrak{P}_i}^3\right)^{\kappa_i} - \prod_{i=1}^{\mathfrak{y}}\left(1-\varrho_{\mathfrak{P}_i}^3-\sigma_{\mathfrak{P}_i}^3\right)^{\kappa_i}\right\}}{\prod_{i=1}^{\mathfrak{y}}\left(1+(\delta-1)\varrho_{\mathfrak{P}_i}^3\right)^{\kappa_i} + (\delta-1)\prod_{i=1}^{\mathfrak{y}}\left(1-\varrho_{\mathfrak{P}_i}^3\right)^{\kappa_i}}}\Bigg\rangle\\
&\quad\oplus\Bigg\langle \sqrt[3]{\frac{\prod_{i=1}^{\mathfrak{y}}\left(1+(\delta-1)\varrho_{\mathscr{T}_i}^3\right)^{\kappa_i} - \prod_{i=1}^{\mathfrak{y}}\left(1-\varrho_{\mathscr{T}_i}^3\right)^{\kappa_i}}{\prod_{i=1}^{\mathfrak{y}}\left(1+(\delta-1)\varrho_{\mathscr{T}_i}^3\right)^{\kappa_i} + (\delta-1)\prod_{i=1}^{\mathfrak{y}}\left(1-\varrho_{\mathscr{T}_i}^3\right)^{\kappa_i}}}, \sqrt[3]{\frac{\delta\left\{\prod_{i=1}^{\mathfrak{y}}\left(1-\varrho_{\mathscr{T}_i}^3\right)^{\kappa_i} - \prod_{i=1}^{\mathfrak{y}}\left(1-\varrho_{\mathscr{T}_i}^3-\sigma_{\mathscr{T}_i}^3\right)^{\kappa_i}\right\}}{\prod_{i=1}^{\mathfrak{y}}\left(1+(\delta-1)\varrho_{\mathscr{T}_i}^3\right)^{\kappa_i} + (\delta-1)\prod_{i=1}^{\mathfrak{y}}\left(1-\varrho_{\mathscr{T}_i}^3\right)^{\kappa_i}}}\Bigg\rangle\\
&= \mathrm{FFHIWA}\left(\mathfrak{P}_1, \mathfrak{P}_2, \ldots, \mathfrak{P}_{\mathfrak{y}}\right) \oplus \mathrm{FFHIWA}\left(\mathscr{T}_1, \mathscr{T}_2, \ldots,\end{aligned} \tag{32}$$

□

Property 7. Let $\mathfrak{P}_{i} = (\varrho_{i}, \sigma_{i})$ and $\mathscr{T} = (\varrho, \sigma)$ be FFNs and $\eta > 0$, then

$$\mathrm{FFHIWA}\left(\eta\mathfrak{P}_1 \oplus \mathscr{T}, \eta\mathfrak{P}_2 \oplus \mathscr{T}, \ldots, \eta\mathfrak{P}_{\mathfrak{y}} \oplus \mathscr{T}\right) = \eta\mathrm{FFHIWA}\left(\mathfrak{P}_1, \mathfrak{P}_2, \ldots, \mathfrak{P}_{\mathfrak{y}}\right) \oplus \mathscr{T}. \tag{33}$$

Proof. By applying the Properties 1, 5, and 6, we can proof it. □

4. Ordered Weighted Averaging Operator

Definition 5. Let $\mathfrak{P}_{\mathfrak{i}} = (\varrho_{\mathfrak{i}}, \sigma_{\mathfrak{i}})$ be a collection of FFNs and $\kappa = (\kappa_1, \kappa_2, \ldots, \kappa_{\mathfrak{y}})^T$ be its WV such that $\kappa_{\mathfrak{i}} > 0$ and $\sum_{\mathfrak{i}=1}^{\mathfrak{y}} \kappa_{\mathfrak{i}} = 1$, then FFHIOWA: $\Omega^{\mathfrak{y}} \longrightarrow \Omega$ is defined as

$$\text{FFHIWA}\left(\mathfrak{P}_1, \mathfrak{P}_2, \ldots, \mathfrak{P}_{\mathfrak{y}}\right) = \kappa_1 \mathfrak{P}_{\sigma(1)} \oplus \kappa_2 \mathfrak{P}_{\sigma(2)} \oplus \cdots \kappa_{\mathfrak{y}} \mathfrak{P}_{\sigma(\mathfrak{y})}, \tag{34}$$

where $(\sigma(1), \sigma(2), \ldots, \sigma(\mathfrak{y}))$ is a permutation of $(1, 2, \ldots, \mathfrak{y})$ such that $\sigma(\mathfrak{i} - 1) \geq \sigma(\mathfrak{i})$ for any $\mathfrak{i}$.

Theorem 3. *Let $\mathfrak{P}_{\mathfrak{i}} = (\varrho_{\mathfrak{i}}, \sigma_{\mathfrak{i}})$ be a collection of FFNs, then*

$$\begin{aligned}\text{FFHIOWA}\left(\left(\mathfrak{P}_1, \mathfrak{P}_2, \ldots, \mathfrak{P}_{\mathfrak{y}}\right) = \Bigg\langle \sqrt[3]{\frac{\prod_{\mathfrak{i}=1}^{\mathfrak{y}}\left(1 + (\delta - 1)\varrho_{\sigma(\mathfrak{i})}^3\right)^{\kappa_{\mathfrak{i}}} - \prod_{\mathfrak{i}=1}^{\mathfrak{y}}\left(1 - \varrho_{\sigma(\mathfrak{i})}^3\right)^{\kappa_{\mathfrak{i}}}}{\prod_{\mathfrak{i}=1}^{\mathfrak{y}}\left(1 + (\delta - 1)\varrho_{\sigma(\mathfrak{i})}^3\right)^{\kappa_{\mathfrak{i}}} + (\delta - 1)\prod_{\mathfrak{i}=1}^{\mathfrak{y}}\left(1 - \varrho_{\sigma(\mathfrak{i})}^3\right)^{\kappa_{\mathfrak{i}}}}},\\ \cdot \sqrt[3]{\frac{\delta\left\{\prod_{\mathfrak{i}=1}^{\mathfrak{y}}\left(1 - \varrho_{\sigma(\mathfrak{i})}^3\right)^{\kappa_{\mathfrak{i}}} - \prod_{\mathfrak{i}=1}^{\mathfrak{y}}\left(1 - \varrho_{\sigma(\mathfrak{i})}^3 - \sigma_{\sigma(\mathfrak{i})}^3\right)^{\kappa_{\mathfrak{i}}} \prod_{\mathfrak{i}=1}^{\mathfrak{y}}\right\}}{\prod_{\mathfrak{i}=1}^{\mathfrak{y}}\left(1 + (\delta - 1)\varrho_{\sigma(\mathfrak{i})}^3\right)^{\kappa_{\mathfrak{i}}} + (\delta - 1)\prod_{\mathfrak{i}=1}^{\mathfrak{y}}\left(1 - \varrho_{\sigma(\mathfrak{i})}^3\right)^{\kappa_{\mathfrak{i}}}}}\Bigg\rangle\end{aligned} \tag{35}$$

Proof. It is similar to Theorem 1. □

Remark 2. We elaborate two cases of the FFHIOWA operator.

(i) For $\delta = 1$, FFHIOWA operator becomes FF interactive ordered weighted averaging (FFIOWA) operator:

$$\text{FFIOWA}\left(\mathfrak{P}_1, \mathfrak{P}_2, \ldots, \mathfrak{P}_{\mathfrak{y}}\right) = \left\langle \sqrt[3]{1 - \prod_{\mathfrak{i}=1}^{\mathfrak{y}}\left(1 - \varrho_{\sigma(\mathfrak{i})}^3\right)^{\kappa_{\mathfrak{i}}}}, \sqrt[3]{\prod_{\mathfrak{i}=1}^{\mathfrak{y}}\left(1 - \varrho_{\sigma(\mathfrak{i})}^3\right)^{\kappa_{\mathfrak{i}}} - \prod_{\mathfrak{i}=1}^{\mathfrak{y}}\left(1 - \varrho_{\sigma(\mathfrak{i})}^3 - \sigma_{\sigma(\mathfrak{i})}^3\right)^{\kappa_{\mathfrak{i}}}}\right\rangle. \tag{36}$$

(ii) For $\delta = 2$, FFHIOWA operator becomes FF Einstein interactive ordered weighted averaging (FFEIOWA) operator:

$$\begin{aligned}\text{FFEIOWA}\left(\mathfrak{P}_1, \mathfrak{P}_2, \ldots, \mathfrak{P}_{\mathfrak{y}}\right) = \Bigg\langle \sqrt[3]{\frac{\prod_{\mathfrak{i}=1}^{\mathfrak{y}}\left(1 + \varrho_{\sigma(\mathfrak{i})}^3\right)^{\kappa_{\mathfrak{i}}} - \prod_{\mathfrak{i}=1}^{\mathfrak{y}}\left(1 - \varrho_{\sigma(\mathfrak{i})}^3\right)^{\kappa_{\mathfrak{i}}}}{\prod_{\mathfrak{i}=1}^{\mathfrak{y}}\left(1 + \varrho_{\sigma(\mathfrak{i})}^3\right)^{\kappa_{\mathfrak{i}}} + \prod_{\mathfrak{i}=1}^{\mathfrak{y}}\left(1 - \varrho_{\sigma(\mathfrak{i})}^3\right)^{\kappa_{\mathfrak{i}}}}},\\ \cdot \sqrt[3]{\frac{2\left\{\prod_{\mathfrak{i}=1}^{\mathfrak{y}}\left(1 - \varrho_{\sigma(\mathfrak{i})}^3\right)^{\kappa_{\mathfrak{i}}} - \prod_{\mathfrak{i}=1}^{\mathfrak{y}}\left(1 - \varrho_{\sigma(\mathfrak{i})}^3 - \sigma_{\sigma(\mathfrak{i})}^3\right)^{\kappa_{\mathfrak{i}}}\right\}}{\prod_{\mathfrak{i}=1}^{\mathfrak{y}}\left(1 + \varrho_{\sigma(\mathfrak{i})}^3\right)^{\kappa_{\mathfrak{i}}} + \prod_{\mathfrak{i}=1}^{\mathfrak{y}}\left(1 - \varrho_{\sigma(\mathfrak{i})}^3\right)^{\kappa_{\mathfrak{i}}}}}.\Bigg\rangle\end{aligned} \tag{37}$$

Property 8. Let $\mathfrak{P}_{\mathfrak{i}} = (\varrho_{\mathfrak{i}}, \sigma_{\mathfrak{i}})$ be a collection of FFNs and $\kappa = (\kappa_1, \kappa_2, \ldots, \kappa_{\mathfrak{y}})^T$ be its WV such that $\kappa_{\mathfrak{i}} > 0$ and $\sum_{\mathfrak{i}=1}^{\mathfrak{y}} \kappa_{\mathfrak{i}} = 1$.

(i) Idempotency: if $\mathfrak{P}_{\mathfrak{i}} = \mathfrak{P}_o = (\varrho_o, \sigma_o)$, $\forall\, \mathfrak{i}$, then

$$\text{FFHIOWA}\left(\mathfrak{P}_1, \mathfrak{P}_2, \ldots, \mathfrak{P}_{\mathfrak{y}}\right) = \mathfrak{P}_o. \tag{38}$$

(ii) Boundedness: let $\mathfrak{P}^- = (\min_{\mathfrak{i}}(\varrho_{\mathfrak{i}}), \max_{\mathfrak{i}}(\sigma_{\mathfrak{i}}))$ and $\mathfrak{P}^+ = (\max_{\mathfrak{i}}(\varrho_{\mathfrak{i}}), \min_{\mathfrak{i}}(\sigma_{\mathfrak{i}}))$, then

$$\mathfrak{P}^- \leq \text{FFHIOWA}\left(\mathfrak{P}_1, \mathfrak{P}_2, \ldots, \mathfrak{P}_{\mathfrak{y}}\right) \leq \mathfrak{P}^+. \tag{39}$$

(iii) Monotonicity: when $\mathfrak{P}_{\mathfrak{i}} \leq \mathcal{T}_{\mathfrak{i}}, \forall\, \mathfrak{i}$, then

$$\begin{aligned}&\text{FFHIOWA}\left(\mathfrak{P}_1, \mathfrak{P}_2, \ldots, \mathfrak{P}_{\mathfrak{y}}\right)\\ &\quad \leq \text{FFHIOWA}\left(\mathcal{T}_1, \mathcal{T}_2, \ldots, \mathcal{T}_{\mathfrak{y}}\right).\end{aligned} \tag{40}$$

(iv) Shift invariance: if $\mathcal{T} = (\varrho_{\mathcal{T}}, \sigma_{\mathcal{T}})$ is another FFN, then

$$\begin{aligned}&\text{FFHIOWA}\left(\mathfrak{P}_1 \oplus \mathcal{T}, \mathfrak{P}_2 \oplus \mathcal{T}, \ldots, \mathfrak{P}_{\mathfrak{y}} \oplus \mathcal{T}\right)\\ &\quad = \text{FFHIOWA}\left(\mathfrak{P}_1, \mathfrak{P}_2, \ldots, \mathfrak{P}_{\mathfrak{y}}\right) \oplus \mathcal{T}.\end{aligned} \tag{41}$$

(v) Homogeneity: let $\beta > 0$, then

$$\begin{aligned}&\mathrm{FFHIOWA}\left(\beta\mathfrak{P}_1,\beta\mathfrak{P}_2,\ldots,\beta\mathfrak{P}_{\mathfrak{y}}\right)\\&\quad=\beta\mathrm{FFHIOWA}\left(\mathfrak{P}_1,\mathfrak{P}_2,\ldots,\mathfrak{P}_{\mathfrak{y}}\right).\end{aligned}\tag{42}$$

Proof. It is similar to the FFHIWA properties. □

5. Hybrid Weighted Averaging Operator

Definition 6. Let $\mathfrak{P}_{\mathfrak{i}} = (\varrho_{\mathfrak{i}}, \sigma_{\mathfrak{i}})$ be a collection of FFNs, then FFHIHWA: $\Omega^{\mathfrak{y}} \longrightarrow \Omega$ is defined as

$$\mathrm{FFHIHWA}\left(\mathfrak{P}_1,\mathfrak{P}_2,\ldots,\mathfrak{P}_{\mathfrak{y}}\right)=\kappa_1\dot{\mathfrak{P}}_1\oplus\kappa_2\dot{\mathfrak{P}}_2\oplus\cdots\kappa_{\mathfrak{y}}\dot{\mathfrak{P}}_{\mathfrak{y}},\tag{43}$$

where $\kappa = (\kappa_1, \kappa_2, \ldots, \kappa_{\mathfrak{y}})^T$ is the WV associated with FFHIHWA operator and $\phi = (\phi_1, \phi_2, \ldots, \phi_{\mathfrak{y}})^T$ is the WV of $\mathfrak{P}_{\mathfrak{i}}$ such that $\phi_{\mathfrak{i}} \in [0, 1]$ and $\sum_{\mathfrak{i}=1}^{\mathfrak{y}} \phi_{\mathfrak{i}} = 1$. Let $\dot{\mathfrak{P}}$ is the $\mathfrak{i}th$ largest of the weighted FFNs $((\dot{\mathfrak{P}} = \mathfrak{y}\phi_{\mathfrak{i}}\mathfrak{P}_{\mathfrak{i}})$ and $(\sigma(1), \sigma(2), \ldots, \sigma(\mathfrak{y}))$ is a permutation of $(1, 2, \ldots, \mathfrak{y})$ such that $\sigma(\mathfrak{i}-1) \geq \sigma(\mathfrak{i})$ for any $\mathfrak{i}$.

Theorem 4. *Let $\mathfrak{P}_{\mathfrak{i}} = (\varrho_{\mathfrak{i}}, \sigma_{\mathfrak{i}})$ be a collection of FFNs, then*

$$\begin{aligned}\mathrm{FFHIHWA}\left(\mathfrak{P}_1,\mathfrak{P}_2,\ldots,\mathfrak{P}_{\mathfrak{y}}\right)=\Bigg\langle&\sqrt[3]{\frac{\prod_{\mathfrak{i}=1}^{\mathfrak{y}}\left(1+(\delta-1)\dot{\varrho}_{\sigma(\mathfrak{i})}^3\right)^{\kappa_{\mathfrak{i}}}-\prod_{\mathfrak{i}=1}^{\mathfrak{y}}\left(1-\dot{\varrho}_{\sigma(\mathfrak{i})}^3\right)^{\kappa_{\mathfrak{i}}}}{\prod_{\mathfrak{i}=1}^{\mathfrak{y}}\left(1+(\delta-1)\dot{\varrho}_{\sigma(\mathfrak{i})}^3\right)^{\kappa_{\mathfrak{i}}}+(\delta-1)\prod_{\mathfrak{i}=1}^{\mathfrak{y}}\left(1-\dot{\varrho}_{\sigma(\mathfrak{i})}^3\right)^{\kappa_{\mathfrak{i}}}}},\\&\cdot\sqrt[3]{\delta\left\{\frac{\prod_{\mathfrak{i}=1}^{\mathfrak{y}}\left(1-\dot{\varrho}_{\sigma(\mathfrak{i})}^3\right)^{\kappa_{\mathfrak{i}}}-\prod_{\mathfrak{i}=1}^{\mathfrak{y}}\left(1-\dot{\varrho}_{\sigma(\mathfrak{i})}^3-\dot{\sigma}_{\sigma(\mathfrak{i})}^3\right)^{\kappa_{\mathfrak{i}}}}{\prod_{\mathfrak{i}=1}^{\mathfrak{y}}\left(1+(\delta-1)\dot{\varrho}_{\sigma(\mathfrak{i})}^3\right)^{\kappa_{\mathfrak{i}}}+(\delta-1)\prod_{\mathfrak{i}=1}^{\mathfrak{y}}\left(1-\dot{\varrho}_{\sigma(\mathfrak{i})}^3\right)^{\kappa_{\mathfrak{i}}}}\right\}}\Bigg\rangle.\end{aligned}\tag{44}$$

Proof. It is similar to Theorem 1. □

Remark 3. FFHIHWA operator also satisfies the same properties as given in Property 8.

6. MAGDM under Fermatean Fuzzy Environment

In MAGDM problem, it is a biggest challenge for decision makers (DMrs) to choose the best alternative among the list of possible alternatives. Let $\{\mathfrak{S}_1, \mathfrak{S}_2, \ldots, \mathfrak{S}_{\mathfrak{y}}\}$ be $\mathfrak{y}$ distinct alternatives which can be classified under the set of m different attributes $\{c_1, c_2, \ldots, c_m\}$ by the DMrs. Suppose that DMrs give their preferences in terms of FFNs $\alpha_{\mathfrak{i}j} = (\varrho_{\mathfrak{i}j}, \sigma_{\mathfrak{i}j})(\mathfrak{i} = 1, 2, \ldots, \mathfrak{y}; j = 1, 2, \ldots, m)$, where $\varrho_{\mathfrak{i}j}$ and $\sigma_{\mathfrak{i}j}$ are the satisfaction and dissatisfaction degrees, respectively, of the alternative corresponding to given parameter given by the DMrs such that $0 \leq {\varrho_{\mathfrak{i}j}}^3 + {\sigma_{\mathfrak{i}j}}^3 \leq 1$. The different steps for MAGDM problem are given as follows:

Step 1. Attain the normalize FF decision matrix by exchanging the assessment value of cost parameter (CP) into benefit parameter (BP) [40], i.e.,

$$\mathscr{P}_{\mathfrak{i}j}=\begin{cases}\alpha_{\mathfrak{i}j}^c; & \text{for CP},\\ \alpha_{\mathfrak{i}j}; & \text{for BP}.\end{cases}\tag{45}$$

Step 2. By using the decision matrix of step 1, the overall aggregated value of alternative $\mathfrak{S}_{\mathfrak{i}}$ under the distinct choices of attributes c_j is obtained by using FFHIWA or FFHIOWA or FFHIHWA operator and get the overall value of them.

Step 3. By using the score function, calculate the score values of all alternatives.

Step 4. Rank the alternatives $\{\mathfrak{S}_1, \mathfrak{S}_2, \ldots, \mathfrak{S}_{\mathfrak{y}}\}$ in the descending order of score values and then select the most suitable alternative.

6.1. Numerical Example. To classify the air quality (AQ) of Guangzhou for the 16th Asian Olympic Games [41] held during November 12-27, 2010, the AQ data in Guangzhou for November 2006, November 2007, November 2008, and November 2009 are collected to find out the trends in the AQ. Suppose that there are three AQ monitoring stations E_1, E_2, and E_3, which are considered as DMrs and suppose that the 0.314, 0.355, and 0.331 are weights of E_1, E_2, and E_3, respectively. There are three measured indexes, namely, SO_2 (c_1), NO_2 (c_2), and PM_10 (c_3), and their weight is $\kappa = (0.40, 0.20, 0.40)^T$. Let $\mathfrak{S}_1$, $\mathfrak{S}_2$, $\mathfrak{S}_3$, and $\mathfrak{S}_4$ be alternatives, where $\mathfrak{S}_1$ = November 2006, $\mathfrak{S}_2$ = November 2007, $\mathfrak{S}_3$ = November 2008, and $\mathfrak{S}_4$ = November 2009. Suppose that the measured values obtained from the AQ monitoring stations E_1, E_2, and E_3 under the measured indexes SO_2 (i.e., the attribute c_1), NO_2 (i.e., the attribute c_2), and PM_10 (i.e., the attribute c_3) in the form of FFNs are shown in Tables 1–3, respectively. We rank the AQ from 2006–2009 by using the proposed method.

By FFHIWA operator, the steps are as follows:

Step 1: as all criteria are of same type, decision matrix cannot be normalized. The aggregated decision matrix by using FF weighted averaging operator with WV $\lambda = (0.314, 0.355, 0.331)^T$ is shown in Table 4.

Step 2: to find the overall assessment of each alternative, we apply the FFHIWA operator for $\delta = 1$ as follows.

For $\mathscr{P}_1$,

Table 1: Air quality data by expert E_1.

E_1	c_1	c_2	c_3
$\mathfrak{S}_1$	(0.5, 0.7)	(0.9, 0.6)	(0.8, 0.7)
$\mathfrak{S}_2$	(0.3, 0.7)	(0.1, 0.9)	(0.66, 0.8)
$\mathfrak{S}_3$	(0.8, 0.3)	(0.7, 0.2)	(0.8, 0.4)
$\mathfrak{S}_4$	(0.9, 0.1)	(0.8, 0.1)	(0.7, 0.3)

Table 2: Air quality data by expert E_2.

E_1	c_1	c_2	c_3
$\mathfrak{S}_1$	(0.6, 0.88)	(0.3, 0.57)	(0.46, 0.76)
$\mathfrak{S}_2$	(0.3, 0.7)	(0.1, 0.99)	(0.4, 0.77)
$\mathfrak{S}_3$	(0.96, 0.3)	(0.7, 0.22)	(0.6, 0.01)
$\mathfrak{S}_4$	(0.88, 0.2)	(0.86, 0.01)	(0.96, 0.33)

Table 3: Air quality data by expert E_3.

E_3	c_1	c_2	c_3
$\mathfrak{S}_1$	(0.1, 0.93)	(0.7, 0.3)	(0.5, 0.3)
$\mathfrak{S}_2$	(0.9, 0.4)	(0.3, 0.56)	(0.47, 0.66)
$\mathfrak{S}_3$	(0.93, 0.3)	(0.76, 0.2)	(0.76, 0.1)
$\mathfrak{S}_4$	(0.97, 0.4)	(0.88, 0.3)	(0.89, 0.5)

Table 4: Aggregated FF decision matrix.

	c_1	c_2	c_3
$\mathfrak{S}_1$	(0.4031, 0.8400)	(0.6208, 0.4901)	(0.5800, 0.5889)
$\mathfrak{S}_2$	(0.4986, 0.6007)	(0.1662, 0.7252)	(0.5048, 0.7430)
$\mathfrak{S}_3$	(0.8998, 0.3000)	(0.7199, 0.2071)	(0.7158, 0.1623)
$\mathfrak{S}_4$	(0.9161, 0.2348)	(0.8478, 0.1343)	(0.8552, 0.3769)

$$\begin{aligned}
&= \text{FFHIWA}\left(\mathfrak{S}_{11}, \mathfrak{S}_{12}, \mathfrak{S}_{13}\right)\\
&= \left\langle \sqrt[3]{1-\left(1-0.4031^3\right)^{0.40}\left(1-0.6208^3\right)^{0.20}\left(1-0.5800^3\right)^{0.40}},\right.\\
&\quad\cdot \sqrt[3]{\left(1-0.4031^3\right)^{0.40}\left(1-0.6208^3\right)^{0.20}\left(1-0.5800^3\right)^{0.40}-\left(1-0.4031^3-0.8400^3\right)^{0.40}\left(1-0.6208^3-0.4901^3\right)^{0.20}\left(1-0.5800^3-0.5889^3\right)^{0.40}}\\
&= (0.5374, 0.7106).
\end{aligned} \tag{46}$$

For $\mathscr{P}_2$,

$$\begin{aligned}
&= \text{FFHIWA}\left(\mathfrak{S}_{11}, \mathfrak{S}_{12}, \mathfrak{S}_{13}\right)\\
&= \left\langle \sqrt[3]{1-\left(1-0.4986^3\right)^{0.40}\left(1-0.1662^3\right)^{0.20}\left(1-0.5048^3\right)^{0.40}},\right.\\
&\quad\left.\cdot \sqrt[3]{\left(1-0.0.4986^3\right)^{0.40}\left(1-0.1662^3\right)^{0.20}\left(1-0.5048^3\right)^{0.40}-\left(1-0.4986^3-0.6007^3\right)^{0.40}\left(1-0.1662^3-0.7252^3\right)^{0.20}\left(1-0.5048^3-0.7430^3\right)^{0.40}}\right\rangle\\
&= (0.4691, 0.6934).
\end{aligned} \tag{47}$$

For $\mathscr{P}_3$,

$$
\begin{aligned}
&= \text{FFHIWA}\left(\mathfrak{S}_{11}, \mathfrak{S}_{12}, \mathfrak{S}_{13}\right) \\
&= \left\langle \sqrt[3]{1-\left(1-0.8998^3\right)^{0.40}\left(1-0.7199^3\right)^{0.20}\left(1-0.7158^3\right)^{0.40}}, \right. \\
&\quad \cdot \left. \sqrt[3]{\left(1-0.8998^3\right)^{0.40}\left(1-0.7199^3\right)^{0.20}\left(1-0.7158^3\right)^{0.40}-\left(1-0.8998^3-0.3000^3\right)^{0.40}\left(1-0.7199^3-0.2071^3\right)^{0.20}\left(1-0.7158^3-0.1623^3\right)^{0.40}} \right\rangle \\
&= (0.8191, 0.2754).
\end{aligned}
\tag{48}
$$

For $\mathscr{P}_4$,

$$
\begin{aligned}
&= \text{FFHIWA}\left(\mathfrak{S}_{11}, \mathfrak{S}_{12}, \mathfrak{S}_{13}\right) \\
&= \left\langle \sqrt[3]{1-\left(1-0.9161^3\right)^{0.40}\left(1-0.8478^3\right)^{0.20}\left(1-0.8552^3\right)^{0.40}}, \right. \\
&\quad \cdot \left. \sqrt[3]{\left(1-0.9161^3\right)^{0.40}\left(1-0.8478^3\right)^{0.20}\left(1-0.8552^3\right)^{0.40}-\left(1-0.9161^3-0.2348^3\right)^{0.40}\left(1-0.8478^3-0.1343^3\right)^{0.20}\left(1-0.8552^3-0.3769^3\right)^{0.40}} \right\rangle \\
&= (0.8831, 0.2951).
\end{aligned}
\tag{49}
$$

Step 3: the score values for alternatives are

$$
\begin{aligned}
S\left(\mathfrak{S}_1\right) &= -0.2036, S\left(\mathfrak{S}_2\right) = -0.2302, S\left(\mathfrak{S}_3\right) \\
&= 0.5287, S\left(\mathfrak{S}_4\right) = 0.6630.
\end{aligned}
\tag{50}
$$

Step 4: as $\mathfrak{S}_4 \succ \mathfrak{S}_3 \succ \mathfrak{S}_1 \succ \mathfrak{S}_2$, the best AQ in Guangzhou is November of 2009.

The whole method which we have adopted in this application is given in Figure 1.

7. Comparison Analysis

For the validity and importance of proposed operators, we aggregate the same information using different operator, namely, FFEWA or FFEOWA operator [30].

Definition 7. (see [30]). The FFEWA operator is as follows:

$$
\text{FFEWA}\left(\mathfrak{P}_1, \mathfrak{P}_2, \ldots, \mathfrak{P}_{\mathfrak{y}}\right) = \left\langle \sqrt[3]{\frac{\prod_{i=1}^{\mathfrak{y}}\left(1+\varrho_i^3\right)^{\kappa_i} - \prod_{i=1}^{\mathfrak{y}}\left(1+\varrho_i^3\right)^{\kappa_i}}{\prod_{i=1}^{\mathfrak{y}}\left(1+\varrho_i^3\right)^{\kappa_i} + \prod_{i=1}^{\mathfrak{y}}\left(1+\varrho_i^3\right)^{\kappa_i}}}, \frac{\sqrt[3]{2}\prod_{i=1}^{\mathfrak{y}}\sigma_i^{\kappa_i}}{\sqrt[3]{\prod_{i=1}^{\mathfrak{y}}\left(2-\sigma_i^3\right)^{\kappa_i} + \prod_{i=1}^{\mathfrak{y}}\left(\sigma_i^3\right)^{\kappa_i}}} \right\rangle. \tag{51}
$$

The FF Einstein ordered weighted averaging (FFEOWA) operator is

$$
\text{FFEOWA}\left(\mathfrak{P}_1, \mathfrak{P}_2, \ldots, \mathfrak{P}_{\mathfrak{y}}\right) = \left\langle \sqrt[3]{\frac{\prod_{i=1}^{\mathfrak{y}}\left(1+\varrho_{\sigma(i)}^3\right)^{\kappa_i} - \prod_{i=1}^{\mathfrak{y}}\left(1-\varrho_{\sigma(i)}^3\right)^{\kappa_i}}{\prod_{i=1}^{\mathfrak{y}}\left(1+\varrho_{\sigma(i)}^3\right)^{\kappa_i} + \prod_{i=1}^{\mathfrak{y}}\left(1-\varrho_{\sigma(i)}^3\right)^{\kappa_i}}}, \frac{\sqrt[3]{2}\prod_{i=1}^{\mathfrak{y}}\sigma_{\sigma(i)}^{\kappa_i}}{\sqrt[3]{\prod_{i=1}^{\mathfrak{y}}\left(2-\sigma_{\sigma(i)}^3\right)^{\kappa_i} + \prod_{i=1}^{\mathfrak{y}}\left(\sigma_{\sigma(i)}^3\right)^{\kappa_i}}} \right\rangle. \tag{52}
$$

By FFEWA operator, the steps are as follows:

Step 1: same as above.

Step 2: to find the overall assessment of each alternative, we apply the FFEWA operator as

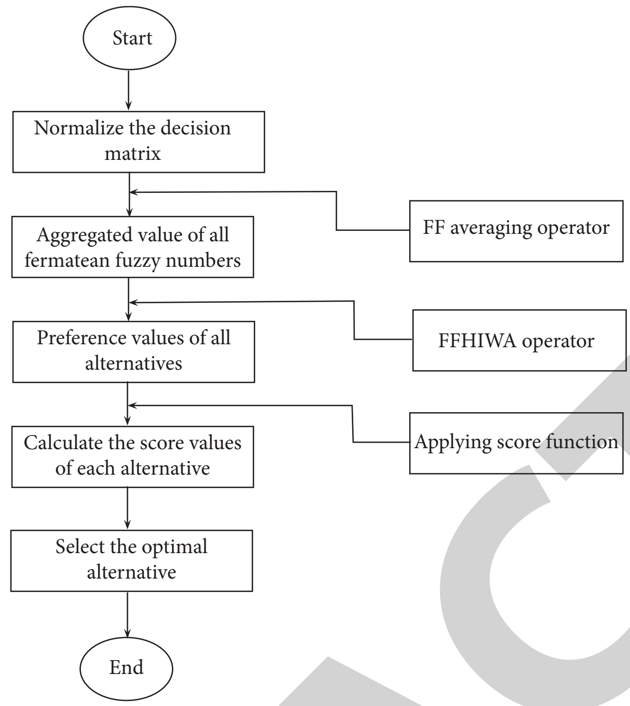

Figure 1: Flow chart for the classification of the air quality of Guangzhou.

$$
\begin{aligned}
\mathscr{P}_1 &= \mathrm{FFEWA}\left(\mathfrak{S}_{11}, \mathfrak{S}_{12}, \mathfrak{S}_{13}\right)\\
&= \left\langle \sqrt[3]{\frac{\left(1+0.4031^3\right)^{0.40}\left(1+0.6208^3\right)^{0.20}\left(1+0.5800^3\right)^{0.40} - \left(1-0.4031^3\right)^{0.40}\left(1-0.6208^3\right)^{0.20}\left(1-0.5800^3\right)^{0.40}}{\left(1+0.4031^3\right)^{0.40}\left(1+0.6208^3\right)^{0.20}\left(1+0.5800^3\right)^{0.40} + \left(1-0.4031^3\right)^{0.40}\left(1-0.6208^3\right)^{0.20}\left(1-0.5800^3\right)^{0.40}}},\right.\\
&\left. \frac{\sqrt[3]{2}\left((0.8400)^{0.40}(0.4901)^{0.20}(0.5889)^{0.40}\right)}{\left(2-0.8400^3\right)^{0.40}\left(2-0.4901^3\right)^{0.20}\left(2-0.5889^3\right)^{0.40} + \left(0.8400^3\right)^{0.40}\left(0.4901^3\right)^{0.20}\left(0.5889^3\right)^{0.40}} \right\rangle\\
&= (0.5347, 0.6628).
\end{aligned}
$$

$$
\begin{aligned}
\mathscr{P}_2 &= \mathrm{FFEWA}\left(\mathfrak{S}_{11}, \mathfrak{S}_{12}, \mathfrak{S}_{13}\right)\\
&= \left\langle \sqrt[3]{\left(\frac{\left(1+0.4986^3\right)^{0.40}\left(1+0.1662^3\right)^{0.20}\left(1+0.5048^3\right)^{0.40} - \left(1-0.4986^3\right)^{0.40}\left(1-0.1662^3\right)^{0.20}\left(1-0.5048^3\right)^{0.40}}{\left(1+0.4986^3\right)^{0.40}\left(1+0.1662^3\right)^{0.20}\left(1+0.5048^3\right)^{0.40} + \left(1-0.4986^3\right)^{0.40}\left(1-0.1662^3\right)^{0.20}\left(1-0.5048^3\right)^{0.40}}\right)},\right.\\
&\left. \cdot \frac{\sqrt[3]{2}\left((0.6007)^{0.40}(0.7252)^{0.20}(0.7430)^{0.40}\right)}{\left(2-0.6007^3\right)^{0.40}\left(2-0.7252^3\right)^{0.20}\left(2-0.7430^3\right)^{0.40} + \left(0.6007^3\right)^{0.40}\left(0.7252^3\right)^{0.20}\left(0.7430^3\right)^{0.40}} \right\rangle\\
&= (0.4674, 0.6810).
\end{aligned}
$$

$$
\mathscr{P}_3 = \mathrm{FFEWA}\left(\mathfrak{S}_{11}, \mathfrak{S}_{12}, \mathfrak{S}_{13}\right)
$$

$$=\left\langle \sqrt[3]{\frac{\left(1+0.8998^3\right)^{0.40}\left(1+0.7199^3\right)^{0.20}\left(1+0.7158^3\right)^{0.40}-\left(1-0.8998^3\right)^{0.40}\left(1-0.7199^3\right)^{0.20}\left(1-0.7158^3\right)^{0.40}}{\left(1+0.8998^3\right)^{0.40}\left(1+0.7199^3\right)^{0.20}\left(1+0.7158^3\right)^{0.40}+\left(1-0.8998^3\right)^{0.40}\left(1-0.7199^3\right)^{0.20}\left(1-0.7158^3\right)^{0.40}}},\right.$$

$$\left.\cdot\frac{\sqrt[3]{2}\left((0.3000)^{0.40}(0.2071)^{0.20}(0.1623)^{0.40}\right)}{\left(2-0.3000^3\right)^{0.40}\left(2-0.2071^3\right)^{0.20}\left(2-0.1623^3\right)^{0.40}+\left(0.3000^3\right)^{0.40}\left(0.2071^3\right)^{0.20}\left(0.1623^3\right)^{0.40}}\right\rangle$$

$$=(0.8137, 0.2180).$$

$$\mathscr{P}_4=\mathrm{FFEWA}\left(\mathfrak{S}_{11},\mathfrak{S}_{12},\mathfrak{S}_{13}\right)$$

$$=\left\langle \sqrt[3]{\frac{\left(1+0.9161^3\right)^{0.40}\left(1+0.8478^3\right)^{0.20}\left(1+0.8552^3\right)^{0.40}-\left(1-0.9161^3\right)^{0.40}\left(1-0.8478^3\right)^{0.20}\left(1-0.8552^3\right)^{0.40}}{\left(1+0.9161^3\right)^{0.40}\left(1+0.8478^3\right)^{0.20}\left(1+0.8552^3\right)^{0.40}+\left(1-0.9161^3\right)^{0.40}\left(1-0.8478^3\right)^{0.20}\left(1-0.8552^3\right)^{0.40}}},\right.$$

$$\left.\cdot\frac{\sqrt[3]{2}\left((0.2348)^{0.40}(0.1343)^{0.20}(0.3769)^{0.40}\right)}{\left(2-0.2348^3\right)^{0.40}\left(2-0.1343^3\right)^{0.20}\left(2-0.3769^3\right)^{0.40}+\left(0.2348^3\right)^{0.40}\left(0.1343^3\right)^{0.20}\left(0.3769^3\right)^{0.40}}\right\rangle$$

$$=(0.8831, 0.2951). \tag{53}$$

Step 3: the score values for alternatives are

$$S(\mathfrak{S}_1)=-0.1383, S(\mathfrak{S}_2)=-0.2137, S(\mathfrak{S}_3)=0.5284, S(\mathfrak{S}_4)=0.6704. \tag{54}$$

Step 4: as $\mathfrak{S}_4 \succ \mathfrak{S}_3 \succ \mathfrak{S}_1 \succ \mathfrak{S}_2$, the best AQ in Guangzhou is November of 2009.

The results obtained from these operators are shown in Table 5 and Figure 2. It is clear that the most suitable alternative obtained by using FFHIWA and FFEWA operators is the same. This implies that our proposed methods are accurate and can be utilized in DM problems.

Advantages of proposed operators: the main reason behind proposed approach is that

(i) We can see the effect of other grades of nonmembership in the aggregated value even if nonmembership of any one alternative is zero.

(ii) We can see that there is a proper interaction between the MD and NMD.

The operators defined in [30] are very concise and have been extensively used, but these operators have certain drawbacks. Few of them have been highlighted as follows:

(1)Let $\mathfrak{P}_1=(0.9,0)$, $\mathfrak{P}_2=(0.77,0.45)$, $\mathfrak{P}_3=(0.80,0.63)$, and $\mathfrak{P}_4=(0.58,0.67)$ be four FFNs and $\kappa=(0.3,03,0.2,0.2)^T$ is the WV corresponding to FFNs. By applying the FFEWA operator, we get FFEWA $(\mathfrak{P}_1,\mathfrak{P}_2,\mathfrak{P}_3,\mathfrak{P}_4)=(0.88,0)$. This shows that NMD of a FFN is independent of the NMD of others FFNs (which are nonzero in $\mathfrak{P}_i's$) and hence does not play a significant role during the AP. The aggregated FFNs as FFHIWA $(\mathfrak{P}_1,\mathfrak{P}_2,\mathfrak{P}_3,\mathfrak{P}_4)=(0.81,0.49)$ for $\delta=1$ and for $\delta=2$ FFHIWA $(\mathfrak{P}_1,\mathfrak{P}_2,\mathfrak{P}_3,\mathfrak{P}_4)=(0.88,0.43)$. It can be seen that NMD is nonzero of the whole aggregated FFNs even if at least one of the NMD of FFNs is zero. Thus, the others nonmembership values of FFNs play a predominant role during the AP in the proposed operator.

(2) Let $\mathfrak{P}_1=(0.53,0.42)$, $\mathfrak{P}_2=(0.98,0.34)$, $\mathfrak{P}_3=(0.61,0.54)$, and $\mathfrak{P}_4=(0.71,0.46)$ be four FFNs and $\kappa=(0.4,0.2,0.3,0.1)^T$ is the WV corresponding to FFNs. By applying the FFEWA operator, we get FFEWA $(\mathfrak{P}_1,\mathfrak{P}_2,\mathfrak{P}_3,\mathfrak{P}_4)=(0.78,0.44)$. If we replace FFNs $\mathfrak{P}_2$ and $\mathfrak{P}_3$ with $\mathscr{T}_2=(0.78,0.34)$ and $\mathscr{T}_3=(0.67,0.54)$, then their corresponding aggregated FFN become $(0.66,0.44)$. Hence, the NMD part of aggregated FFN becomes independent of the change in MD's values. That is why it is incompatible and does not produce an accurate information to the decision maker. The aggregated FFNs as FFHIWA $(\mathfrak{P}_1,\mathfrak{P}_2,\mathfrak{P}_3,\mathfrak{P}_4)=(0.80,0.53)$ for $\delta=1$ and FFHIWA $(\mathfrak{P}_1,\mathfrak{P}_2,\mathfrak{P}_3,\mathfrak{P}_4)=(0.78,0.55)$ for $\delta=2$, and if we consider modified FFNs, then $FFHIWA(\mathfrak{P}_1,\mathscr{T}_2,\mathscr{T}_3,\mathfrak{P}_4)=(0.67,0.45)$ for $\delta=1$ and FFHIWA $(\mathfrak{P}_1,\mathscr{T}_2,\mathscr{T}_3,\mathfrak{P}_4)=(0.66,0.45)$ for $\delta=2$. It can be seen that the modification in membership function will affect aggregated value of nonmembership function and is nonzero. That is why, there is a proper interaction between the MD and NMD, and hence, the results are unchangeable and more realistic than the existing operators results.

Table 5: Comparison analysis with FFEWA operator.

Methods	$S(\mathfrak{S}_1)$	$S(\mathfrak{S}_2)$	$S(\mathfrak{S}_3)$	$S(\mathfrak{S}_4)$	Ranking order
FFEWA operator	−0.1383	−0.2137	0.5284	0.6704	$\mathfrak{S}_4 \succ \mathfrak{S}_3 \succ \mathfrak{S}_1 \succ \mathfrak{S}_2$
FFHIWA operator (proposed)	−0.2036,	−0.2302	0.5287	0.6630	$\mathfrak{S}_4 \succ \mathfrak{S}_3 \succ \mathfrak{S}_1 \succ \mathfrak{S}_2$

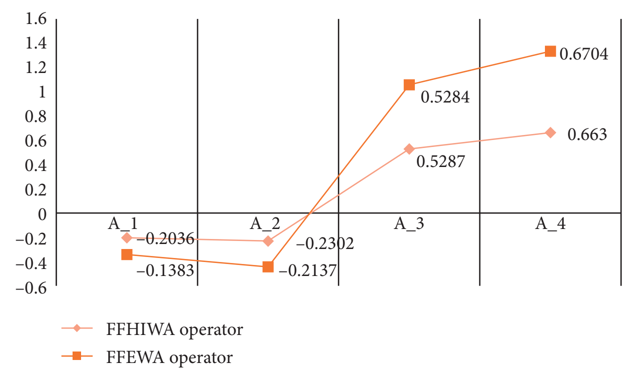

Figure 2: Comparison with FFEWA operator.

8. Conclusions

FFS is a generalized structure of IFS and PFS. It is more powerful tool to solve DM problems involving uncertainty and satisfies the condition $0 \leq \varrho^3 + \sigma^3 \leq 1$. The structure of Hamacher's t-norm and t-conorm is more generalized that effectively integrates the complex information. The shortcomings of the existing methods and beneficial characteristics of Hamacher AOs motivate us to endeavor for the development of a fruitful fusion with FFNs. In this research article, we have developed a group of novel FF Hamacher interactive averaging AOs, such as FFHIWA, FFHIOWA, and FFHIHWA operators. These proposed operators have the characteristic of idempotency, boundedness, monotonicity, homogeneity, and shift invariance. These operators reduce the shortcomings of FFEWA operators. We have also discussed some particular cases of proposed operators. Moreover, the developed operators study the interaction between membership and nonmembership grades. We have presented an algorithm to deal with MAGDM problems. For the validity and flexibility of proposed work, we have given the comparison analysis. In short, this work focuses on role of Hamacher interactive AOs as well as the propitious characteristics of FFNs. It is concluded that the new model of uncertain data is flexible which aptly depicts imprecise and inexact information in complicated scenarios. Thus, the operators serve as a powerful tool with further applications due to their highly adaptable nature. In future, we will work on the following topics:

(1) Neutrality aggregation operators for Fermatean fuzzy sets.

(2) Fermatean fuzzy power aggregation operators.

(3) Fermatean fuzzy Hamy mean aggregation operators and their application in multiattribute decision making.

(4) Fermatean fuzzy soft Dombi aggregation operators.

Data Availability

No data were used to support this study.

Disclosure

This article does not contain any studies with human participants or animals performed by any of the authors.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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