Research Article

More on Selective Covering Properties in Bitopological Spaces

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In this study, we continue our investigation of selective covering properties in bitopological spaces. We discuss their behaviour under certain kinds of mappings. We also introduce selective versions of the ccc property and the star-ccc property in bitopological spaces and give few of their relations with other selective properties. Also, we consider preservation of selective covering properties of bitopological spaces under some known relations in bitopological context.

1. Introduction

Throughout the study, we use the standard topological notation and terminology, mainly as in [1]. By $\mathbb{N}$ and $\mathbb{R}$, we denote the set of natural numbers and the set of real numbers, respectively. By X or (X, τ) , we denote a topological space, while (X, τ_1, τ_2) denotes a bitopological space [2] (called also bispaces), that is a set X equipped with two (in general, unrelated) topologies. The closure and interior of a subset A of a space (X, τ) are denoted by $\text{Cl}(A)$ and $\text{Int}(A)$, respectively. When (X, τ_1, τ_2) is a bitopological space and $A \subset X$, then $\text{Cl}_{\tau_i}(A)$ and $\text{Int}_{\tau_i}(A)$, $i = 1, 2$, denote the closure and interior in the space (X, τ_i) .

If A is a subset of a topological space X , $x \in X$, and $\mathcal{F}$ is a family of subsets of X , then $\text{St}(A, \mathcal{F}) = \bigcup \{F \in \mathcal{F} : A \cap F \neq \emptyset\}$, and $\text{St}(x, \mathcal{F}) = \text{St}(\{x\}, \mathcal{F})$ denote the star of A and x with respect to $\mathcal{F}$.

For a space (X, τ) , we also use the following notations:

- (i) $\mathcal{O}$ is the collection of all open covers of X
- (ii) $\mathcal{O}_D = \{\mathcal{U} \subset \tau : \bigcup \mathcal{U} \text{ is dense in } X\}$ ($\mathcal{U}$ with this property is called an open weak cover of X)
- (iii) $\overline{\mathcal{O}} = \{\mathcal{U} \subset \tau : \bigcup \{\overline{U} : U \in \mathcal{U}\} = X\}$ ($\mathcal{U}$ with this property is called an open almost cover of X)

In case of bitopological spaces, we write $\mathcal{O}^{(i)}$, $\mathcal{O}_D^{(i)}$, $\overline{\mathcal{O}}^{(i)}$ to indicate that $\mathcal{O}$, $\mathcal{O}_D$, $\overline{\mathcal{O}}$ is related to the space (X, τ_i) , $i = 1, 2$.

If $\mathcal{A}$ and $\mathcal{B}$ are families of sets, then

- (1) The symbol $S_{\text{fin}}(\mathcal{A}, \mathcal{B})$ denotes the selection hypothesis: for each sequence $(A_1, A_2, \dots)$ of elements of $\mathcal{A}$, there is a sequence $(B_1, B_2, \dots)$, such that for each $n \in \mathbb{N}$, B_n is a finite subset of A_n , and $\bigcup_{n \in \mathbb{N}} B_n \in \mathcal{B}$ (see [3]);
- (2) The symbol $S_1(\mathcal{A}, \mathcal{B})$ denotes the selection hypothesis: for each sequence $(A_1, A_2, \dots)$ of elements of $\mathcal{A}$, there is a sequence $(b_1, b_2, \dots)$, such that $b_n \in A_n$ for each $n \in \mathbb{N}$, and $\{b_n : n \in \mathbb{N}\} \in \mathcal{B}$ (see [3]).

The following covering properties are defined in this way:

- (i) $S_{\text{fin}}(\mathcal{O}, \mathcal{O})$ is the Menger property [4]
- (ii) $S_1(\mathcal{O}, \mathcal{O})$ is the Rothberger property [5]
- (iii) $S_{\text{fin}}(\mathcal{O}, \mathcal{O}_D)$ is the weak Menger property [6] (see also [7, 8])
- (iv) $S_{\text{fin}}(\mathcal{O}, \overline{\mathcal{O}})$ is the almost Menger property [9] (see also [10])
- (v) $S_1(\mathcal{O}, \mathcal{O}_D)$ is the weak Rothberger property [6]
- (vi) $S_1(\mathcal{O}, \overline{\mathcal{O}})$ is the almost Rothberger property [9]

A space X has the Hurewicz property if for each sequence $(\mathcal{U}_1, \mathcal{U}_2, \dots)$ of open covers of X , there is a sequence

$(\mathcal{V}_1, \mathcal{V}_2, \dots)$, such that for each $n \in \mathbb{N}$, $\mathcal{V}_n$ is a finite subset of $\mathcal{U}_n$, and each $x \in X$ belongs to $\bigcup \mathcal{V}_n$ for all but finitely many n [11].

For selection principles theory see [12].

Recall that a family of pairwise disjoint open sets in a topological space X is called a cellular open family. X is said to be a ccc space if every cellular open family in it is countable.

Observe that any separable space, clearly, is ccc.

Selective version of the ccc space known as “selectively ccc spaces” was first introduced (under this name) by Aurichi [13].

Definition 1. (see [13]) Let (X, τ) be a topological space. X is selectively ccc if for every sequence $(\mathcal{A}_n: n \in \mathbb{N})$ of maximal cellular open families in X , there is a sequence $(A_n: n \in \mathbb{N})$, such that for each n , $A_n \in \mathcal{A}_n$ and $\bigcup_{n \in \mathbb{N}} A_n$ is dense in X .

In fact, these spaces were considered in the literature under different names. In [14, 15], Scheepers (first) studied the selectively ccc spaces under the notation $S_1(\mathcal{D}, \mathcal{D})$, where $\mathcal{D}$ is the collection of open families with a dense union (i.e., $\mathcal{D}$ is $\mathcal{O}_D$ in our notation). In [16], these spaces have been studied under the notation $S_1(\mathcal{D}_O, \mathcal{D})$, where $\mathcal{D}_O$ denotes the collection of open families with a dense union. In both studies [13, 16], it was explained that all these three definitions are equivalent.

In this study, we consider selectively ccc spaces as spaces belonging to the class $S_1(\mathcal{O}_D, \mathcal{O}_D)$.

If $\mathcal{A}$ and $\mathcal{B}$ are collections of covers of a space X , then

- (1) The symbol $SS_1(\mathcal{A}, \mathcal{B})$ denotes the selection hypotheses that for each sequence $(\mathcal{U}_1, \mathcal{U}_2, \dots)$ of elements of $\mathcal{A}$, there is a sequence $(x_1, x_2, \dots)$ of elements of X such that $\{\text{St}(x_n, \mathcal{U}_n): n \in \mathbb{N}\} \in \mathcal{B}$ (see [17])
- (2) The symbol $SS_{\mathcal{O},1}^*(\mathcal{A}, \mathcal{B})$ denotes the selection hypotheses: for each open cover $\mathcal{U}$ of X and each sequence $(\mathcal{U}_1, \mathcal{U}_2, \dots)$ of elements of $\mathcal{A}$, there is a sequence $(U_1, U_2, \dots)$, such that for each n , $U_n \in \mathcal{U}_n$ and $\text{St}(\bigcup_{n \in \mathbb{N}} U_n, \mathcal{U}) \in \mathcal{B}$ (see [18])

In [19], the following spaces have been defined and studied.

Definition 2. (see [19]) A topological space X is said to be selectively star-ccc if it belongs to the class $SS_{\mathcal{O},1}^*(\mathcal{O}_D, \mathcal{O})$.

In this study, we introduce and study the selective versions of the ccc property and selectively star-ccc property in bitopological spaces. We establish some relations with other selective bitopological properties.

The study is organized as follows. In Section 2, we consider preservation of the bitopological almost Menger and almost Rothberger properties under certain classes of mappings. Section 3 deals with selectively ccc and selectively star-ccc bitopological spaces. Finally, in Section 4, we discuss the behaviour of several topological and bitopological covering properties under known relations between the two topologies.

2. Mappings and Almost Menger Bispaces

The first study of weak covering properties in the bitopological context began with the study [20] on the almost Menger property in bitopological spaces and continued in [21, 22] where some results on almost Menger and weak Menger properties in the bitopological spaces were obtained. Selective properties in bitopological spaces have been studied in [23–25].

Let us recall the following definition.

Definition 3. (see [20]). A bitopological space (X, τ_1, τ_2) is said to be (i, j) -almost Menger, $i, j = 1, 2$, if for each sequence $(\mathcal{U}_n: n \in \mathbb{N})$ of τ_i -open covers of X , there exists a sequence $(\mathcal{V}_n: n \in \mathbb{N})$ of finite families, such that for each n , $\mathcal{V}_n \subset \mathcal{U}_n$ and $X = \bigcup_{n \in \mathbb{N}} \bigcup_{V \in \mathcal{V}_n} \text{Cl}_{\tau_j}(V)$.

This section focuses on the behaviour of almost Menger, almost Rothberger, and almost Hurewicz properties under some kinds of mappings between bitopological spaces.

We recall the following definitions. A mapping $f: (X, \tau_1, \tau_2) \longrightarrow (Y, \sigma_1, \sigma_2)$ is called

- (1) (i, j) - θ -continuous [26] (respectively, (i, j) -strongly θ -continuous [27]) if for each $x \in X$ and each σ_i -open set $V \subset Y$ containing $f(x)$, there exists a τ_i -open set $U \subset X$ containing x , such that $f(\text{Cl}_{\tau_j}(U)) \subset \text{Cl}_{\sigma_j}(V)$ (respectively, $f(\text{Cl}_{\tau_j}(U)) \subset V$). It is obvious that each (i, j) -strongly θ -continuous mapping is (i, j) - θ -continuous.
- (2) (i, j) -almost continuous [28] if $f^{-}(B)$ is the τ_i -open set in X for every (i, j) -regular open set B in Y . In addition, f is called p -almost continuous if it is (1, 2) and (2, 1)-almost continuous. Recall that a set B in a bitopological space (X, τ_1, τ_2) is (i, j) -regular open [29] if $B = \text{Int}_{\tau_i} \text{Cl}_{\tau_j}(B)$.

We note that p -almost continuous mappings are (i, j) - θ -continuous [30], so the following result generalizes the result in [20], which states that the (i, j) -almost Menger property is preserved by p -almost continuous mappings.

Theorem 1 (see [21]). *An (i, j) – θ -continuous image of an (i, j) -almost Menger bitopological space is (i, j) -almost Menger.*

In [21], the definitions of (i, j) -almost γ -sets and the (i, j) -almost Hurewicz property were introduced, and the following theorem was obtained.

Theorem 2 (see [21]). *Let (X, τ_1, τ_2) be an (i, j) -almost γ -set and (Y, σ_1, σ_2) be a bitopological space. If $f: X \longrightarrow Y$ is an (i, j) - θ -continuous surjection, then (Y, σ_1, σ_2) is (i, j) -almost Hurewicz.*

Now, we consider the behaviour of the (i, j) -almost Rothberger property under (i, j) - θ -continuous mappings.

A bitopological space (X, τ_1, τ_2) is said to be (i, j) -almost Rothberger, $(i, j = 1, 2)$, if for each sequence

$(\mathcal{U}_n: n \in \mathbb{N})$ of τ_i -open covers of X , there is a sequence $(U_n: n \in \mathbb{N})$, such that for each n , $U_n \in \mathcal{U}_n$ and $X = \bigcup_{n \in \mathbb{N}} \text{Cl}_{\tau_j}(U_n)$ [20].

Theorem 3. *An (i, j) - θ -continuous image of an (i, j) -almost Rothberger bitopological space is (i, j) -almost Rothberger.*

Proof. Let (X, τ_1, τ_2) be an (i, j) -almost Rothberger bitopological space. Let $Y = f(X)$ be the image of X , where $f: (X, \tau_1, \tau_2) \longrightarrow (Y, \sigma_1, \sigma_2)$ is an (i, j) - θ -continuous mapping.

Let $(\mathcal{V}_n: n \in \mathbb{N})$ be a sequence of σ_i -open covers of Y . For each $n \in \mathbb{N}$ and fixed $x \in X$, there is a set $V_{x,n} \in \mathcal{V}_n$, such that $f(x) \in V_{x,n}$. Since f is (i, j) - θ -continuous, there exists a τ_i -open set $U_{x,n} \subset X$ containing x , such that $f(\text{Cl}_{\tau_j}(U_{x,n})) \subset \text{Cl}_{\sigma_j}(V_{x,n})$. For each n , $\mathcal{U}_n = \{U_{x,n}: x \in X\}$ is a τ_i -open cover of X . Apply the fact that X is (i, j) -almost Rothberger. There is a sequence $(x_n: n \in \mathbb{N})$, such that for each $n \in \mathbb{N}$, $U_{x_n,n} \in \mathcal{U}_n$ and $X = \bigcup_{n \in \mathbb{N}} \text{Cl}_{\tau_j}(U_{x_n,n})$. For each n and $U_{x_n,n}$, we assign a set $V_{x_n,n} \in \mathcal{V}_n$, such that $f(\text{Cl}_{\tau_j}(U_{x_n,n})) \subset \text{Cl}_{\sigma_j}(V_{x_n,n})$. We obtain the sequence $(V_{x_n,n}: n \in \mathbb{N})$, with $V_{x_n,n} \in \mathcal{V}_n$ and

$$\begin{aligned} Y = f(X) &= f\left(\bigcup_{n \in \mathbb{N}} \text{Cl}_{\tau_j}(U_{x_n,n})\right) \\ &= \bigcup_{n \in \mathbb{N}} f\left(\text{Cl}_{\tau_j}(U_{x_n,n})\right) \subset \bigcup_{n \in \mathbb{N}} \text{Cl}_{\sigma_j}(V_{x_n,n}). \end{aligned} \quad (1)$$

This means that (Y, σ_1, σ_2) is (i, j) -almost Rothberger. $\square$

Corollary 1. *The (i, j) -almost Rothberger property is preserved by p -almost continuous mappings.*

Theorem 4. *If $f: (X, \tau_1, \tau_2) \longrightarrow (Y, \sigma_1, \sigma_1)$ is an (i, j) -strongly θ -continuous mapping and X is (i, j) -almost Menger, then (Y, σ_i) is Menger.*

Proof. Let $(\mathcal{V}_n: n \in \mathbb{N})$ be a sequence of σ_i -open covers of Y and $x \in X$. For each $n \in \mathbb{N}$, there exists a set $V_{x,n} \in \mathcal{V}_n$ with $f(x) \in V_{x,n}$. Since f is (i, j) -strongly θ -continuous, there is a τ_i -open set $U_{x,n} \subset X$ with $x \in U_{x,n}$, such that $f(\text{Cl}_{\tau_j}(U_{x,n})) \subset V_{x,n}$. Then, $\mathcal{U}_n = \{U_{x,n}: x \in X\}$ is a τ_i -open cover of X . Since X is (i, j) -almost Menger bitopological space, there exists a sequence $(\mathcal{H}_n: n \in \mathbb{N})$, such that for each $n \in \mathbb{N}$, $\mathcal{H}_n$ is a finite subset of $\mathcal{U}_n$, and $X = \bigcup_{n \in \mathbb{N}} \bigcup \left\{ \text{Cl}_{\tau_j}(H): H \in \mathcal{H}_n \right\}$. Now, for each $n \in \mathbb{N}$ and $H \in \mathcal{H}_n$, we can choose $V_H \in \mathcal{V}_n$, such that $f(\text{Cl}_{\tau_j}(H)) \subset V_H$. Let $\mathcal{F}_n = \{V_H: H \in \mathcal{H}_n\}$; we obtain a sequence $(\mathcal{F}_n: n \in \mathbb{N})$ of finite subsets of $\mathcal{V}_n$, $n \in \mathbb{N}$. We will show $\bigcup \mathcal{F}_n$ is a σ_i -cover of Y . Indeed,

$$\begin{aligned} Y = f(X) &= f\left(\bigcup_{n \in \mathbb{N}} \bigcup \left\{ \text{Cl}_{\tau_j}(H): H \in \mathcal{H}_n \right\}\right), \\ &= \bigcup_{n \in \mathbb{N}} \bigcup \left\{ f\left(\text{Cl}_{\tau_j}(H)\right): H \in \mathcal{H}_n \right\}, \\ &\subset \bigcup_{n \in \mathbb{N}} \bigcup \{V_H: H \in \mathcal{H}_n\}. \end{aligned} \quad (2)$$

This shows that (Y, σ_i) is Menger. $\square$

Remark 1. It is obvious that if $f: (X, \tau_1, \tau_2) \longrightarrow (Y, \sigma_1, \sigma_2)$ is an (i, j) -strongly θ -continuous mapping, then f is i -continuous; furthermore, if (X, τ_1, τ_2) is the (i, j) -regular bitopological space, then the reverse implication is also true.

Corollary 2. *If $f: (X, \tau_1, \tau_2) \longrightarrow (Y, \sigma_1, \sigma_1)$ is i -continuous mapping and X is (i, j) -regular, (i, j) -almost Menger bitopological space, then (Y, σ_i) is Menger.*

Now, recall the notions of contra continuous and precontinuous mappings in bitopological spaces. A mapping $f: (X, \tau_1, \tau_2) \longrightarrow (Y, \sigma_1, \sigma_2)$ is said to be (i, j) -contracontinuous and (i, j) -precontinuous [29], if the preimage of each σ_i -open set V of Y is τ_j -closed in X ((i, j) -preopen in X , i.e., $f^{-1}(V) \subset \text{Int}_{\tau_i} \text{Cl}_{\tau_j}(f^{-1}(V))$), where $i, j = 1, 2$.

Theorem 5. *If (X, τ_1, τ_2) is the (i, j) almost Menger bitopological space and $f: (X, \tau_1, \tau_2) \longrightarrow (Y, \sigma_1, \sigma_1)$ is an (i, j) -contracontinuous and (i, j) -precontinuous mapping, then (Y, σ_i) is a Menger space.*

Proof. Let $(\mathcal{V}_n: n \in \mathbb{N})$ be a sequence of covers of Y by σ_i -open sets. Since f is (i, j) -contra continuous, for each $n \in \mathbb{N}$ and $V \in \mathcal{V}_n$, the set $f^{-1}(V)$ is τ_j -closed in X . Then, we have $f^{-1}(V) \subset \text{Int}_{\tau_i} \text{Cl}_{\tau_j}(f^{-1}(V))$, since f is (i, j) -precontinuous. Now, we obtain $f^{-1}(V) \subset \text{Int}_{\tau_i}(f^{-1}(V))$, which gives $f^{-1}(V) = \text{Int}_{\tau_i}(f^{-1}(V))$. Then, for each $n \in \mathbb{N}$, we set $\mathcal{U}_n = \{f^{-1}(V): V \in \mathcal{V}_n\}$. Then, $(\mathcal{U}_n: n \in \mathbb{N})$ is a sequence of τ_i -open covers of X . Since X is (i, j) -almost Menger, there exists a sequence $(\mathcal{H}_n: n \in \mathbb{N})$, such that for each $n \in \mathbb{N}$, $\mathcal{H}_n$ is a finite subset of $\mathcal{U}_n$ and $\bigcup_{n \in \mathbb{N}} \bigcup \left\{ \text{Cl}_{\tau_j}(H): H \in \mathcal{H}_n \right\} = X$. Let $\mathcal{F}_n = \{f(H): H \in \mathcal{H}_n\}$. It is obvious that $(\mathcal{F}_n: n \in \mathbb{N})$ is a sequence of finite subsets of $\mathcal{V}_n$ for each $n \in \mathbb{N}$, and $\bigcup_{n \in \mathbb{N}} \mathcal{F}_n$ is an σ_i -open cover of Y which means that (Y, σ_i) is a Menger space.

A mapping $f: (X, \tau_1, \tau_2) \longrightarrow (Y, \sigma_1, \sigma_2)$ is said to be (i, j) -weakly continuous [26] if for each $x \in X$ and each σ_i -open subset $V \subset Y$ containing $f(x)$, there exist a τ_i -open set $U \subset X$ with $x \in U$, such that $f(U) \subset \text{Cl}_{\sigma_j}(V)$. $\square$

Theorem 6. *If $f: (X, \tau_1, \tau_2) \longrightarrow (Y, \sigma_1, \sigma_2)$ is an (i, j) -weakly continuous mapping and (X, τ_i) is Menger, then (Y, σ_1, σ_2) is an (i, j) -almost Menger space.*

Proof. Let (X, τ_i) be a Menger space and $(\mathcal{V}_n: n \in \mathbb{N})$ be a sequence of σ_i -open covers of Y . Let $x \in X$, then for each $n \in \mathbb{N}$, there exists $V \in \mathcal{V}_n$ with $f(x) \in V$. Since f is (i, j) -weakly continuous, there is a τ_i -open set $U_{x,n} \subset X$ with $x \in U_{x,n}$ and $f(U_{x,n}) \subset \text{Cl}_{\sigma_j}(V)$. Now, let $\mathcal{U}_n = \{U_{x,n}: x \in X\}$. Clearly, $\mathcal{U}_n$ is a τ_i -open cover of X . Since (X, τ_i) is Menger, we find a sequence $(\mathcal{F}_n: n \in \mathbb{N})$ of finite sets, such that for each $n \in \mathbb{N}$, $\mathcal{F}_n \subset \mathcal{U}_n$ and $\bigcup_{n \in \mathbb{N}} \mathcal{F}_n$ is a τ_i -open cover of X . For each $n \in \mathbb{N}$ and $U \in \mathcal{F}_n$, we can choose a set $V_U \in \mathcal{V}_n$, such that $f(U) \subset \text{Cl}_{\sigma_j}(V_U)$. Set $\mathcal{H}_n = \{V_U: U \in \mathcal{F}_n\}$. We show that $\bigcup \mathcal{H}_n$ is a cover of Y . Indeed, as $X = \bigcup_{n \in \mathbb{N}} \bigcup \{U: U \in \mathcal{F}_n\}$, we have

$$\begin{aligned} Y &= f(X) = f\left(\bigcup_{n \in \mathbb{N}} \bigcup \{U: U \in \mathcal{F}_n\}\right) \\ &\subset \bigcup_{n \in \mathbb{N}} \bigcup \{f(U): U \in \mathcal{F}_n\} \\ &\subset \bigcup_{n \in \mathbb{N}} \bigcup \{\text{Cl}_{\sigma_j}(V_U): U \in \mathcal{F}_n\}. \end{aligned} \quad (3)$$

So, (Y, σ_1, σ_2) is (i, j) -almost Menger. $\square$

Remark 2. Since (i, j) - θ -continuous mappings are (i, j) -weakly continuous, Theorem 6 is also true under θ -continuous mappings.

3. Selectively ccc and Selectively Star-ccc Bispaces

3.1. Selectively ccc Bispaces

Definition 4. A bitopological space (X, τ_1, τ_2) is said to satisfy i -countable chain condition (for short i -ccc), $i = 1, 2$, if the topological space (X, τ_i) satisfies countable chain condition. X satisfies countable chain condition if it satisfies i -countable chain condition for $i = 1, 2$.

Definition 5. A bitopological space (X, τ_1, τ_2) is (i, j) -selectively ccc ($i, j = 1, 2$) if for every sequence $(\mathcal{U}_n: n \in \mathbb{N})$ of elements of $\mathcal{O}_D^{(i)}$, there is a sequence $(U_n: n \in \mathbb{N})$, such that $U_n \in \mathcal{U}_n$, $n \in \mathbb{N}$, and $\bigcup_{n \in \mathbb{N}} U_n$ is dense in (X, τ_j) , i.e., if $S_1(\mathcal{O}_D^{(i)}, \mathcal{O}_D^{(j)})$ is satisfied.

Proposition 1. Let (X, τ_1, τ_2) be a bitopological space such that $\tau_1 \leq \tau_2$ (i.e., the topology τ_1 is weaker than τ_2).

- (1) If (X, τ_1, τ_2) is $(1, 2)$ -selectively ccc, then (X, τ_1) is selectively ccc
- (2) If (X, τ_2) is selectively ccc, then (X, τ_1, τ_2) is $(2, 1)$ -selectively ccc

Since the selectively ccc property is stronger than the ccc property, we have the following.

Corollary 3. Let (X, τ_1, τ_2) be a bitopological space such that $\tau_1 \leq \tau_2$. If X is $(1, 2)$ -selectively ccc bispaces, then (X, τ_1) has the ccc property.

Proposition 2. Let (X, τ_1, τ_2) be a bitopological space and $Y \subset X$ be τ_i -closed and τ_j -open subspace in X . If X is (i, j) -selectively ccc, then $(Y, \tau_1|_Y, \tau_2|_Y)$ is $(i|Y, j|Y)$ -selectively ccc.

Proof. (For $i = 1$ and $j = 2$). Let Y be a τ_1 -closed and τ_2 -open subset of X and let $(\mathcal{U}_n: n \in \mathbb{N})$ be a sequence of members of $\mathcal{O}_D^{(1|Y)}$. Thus, for each $n \in \mathbb{N}$, $\mathcal{V}_n = \mathcal{U}_n \cup \{X \setminus Y\}$ is an element of $\mathcal{O}_D^{(1)}$. Since X is $(1, 2)$ -selectively ccc, there is a sequence $(V_n: n \in \mathbb{N})$, such that for every $n \in \mathbb{N}$, $V_n \in \mathcal{V}_n$ and $\text{Cl}_2(\bigcup_{n \in \mathbb{N}} V_n) = X$. If $V_n \in \mathcal{U}_n$, we set $W_n = V_n$; if $V_n = X \setminus Y$, then we take W_n to be any element from $\mathcal{U}_n$. Then, it is clear that $\text{Cl}_{2|Y}(\bigcup_{n \in \mathbb{N}} W_n) = Y$.

Recall that a subset A of a bispaces (X, τ_1, τ_2) is called bidense (here called double dense or shortly d -dense) in X if A is dense in both (X, τ_1) and (X, τ_2) . X is d -separable if there is a countable set A which is d -dense in X . $\square$

Theorem 7. Let $Y \subset X$ be a d -dense set in a bispaces (X, τ_1, τ_2) . X is (i, j) -selectively ccc if and only if Y is $(i|Y, j|Y)$ selectively ccc.

Proof. (For $i = 1$ and $j = 2$). Let $(\mathcal{V}_n: n \in \mathbb{N})$ be a sequence of $\tau_1|_Y$ -open families such that $\bigcup \mathcal{V}_n$ is dense in $(Y, \tau_1|_Y)$. Since Y is τ_1 -dense, then each $\bigcup \mathcal{V}_n$ is dense in (X, τ_1) . As X is $(1, 2)$ -selectively ccc, there is a sequence $(V_n: n \in \mathbb{N})$, such that for each $V_n \in \mathcal{V}_n$, $n \in \mathbb{N}$, and $\bigcup_{n \in \mathbb{N}} V_n$ is dense in (X, τ_2) . Since Y is dense in (X, τ_2) , then $\bigcup_{n \in \mathbb{N}} V_n$ is dense in $(Y, \tau_2|_Y)$.

Let $(\mathcal{U}_n: n \in \mathbb{N})$ be a sequence of τ_1 -open families, such that each $\bigcup \mathcal{U}_n$ is dense in (X, τ_1) . The restriction to Y , $\mathcal{U}_n|_Y = \{U \cap Y: U \in \mathcal{U}_n\}$ is a $\tau_1|_Y$ -open family for each n . Since Y is $(1|Y, 2|Y)$ -selectively ccc, there is a sequence $(V_n: n \in \mathbb{N})$, such that $V_n \in \mathcal{U}_n|_Y$ and $\bigcup_{n \in \mathbb{N}} V_n$ is dense in $(Y, \tau_2|_Y)$. But Y is dense in (X, τ_2) , so $\bigcup_{n \in \mathbb{N}} V_n$ is dense in (X, τ_2) . $\square$

Theorem 8. Let (X, τ_1, τ_2) and (Y, σ_1, σ_2) be bitopological spaces and $f: X \rightarrow Y$ be a $(\tau_i - \sigma_i)$ -continuous, $(\tau_i - \sigma_i)$ -open and $(\tau_j - \sigma_j)$ continuous, $(\tau_j - \sigma_j)$ -closed surjection. If X is (i, j) -selectively ccc, then Y is (i, j) -selectively ccc.

Proof. (For $i = 1$ and $j = 2$). Let $(\mathcal{V}_n: n \in \mathbb{N})$ be a sequence of σ_1 -open families in Y such that $\bigcup \mathcal{V}_n$ is σ_1 -dense in Y . Since f is $(\tau_1 - \sigma_1)$ continuous and $(\tau_1 - \sigma_1)$ open, for each $n \in \mathbb{N}$, $\mathcal{U}_n = \{f^{-1}(V): V \in \mathcal{V}_n\}$ is τ_1 -open family in X and $\bigcup \mathcal{U}_n$ is τ_1 -dense in X . Because X is $(1, 2)$ -selectively ccc, we can pick $U_n \in \mathcal{U}_n$ with $\bigcup \{U_n: n \in \mathbb{N}\}$ dense in (X, τ_2) . For the sequence $(V_n: n \in \mathbb{N})$, such that $f^{-1}(V) = U_n$, we have $\bigcup_{n \in \mathbb{N}} V_n$ dense in (Y, σ_2) , since f is a $(\tau_2 - \sigma_2)$ continuous and $(\tau_2 - \sigma_2)$ closed mapping. Thus, (Y, σ_1, σ_2) is $(1, 2)$ selectively ccc.

Bitopological R -separability has been studied by Kočinac and Özçağ in [31].

Denote by $\mathcal{D}_1$ and $\mathcal{D}_2$ the collection of all dense subsets of (X, τ_1) and (X, τ_2) , respectively. We say that X is $R_{(\tau_i, \tau_j)}$ -separable ($i, j = 1, 2$), if for each sequence

$(D_n: n \in \mathbb{N})$ of elements of $\mathcal{D}_i$, there are $d_n \in D_n$, $n \in \mathbb{N}$, such that $\{d_n: n \in \mathbb{N}\}$ is an element of $\mathcal{D}_j$, i.e., if $S_1(\mathcal{D}_i, \mathcal{D}_j)$ holds.

We have the following fact. $\square$

Proposition 3. Every $R_{(\tau_i, \tau_j)}$ -separable bispase is (i, j) -selectively ccc.

Corollary 4. If the bitopological space (X, τ_1, τ_2) has a τ_i -closed and τ_j -open $R_{(\tau_i, \tau_j)}$ -separable subspace, then X is (τ_i, τ_j) selectively ccc.

Proof. It follows by Proposition 2 and the previous result.

Let $\mathcal{K}$ denote the collection of all k -covers of a space X (open covers such that every compact subset of X is contained in a member of the cover), and let Ω denote the collection of all ω -covers of X (open covers such that every finite subset of X is contained in a member of the cover and X itself is not a member of the cover). $\square$

Example 1. If a Tychonoff space X with a countable base belongs to the class $S_1(\mathcal{K}, \Omega)$, then ([31], Theorem 4) the function bispase $(C(X), \tau_p, \tau_k)$ is $R_{(\tau_k, \tau_p)}$ -separable. Therefore, this function bispase is selectively (τ_k, τ_p) -star-ccc, where τ_p denotes the pointwise topology, and τ_k denotes the compact-open topology on the set $C(X)$ of all continuous real-valued functions on a Tychonoff space X .

3.2. Selectively Star-ccc Bispases. We now present a bitopological version of selectively star-ccc spaces introduced and studied in [19] (see also [32–34]).

Definition 6. A bispase (X, τ_1, τ_2) is said to be (i, j) -selectively star-ccc, $i, j = 1, 2$, if for every sequence $(\mathcal{U}_n: n \in \mathbb{N})$ of τ_i -open weak covers and every open cover $\mathcal{C}$ of (X, τ_j) , there is a sequence $(U_n: n \in \mathbb{N})$, such that for every n , $U_n \in \mathcal{U}_n$ and $\text{St}(\cup_{n \in \mathbb{N}} U_n, \mathcal{C}) = X$.

The following result shows that the class of selectively star-ccc spaces is big enough.

Theorem 9. Let (X, τ_1, τ_2) be a bispase, such that $\tau_1 \geq \tau_2$. If (X, τ_2) is a Lindelöf space, then (X, τ_1, τ_2) is a $(1, 2)$ -selectively star-ccc space.

Proof. Let $(\mathcal{U}_n: n \in \mathbb{N})$ be a sequence of τ_1 -open weak covers of X , and let $\mathcal{C}$ be an open cover of (X, τ_2) . There exists a countable set $\mathcal{V} = \{C_1, C_2, \dots\} \subset \mathcal{C}$, such that $X = \bigcup \mathcal{V}$. Since each C_i , $i \in \mathbb{N}$, is τ_1 -open, for every $n \in \mathbb{N}$, there exists $U_n \in \mathcal{U}_n$, such that $U_n \cap C_n \neq \emptyset$. It follows that $\text{St}(\cup_{n \in \mathbb{N}} U_n, \mathcal{V}) = \bigcup \mathcal{V} = X$. This proves the theorem. $\square$

Corollary 5. The bispase $(\mathbb{R}, \tau_s, \tau)$, where $\mathbb{R}$ is the set of real numbers, τ is the usual metric topology, and τ_s is the Sorgenfrey topology, is (τ_s, τ) -selectively star-ccc.

Recall that a space X is said to be strongly star Lindelöf, if for each open cover $\mathcal{U}$ of X , there is a countable set $C \subset \mathcal{U}$, such that $\text{St}(C, \mathcal{U}) = X$.

Call a bispase (X, τ_1, τ_2) (i, j) -selectively 2-star-ccc if for every τ_j -open cover $\mathcal{U}$ of X and every sequence $(\mathcal{V}_n: n \in \mathbb{N})$ of τ_i -open weak covers of X , there is a sequence $(V_n: n \in \mathbb{N})$, such that $V_n \in \mathcal{V}_n$, $n \in \mathbb{N}$, and $\text{St}^2(\cup_{n \in \mathbb{N}} V_n, \mathcal{U}) = X$, where $\text{St}^2(V, \mathcal{U}) = \text{St}(\text{St}(V, \mathcal{U}), \mathcal{U})$.

Theorem 10. Let (X, τ_1, τ_2) be a bispase such that $\tau_1 \geq \tau_2$. If (X, τ_2) is a strongly star Lindelöf space, then (X, τ_1, τ_2) is an $(1, 2)$ -selectively 2-star-ccc space.

Proof. Let $\mathcal{U}$ be an arbitrary τ_2 -open cover of X and $(\mathcal{V}_n: n \in \mathbb{N})$ be a sequence of τ_1 -open weak covers of X . Since (X, τ_2) is strongly star Lindelöf, there exists a countable set $A = \{a_1, a_2, \dots\} \subset X$, such that $\text{St}(A, \mathcal{U}) = X$. Therefore, $\{\text{St}(a_n, \mathcal{U}): n \in \mathbb{N}\}$ is a τ_2 -open cover of X . For each $n \in \mathbb{N}$, there exists $U_n \in \mathcal{U}$, such that $a_n \in U_n \subset \text{St}(a_n, \mathcal{U})$. So, $U_n \subset \text{St}(a_n, \mathcal{U}) \subset \text{St}(U_n, \mathcal{U})$. For every $n \in \mathbb{N}$, the set $\cup_{n \in \mathbb{N}} U_n$ is dense in (X, τ_1) and hence in (X, τ_2) , so that $U_n \cap (\cup_{n \in \mathbb{N}} \mathcal{V}_n) \neq \emptyset$. This implies that for each $n \in \mathbb{N}$, there exists $V_n \in \mathcal{V}_n$, such that $U_n \cap V_n \neq \emptyset$. Therefore, for each $n \in \mathbb{N}$,

$$\text{St}(U_n, \mathcal{U}) \subset \text{St}(\text{St}(V_n, \mathcal{U}), \mathcal{U}) = \text{St}^2(V_n, \mathcal{U}), \quad (4)$$

which implies

$$X = \text{St}(A, \mathcal{U}) \subset \text{St}^2\left(\cup_{n \in \mathbb{N}} V_n, \mathcal{U}\right). \quad (5)$$

This shows that (X, τ_1, τ_2) is $(1, 2)$ -selectively 2-star-ccc. $\square$

Theorem 11. Every (i, j) -selectively ccc bispase (X, τ_1, τ_2) is (i, j) -selectively star-ccc.

Proof. (For $i = 1, j = 2$.) Let $\mathcal{C}$ be a τ_2 -open cover of X and $(\mathcal{U}_n: n \in \mathbb{N})$ be a sequence of τ_1 -open weak covers of X . Since X is $(1, 2)$ -selectively ccc, there exists a sequence $(U_n: n \in \mathbb{N})$, such that for every n , $U_n \in \mathcal{U}_n$, and $M = \cup_{n \in \mathbb{N}} U_n$ is dense in (X, τ_2) . So, M intersects each element of $\mathcal{C}$, and thus,

$$\text{St}\left(\cup_{n \in \mathbb{N}} U_n, \mathcal{C}\right) = X, \quad (6)$$

where X is a $(1, 2)$ -selectively star-ccc space.

From Proposition 3 and Theorem 11, we have the following corollary. $\square$

Corollary 6. Every $R_{(\tau_i, \tau_j)}$ -separable bispase is (i, j) -selectively star-ccc.

We show now that the converse in this corollary is not always true.

Example 2. Let (X, τ_1, τ_2) be a bispase with $\tau_1 \geq \tau_2$, such that (X, τ_2) is a compact and nonseparable space. By Theorem 9, the bitopological space (X, τ_1, τ_2) is $(1, 2)$ -selectively

star-ccc. But it is not $R_{(\tau_1, \tau_2)}$ -separable by the following fact: $R_{(\tau_1, \tau_2)}$ -separability implies separability of (X, τ_2) .

4. Covering Properties and Relations on Bispace

In this section, we will investigate the behaviour and preservation of covering properties studied in this study under various relations (as $<_C$ and $<_N$) between two topologies on a bitopological space (X, τ_1, τ_2) .

In 1957, Weston [35] introduced the C-relation, while in [30], the N-relation was defined. The abovementioned relations and their combinations with the set-theoretic operation inclusion are considered in [30].

Let (X, τ_1, τ_2) be a bitopological space. Obviously, $\tau_2 < \tau_1$ implies $Cl_{\tau_1}(A) \subset Cl_{\tau_2}(A)$ for each $A \subset X$.

Definition 7 (see [30, 35]). The topology τ_1 is coupled to the topology τ_2 on a set X , denoted by $\tau_1 C \tau_2$, if $Cl_{\tau_1}(A) \subset Cl_{\tau_2}(A)$ for each $A \in \tau_1$.

Remark 3. (see [30]). If the open sets and closed sets are equal for a topology τ_1 on a set X , then τ_1 is coupled to every topology on X . For example, the discrete and indiscrete topologies are coupled to each topology on X .

The C-relation defines a partial order $\leq$ (we use the notation $<_C$) on the family of all topologies on X by

$$\tau_1 <_C \tau_2 \Leftrightarrow (\tau_1 C \tau_2 \wedge \tau_1 < \tau_2). \quad (7)$$

It is easy to see that $\tau_1 <_C \tau_2$ implies $(\tau_1 C \tau_2 \wedge \tau_2 C \tau_1)$, but the reverse implication is not correct.

Example 3. Let $X = \{a, b, c, d, e\}$, $\tau_1 = \{\emptyset, \{e\}, \{a, b, c, d\}, X\}$, and $\tau_2 = \{\emptyset, \{c\}, \{a, b, d, e\}, X\}$. It is clear that $\tau_1 <_C \tau_2$ and $\tau_2 <_C \tau_1$, but these topologies are not comparable by inclusion.

A set X together with the topologies $\tau_1 C \tau_2$ ($\tau_1 <_C \tau_2$) is denoted by $(X, \tau_1 C \tau_2)$ ($(X, \tau_1 <_C \tau_2)$).

Now, we consider the N-relation between two topologies on a set X .

Definition 8. (see [30]). A topology τ_1 is near to a topology τ_2 on a set X (briefly $\tau_1 N \tau_2$) if $Cl_{\tau_1}(A) \subset Cl_{\tau_2}(A)$ for each $A \in \tau_2$.

The discrete topology on X is near any topology on X , and every topology on X is near the indiscrete topology on X .

Similar to the C-relation, the N-relation defines a partial order on the family of all topologies on X by

$$\tau_1 <_N \tau_2 \Leftrightarrow (\tau_1 N \tau_2 \wedge \tau_1 < \tau_2). \quad (8)$$

We mention that the notions of coupling and nearness of topologies are independent of each other; however, we have the following implication:

$$\tau_1 <_N \tau_2 \Rightarrow \tau_1 <_C \tau_2. \quad (9)$$

A set X together with the topologies $\tau_1 N \tau_2$ ($\tau_1 <_N \tau_2$) is denoted by $(X, \tau_1 N \tau_2)$ ($(X, \tau_1 <_N \tau_2)$).

We omit the proof of the following proposition.

Proposition 4. Let $(X, \tau_1 C \tau_2)$ be a bitopological space. If (X, τ_1) is almost Menger, then $(X, \tau_1 C \tau_2)$ is $(1, 2)$ -almost Menger.

However, in the following example, we show that the reverse implication is not true in general.

Example 4. Let X be the Euclidean plane $\mathbb{R}^2$, τ_1 be the Sorgenfrey topology, and τ_2 be the ordinary metric topology on X . Clearly, $(X, \tau_1 C \tau_2)$ is $(1, 2)$ -almost Menger; however, (X, τ_1) does not have the almost Menger property, since (X, τ_1) is not almost Lindelöf.

On the other hand, we have the following result.

Theorem 12. A bispace $(X, \tau_1 <_C \tau_2)$ is $(1, 2)$ -almost Menger if and only if (X, τ_1) is almost Menger.

Proof. Let $(\mathcal{U}_n: n \in \mathbb{N})$ be a sequence of τ_1 -open covers of X . Since $(X, \tau_1 <_C \tau_2)$ is $(1, 2)$ -almost Menger, there exists a sequence $(\mathcal{V}_n: n \in \mathbb{N})$ of finite sets with $\mathcal{V}_n \subset \mathcal{U}_n$ and

$$\bigcup_{n \in \mathbb{N}} \bigcup \{Cl_{\tau_2}(V): V \in \mathcal{V}_n\} = X. \quad (10)$$

By $\tau_1 <_C \tau_2$, we have $\tau_1 C \tau_2$ and $\tau_1 < \tau_2$. For $V \in \tau_1 < \tau_2$, we have $Cl_{\tau_2}(V) \subset Cl_{\tau_1}(V)$. Therefore, $\bigcup_{n \in \mathbb{N}} \bigcup \{Cl_{\tau_1}(V): V \in \mathcal{V}_n\} = X$ gives that (X, τ_1) is almost Menger.

Let $(\mathcal{U}_n: n \in \mathbb{N})$ be a sequence of τ_1 -open covers of X . By the almost Menger property of (X, τ_1) , one can find finite sets $\mathcal{V}_n \subset \mathcal{U}_n$, $n \in \mathbb{N}$, such that $\bigcup_{n \in \mathbb{N}} \bigcup \{Cl_{\tau_1}(V): V \in \mathcal{V}_n\} = X$. Because $\tau_1 < \tau_2$, each $\mathcal{V}_n$ is a family of τ_2 -open sets. So, for each n and each $V \in \mathcal{V}_n$, $\tau_1 <_C \tau_2$ implies $Cl_{\tau_1}(V) \subset Cl_{\tau_2}(V)$. It follows from here that

$$\begin{aligned} X &= \bigcup_{n \in \mathbb{N}} \bigcup \{Cl_{\tau_1}(V): V \in \mathcal{V}_n\} \\ &\subset \bigcup_{n \in \mathbb{N}} \bigcup \{Cl_{\tau_2}(V): V \in \mathcal{V}_n\} = X. \end{aligned} \quad (11)$$

Therefore, (X, τ_1, τ_2) is (τ_1, τ_2) -almost Menger.

It should be noted that the Proposition 4 extends and generalizes the following result obtained by Özçağ and Eysen in [20]. $\square$

Corollary 7. If (X, τ_1) is almost Menger and $\tau_2 < \tau_1$, then the bitopological space (X, τ_1, τ_2) is $(1, 2)$ -almost Menger.

Similar to the proof of the first part of Theorem 12, we prove the following.

Theorem 13. If $(X, \tau_1 <_N \tau_2)$ is $(1, 2)$ -almost Menger, then (X, τ_1) is almost Menger.

Since our main aim was, as mentioned in the introduction, to investigate the behaviour of some covering properties under several functions, we turn to mappings between bitopological spaces equipped with some relations.

The classes of all (i, j) - θ -continuous functions of X to Y are denoted by $(i, j) - \theta C(X, Y)$.

Corollary 8 (see [30]). *Let (X, τ_1, τ_2) and (Y, σ_1, σ_2) be bitopological spaces. For $(X, \tau_1 <_C \tau_2)$ and $(Y, \sigma_1 <_C \sigma_2)$, we have*

$$(1, 2) - \theta C(X, Y) = 1 - \theta C(X, Y). \quad (12)$$

Theorem 14. *Let $f: (X, \tau_1 <_C \tau_2) \longrightarrow (Y, \sigma_1 <_C \sigma_2)$ be $(1, 2)$ - θ -continuous mapping. If $(X, \tau_1 <_C \tau_2)$ is $(1, 2)$ -almost Menger, then*

Proof. Let $(X, \tau_1 <_C \tau_2)$ be an $(1, 2)$ -almost Menger bitopological space. By Theorem 12, (X, τ_1) is almost Menger, and the mapping $f_i: (X, \tau_1) \longrightarrow (Y, \sigma_1)$ is θ -continuous by Corollary 8. Since θ -continuous image of an almost Menger space is almost Menger ([36], Theorem 2.1), then (Y, σ_1) is almost Menger. Thus, $(Y, \sigma_1 <_C \sigma_2)$ is $(1, 2)$ -almost Menger by Theorem 12. $\square$

Corollary 9 (see [30]). *The following conditions are satisfied for a bitopological space $(X, \tau_1 C \tau_2)$.*

- (1) *If (X, τ_1, τ_2) is $(1, 2)$ -regular, then (X, τ_1, τ_2) is 1-regular*
Moreover, for a bitopological space $(X, \tau_1 <_C \tau_2)$, we have
- (2) *(X, τ_1, τ_2) is $(1, 2)$ -regular $\Leftrightarrow (X, \tau_1, \tau_2)$ is 1-regular*

Theorem 15. *Let $(X, \tau_1 <_C \tau_2)$ be $(1, 2)$ -regular and $(1, 2)$ -almost Menger bitopological space. If $f: (X, \tau_1, \tau_2) \longrightarrow (Y, \sigma_1, \sigma_2)$ is 1-continuous mapping, then $(Y, \sigma_1 <_C \sigma_2)$ is $(1, 2)$ -almost Menger.*

Proof. Let $(X, \tau_1 <_C \tau_2)$ be $(1, 2)$ -almost Menger and $(1, 2)$ -regular bitopological space, then (X, τ_1) is almost Menger by Theorem 12 and (X, τ_1) is regular by Corollary 9; thus, we obtain that (X, τ_1) is Menger ([10], Theorem 2.1). Since $f: (X, \tau_1) \longrightarrow (Y, \sigma_1)$ is continuous and continuous image of a Menger space is Menger, then (Y, σ_1) is Menger and (Y, σ_1) is almost Menger. By using Theorem 12, $(Y, \sigma_1 <_C \sigma_2)$ is $(1, 2)$ -almost Menger. $\square$

Proposition 5. *If $f: (X, \tau_1 C \tau_2) \longrightarrow (Y, \sigma_1, \sigma_2)$ is $(1, 2)$ -strongly θ -continuous mapping, then f is 1-strongly θ -continuous.*

Proof. Let $f: (X, \tau_1 C \tau_2) \longrightarrow (Y, \sigma_1, \sigma_2)$ be $(1, 2)$ -strongly θ -continuous mapping. Let $x \in X$ and V be σ_1 -open set in Y containing $f(x)$. Since f is $(1, 2)$ -strongly θ -continuous mapping, there is a τ_1 -open set U in X containing x , such that $f(\text{Cl}_{\tau_2}(U)) \subseteq V$. Since $\tau_1 C \tau_2$, we have $f(\text{Cl}_{\tau_1}(U)) \subseteq f(\text{Cl}_{\tau_2}(U)) \subseteq V$; thus, f is 1-strongly θ -continuous. $\square$

Proposition 6. *$f: (X, \tau_1 <_C \tau_2) \longrightarrow (Y, \sigma_1, \sigma_2)$ is $(1, 2)$ -strongly θ -continuous mapping if and only if f is 1-strongly θ -continuous.*

Proof. Immediate. $\square$

Corollary 10. *Let $f: (X, \tau_1 <_C \tau_2) \longrightarrow (Y, \sigma_1, \sigma_1)$ be (i, j) -strongly θ -continuous mapping. If $(X, \tau_1 <_C \tau_2)$ is $(1, 2)$ -almost Menger, then (Y, σ_1) is Menger.*

Proof. If $(X, \tau_1 <_C \tau_2)$ is (i, j) -almost Menger, then by Proposition 4, (X, τ_1) is almost Menger. Since $f: (X, \tau_1) \longrightarrow (Y, \sigma_1)$ is strongly θ -continuous by the Proposition 5, we obtain that (Y, σ_1) is Menger by ([36], Theorem 2.7). $\square$

4.1. More on Relations and Covering Properties

Example 5. Lindelöfness is not preserved by the order relation $<_C$.

- (1) Let (X, τ_1, τ_2) be the plane $\mathbb{R}^2$ equipped with the following two topologies: τ_1 is the usual Euclidean topology, and τ_2 is the deleted radius topology ([37], example 77). Then, $\tau_1 < \tau_2$, and for any open set $U \in \tau_1$, $\text{Cl}_{\tau_1}(U) = \text{Cl}_{\tau_2}(U)$, i.e., $\tau_1 <_C \tau_2$. (X, τ_1) is a Lindelöf spaces, while (X, τ_2) is not ([37], example 77).
- (2) Another such example is the set $X = \{(x, y) \in \mathbb{R}^2: y \geq 0\}$ with the Euclidean topology (τ_1) and the Niemytzki plane topology (τ_2)

Example 6. σ -compactness is not preserved by the order relation $<_C$

Let (X, τ_1, τ_2) be the plane $\mathbb{R}^2$ equipped with the following two topologies: τ_1 is the usual Euclidean topology, and τ_2 is the deleted radius topology ([37], example 77). It is clear that $\tau_1 <_C \tau_2$. (X, τ_1) is σ -compact, but (X, τ_2) is not σ -compact.

Remark 4. Being σ -compact, the space (X, τ_1) in the previous example is Alster [8]. However, (X, τ_2) is not Alster because it is not Menger (Alster spaces are Hurewicz; hence, Menger in all finite powers).

Example 7. Mengeress is not preserved by the order relation $<_N$.

Let $X = \{(x, y) \in \mathbb{R}^2: y \geq 0\}$. Let $Y = \{(x, y) \in \mathbb{R}^2: y > 0\}$. Let τ_1 be the usual metric topology on X , and τ_2 , the half-disk topology ([37], example 78), is defined as follows: any point $p \in Y$ has the Euclidean local base, while the point $p = (x, 0) \in X \setminus Y$ has basic neighbourhoods of the form $B(p, \varepsilon) \cap Y \cup \{p\}$. Then, $\tau_1 < \tau_2$, and for any τ_2 -open set U , $\text{Cl}_{\tau_2}(U) = \text{Cl}_{\tau_1}(U)$, so that $\tau_1 <_N \tau_2$. The space (X, τ_1) is Menger, while (X, τ_2) is not Menger because it is not Lindelöf.

TABLE 1: The preservation of some covering properties under various relations.

	Property	$\tau_1 \subset \tau_2$	$\tau_1 \subset_N \tau_2$	$\tau_1 \leq_C \tau_2$	$\tau_1 \leq_N \tau_2$
1	τ_1 L			τ_2 L	
2	$\tau_1 \sigma$ -com			$\tau_2 \sigma$ -com	
3	τ_1 M			τ_2 M	
4	τ_2 aM				τ_1 aM
5	τ_2 wM				τ_1 wM
6	τ_1 H			τ_2 H	
7	τ_2 R		τ_1 R		
8	τ_1 Alster			τ_2 Alster	
9	τ_2 sel ccc			τ_1 sel ccc	

Example 8. Hurewicz property is not preserved by the order relation $<_C$.

Let (X, τ_1, τ_2) be the plane $\mathbb{R}^2$ equipped with the following two topologies: τ_1 is the usual Euclidean topology, and τ_2 is the deleted diameter topology ([37], Example 77). It is clear that $\tau_1 <_C \tau_2$. (X, τ_1) is the Hurewicz, but (X, τ_2) is not Hurewicz because it is not Lindelöf.

Another such example is the closed upper half-plane with the Euclidean topology τ_1 and the half-disc topology τ_2 ([37], example 78).

Example 9. Let X be any uncountable scattered compact space (for instance, the product $[0, \omega_1] \times [0, \omega]$ of ordinal spaces) with the topology τ_2 , and let τ_1 be the discrete topology on X . Then, $\tau_1 \subset_N \tau_2$, (X, τ_2) is a Rothberger space, and (X, τ_1) is not Rothberger.

Theorem 16. *If $\tau_1 <_N \tau_2$ and (X, τ_2) is almost Menger, then (X, τ_1) is almost Menger.*

Proof. Let $(\mathcal{U}_n; n \in \mathbb{N})$ be a sequence of open covers of (X, τ_1) . As $\tau_1 <_N \tau_2$, it is a sequence of open covers of (X, τ_2) . By assumption, there are finite sets $\mathcal{V}_n \subset \mathcal{U}_n, n \in \mathbb{N}$, so that $X = \bigcup_{n \in \mathbb{N}} \text{Cl}_{\tau_2}(\bigcup \mathcal{V}_n)$. From $\tau_1 <_N \tau_2$ it follows $\text{Cl}_{\tau_2}(\bigcup \mathcal{V}_n) = \text{Cl}_{\tau_1}(\bigcup \mathcal{V}_n)$ for each $n \in \mathbb{N}$, which means that (X, τ_1) is almost Menger.

Similarly, one obtains. $\square$

Theorem 17. *If $\tau_1 <_N \tau_2$ and (X, τ_2) is weakly Menger, then (X, τ_1) is weakly Menger.*

Theorem 18. *Let (X, τ_1, τ_2) be a bitopological space. If $\tau_1 <_C \tau_2$ and (X, τ_2) is selectively ccc, then (X, τ_1) is selectively ccc.*

Proof. Let $(\mathcal{U}_n; n \in \mathbb{N})$ be a sequence of τ_1 -open weak cover of X . Then, $\bigcup \mathcal{U}_n \in \tau_1$, $\tau_1 <_C \tau_2$ and $\text{Cl}_{\tau_1}(\bigcup \mathcal{U}_n) = X$, $n \in \mathbb{N}$, imply that for each n , $\mathcal{U}_n \in \tau_2$, and $\text{Cl}_{\tau_2}(\bigcup \mathcal{U}_n) = X$. As (X, τ_2) is selectively ccc, there are $U_n \in \mathcal{U}_n, n \in \mathbb{N}$, such that $\text{Cl}_{\tau_2}(\bigcup_{n \in \mathbb{N}} U_n) = X$. From $\tau_1 <_C \tau_2$, it follows $\text{Cl}_{\tau_1}(\bigcup_{n \in \mathbb{N}} U_n) = \text{Cl}_{\tau_2}(\bigcup_{n \in \mathbb{N}} U_n) = X$, which means that (X, τ_1) is selectively ccc. $\square$

Theorem 19. *Let (X, τ_1, τ_2) be a bitopological space. If $\tau_1 <_C \tau_2$ and (X, τ_2) is selectively star-ccc, then (X, τ_1) is selectively star-ccc.*

Proof. Let $(\mathcal{U}_n)_{n \in \mathbb{N}}$ be a sequence of open weak covers of (X, τ_1) and $\mathcal{U}$ a τ_1 -open cover of X . Since $\bigcup \mathcal{U}_n \in \tau_1, n \in \mathbb{N}$, and $\tau_1 <_C \tau_2$, we have $\text{Cl}_{\tau_1}(\bigcup \mathcal{U}_n) \subset \text{Cl}_{\tau_2}(\bigcup \mathcal{U}_n) = X$. Also, $\mathcal{U}$ is a τ_2 -open cover of X . Using the fact that (X, τ_2) is selectively star-ccc, we find $U_n \in \mathcal{U}_n, n \in \mathbb{N}$, such that $\text{St}(\bigcup_{n \in \mathbb{N}} U_n, \mathcal{U}) = X$, i.e., (X, τ_1) is selectively star-ccc.

Table 1 summarizes the above results. Empty fields in the table may serve as open questions. $\square$

5. Conclusion

This study is a continuation and extension of our research in the field of selective covering properties in bitopological spaces. Complementing some earlier studies on almost Menger and almost Rothberger bitopological spaces, we investigated preservation of these properties under various classes of mappings between bitopological spaces. Next, we introduced and studied in detail the selective bitopological versions of the topological ccc property called selectively ccc and selectively star-ccc property. Relations of these bitopological properties with other known properties are established. In the last part of the study, we discussed (non) preservation of a number of topological and bitopological covering properties under known relations between the two topologies. We believe that a natural continuation of our study may be findings of relations of the properties studied in this study with game theory, function spaces, and hyperspaces.

Data Availability

The data used to support the findings of this study are included within the article as references and are available from the corresponding author upon request.

Conflicts of Interest

The authors declare that there are no conflicts of interest.

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