Research Article

(α, BK) -Contractions in Bipolar Metric Spaces

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In this paper, we introduce the concept of (α, BK) -contractions and prove the existence of fixed points for contravariant mappings on bipolar metric spaces.

1. Introduction and Preliminaries

The notion of a metric space has many generalizations in literature. One of the most recent generalizations is that of a *bipolar metric space*, introduced by Mutlu and Gürdal [1], the motivation being that in many real-life applications, distances arise between elements of two different sets, rather than between points of a unique set. Hence, bipolar metrics came to formalize these types of distances. Some basic examples are distance between lines and points in an Euclidean space, distance between points and sets in a metric spaces, affinity between a class of students and a set of activities, lifetime mean distances between people and places, and many more. Many investigators have published various papers on fixed point in bipolar metric spaces, and these can be read in [2–6] and the references therein, just to name a few. This new notion of generalization and improvement of metric spaces leads to give existence and development of fixed-point theorems. In bipolar metric spaces, a lot of significant work has been done (see [3, 7–9]). More recently, Kishore et al. [7] established the existence and uniqueness of common coupled fixed-point results for three covariant mappings in bipolar metric spaces, while Mutlu [8] extended

certain coupled fixed-point theorems, which can be considered as generalization of Banach fixed-point theorem, to bipolar metric spaces. Hence, bipolar metric fixed-point theory is an active research area which is capturing a lot of attention and seems to be a prospective flourishing domain.

In the present paper, we introduce the so-called (α, BK) -contractions and prove the existence of fixed points for contravariant mappings on bipolar metric spaces. We illustrate, how, under the choice of convenient constants, we can easily recover some well-known classical results. A detailed account of the theory of bipolar metric spaces can be read in the paper by Mutlu and Gürdal [1], but for the sake of completeness, we recall the definitions of some fundamental notions related to bipolar metric spaces.

Definition 1 (see [1], Definition 2.1). A bipolar metric space is a triple (X, Y, d) such that X, Y are two nonempty sets and $d: X \times Y \longrightarrow \mathbb{R}_+ = [0, +\infty)$ is a function satisfying the following properties:

- (1) $d(x, y) = 0 \Leftrightarrow x = y$, whenever $(x, y) \in X \times Y$
- (2) $d(x, y) = d(y, x)$, whenever $x, y \in X \cap Y$

- (3) $d(x_1, y_2) \leq d(x_1, y_1) + d(x_2, y_1) + d(x_2, y_2)$, whenever $(x_1, y_1), (x_2, y_2) \in X \times Y$

The pair (X, Y) is called a bipolar metric.

Definition 2 (see [1], Definition 2.3). Let (X_1, Y_1) and (X_2, Y_2) be pairs of sets and given a function $f: X_1 \cup Y_1 \longrightarrow X_2 \cup Y_2$.

- (1) If $f(X_1) \subseteq Y_2$ and $f(Y_1) \subseteq X_2$, we call f a *contravariant map* from (X_1, Y_1) to (X_2, Y_2) and denote this with $f: (X_1, Y_1)(X_2, Y_2)$.
- (2) Moreover, if d_1 and d_2 are bipolar metrics on (X_1, Y_1) and (X_2, Y_2) , respectively, then the notation $f: (X_1, Y_1, d_1) \bowtie (X_2, Y_2, d_2)$ denotes a *contravariant map* from (X_1, Y_1, d_1) to (X_2, Y_2, d_2) .

Definition 3 (see [1], Definition 4.1). Let (X, Y, d) be a bipolar metric space.

- (1) We have

$$\begin{aligned} X &= \text{set of left points;} \\ Y &= \text{set of right points;} \\ X \cap Y &= \text{set of central points.} \end{aligned} \quad (1)$$

In particular, if $X \cap Y = \emptyset$, the space is called **disjoint**, and otherwise it is called **joint**. Unless otherwise stated, we shall work with joint spaces.

- (2) A sequence (x_n) on the set X is called a *left sequence* and a sequence (y_n) on Y is called a *right sequence*. In a bipolar metric space, a left or a right sequence is called simply a sequence.
- (3) A sequence (u_n) is said to be convergent to a point u , if and only if (u_n) is a left sequence, $\lim_{n \rightarrow \infty} d(u_n, u) = 0$ and $u \in Y$, or (u_n) is a right sequence, $\lim_{n \rightarrow \infty} d(u, u_n) = 0$ and $u \in X$.
- (4) A *bisequence* (x_n, y_n) on (X, Y, d) is a sequence on the set $X \times Y$. Furthermore, if the sequences (x_n) and (y_n) are convergent, then the bisequence (x_n, y_n) is said to be convergent. In addition, if (x_n) and (y_n) converge to a common point $t \in X \cap Y$, then (x_n, y_n) is called *biconvergent*.
- (5) A bisequence (x_n, y_n) is a Cauchy bisequence, if $\lim_{n \rightarrow \infty, m \rightarrow \infty} d(x_n, y_m) = 0$.

Remark 1 (see [1], Proposition 4.3). In a bipolar metric space, every convergent Cauchy bisequence is biconvergent.

Definition 4 (see [1], Definition 4.4). A bipolar metric space is called complete, if every Cauchy bisequence is convergent, hence biconvergent.

Proposition 1 (see [1]). *If a central point is a limit of a sequence, then this sequence has a unique limit.*

Example 1 (see [1], Example 5.7). Let X be the class of all singleton subsets of $\mathbb{R}$ and Y be the class of all nonempty compact subsets of $\mathbb{R}$. We define $d: X \times Y \longrightarrow \mathbb{R}_+$ as

$$d(x, A) = |x - \inf(A)| + |x - \sup(A)|. \quad (2)$$

Then, the triple (X, Y, d) is a complete bipolar metric space.

Definition 5 (see [1], Definition 3.4). A covariant or a contravariant map f from the bipolar metric space (X_1, Y_1, d_1) to the bipolar metric space (X_2, Y_2, d_2) is continuous, if and only if $(u_n) \longrightarrow v$ on (X_1, Y_1, d_1) implies $(f(u_n)) \longrightarrow f(v)$ on (X_2, Y_2, d_2) .

Proposition 2 (see [1]). *If the point to which a covariant or contravariant map f is left and right continuous is central point, then the map f is continuous at this point.*

2. First Results

In this section, we express a theorem based on Reich-type contractions on bipolar metric spaces. The results presented here will be generalized in the next section.

Theorem 1 (Reich-type contraction). *Let (X, Y, d) be a complete bipolar metric space and $f: (X, Y, d) \bowtie (X, Y, d)$ be a contravariant map such that there exist constants $a_1, a_2, a_3 \geq 0$ with $a_1 + a_2 + a_3 < 1$ so that*

$$d(fy, fx) \leq a_1 d(x, y) + a_2 d(x, fx) + a_3 d(fy, y), \quad (3)$$

whenever $(x, y) \in X \times Y$. Then the function $f: X \cup Y \longrightarrow X \cup Y$ has a unique fixed point.

Proof. Let $x_0 \in X$; for each nonnegative integer n , we define $y_n = f(x_n)$ and $x_{n+1} = f(y_n)$. Then, we have

$$\begin{aligned} d(x_n, y_n) &= d(fy_{n-1}, fx_n) \\ &\leq a_1 d(x_n, y_{n-1}) + a_2 d(x_n, fx_n) \\ &\quad + a_3 d(fy_{n-1}, y_{n-1}) \\ &= a_1 d(x_n, y_{n-1}) + a_2 d(x_n, y_n) + a_3 d(x_n, y_{n-1}) \\ &= (a_1 + a_3) d(x_n, y_{n-1}) + a_2 d(x_n, y_n), \end{aligned} \quad (4)$$

for all integers $n \geq 1$. Then,

$$d(x_n, y_n) \leq \frac{a_1 + a_3}{1 - a_2} d(x_n, y_{n-1}),$$

$$\begin{aligned} d(x_n, y_{n-1}) &= d(fy_{n-1}, fx_{n-1}) \\ &\leq a_1 d(x_{n-1}, y_{n-1}) + a_2 d(x_{n-1}, fx_{n-1}) \\ &\quad + a_3 d(fy_{n-1}, y_{n-1}) \\ &= a_1 d(x_{n-1}, y_{n-1}) + a_2 d(x_{n-1}, y_{n-1}) \\ &\quad + a_3 d(x_n, y_{n-1}), \end{aligned} \quad (5)$$

so that

$$d(x_n, y_{n-1}) \leq \frac{a_1 + a_2}{1 - a_3} d(x_{n-1}, y_{n-1}). \quad (6)$$

By setting

$$\begin{aligned} \lambda &= \frac{a_1 + a_3}{1 - a_2}, \\ \beta &= \frac{a_1 + a_2}{1 - a_3}, \end{aligned} \quad (7)$$

we see that $\lambda < 1$ and $\beta < 1$ since $a_1 + a_2 + a_3 = 1$.

Moreover, it is easy to see that

$$\begin{aligned} d(x_n, y_n) &\leq \lambda^{2n} d(x_0, y_0), \\ d(x_n, y_{n-1}) &\leq \beta^{2n-1} d(x_0, y_0). \end{aligned} \quad (8)$$

Hence, for all positive integers m and n , we have

(1) If $m > n$,

$$\begin{aligned} d(x_n, y_m) &\leq d(x_n, y_n) + d(x_{n+1}, y_n) + d(x_{n+1}, y_m) \\ &\leq (\lambda^{2n} + \beta^{2n+1}) d(x_0, y_0) + d(x_{n+1}, y_m) \\ &\vdots \\ &\leq (\lambda^{2n} + \beta^{2n+1} + \dots + \beta^{2m}) d(x_0, y_0). \end{aligned} \quad (9)$$

(2) If $m < n$,

$$\begin{aligned} d(x_n, y_m) &\leq d(x_{m+1}, y_m) + d(x_{m+1}, y_{m+1}) + d(x_n, y_{m+1}) \\ &\leq (\beta^{2m+1} + \lambda^{2m+2}) d(x_0, y_0) + d(x_n, y_{m+1}) \\ &\vdots \\ &\leq (\beta^{2m+1} + \lambda^{2m+2} + \beta^{2m+1} + \dots + \beta^{2n}) d(x_0, y_0) \\ &\quad + d(x_n, y_n) \\ &\leq (\beta^{2m+1} + \lambda^{2m+2} + \beta^{2m+1} + \dots + \beta^{2n} + \lambda^{2n}) \\ &\quad d(x_0, y_0). \end{aligned} \quad (10)$$

Since $\lambda < 1$ and $\beta < 1$, this means that $d(x_n, y_m)$ can be made arbitrarily small by larger m and n , and hence (x_n, y_m) is a Cauchy bisequence. Since (X, Y, d) is complete, (x_n, y_m) is convergent, and in fact biconvergent, since it is a convergent Cauchy bisequence. Let u be the point to which (x_n, y_m) biconverges. Then, $(x_n) \rightarrow u$, $(y_n) \rightarrow u$, and $u \in X \cap Y$. Also, $(y_n) = (fx_n) \rightarrow Tu$. Since (y_n) has a limit in $X \cap Y$, this limit is unique. Hence, $fu = u$, and so f has a fixed point.

If v is any fixed point of f , then $fv = v$ implies that v belongs to $X \cap Y$. Then,

$$\begin{aligned} d(u, v) &= d(Tu, Tv) \leq a_1 d(v, u) + a_2 d(u, Tu) + a_3 d(Tv, v) \\ &= a_1 d(v, u). \end{aligned} \quad (11)$$

It is a contradiction, except $u = v$. $\square$

Example 2. Let X be the class of all singleton subsets of $\mathbb{R}$ and Y be the class of all nonempty compact subsets of $\mathbb{R}$. We define $d: X \times Y \rightarrow \mathbb{R}$ as

$$d(x, A) = |x - \inf(A)| + |x + \sup(A)|. \quad (12)$$

Then, (X, Y, d) is a complete bipolar metric space. The contravariant map $f: (X, Y, d) \bowtie (X, Y, d)$, defined as

$$fA = \frac{\inf A + \sup A + 6}{8} \in \mathbb{R}, \quad (13)$$

for all $A \in X \cup Y$ satisfies inequality (3) with $a_1 = (11/15)$, $a_2 = (1/3)$, $a_3 = (2/5)$. Hence, f must have a unique fixed point, which in fact is the set $\{1\} \in X \cap Y$.

Now, we present some natural corollaries derived from Theorem 1.

Corollary 1 (Reich–Rus–Ćirić-type contraction). *Let (X, Y, d) be a complete bipolar metric space and $f: (X, Y, d) \bowtie (X, Y, d)$ be a contravariant map such that for some $\alpha < (1/3)$,*

$$d(fy, fx) \leq \alpha(d(x, y) + d(x, fx) + d(fy, y)), \quad (14)$$

whenever $(x, y) \in X \times Y$. Then, the function $f: X \cup Y \rightarrow X \cup Y$ has a unique fixed point.

Proof. Take $a_1 = a_2 = a_3 = \alpha$ in Theorem 1. $\square$

Corollary 2 (Kannan-type contraction). *Let (X, Y, d) be a complete bipolar metric space and $f: (X, Y, d) \bowtie (X, Y, d)$ be a contravariant map such that for some $\alpha < (1/2)$,*

$$d(fy, fx) \leq \alpha(d(x, fx) + d(fy, y)), \quad (15)$$

whenever $(x, y) \in X \times Y$. Then, the function $f: X \cup Y \rightarrow X \cup Y$ has a unique fixed point.

Proof. Take $a_1 = 0$, $a_2 = a_3 = \alpha$ in Theorem 1.

We conclude this section with the following natural extension of Theorem 1. The result came from the observation that

$$\begin{aligned} d(fy, fx) &\leq a_1 d(x, y) + a_2 d(x, fx) + a_3 d(fy, y) \\ &\leq (a_1 + a_2 + a_3) \max\{d(x, y), d(x, fx), d(fy, y)\}, \end{aligned} \quad (16)$$

i.e.,

$$d(fy, fx) \leq (a_1 + a_2 + a_3) \max\{d(x, y), d(x, fx), d(fy, y)\}. \quad (17) \quad \square$$

Theorem 2. *Let (X, Y, d) be a complete bipolar metric space and $f: (X, Y, d) \bowtie (X, Y, d)$ be a contravariant map such that there exists $0 < \lambda < 1$ so that*

$$d(fy, fx) \leq \max \lambda \{d(x, y), d(x, fx), d(fy, y)\}, \quad (18)$$

whenever $(x, y) \in X \times Y$. Then, the function $f: X \cup Y \rightarrow X \cup Y$ has a unique fixed point.

Proof. The proof follows closely the steps of the proof of Theorem 1.

Let $x_0 \in X$; for each nonnegative integer n , we define $y_n = f(x_n)$ and $x_{n+1} = f(y_n)$. Then, we have

$$\begin{aligned} d(x_n, y_n) &= d(fy_{n-1}, fx_n) \\ &\leq \lambda \max\{d(x_n, y_{n-1}), d(x_n, fx_n), d(fy_{n-1}, y_{n-1})\} \\ &= \lambda \max\{d(x_n, y_{n-1}), d(x_n, y_n), d(x_n, y_{n-1})\} \\ &= \lambda d(x_n, y_{n-1}). \end{aligned} \quad (19)$$

Similarly,

$$\begin{aligned} d(x_n, y_{n-1}) &= d(fy_{n-1}, fx_{n-1}) \\ &\leq \lambda \max\{d(x_{n-1}, y_{n-1}), d(x_{n-1}, fx_{n-1}), d(fy_{n-1}, y_{n-1})\} \\ &= \lambda \max\{d(x_{n-1}, y_{n-1}), d(x_{n-1}, y_n), d(x_n, y_{n-1})\} \\ &= \lambda d(x_{n-1}, y_{n-1}). \end{aligned} \quad (20)$$

Therefore, it is easy to see that

$$\begin{aligned} d(x_n, y_n) &\leq \lambda^{2n} d(x_0, y_0), \\ d(x_n, y_{n-1}) &\leq \lambda^{2n-1} d(x_0, y_0). \end{aligned} \quad (21)$$

The remaining part of the proof follows from that of Theorem 1. $\square$

3. Main Results

In this section, we present the generalization of the Reich-type contraction proved in the previous section. As we have pointed out in the introduction, the contractivity condition we will employ is quite general. Our framework is that of bipolar metric spaces. In the what comes next, we introduce the following fundamental definition.

Definition 6. Let (X, Y, d) be a bipolar metric space, $T: (X, Y, d) \bowtie (X, Y, d)$ be a contravariant map, and $\alpha = (\alpha_1, \alpha_2, \alpha_3)$, $\alpha_1, \alpha_2, \alpha_3 \geq 0$ be constants. Then, T is called an (α, BK) -contraction if there exist two functions $\psi, \phi: (0, \infty) \rightarrow \mathbb{R}$:

$$\psi(d(Ty, Tx)) \leq \phi[a_1(d(x, y)) + a_2(d(x, Tx)) + a_3(d(Ty, y))], \quad (22)$$

whenever $(x, y) \in X \times Y$ and $d(Ty, Tx) > 0$.

For simplicity, we shall put

$$\text{BK}_T(\alpha, x, y) = \alpha_1(d(x, y)) + \alpha_2(d(x, Tx)) + \alpha_3(d(Ty, y)), \quad (23)$$

where $\alpha = (\alpha_1, \alpha_2, \alpha_3)$ so that inequality (22) amounts to

$$\psi(d(Ty, Tx)) \leq \phi[\text{BK}_T(\alpha, x, y)], \quad (24)$$

whenever $(x, y) \in X \times Y$ and $d(Ty, Tx) > 0$. In the first part of this section, we shall study, in relation with the functions ψ and ϕ , the existence of a fixed point for an (α, BK) -contraction with respect to the sign of $\text{BK}_T(x, y)$.

We begin with the following proposition.

Proposition 3. Let (X, Y, d) be a bipolar metric space and $T: (X, Y, d) \bowtie (X, Y, d)$ be a contravariant map. Suppose that there are two distinct points $x_0 \in X, y_0 \in Y$ such that $\text{BK}_T(\alpha, x_0, y_0) = 0$, where BK_T is defined in (23). Then, $\alpha_1 = 0$, and at least one of the following three statements holds:

- (1) $\alpha_i = 0$ for all $i \in \{1, 2, 3\}$. Then, $\text{BK}_T(\alpha, x, y) = 0$ whenever $(x, y) \in X \times Y$.
- (2) $\alpha_2 \neq 0$ and x_0 is a fixed point of T .
- (3) $\alpha_3 \neq 0$ and y_0 is a fixed point of T .

Proof

- (1) If $\alpha_i = 0$ for all $i \in \{1, 2, 3\}$, then the first case holds.
- (2) By the way of contradiction, assume that some α_i is distinct from zero. Since $\text{BK}_T(\alpha, x_0, y_0) = 0$ and $\alpha_i \geq 0$ for all $i \in \{1, 2, 3\}$, then

$$\alpha_1 d(x_0, y_0) = \alpha_2 d(x_0, Tx_0) = \alpha_3 d(Ty_0, y_0) = 0. \quad (25)$$

Since $d(x_0, y_0) > 0$, then necessarily $\alpha_1 = 0$ (observe that this is already an hypothesis). If $\alpha_2 \neq 0$, then $d(x_0, Tx_0) = 0$, so x_0 is a fixed point of T and the second case holds.

- (3) If $\alpha_2 = 0$ and $\alpha_3 \neq 0$, then $d(Ty_0, y_0) = 0$, so y_0 is a fixed point of T and the third case holds.

From the above discussion, the following corollaries arise naturally. $\square$

Corollary 3. Let (X, Y, d) be a bipolar metric space, $T: (X, Y, d) \bowtie (X, Y, d)$ be a contravariant map, and $\text{BK}_T(\alpha, x, y)$ be as defined in (23). Then, the following hold:

- (1) If $\alpha_1 > 0$, then $\text{BK}_T(\alpha, x, y) > 0$ for all distinct points $(x, y) \in X \times Y$.
- (2) If $\alpha_2 > 0$ and there are $x_0 \in X, y_0 \in Y$ such that $\text{BK}_T(\alpha, x_0, y_0) = 0$, then x_0 is a fixed point of T .
- (3) If $\alpha_3 > 0$ and there are $x_0 \in X, y_0 \in Y$ such that $\text{BK}_T(\alpha, x_0, y_0) = 0$, then y_0 is a fixed point of T .
- (4) If $\alpha_1 + \alpha_2 + \alpha_3 > 0$, then either T admits a fixed point or $\text{BK}_T(\alpha, x, y) > 0$ for all distinct points $(x, y) \in X \times Y$.

The two preceding results give us a measure of the cardinality of $F(T) = \{x \in X \cup Y: Tx = x\}$, although inequality (22) could be satisfied. Indeed, if we let $X = \{x_0\}, Y = \{y_0\}$, where $x_0 \neq y_0$, $d(x, y) = 2$, and let us define $T: (X, Y, d) \bowtie (X, Y, d)$ by $Tx_0 = y_0$ and $Ty_0 = x_0$. Then, T is fixed-point-free. However, if $\alpha_1 = \alpha_2 = \alpha_3 = 0$, then $\text{BK}_T(\alpha, x_0, y_0) = 0$.

Corollary 4. Let (X, Y, d) be a bipolar metric space, $T: (X, Y, d) \bowtie (X, Y, d)$ be a contravariant map, and $BK_T(\alpha, x, y)$ be as defined in (23). Suppose that T is fixed-point-free. If $\alpha_1 + \alpha_2 + \alpha_3 > 0$, then $BK_T(\alpha, x, y) > 0$ for all distinct $(x, y) \in X \times Y$.

Remark 2. When $\alpha_1 = 0$, the equality $BK_T(\alpha, x_0, y_0) = 0$ implies that x_0 or y_0 depending on whether $\alpha_2 \neq 0$ or $\alpha_3 \neq 0$. Hence, an idea to guarantee that $BK_T(\alpha, x, y) > 0$ for all for all $(x, y) \in X \times Y$ such that $d(Ty, Tx) > 0$ follows from the assumption that $\alpha_1 \neq 0$.

We have described the cautions to be observed when applying contractivity condition (22). In the last part of this section, we give and prove the two main theorems of this paper.

Theorem 3. Let (X, Y, d) be a bipolar metric space, $T: (X, Y, d) \bowtie (X, Y, d)$ be a contravariant map, and $\alpha = (\alpha_1, \alpha_2, \alpha_3), \alpha_1, \alpha_2, \alpha_3 \geq 0$ be constants. Let T be an (α, BK) -contraction, where the two functions $\psi, \phi: (0, \infty) \rightarrow \mathbb{R}$ are such that $\phi(t) < \psi(t)$ for any $t > 0$ and ψ is nondecreasing and $0 < \alpha_1 \leq 1$. Then, T admits at most one fixed point.

Proof. Observe that for the contravariant map T , any fixed point x^* necessarily belongs to $X \cap Y$. Indeed if $x^* \in X$ is a fixed point for T , then $x^* \in X$ and $x^* = Tx^* \in Y$ and similarly if we assume that $x^* \in Y$. For uniqueness of the fixed point, in case of existence, suppose that T admits two distinct fixed points, that is, there are $a, b \in X \cap Y$ such that $Ta = a \neq b = Tb$. Then, $d(a, b) > 0$ and

$$\begin{aligned} BK_T(\alpha, a, b) &= \alpha_1 d(a, b) + \alpha_2 d(a, Ta) + \alpha_3 d(Tb, b) \\ &= \alpha_1 d(a, b). \end{aligned} \quad (26)$$

Therefore, $BK_T(\alpha, a, b) > 0$ because $\alpha_1 > 0$ and $d(a, b) > 0$. Hence, condition (22) can be applied because $d(Tb, Ta) = d(Ta, Tb) > 0$, and it guarantees that

$$\psi(d(Tb, Ta)) \leq \phi(BK_T(\alpha, a, b)). \quad (27)$$

Since $0 < \alpha_1 \leq 1$ and using the hypotheses on ψ and ϕ , one has

$$\begin{aligned} \psi(d(Tb, Ta)) &= \psi(d(Ta, Tb)) = \psi(d(a, b)) \leq \phi(BK_T(\alpha, a, b)) \\ &= \phi(\alpha_1 d(a, b)) \\ &\leq \psi(\alpha_1 d(a, b)) \\ &\leq \psi(d(a, b)), \end{aligned} \quad (28)$$

which is a contradiction. Hence, we can conclude that T admits, at most, a unique fixed point.

In the following result, the uniqueness of the fixed point will be deduced from Theorem 3. $\square$

Theorem 4. Let (X, Y, d) be a bipolar metric space, $T: (X, Y, d) \bowtie (X, Y, d)$ be a contravariant map, and $\alpha = (\alpha_1, \alpha_2, \alpha_3), \alpha_1, \alpha_2, \alpha_3 \geq 0$ be constants. Let T be an (α, BK) -contraction, where the two functions $\psi, \phi: (0, \infty) \rightarrow \mathbb{R}$ are

such that $\phi(t) < \psi(t)$ for any $t > 0$ and ψ is nondecreasing and $0 < \alpha_1 + \alpha_2 + \alpha_3 < 1$. Then, T admits exactly one fixed point.

4. Conclusion

In this paper, using the new framework of bipolar metric spaces, we introduced the concept of (α, BK) -contractions and proved the existence of fixed points for contravariant mappings. It was pointed out that (α, BK) -contractions are a special extension of Reich-type contractions for which the existence of fixed point need not follow the traditional routine. An interesting future study could then be to look at the implications of these results in the induced metric space. We might get interesting feedback. Moreover, one could also be interested in the formulation of (α, BK) -contractions for covariant maps.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that they have no conflicts of interest concerning the publication of this article.

Authors' Contributions

All authors contributed equally and significantly in writing this article.

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Y. U. Gaba, dedicates this paper to his mother, Gilberte A. Epse G., on the occasion of her 65th birthday.

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