Research Article

Recent Fixed-Point Results for θ – Contraction Mappings in Rectangular M – Metric Spaces with Supportive Application

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The goal of this manuscript is to present a new fixed-point theorem on θ –contraction mappings in the setting of rectangular M -metric spaces (RMMSs). Also, a nontrivial example to illustrate our main result has been given. Moreover, some related sequences with θ –contraction mappings have been discussed. Ultimately, our theoretical result has been implicated to study the existence and uniqueness of the solution to a nonlinear integral equation (NIE).

1. Introduction

Fixed-point theory is one of the branches of functional analysis known as “Banach contraction principle (BCP)” [1] which plays a main role in several branches of mathematics and applied sciences. Its core subject is concerned with the stipulations for the existence of one or more fixed points of a mapping T from a topological space $\aleph$ into itself, whereas we can obtain a fixed point such that $T1 = 1$ and $1 \in \aleph$. Later, this principle has been extended by using different forms of contractive conditions in various spaces (for example, see [2–7]).

The idea of a generalized metric space (GMS) (rectangular metric space or Branciari metric space) was introduced by Branciari [8] in 2000, where the triangle inequality is replaced by the inequality $d(1, \mathcal{J}) \leq d(1, \ell) + d(\ell, \hbar) + d(\hbar, \mathcal{J})$ for all pairwise distinct points $1, \mathcal{J}, \ell, \hbar \in \aleph$. Under this space, many fixed-point results were proposed and nice applications were discussed (see [9–13]).

Recently, GMSs extended to recent versions such as M –metric spaces (MMSs), rectangular metric spaces

(RMSs), extended rectangular b-metric spaces (ERbMSs), partial rectangular b-metric spaces (PRbMSs), and partial rectangular metric spaces (PRMSs) (see [8, 14–17]).

In 2018, a new version of a metric space was introduced by Özgür et al. [18], called a rectangular M –metric space (RMMS) which is also extended (MMS). Shukla [17] defined a PRMS which is the generalized RMS. In such spaces, the authors [17, 18] discussed and presented some fixed-point theorems for self-mappings. They presented some topological concepts about open balls and convergence and defined the notions of a circle and a fixed circle using these concepts with an application to fixed-circle problems. For more details of generalizing fixed-point results, see [8, 14, 17, 19–27].

Continuing in the same direction, in this manuscript, a new fixed-point theorem for θ –contraction mapping in RMMSs is proved and some related sequences are introduced. To support the theoretical result, a nontrivial example is obtained and the existence of a solution to an NIE has been found.

2. Preliminaries

In this part, we consider for all $1, \mathcal{J} \in \mathfrak{N}$, $m_{1,\mathcal{J}} = \min\{m(1, 1), m(\mathcal{J}, \mathcal{J})\}$, $M_{1,\mathcal{J}} = \max\{m(1, 1), m(\mathcal{J}, \mathcal{J})\}$, $m_{r,1,\mathcal{J}} = \min\{m_r(1, 1), m_r(\mathcal{J}, \mathcal{J})\}$, and $M_{r,1,\mathcal{J}} = \max\{m_r(1, 1), m_r(\mathcal{J}, \mathcal{J})\}$.

Now, we give some previous results in the gradation of spaces.

Definition 1 (see [17], PRMS). Let $\mathfrak{N}$ be a nonempty set. A mapping $p: \mathfrak{N} \times \mathfrak{N} \rightarrow \mathbb{R}^+$ is said to be a PRM on $\mathfrak{N}$ if for any $1, \mathcal{J} \in \mathfrak{N}$ and all distinct points $(\ell, h \in \mathfrak{N}/\{1, \mathcal{J}\})$, the following stipulations hold:

- (i) (RP₁) $1 = \mathcal{J} \Leftrightarrow p(1, \mathcal{J}) = p(1, 1) = p(\mathcal{J}, \mathcal{J})$
- (ii) (RP₂) $p(1, 1) \leq p(1, \mathcal{J})$
- (iii) (RP₃) $p(1, \mathcal{J}) = p(\mathcal{J}, 1)$
- (iv) (RP₄) $p(1, \mathcal{J}) \leq p(1, \ell) + p(\ell, h) + p(h, \mathcal{J}) - p(\ell, \ell) - p(h, h)$

The pair $(\mathfrak{N}, p)$ is called a PRMS.

The above space was extended to an MMS by Asadi et al. [14] as follows.

Definition 2 (MMS). Assume that $\mathfrak{N}$ is a nonempty set. If the function $m: \mathfrak{N} \times \mathfrak{N} \rightarrow \mathbb{R}^+$ satisfies the following hypotheses, for all $1, \mathcal{J}, \ell \in \mathfrak{N}$, then the pair $(\mathfrak{N}, m)$ is called an MMS:

- (i) (MM₁) $m(1, 1) = m(\mathcal{J}, \mathcal{J}) = m(1, \mathcal{J}) \Leftrightarrow 1 = \mathcal{J}$
- (ii) (MM₂) $m_{1,\mathcal{J}} \leq m(1, \mathcal{J})$
- (iii) (MM₃) $m(1, \mathcal{J}) = m(\mathcal{J}, 1)$
- (iv) (MM₄) $(m(1, \mathcal{J}) - m_{1,\mathcal{J}}) \leq (m(1, \ell) - m_{1,\ell}) + (m(\ell, \mathcal{J}) - m_{\ell,\mathcal{J}})$

Özgür et al. [18] generalized an MMS to a RMMS as follows.

Definition 3 (RMMS). Let $\mathfrak{N}$ be a nonempty set. If the function $m_r: \mathfrak{N} \times \mathfrak{N} \rightarrow \mathbb{R}^+$ verifies the following assumptions, for all $1, \mathcal{J} \in \mathfrak{N}$, then the pair $(\mathfrak{N}, m_r)$ is called a RMMS:

- (i) (RM₁) $m_r(1, \mathcal{J}) = m_{r,1,\mathcal{J}} = M_{r,1,\mathcal{J}} \Leftrightarrow 1 = \mathcal{J}$
- (ii) (RM₂) $m_{r,1,\mathcal{J}} \leq m_r(1, \mathcal{J})$
- (iii) (RM₃) $m_r(1, \mathcal{J}) = m_r(\mathcal{J}, 1)$
- (iv) (RM₄) $m_r(1, \mathcal{J}) - m_{r,1,\mathcal{J}} \leq m_r(1, \ell) - m_{r,1,\ell} + m_r(\ell, h) - m_{r,\ell,h} + m_r(h, \mathcal{J}) - m_{r,h,\mathcal{J}}$, for all $\ell, h \in (\mathfrak{N}/\{1, \mathcal{J}\})$

It is clear that every MMS is a RMMS.

The results given follow from Özgür et al. [18].

Remark 1. Let $(\mathfrak{N}, m_r)$ be a RMMS. Clearly, for every $1, \mathcal{J} \in \mathfrak{N}$, we have

- (1) $0 \leq M_{r,1,\mathcal{J}} + m_{r,1,\mathcal{J}} = m_r(1, 1) + m_r(\mathcal{J}, \mathcal{J})$
- (2) $0 \leq M_{r,1,\mathcal{J}} - m_{r,1,\mathcal{J}} = |m_r(1, 1) - m_r(\mathcal{J}, \mathcal{J})|$

Also, it can be easily verified under special cases that

$$(3) \quad M_{r,1,\mathcal{J}} - m_{r,1,\mathcal{J}} \leq (M_{r,1,\ell} - m_{r,1,\ell}) + (M_{r,\ell,h} - m_{r,\ell,h}) + (M_{r,h,\mathcal{J}} - m_{r,h,\mathcal{J}})$$

Example 1. Assume that $\mathbb{C}$ is the set of all complex numbers, and consider the set $\mathfrak{N}_\varphi = \{z \in \mathbb{C} : \arg(z) = \varphi\} \cup \{0\}$ for a fixed $\varphi \in [0, 2\pi]$ and define the function $m_r: \mathfrak{N} \times \mathfrak{N} \rightarrow [0, \infty)$ by $m_r(1, \mathcal{J}) = (|1| + |\mathcal{J}|)/2$ for all $1, \mathcal{J} \in \mathfrak{N}_\varphi$. Then, $(\mathfrak{N}_\varphi, m_r)$ is a RMMS.

Proposition 1 (see [18]). Suppose that $(\mathfrak{N}, d)$ is a RMS and a function $\xi: [0, \infty) \rightarrow [\alpha, \infty)$ is a one-to-one and nondecreasing function with $\xi(0) = \alpha$ so that $\xi(1 + \mathcal{J} + h) = \xi(1) + \xi(\mathcal{J}) + \xi(h) - 2\alpha$, for all $1, \mathcal{J}, h \in \mathfrak{N}$. Then, the function $m_r: \mathfrak{N} \times \mathfrak{N} \rightarrow [0, \infty)$ which is defined by $m_r(1, \mathcal{J}) = \xi(d(1, \mathcal{J}))$, for all $1, \mathcal{J} \in \mathfrak{N}$ with the set $\mathfrak{N}$, is a RMMS.

Example 2. Let $(\mathfrak{N}, d)$ be a RMS and a function $\xi: [0, \infty) \rightarrow [\alpha, \infty)$ be defined as $\xi(t) = mt + n$, with $\xi(0) = \alpha$ for all $t \in [0, \infty)$, and $m_r(1, \mathcal{J}) = m d(1, \mathcal{J}) + n$ is a RMMS with the set $\mathfrak{N}$.

Definition 4. Assume that $(\mathfrak{N}, m_r)$ is a RMMS; a sequence $\{1_n\} \in \mathfrak{N}$ is said to be

- (1) Converging to a point 1 iff

$$\lim_{n \rightarrow \infty} (m_r(1_n, 1) - m_{r,1_n,1}) = 0, \quad 1 = 0. \quad (1)$$

- (2) m_r -Cauchy sequence iff

$$\lim_{n,m \rightarrow \infty} (m_r(1_n, 1_m) - m_{r,1_n,1_m}) = 0, \quad (2)$$

$$\lim_{n,m \rightarrow \infty} (M_{r,1_n,1_m} - m_{r,1_n,1_m}) = 0,$$

exist and is finite.

- (3) A RMMS called m_r -complete if every m_r -Cauchy sequence $\{1_n\}$ converges to a point 1 so that

$$\lim_{n \rightarrow \infty} (m_r(1_n, 1) - m_{r,1_n,1}) = 0, \quad (3)$$

$$\lim_{n \rightarrow \infty} (M_{r,1_n,1} - m_{r,1_n,1}) = 0.$$

Lemma 1 (see [18]). Assume that $1_n \rightarrow 1$ and $\mathcal{J}_n \rightarrow \mathcal{J}$ as $n \rightarrow \infty$ in a RMMS $(\mathfrak{N}, m_r)$. Then,

$$\lim_{n \rightarrow \infty} (m_r(1_n, \mathcal{J}_n) - m_{r,1_n,\mathcal{J}_n}) = m_r(1, \mathcal{J}) - m_{r,1,\mathcal{J}}. \quad (4)$$

Lemma 2 (see [18]). Assume that $1_n \rightarrow 1$ as $n \rightarrow \infty$ in a RMMS $(\mathfrak{N}, m_r)$. Then,

$$\lim_{n \rightarrow \infty} (m_r(i_n, \mathcal{J}) - m_{r_{i_n, \mathcal{J}}}) = m_r(i, \mathcal{J}) - m_{r_{i, \mathcal{J}}} \text{ for all } \mathcal{J} \in \mathbb{N}. \quad (5)$$

Lemma 3 (see [18]). Suppose that $i_n \rightarrow i$ and $\mathcal{J}_n \rightarrow \mathcal{J}$ as $n \rightarrow \infty$ in a RMMS $(\mathbb{N}, m_r)$. Then, $m_r(i, \mathcal{J}) = m_{r_{i, \mathcal{J}}}$. Furthermore, if $m_r(i, i) = m_r(\mathcal{J}, \mathcal{J})$, then $i = \mathcal{J}$.

Lemma 4 (see [18]). Let $\{i_n\}$ be a sequence in a RMMS $(\mathbb{N}, m_r)$. If there is $r \in [0, 1)$ so that $m_r(i_{n+1}, i_n) \leq r m_r(i_n, i_{n-1})$ for all $n \in \mathbb{N}$, then

- (i) $\lim_{n \rightarrow \infty} m_r(i_n, i_{n-1}) = 0$
- (ii) $\lim_{n \rightarrow \infty} m_r(i_n, i_n) = 0$
- (iii) $\lim_{n, m \rightarrow \infty} m_{r_{i_n, i_m}} = 0$
- (iv) $\{i_n\}$ is an m_r -Cauchy sequence

Lemma 5 (see [18]). Let $(\mathbb{N}, m_r)$ be a RMMS and T be a self-mapping on $\mathbb{N}$. If there is $k \in [0, 1)$ so that $m_r(Ti, T\mathcal{J}) \leq k m_r(i, \mathcal{J})$ for all $i, \mathcal{J} \in \mathbb{N}$, then $Ti_n \rightarrow Tu$ as $n \rightarrow \infty$, where $\{i_n\}_{n \geq 0}$ is a sequence defined by $i_{n+1} = Ti_n$ so that $i_n \rightarrow u$ as $n \rightarrow \infty$.

Theorem 1 (see [18]). Suppose that T is a self-mapping on a RMMS $(\mathbb{N}, m_r)$. If there is $k \in (0, 1)$ so that

$$m_r(Ti, T\mathcal{J}) \leq k m_r(i, \mathcal{J}) \text{ for all } i, \mathcal{J} \in \mathbb{N}, \quad (6)$$

then T has a unique fixed point u in $\mathbb{N}$, where $m_r(u, u) = 0$.

Recently, Jleli and Samet [10] established the fixed-point theorem as follows.

Theorem 2 (see [10]). Let $(\mathbb{N}, d)$ be a complete GMS and $T: \mathbb{N} \rightarrow \mathbb{N}$ be a given map. Suppose that there is $\theta \in \Theta$ and $k \in (0, 1)$ such that

$$i, \mathcal{J} \in \mathbb{N}, \quad d(Ti, T\mathcal{J}) \neq 0 \Rightarrow \theta(d(Ti, T\mathcal{J})) \leq [\theta(d(i, \mathcal{J}))]^k, \quad (7)$$

then T has a unique fixed point.

Theorem 2 was extended by Jleli et al. [11] as follows.

Theorem 3 (see [11]). Let $(\mathbb{N}, d)$ be a complete GMS and $T: \mathbb{N} \rightarrow \mathbb{N}$ be a given map. Suppose that there is $\theta \in \Theta$ and $k \in (0, 1)$ so that

$$i, \mathcal{J} \in \mathbb{N}, \quad d(Ti, T\mathcal{J}) \neq 0 \Rightarrow \theta(d(Ti, T\mathcal{J})) \leq [\theta(M(i, \mathcal{J}))]^k, \quad (8)$$

where

$$M(i, \mathcal{J}) = \max\{d(i, \mathcal{J}), d(i, Ti), d(\mathcal{J}, T\mathcal{J})\}, \quad (9)$$

then T has a unique fixed point.

In Theorems 2 and 3, the authors considered Θ is the set of all functions $\theta: (0, \infty) \rightarrow (1, \infty)$ so that the following conditions hold:

- (i) (Θ_1) θ is nondecreasing
- (ii) (Θ_2) for each sequence $\{t_n\} \subset (0, \infty)$, $\lim_{n \rightarrow \infty} \theta(t_n) = 1$ iff $\lim_{n \rightarrow \infty} t_n = 0^+$
- (iii) (Θ_3) there is $r \in (0, 1)$ and $L \in (0, \infty]$ so that $\lim_{t \rightarrow 0^+} (\theta(t) - 1/t^r) = L$
- (iv) (Θ_4) θ is continuous

3. Main Result

We introduce in this part some new fixed-point results via θ -contraction mapping in RMMSs which generalizes the results of Jleli et al. [11].

Theorem 4. Let $(\mathbb{N}, m_r)$ be a complete RMMS and $T: \mathbb{N} \rightarrow \mathbb{N}$ be a given mapping. Suppose that there is $\theta \in \Theta$ and $k \in (0, 1)$ so that

$$i, \mathcal{J} \in \mathbb{N}, \quad m_r(Ti, T\mathcal{J}) \neq 0 \Rightarrow \theta(m_r(Ti, T\mathcal{J})) \leq [\theta(N(i, \mathcal{J}))]^k, \quad (10)$$

where

$$N(i, \mathcal{J}) = \max\{m_r(i, \mathcal{J}), m_r(i, Ti), m_r(\mathcal{J}, T\mathcal{J})\}, \quad (11)$$

and θ is as defined in the above section. Then, T has a unique fixed point.

Proof. Let $i_0 \in \mathbb{N}$ be an arbitrary value and define a sequence $\{i_n\}_{n \in \mathbb{N}}$ by letting $i_1 = Ti_0$, $i_2 = Ti_1$, $i_3 = Ti_2, \dots, i_{n+1} = Ti_n = T^{n+1}i_0$. If for some $q \in \mathbb{N}$, we have $T^q i_1 = T^{q+1} i_1$, then $T^q i_1$ is a FP of T . So we suppose that $m_r(T^n i_1, T^{n+1} i_1) > 0 \forall n \in \mathbb{N}$. Now, from (10), we have

$$\theta(m_r(T^n i_1, T^{n+1} i_1)) \leq [\theta(N(T^{n-1} i_1, T^n i_1))]^k, \quad (12)$$

$$N(T^{n-1} i_1, T^n i_1) = \max \left\{ \begin{array}{l} m_r(T^{n-1} i_1, T^n i_1), m_r(T^{n-1} i_1, T^{n+1} i_1) \\ m_r(T^n i_1, T^{n+1} i_1) \end{array} \right\}, \quad (13)$$

$$\begin{aligned} N(T^{n-1} i_1, T^n i_1) &= \max \left\{ \begin{array}{l} m_r(T^{n-1} i_1, T^n i_1), m_r(T^{n-1} i_1, T^{n+1} i_1) \\ m_r(T^n i_1, T^{n+1} i_1) \end{array} \right\}, \\ &= \max\{m_r(T^{n-1} i_1, T^n i_1), m_r(T^n i_1, T^{n+1} i_1)\}. \end{aligned} \quad (14)$$

If $N(T^{n-1} i_1, T^n i_1) = m_r(T^n i_1, T^{n+1} i_1)$, then from (12), one can write

$$\theta(m_r(T^n i_1, T^{n+1} i_1)) \leq [\theta(m_r(T^n i_1, T^{n+1} i_1))]^k. \quad (15)$$

Taking $\ln$ in both sides, we obtain

$$\ln[\theta(m_r(T^n i_1, T^{n+1} i_1))] \leq k \ln[\theta(m_r(T^n i_1, T^{n+1} i_1))], \quad (16)$$

which is a contradiction because $k \in (0, 1)$. Thus, we have

$$N(T^{n-1}1, T^n1) = m_r(T^{n-1}1, T^n1). \quad (17)$$

It follows from (12) that

$$\begin{aligned} \theta(m_r(T^n1, T^{n+1}1)) &\leq [\theta(m_r(T^{n-1}1, T^n1))]^k \\ &\leq [\theta(m_r(T^{n-2}1, T^{n-1}1))]^{k^2} \\ &\vdots \\ &\leq [\theta(m_r(1, T1))]^{k^n}. \end{aligned} \quad (18)$$

This leads to

$$1 \leq \theta(m_r(T^n1, T^{n+1}1)) \leq [\theta(m_r(1, T1))]^{k^n}, \quad \forall n \in \mathbb{N}. \quad (19)$$

As $n \rightarrow \infty$ in (19), we obtain that

$$\theta(m_r(T^n1, T^{n+1}1)) \rightarrow 1, \quad \text{as } n \rightarrow \infty, \quad (20)$$

which implies from (Θ_2) that $\lim_{n \rightarrow \infty} m_r(T^n1, T^{n+1}1) = 0$. Now, from condition (Θ_3) , there are $r \in (0, 1)$ and $L \in (0, \infty]$ so that

$$\lim_{n \rightarrow \infty} \frac{\theta(m_r(T^n1, T^{n+1}1)) - 1}{[m_r(T^n1, T^{n+1}1)]^r} = L. \quad (21)$$

Suppose that $L < \infty$. In this case, let $B = (L/2) > 0$. From the definition of the limit, there is $n_0 \in \mathbb{N}$ such that

$$\left| \frac{\theta(m_r(T^n1, T^{n+1}1)) - 1}{[m_r(T^n1, T^{n+1}1)]^r} - L \right| \leq B, \quad \forall n \geq n_0. \quad (22)$$

This implies that

$$\frac{\theta(m_r(T^n1, T^{n+1}1)) - 1}{[m_r(T^n1, T^{n+1}1)]^r} \geq L - B = B, \quad \forall n \geq n_0. \quad (23)$$

Thus,

$$n[m_r(T^n1, T^{n+1}1)]^r \leq An[\theta(m_r(T^n1, T^{n+1}1)) - 1], \quad \forall n \geq n_0, \quad (24)$$

where $A = (1/B)$.

Now, consider $L = \infty$ and $B > 0$ as an arbitrary positive number. From the definition of the limit, there is $n_0 \in \mathbb{N}$ so that

$$\frac{\theta(m_r(T^n1, T^{n+1}1)) - 1}{[m_r(T^n1, T^{n+1}1)]^r} \geq B, \quad \forall n \geq n_0. \quad (25)$$

This leads to

$$n[m_r(T^n1, T^{n+1}1)]^r \leq An[\theta(m_r(T^n1, T^{n+1}1)) - 1], \quad \forall n \geq n_0, \quad (26)$$

where $A = (1/B)$.

Thus, in both cases, there exist $A > 0$ and $n_0 \in \mathbb{N}$ so that

$$n[m_r(T^n1, T^{n+1}1)]^r \leq An[\theta(m_r(T^n1, T^{n+1}1)) - 1], \quad \forall n \geq n_0. \quad (27)$$

Using (14), we obtain

$$n[m_r(T^n1, T^{n+1}1)]^r \leq An[\theta(m_r(1, T1))]^{k^n} - 1, \quad \forall n \geq n_0. \quad (28)$$

Taking limit as $n \rightarrow \infty$ in the above inequality, one can obtain

$$\lim_{n \rightarrow \infty} n[m_r(T^n1, T^{n+1}1)]^r = 0. \quad (29)$$

Thus, there is $n_1 \in \mathbb{N}$ so that

$$m_r(T^n1, T^{n+1}1) \leq \frac{1}{n^{(1/r)}}, \quad \forall n \geq n_1. \quad (30)$$

Now, we will discuss the following two steps.

St₁. When $T^n1 = T^m1$ for some integers $n \neq m$ with $m > n$, we have $T^{m-n}(T^n1) = T^m1$. Choose $\mathcal{J} = T^n1$ and $p = m - n$. Then, $T^p\mathcal{J} = \mathcal{J}$ and $\mathcal{J}$ is a periodic point of T . By (10) and (14), we obtain

$$\theta(m_r(\mathcal{J}, T\mathcal{J})) = \theta(m_r(T^p\mathcal{J}, T^{p+1}\mathcal{J})) \leq [\theta(m_r(\mathcal{J}, T\mathcal{J}))]^{k^p}, \quad (31)$$

As $p \rightarrow \infty$, we have $m_r(\mathcal{J}, T\mathcal{J}) = 0$. On the other hand, we can write

$$\begin{aligned} \theta(m_r(\mathcal{J}, \mathcal{J})) &= \theta(m_r(T^p\mathcal{J}, T^p\mathcal{J})) \\ &\leq [\theta(m_r(T^{p-1}\mathcal{J}, T^{p-1}\mathcal{J}))]^k \\ &\vdots \\ &\leq [\theta(m_r(T\mathcal{J}, T\mathcal{J}))]^{k^{p-1}} \\ &\leq [\theta(m_r(\mathcal{J}, \mathcal{J}))]^{k^p}, \end{aligned} \quad (32)$$

which leads to

$$\begin{aligned} \ln \theta(m_r(\mathcal{J}, \mathcal{J})) &\leq k^{p-1} \ln [\theta(m_r(T\mathcal{J}, T\mathcal{J}))] \\ &\leq k^p \ln [\theta(m_r(\mathcal{J}, \mathcal{J}))], \end{aligned} \quad (33)$$

or equivalently

$$\theta(m_r(\mathcal{J}, \mathcal{J})) = \theta(m_r(T\mathcal{J}, T\mathcal{J})). \quad (34)$$

This yields $m_r(\mathcal{J}, \mathcal{J}) = m_r(T\mathcal{J}, T\mathcal{J}) = 0$. Hence, $T\mathcal{J} = \mathcal{J}$, that is, $\mathcal{J}$ is a fixed point of T .

St₂. When $T^n1 \neq T^m1$ for every $n, m \in \mathbb{N}$ with $n \neq m$, we shall prove that T has a periodic point. Suppose on the contrary, utilizing (10) and (11), we obtain

$$\theta(m_r(T^n1, T^{n+2}1)) \leq [\theta(N(T^{n-1}1, T^{n+1}1))]^k, \quad (35)$$

where

$$N(T^{n-1}1, T^{n+1}1) = \max \left\{ \begin{array}{l} m_r(T^{n-1}1, T^{n+1}1), m_r(T^{n-1}1, T^n1) \\ m_r(T^{n+1}1, T^{n+2}1) \end{array} \right\}. \quad (36)$$

Since θ is continuous and nondecreasing, using (35) and (36), one can write

$$\theta(m_r(T^n_1, T^{n+2}_1)) \leq \left(\max \left\{ \begin{array}{c} \theta(m_r(T^{n-1}_1, T^{n+1}_1)) \\ \theta(m_r(T^{n-1}_1, T^n_1)) \\ \theta(m_r(T^{n+1}_1, T^{n+2}_1)) \end{array} \right\} \right)^k. \quad (37)$$

Let I be the set of $n \in \mathbb{N}$ such that

$$\begin{aligned} u_n &= \max\{\theta(m_r(T^{n-1}_1, T^{n+1}_1)), \theta(m_r(T^{n-1}_1, T^n_1)), \\ &\quad \theta(m_r(T^{n+1}_1, T^{n+2}_1))\} \\ &= \theta(m_r(T^{n-1}_1, T^{n+1}_1)). \end{aligned} \quad (38)$$

If $|I| < \infty$, there exists $N \in \mathbb{N}$ such that for every $n \geq N$,

$$\begin{aligned} \max\{\theta(m_r(T^{n-1}_1, T^{n+1}_1)), \theta(m_r(T^{n-1}_1, T^n_1)), \\ \theta(m_r(T^{n+1}_1, T^{n+2}_1))\} \\ = \max\{\theta(m_r(T^{n-1}_1, T^n_1)), \theta(m_r(T^{n+1}_1, T^{n+2}_1))\}. \end{aligned} \quad (39)$$

In this case, we obtain from (37) that

$$\begin{aligned} 1 &\leq \theta(m_r(T^n_1, T^{n+2}_1)) \\ &\leq \left[\max\{\theta(m_r(T^{n-1}_1, T^n_1)), \theta(m_r(T^{n+1}_1, T^{n+2}_1))\} \right]^k, \end{aligned} \quad (40)$$

for all $n \geq N$. Letting $n \rightarrow \infty$ in the above inequality and using (20), we obtain

$$\theta(m_r(T^n_1, T^{n+2}_1)) \rightarrow 1, \quad \text{as } n \rightarrow \infty. \quad (41)$$

If $|I| = \infty$, we can find a subsequence of $\{u_n\}$, which we denote also by $\{u_n\}$, such that

$$u_n = \theta(m_r(T^{n-1}_1, T^{n+1}_1)), \quad \text{for } n \text{ large enough.} \quad (42)$$

In this case, we obtain from (37) that

$$\begin{aligned} 1 &\leq \theta(m_r(T^n_1, T^{n+2}_1)) \leq \left[\theta(m_r(T^{n-1}_1, T^{n+1}_1)) \right]^k \\ &\leq \left(\theta(m_r(T^{n-2}_1, T^n_1)) \right)^{k^2} \leq \dots \leq \left[\theta(m_r(T^2_1, T^1_1)) \right]^{k^n}, \end{aligned} \quad (43)$$

for n large enough. Letting $n \rightarrow \infty$ in the above inequality, we have

$$\theta(m_r(T^n_1, T^{n+2}_1)) \rightarrow 1, \quad \text{as } n \rightarrow \infty. \quad (44)$$

Then, in both cases, (44) holds. Using (44) and property (Θ_2) , we obtain

$$\lim_{n \rightarrow \infty} m_r(T^n_1, T^{n+2}_1) = 0. \quad (45)$$

Similarly, from condition (Θ_3) , there exists $n_2 \in \mathbb{N}$ such that

$$m_r(T^n_1, T^{n+2}_1) \leq \frac{1}{n^{(1/r)}}, \quad \text{for all } n \geq n_2. \quad (46)$$

Let $N' = \max\{n_0, n_1\}$. If $m > 2$ is odd, then we consider $m = 2q + 1$, where $q \geq 1$. Using (30) for all $n \geq N$, we can obtain

$$\begin{aligned} m_r(T^n_1, T^{n+m}_1) - m_{r_{T^n_1 T^{n+m}_1}} &\leq m_r(T^n_1, T^{n+1}_1) - m_{r_{T^n_1 T^{n+1}_1}} \\ &\quad + m_r(T^{n+1}_1, T^{n+2}_1) - m_{r_{T^{n+1}_1 T^{n+2}_1}} \\ &\quad + \dots + m_r(T^{n+2q}_1, T^{n+2q+1}_1) \\ &\quad - m_{r_{T^{n+2q}_1 T^{n+2q+1}_1}} \\ &\leq m_r(T^n_1, T^{n+1}_1) + m_r(T^{n+1}_1, T^{n+2}_1) \\ &\quad + \dots + m_r(T^{n+2q}_1, T^{n+2q+1}_1) \\ &\leq \frac{1}{n^{(1/r)}} + \frac{1}{(n+1)^{(1/r)}} + \dots \\ &\quad + \frac{1}{(n+2q)^{(1/r)}} \\ &\leq \sum_{i=n}^{\infty} \frac{1}{i^{(1/r)}}. \end{aligned} \quad (47)$$

Now, if $m > 2$ is even, then writing $m = 2q$, where $q \geq 2$ and using (30) and (46), for all $n \geq N$, we obtain

$$\begin{aligned} m_r(T^n x, T^{n+m} x) - m_{r_{T^n x T^{n+m} x}} &\leq m_r(T^n x, T^{n+2} x) - m_{r_{T^n x T^{n+2} x}} \\ &\quad + m_r(T^{n+2} x, T^{n+3} x) \\ &\quad - m_{r_{T^{n+2} x T^{n+3} x}} \\ &\quad + \dots + m_r(T^{n+2q-1} x, T^{n+2q} x) \\ &\quad - m_{r_{T^{n+2q-1} x T^{n+2q} x}}, \\ &\leq m_r(T^n x, T^{n+2} x) \\ &\quad + m_r(T^{n+2} x, T^{n+3} x) \\ &\quad + \dots + m_r(T^{n+2q-1} x, T^{n+2q} x) \\ &\leq \frac{1}{n^{(1/r)}} + \frac{1}{(n+2)^{(1/r)}} + \dots \\ &\quad + \frac{1}{(n+2q-1)^{(1/r)}} \\ &\leq \sum_{i=n}^{\infty} \frac{1}{i^{(1/r)}}. \end{aligned} \quad (48)$$

Combining (47) and (48), we have

$$m_r(T^n 1, T^{n+m} 1) - m_{r_{T^n 1, T^{n+m} 1}} \leq \sum_{i=n}^{\infty} \frac{1}{i^{(1/r)}}, \quad \text{for all } n \geq N', m \in \mathbb{N}. \quad (49)$$

Since the series $\sum_{i=n}^{\infty} (1/i^{(1/r)})$ is convergent, we find that $m_r(T^n 1, T^{n+m} 1) - m_{r_{T^n 1, T^{n+m} 1}}$ is also convergent as $n, m \rightarrow \infty$, which implies that

$$m_r(T^n 1, T^{n+m} 1) - m_{r_{T^n 1, T^{n+m} 1}} \leq m_r(T^n 1, T^{n+m} 1). \quad (50)$$

Thus, the sequence $\{T^n 1\}$ is m_r -Cauchy in the m_r -complete RMMS $(\aleph, m_r)$. Hence, there is some $u \in \aleph$ so that $T^n 1 \rightarrow u$ as $n \rightarrow \infty$. Without restriction of the generality, we can suppose that $T^n 1 \neq u$ for all n .

Assume that $m_r(u, Tu) > 0$. Using (10), we obtain

$$\theta(m_r(T^{n+1} 1, Tu)) \leq [\theta(N(T^n 1, u))]^k, \quad \text{for all } n \in \mathbb{N}, \quad (51)$$

where

$$N(T^n 1, u) = \max\{m_r(T^n 1, u), m_r(T^n 1, T^{n+1} 1), m_r(u, Tu)\}. \quad (52)$$

Taking limit as $n \rightarrow \infty$ in the above inequality and using (Θ_4) and Lemmas 1–3, we have

$$\theta(m_r(u, Tu)) \leq [\theta(m_r(u, Tu))]^k < \theta(m_r(u, Tu)), \quad (53)$$

which is a contradiction; therefore, $Tu = u$, which is also a contradiction with the supposition that T does not have a periodic point. Thus, T has a periodic point, say u , of period q . Assume that the set of all fixed points of T is empty. Then, we have

$$q > 1 \text{ and } m_r(u, Tu) > 0. \quad (54)$$

Using (10), we obtain that

$$\begin{aligned} \theta(m_r(u, Tu)) &= \theta(m_r(T^q u, T^{q+1} u)) \\ &\leq [\theta(m_r(u, Tu))]^{k^q} < \theta(m_r(u, Tu)), \end{aligned} \quad (55)$$

which is a contradiction. Thus, the set of fixed points of T is nonempty, that is, T has more than one fixed point.

Now, assume that $u, v \in \aleph$ are two fixed points of T so that $m_r(u, v) = m_r(Tu, Tv) > 0$. Utilizing (10), we obtain

$$\begin{aligned} \theta(m_r(u, v)) &= \theta(m_r(Tu, Tv)) \leq [\theta(m_r(u, v))]^k \\ &< \theta(m_r(u, v)), \end{aligned} \quad (56)$$

which is a contradiction. Then, T has a unique fixed point. This completes the proof. $\square$

Theorem 4 is still valid if we replace condition (10) with one of the following stipulations: for all $1, \mathcal{J}, u \in \aleph$ with $m_r(u, u) = 0$,

(i) There is $0 < k < 1$ so that

$$m_r(T1, T\mathcal{J}) \leq k \max\{m_r(1, \mathcal{J}), m_r(1, T1), m_r(\mathcal{J}, T\mathcal{J})\}. \quad (57)$$

(ii) There is $0 \leq k < (1/2)$ so that

$$m_r(T1, T\mathcal{J}) \leq k[m_r(1, T1) + m_r(\mathcal{J}, T\mathcal{J})]. \quad (58)$$

(iii) There is $0 < k < (\sqrt{3} - 1/2)$ so that

$$m_r(T1, T\mathcal{J}) \leq k[m_r(1, T\mathcal{J}) + m_r(\mathcal{J}, T1)]. \quad (59)$$

(iv) There are $\theta \in \Theta$ and $k \in (0, 1)$ so that

$$m_r(T1, T\mathcal{J}) \neq 0 \implies \theta(m_r(T1, T\mathcal{J})) \leq [\theta(m_r(1, \mathcal{J}))]^k. \quad (60)$$

We note that Θ contains a large class of functions. For instance,

$$\theta(t) = 2 - \frac{2}{\pi} \arctan\left(\frac{1}{t^\alpha}\right), \quad 0 < \alpha < 1, t > 0. \quad (61)$$

So we conclude the following corollary from Theorem 4.

Corollary 1. Let $(\aleph, m_r)$ be a complete RMMS and $T: \aleph \rightarrow \aleph$ be a self-mapping. If there exists $\theta \in \Theta$ and $\alpha, k \in (0, 1)$ such that for all $1, \mathcal{J} \in \aleph$ and $T1 \neq T\mathcal{J}$,

$$\begin{aligned} &2 - \frac{2}{\pi} \arctan\left(\frac{1}{[m_r(T1, T\mathcal{J})]^\alpha}\right) \\ &\leq \left[2 - \frac{2}{\pi} \arctan\left(\frac{1}{[N(1, \mathcal{J})]^\alpha}\right)\right]^k, \end{aligned} \quad (62)$$

where

$$N(1, \mathcal{J}) = \max\{m_r(1, \mathcal{J}), m_r(1, T\mathcal{J}), m_r(\mathcal{J}, T\mathcal{J})\}. \quad (63)$$

Then, T has a unique fixed point.

The example below justifies all requirements of Theorem 4.

Example 3. Let $\aleph_\varphi = \{0\} \cup \{z \in \mathbb{C}: \arg(z) = \varphi\}$ for a fixed φ , where $0 \leq \varphi < 2\pi$. Define the function $m_r: \aleph \times \aleph \rightarrow [0, \infty)$ by $m_r(1, \mathcal{J}) = (|1| + |\mathcal{J}|/2)$ for all $1, \mathcal{J} \in \aleph_\varphi$. Then, $(\aleph_\varphi, m_r)$ is a RMMS. Take $\varphi = 0, (\pi/2)$ and the RMMS becomes $(\aleph_{0, (\pi/2)}, m_r)$ such that $\aleph_{0, (\pi/2)} = [0, \infty) \cup (A_i = \{\tilde{z} = ni: 1 \leq n < \infty\})$ and $m_r(\tilde{z}_1, \tilde{z}_2) = (|\tilde{z}_1| + |\tilde{z}_2|/2)$ for all $\tilde{z}_1, \tilde{z}_2 \in \aleph_{0, (\pi/2)}$. Let $\theta: (0, \infty) \rightarrow (1, \infty)$ be defined by $\theta(t) = e^{\sqrt{t}}$; it is clear that $\theta \in \Theta$. Define $T: \aleph_{0, (\pi/2)} \rightarrow \aleph_{0, (\pi/2)}$ by

$$T\tilde{z} = \begin{cases} \frac{\tilde{z}}{2+n}, & \tilde{z} \in A_i, 1 \leq n < \infty, \\ \frac{\tilde{z}}{2}, & \tilde{z} \in [0, 1], \\ \frac{\tilde{z}}{2^n}, & \tilde{z} \in (1, \infty), 2 \leq n < \infty. \end{cases} \quad (64)$$

Now, for $\tilde{z} \in A_i, 1 \leq n < \infty$, we have

$$\begin{aligned}
\theta(m_r(T\tilde{z}_1, T\tilde{z}_2)) &= e^{\sqrt{m_r(T\tilde{z}_1, T\tilde{z}_2)}} = \theta\left(\frac{|\tilde{z}_1/2 + n| + |\tilde{z}_2/2 + n|}{2}\right) = \theta\left(\frac{1}{2+n}\left(\frac{|\tilde{z}_1| + |\tilde{z}_2|}{2}\right)\right) \\
&= \theta\left(\frac{1}{2+n}m_r(\tilde{z}_1, \tilde{z}_2)\right) = e^{\sqrt{1/2+n}\sqrt{m_r(\tilde{z}_1, \tilde{z}_2)}} = \left(e^{\sqrt{m_r(\tilde{z}_1, \tilde{z}_2)}}\right)^{\sqrt{1/2+n}} \\
&\leq \left(e^{\sqrt{N(\tilde{z}_1, \tilde{z}_2)}}\right)^{\sqrt{1/2+n}} = \left[\theta(N(\tilde{z}_1, \tilde{z}_2))\right]^k.
\end{aligned} \tag{65}$$

Then, T satisfies the contraction with $0 < k = \sqrt{(1/2+n)} < 1$.

On the other hand, for $\tilde{z} \in [0, 1]$, we have

$$\begin{aligned}
\theta(m_r(T\tilde{z}_1, T\tilde{z}_2)) &= \theta\left(\frac{|\tilde{z}_1/2| + |\tilde{z}_2/2|}{2}\right) = \theta\left(\frac{1}{2}\frac{|\tilde{z}_1| + |\tilde{z}_2|}{2}\right) \\
&= \theta\left(\frac{1}{2}m_r(\tilde{z}_1, \tilde{z}_2)\right) \\
&= e^{(1/2)\sqrt{m_r(\tilde{z}_1, \tilde{z}_2)}} = \left(e^{\sqrt{m_r(\tilde{z}_1, \tilde{z}_2)}}\right)^{(1/2)} \\
&\leq \left(e^{\sqrt{N(\tilde{z}_1, \tilde{z}_2)}}\right)^{(1/2)} = \left[\theta(N(\tilde{z}_1, \tilde{z}_2))\right]^k.
\end{aligned} \tag{66}$$

Similarly, for $\tilde{z} \in (1, \infty)$, $2 \leq n < \infty$, we obtain

$$\begin{aligned}
\theta(m_r(T\tilde{z}_1, T\tilde{z}_2)) &= \theta\left(\frac{|\tilde{z}_1/2^n| + |\tilde{z}_2/2^n|}{2}\right) \\
&= \theta\left(\frac{1}{2^n}\frac{|\tilde{z}_1| + |\tilde{z}_2|}{2}\right) \\
&= \theta\left(\frac{1}{2^n}m_r(\tilde{z}_1, \tilde{z}_2)\right) \\
&= e^{(1/2^n)\sqrt{m_r(\tilde{z}_1, \tilde{z}_2)}} = \left(e^{\sqrt{m_r(\tilde{z}_1, \tilde{z}_2)}}\right)^{(1/2^n)} \\
&\leq \left(e^{\sqrt{N(\tilde{z}_1, \tilde{z}_2)}}\right)^{(1/2^n)} = \left[\theta(N(\tilde{z}_1, \tilde{z}_2))\right]^k,
\end{aligned} \tag{67}$$

where $k = (1/2^n) \in (0, 1)$. Hence, the conditions of Theorem 4 are fulfilled and $\tilde{z} = 0$ is a unique fixed point of T .

4. Supportive Application

In this section, we apply the theoretical results of Theorem 4 to discuss the existence of solution to a NIE:

$$1(t) = \int_0^t \tilde{E}(t, r)\tilde{\omega}(r, 1(r))dr, \quad t \in [0, D], D > 0, \tag{68}$$

where $\tilde{\omega}$ pounds: $[0, D] \times [0, D] \longrightarrow \mathbb{R}$ and $\tilde{\omega}: [0, D] \times \mathbb{R} \longrightarrow \mathbb{R}$ are continuous functions.

Define the operator $T: \mathfrak{N} \longrightarrow \mathfrak{N}$ by

$$T1(t) = \int_0^t \tilde{E}(t, r)\tilde{\omega}(r, 1(r))dr, \tag{69}$$

where $\mathfrak{N} = C([0, D])$ is the space of all continuous functions defined from $[0, D]$ to $\mathbb{R}$ equipped with

$$\begin{aligned}
m_r(1, \mathcal{J}) &= \frac{\|1\| + \|\mathcal{J}\|}{2} = \max_{t \in [0, D]} \left(\frac{|1(t)| + |\mathcal{J}(t)|}{2} \right), \\
&\forall 1, \mathcal{J} \in \mathfrak{N}.
\end{aligned} \tag{70}$$

Obviously, $(\mathfrak{N}, m_r)$ is a complete RMMS.

We will consider Problem (68) under the following assumptions:

(i) The function pounds: $[0, D] \times [0, D] \longrightarrow \mathbb{R}$ is continuous and $\max_{t, r \in [0, D]} |\text{pounds}(t, r)| \leq 1$

(ii) The function $\tilde{\omega}: [0, D] \times \mathbb{R} \longrightarrow \mathbb{R}$ is continuous,

$$\begin{aligned}
|\tilde{\omega}(r, 1(r))| + |\tilde{\omega}(r, \mathcal{J}(r))| &< \lambda^2 e^{-\lambda} P(1, \mathcal{J}), \\
&\forall 1, \mathcal{J} \in \mathfrak{N}, r \in [0, D],
\end{aligned} \tag{71}$$

where $\lambda \geq 1$ and $P(1, \mathcal{J}) = \max\{(|1| + |\mathcal{J}|/2), (|1| + |T1|/2), (|\mathcal{J}| + |T\mathcal{J}|/2)\}$

(iii) For all $1, \mathcal{J} \in \mathfrak{N}$, $\|1\|_\lambda = \max_{t \in [0, D]} |1(t)|e^{-\lambda t}$ and

$$m_{r\lambda}(1, \mathcal{J}) = \frac{\|1\|_\lambda + \|\mathcal{J}\|_\lambda}{\lambda} = \max_{t \in [0, D]} \left(\frac{|1(t)| + |\mathcal{J}(t)|}{\lambda} \right) e^{-\lambda t}, \tag{72}$$

so that the pair $(\mathfrak{N}, m_{r\lambda})$ is a complete RMMS with $\lambda = 2$

(iv) For all $1, \mathcal{J} \in \mathfrak{N}$, define the function $\theta \in \Theta$ by $\theta(t) = e^{\sqrt{t}}$, where $t > 0$

Now, we can state and prove the important theorem in this part as follows.

Theorem 5. Via hypotheses (i)–(iv), Problem (68) has a unique solution in $\mathfrak{N}$.

Proof. Define the operator $T: \mathfrak{N} \longrightarrow \mathfrak{N}$ as in (69). If T has a unique fixed point, then the NIE (68) has a unique solution. From Hypotheses (i)–(iii), we can write

$$\begin{aligned}
|T_1(r)| + |T\mathcal{J}(r)| &= \left| \int_0^t E(t,r)\varpi(r,1(r))dr \right| + \left| \int_0^t E(t,r)\varpi(r,\mathcal{J}(r))dr \right| \\
&\leq \int_0^t |E(t,r)\varpi(r,1(r))|dr + \int_0^t |\tilde{E}(t,r)\varpi(r,\mathcal{J}(r))|dr \\
&\leq \int_0^t |E(t,r)|(|\varpi(r,1(r))| + |\varpi(r,\mathcal{J}(r))|)dr \\
&\leq \int_0^t (|\varpi(r,1(r))| + |\varpi(r,\mathcal{J}(r))|)dr \\
&< \int_0^t \lambda^2 e^{-\lambda} P(1(r), \mathcal{J}(r))dr \\
&= \lambda^2 e^{-\lambda} \int_0^t e^{r\lambda} \max \left\{ \frac{|1(r)| + |\mathcal{J}(r)|}{\lambda} e^{-r\lambda}, \frac{|1(r)| + |T_1(r)|}{\lambda} e^{-r\lambda}, \frac{|\mathcal{J}(r)| + |T\mathcal{J}(r)|}{\lambda} e^{-r\lambda} \right\} dr \\
&\leq \lambda^2 e^{-\lambda} \int_0^t e^{r\lambda} \max\{m_{r\lambda}(1, \mathcal{J}), m_{r\lambda}(1, T_1), m_{r\lambda}(\mathcal{J}, T\mathcal{J})\} dr \\
&= \lambda^2 e^{-\lambda} N(1, \mathcal{J}) \int_0^t e^{r\lambda} dr \\
&= \lambda^2 e^{-\lambda} N(1, \mathcal{J}) \frac{1}{\lambda} (e^{\lambda t} - 1) \\
&= \lambda e^{-\lambda} N(1, \mathcal{J}) e^{\lambda t} - \lambda e^{-\lambda} N(1, \mathcal{J}) \\
&\leq \lambda e^{-\lambda} N(1, \mathcal{J}) e^{\lambda t}.
\end{aligned} \tag{73}$$

Consequently, we have

$$\frac{|T_1(r)| + |T\mathcal{J}(r)|}{\lambda} e^{-\lambda t} \leq e^{-\lambda} N(1, \mathcal{J}). \tag{74}$$

This implies that

$$m_{r\lambda}(T_1, T\mathcal{J}) = \max_{t \in [0, D]} \frac{|T_1(r)| + |T\mathcal{J}(r)|}{\lambda} e^{-\lambda t} \leq e^{-\lambda} N(1, \mathcal{J}). \tag{75}$$

By Hypothesis (iv), we obtain

$$e^{\sqrt{m_{r\lambda}(T_1, T\mathcal{J})}} \leq e^{\sqrt{e^{-\lambda} N(1, \mathcal{J})}} = \left(e^{\sqrt{N(1, \mathcal{J})}} \right)^{e^{-(\lambda/2)}}, \tag{76}$$

yielding

$$\theta(m_{r\lambda}(T_1, T\mathcal{J})) \leq [\theta(N(1, \mathcal{J}))]^k, \quad \forall 1, \mathcal{J} \in \mathbb{N}, \tag{77}$$

where $k = e^{-(\lambda/2)} \in (0, 1)$ with $\lambda = 2$. Thus, all conditions of Theorem 4 are fulfilled. So T has a unique fixed point which is a unique solution to NIE (68). $\square$

5. Conclusions

In this paper, we established fixed points for θ -contraction self-mapping in RMMS. Our new results are extensions of recent fixed-point theorems of Jalili et al. [11]. Also, we gave an example to clarify the obtained results. Finally, we applied our main result to study the existence and uniqueness of a solution for a NIE. The new concepts lead to further investigations and applications.

Data Availability

The data used to support the findings of this study are available from the corresponding author upon request.

Conflicts of Interest

The authors declare that they have no conflicts of interests regarding the publication of this article.

Authors' Contributions

All authors contributed equally and significantly in writing this article.

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