Research Article

Uniform Treatment of Jensen's Inequality by Montgomery Identity

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We generalize Jensen's integral inequality for real Stieltjes measure by using Montgomery identity under the effect of n -convex functions; also, we give different versions of Jensen's discrete inequality along with its converses for real weights. As an application, we give generalized variants of Hermite–Hadamard inequality. Montgomery identity has a great importance as many inequalities can be obtained from Montgomery identity in q -calculus and fractional integrals. Also, we give applications in information theory for our obtained results, especially for Zipf and Hybrid Zipf–Mandelbrot entropies.

1. Introduction

Convex functions have a great importance in mathematical inequalities, and the well-known Jensen's inequality is the characterization of convex functions. Jensen's inequality for differentiable convex functions plays a significant role in the field of inequalities as several other inequalities can be seen as special cases of it. One can find the application of Jensen's discrete inequality in discrete-time delay systems in [1].

Taking into consideration the tremendous applications of Jensen's inequality in various fields of mathematics and other applied sciences, the generalizations and improvements of Jensen's inequality have been a topic of supreme interest for the researchers during the last few decades as evident from a large number of publications on the topic (see [2–4] and the references therein).

The well-known Jensen's inequality asserts that for the function Γ it holds that

$$\Psi\left(\frac{1}{P_m} \sum_{j=1}^m p_j x_j\right) \leq \frac{1}{P_m} \sum_{j=1}^m p_j \Psi(x_j), \quad (1)$$

if Ψ is a convex function on interval $I \subset \mathbb{R}$, where p_j are positive real numbers and $x_j \in I$ ($j = 1, \dots, m$), while $P_m = \sum_{j=1}^m p_j$.

However, the well-known integral analogue of Jensen's inequality is as follows.

Theorem 1. Let $h: [a, b] \rightarrow [\alpha, \beta]$ be a continuous function and $\lambda: [a, b] \rightarrow \mathbb{R}$ be an increasing and bounded function with $\lambda(a) \neq \lambda(b)$. Then, for every continuous convex function $\Psi: [\alpha, \beta] \rightarrow \mathbb{R}$, the following inequality holds:

$$\Psi(\tilde{h}) \leq \frac{\int_a^b \Psi(h(\zeta)) d\lambda(\zeta)}{\int_a^b d\lambda(\zeta)}, \quad (2)$$

where

$$\tilde{h} = \frac{\int_a^b h(\zeta) d\lambda(\zeta)}{\int_a^b d\lambda(\zeta)} \in [\alpha, \beta]. \quad (3)$$

There are several inequalities coming from Jensen's inequality both in integral and discrete cases which can be

obtained by varying conditions on the function h and measure λ defined in Theorem 1.

Montgomery identity is used in quantum calculus or q -calculus. There are different identities of Montgomery, and several inequalities of Ostrowski type were formulated by using these identities. Budak and Sarikaya established the generalized Montgomery-type identities for differential mappings in [5]. Applications of Montgomery identity can be found in fractional integrals as well as in quantum integral operators. Here we utilize Montgomery's identity for the generalization of Jensen's inequality. In [6], Cerone and Dragomir developed a systematic study which produced some novel inequalities. Several interesting results related to inequalities and different types of convexity can be found in [7–21]. The class of convex functions is a very useful concept that has become a focus of interest for researchers in statistics, convex programming, and many other applied disciplines, as well as in inequality theory. The readers can find

some motivated findings related to convex functions and some new integral inequalities in [22–27].

In [28], Khan et al. have mentioned about n -convex functions as follows.

Definition 1. A function $f: I \rightarrow \mathbb{R}$ is called convex of order n or n -convex if for all choices of $(n+1)$ distinct points $x_i, \dots, x_{i+n}$ we have $\Delta_{(n)} f(x_i) \geq 0$.

If n -th order derivative $f^{(n)}$ exists, then f is n -convex if and only if $f^{(n)} \geq 0$. For $1 \leq k \leq (n-2)$, a function f is n -convex if and only if $f^{(k)}$ exists and is $(n-k)$ -convex.

In the present paper, we will use Montgomery identity that is presented as following.

Theorem 2. Let $n \in \mathbb{N}$, $\Psi: I \rightarrow \mathbb{R}$ be such that $\Psi^{(n-1)}$ is absolutely continuous, $I \subset \mathbb{R}$ is an open interval, and $\alpha, \beta \in I$, $\alpha < \beta$. Then, the following identity holds:

$$\Psi(x) = \frac{1}{\beta - \alpha} \int_{\alpha}^{\beta} \Psi(t) dt + \sum_{\ell=0}^{n-2} \frac{\Psi^{(\ell+1)}(\alpha)}{\ell! (\ell+2)} \frac{(x-\alpha)^{\ell+2}}{\beta - \alpha} - \sum_{\ell=0}^{n-2} \frac{\Psi^{(\ell+1)}(\beta)}{\ell! (\ell+2)} \frac{(x-\beta)^{\ell+2}}{\beta - \alpha} + \frac{1}{(n-1)!} \int_{\alpha}^{\beta} R_n(x, s) \Psi^{(n)}(s) ds, \quad (4)$$

where

$$R_n(x, s) = \begin{cases} -\frac{(x-s)^n}{n(\beta-\alpha)} + \frac{x-\alpha}{\beta-\alpha} (x-s)^{n-1}, & \alpha \leq s \leq x, \\ -\frac{(x-s)^n}{n(\beta-\alpha)} + \frac{x-\beta}{\beta-\alpha} (x-s)^{n-1}, & x < s \leq \beta. \end{cases} \quad (5)$$

2. Generalization of Jensen's Integral Inequality by Using Montgomery Identity

Before giving our main results, we consider the following assumptions that we use throughout our paper:

A_1 Let $h: [a, b] \rightarrow \mathbb{R}$ be continuous function.

A_2 Let $\lambda: [a, b] \rightarrow \mathbb{R}$ be a continuous function or the functions of bounded variation such that $\lambda(a) \neq \lambda(b)$.

2.1. New Generalization of Jensen's Integral Inequality. In our first main result, we employ Montgomery identity to obtain the following real Stieltjes measure's theoretical representations of Jensen's inequality.

Theorem 3. Let g, λ be as defined in A_1, A_2 such that $h([a, b]) \subset [\alpha, \beta]$. Also, let $\Psi: [\alpha, \beta] \rightarrow \mathbb{R}$ be such that for $n \geq 1$, $\Psi^{(n-1)}$ is absolutely continuous. If Ψ is n -convex such that

$$R_n(\tilde{h}, s) \leq \frac{\int_a^b R_n(\tilde{h}(\zeta), s) d\lambda(\zeta)}{\int_a^b d\lambda(\zeta)}, \quad s \in [\alpha, \beta], \quad (6)$$

with $\tilde{h}$ and $R_n(x, s)$ as defined in (3) and (5), respectively, then we have

$$\begin{aligned} \Psi(\tilde{h}) - \frac{\int_a^b \Psi(\tilde{h}(\zeta)) d\lambda(\zeta)}{\int_a^b d\lambda(\zeta)} &\leq \sum_{\ell=0}^{n-2} \frac{1}{\ell! (\ell+2) (\beta - \alpha)} \times \\ &\left[\Psi^{(\ell+1)}(\alpha) \left((\tilde{h} - \alpha)^{(\ell+2)} - \frac{\int_a^b (\tilde{h}(\zeta) - \alpha)^{(\ell+2)} d\lambda(\zeta)}{\int_a^b d\lambda(\zeta)} \right) - \Psi^{(\ell+1)}(\beta) \left((\tilde{h} - \beta)^{(\ell+2)} + \frac{\int_a^b (\tilde{h}(\zeta) - \beta)^{(\ell+2)} d\lambda(\zeta)}{\int_a^b d\lambda(\zeta)} \right) \right]. \end{aligned} \quad (7)$$

Proof. As $\Psi^{(n-1)}$ is absolutely continuous for $(n \geq 1)$, we can use the representation of Ψ using Montgomery identity (4) and can calculate

$$\Psi(\tilde{h}) = \frac{1}{\beta - \alpha} \int_{\alpha}^{\beta} \Psi(\zeta) d(\zeta) + \sum_{\ell=0}^{n-2} \frac{\Psi^{(\ell+1)}(\alpha) (\tilde{h} - \alpha)^{\ell+2}}{\ell! (\ell+2) (\beta - \alpha)} - \sum_{\ell=0}^{n-2} \frac{\Psi^{(\ell+1)}(\beta) (\tilde{h} - \beta)^{\ell+2}}{\ell! (\ell+2) (\beta - \alpha)} + \frac{1}{(n-1)!} \int_{\alpha}^{\beta} R_n(\tilde{h}, s) \Psi^{(n)}(s) ds. \quad (8)$$

The integration of the composition of functions $\Psi \circ h$ for the real measure λ on $[a, b]$ gives

$$\begin{aligned} \frac{\int_a^b \Psi(h(\zeta)) d\lambda(\zeta)}{\int_a^b d\lambda(\zeta)} &= \frac{1}{\beta - \alpha} \int_{\alpha}^{\beta} \Psi(\zeta) d(\zeta) + \sum_{\ell=0}^{n-2} \frac{\Psi^{(\ell+1)}(\alpha) \int_a^b (h(\zeta) - \alpha)^{\ell+2} d\lambda(\zeta)}{\ell! (\ell+2) (\beta - \alpha) \int_a^b d\lambda(\zeta)} \\ &\quad - \sum_{\ell=0}^{n-2} \frac{\Psi^{(\ell+1)}(\beta) \int_a^b (h(\zeta) - \beta)^{\ell+2} d\lambda(\zeta)}{\ell! (\ell+2) (\beta - \alpha) \int_a^b d\lambda(\zeta)} + \frac{1}{(n-1)!} \int_{\alpha}^{\beta} R_n(h(\zeta), s) \Psi^{(n)}(s) ds. \end{aligned} \quad (9)$$

Now computing the difference $\Psi(\tilde{h}) - \frac{\int_a^b \Psi(h(\zeta)) d\lambda(\zeta)}{\int_a^b d\lambda(\zeta)}$, we get the following generalized identity involving real Stieltjes measure:

$$\begin{aligned} \Psi(\tilde{h}) - \frac{\int_a^b \Psi(h(\zeta)) d\lambda(\zeta)}{\int_a^b d\lambda(\zeta)} &= \sum_{\ell=0}^{n-2} \frac{1}{\ell! (\ell+2) (\beta - \alpha)} \\ &\quad \times \left[\Psi^{(\ell+1)}(\alpha) \left((\tilde{h} - \alpha)^{\ell+2} - \frac{\int_a^b (h(\zeta) - \alpha)^{\ell+2} d\lambda(\zeta)}{\int_a^b d\lambda(\zeta)} \right) - \Psi^{(\ell+1)}(\beta) \left((\tilde{h} - \beta)^{\ell+2} + \frac{\int_a^b (h(\zeta) - \beta)^{\ell+2} d\lambda(\zeta)}{\int_a^b d\lambda(\zeta)} \right) \right] \\ &\quad + \frac{1}{(n-1)!} \int_{\alpha}^{\beta} \left(R_n(\tilde{h}, s) - \frac{\int_a^b R_n(h(\zeta), s) d\lambda(\zeta)}{\int_a^b d\lambda(\zeta)} \right) \Psi^{(n)}(s) ds. \quad (\text{GI.1}). \end{aligned} \quad (10)$$

Finally, by our assumption, $\Psi^{(n-1)}$ is absolutely continuous on $[\alpha, \beta]$; as a result, $\Psi^{(n)}$ exists almost everywhere. Moreover, Ψ is supposed to be n -convex, so we have $\Psi^{(n)}(x) \geq 0$ almost everywhere on $[\alpha, \beta]$. Therefore, by taking into account the last term in generalized identity (GI.1) and integral analogue of Jensen's inequality that is given in (6), we get (7). $\square$

In the later part of this section, we will vary our conditions on functions g and Stieltjes measure $d\lambda$ to obtain generalized variants of Jensen–Steffensen, Jensen–Boas, Jensen–Brunk, and Jensen-type inequalities. We start with

the following generalization of Jensen–Steffensen inequality for n -convex functions.

Theorem 4. Let Ψ defined in Theorem 3 be n -convex and h defined in M_1 be monotonic. Then, the following results hold.

(i) If λ defined in M_2 satisfies

$$\lambda(a) \leq \lambda(x) \leq \lambda(b), \quad \forall x \in [a, b], \lambda(b) > \lambda(a), \quad (11)$$

then for even $n \geq 3$, (6) is valid.

(ii) Moreover, if (6) is valid and the function

$$H(x) := \sum_{\ell=0}^{n-2} \left(\frac{1}{\ell! (\ell+2) (\beta - \alpha)} \right) \left(\Psi^{(\ell+1)}(\alpha) (x - \alpha)^{\ell+2} - \Psi^{(\ell+1)}(\beta) (x - \beta)^{\ell+2} \right) \quad (12)$$

is convex, then we get inequality (2) which is called generalized Jensen–Steffensen inequality for n -convex function.

Proof. (i) By applying second derivative test, we can show that the function

$R_n(x, s)$ is convex for even $n > 3$. Now using the assumed conditions, one can employ Jensen–Steffensen inequality given by Boas (see [29] or [30], p. 59) for convex function $R_n(x, s)$ to obtain (6).

(ii) Since we can rewrite the R.H.S. of (7) in the difference

$$H(\tilde{h}) - \frac{\int_a^b H(\tilde{h}(\zeta))d\lambda(\zeta)}{\int_a^b d\lambda(\zeta)}, \quad (13)$$

$$\lambda(a) \leq \lambda(x_1) \leq \lambda(y_1) \leq \lambda(x_2) \leq \lambda(y_2) \leq \dots \leq \lambda(y_{m-1}) \leq \lambda(x_m) \leq \lambda(b), \quad (14)$$

$\forall x_k \in (y_{k-1}, y_k)$ and $\lambda(b) > \lambda(a)$, then for even $n \geq 3$, (6) is valid.

(ii) Moreover, if (6) is valid and the function $H(\cdot)$ defined in (18) is convex, then again inequality (2) holds and is called Jensen–Boas inequality for n -convex function.

Proof. We follow the similar argument as in the proof of Theorem 4, but under the conditions of this corollary, we utilize Jensen–Boas inequality (see [29] or [24], p. 59) instead of Jensen–Steffensen inequality.

Next, we give results for Jensen–Brunk inequality. $\square$

Corollary 2. Let Ψ defined in Theorem 3 be n -convex and $\tilde{h}$ defined in M_1 be an increasing function. Then, the following results hold.

(i) If λ defined in M_2 with $\lambda(b) > \lambda(a)$ and

$$\int_a^x (\tilde{h}(x) - \tilde{h}(\zeta))d\lambda(\zeta) \geq 0, \quad (15)$$

and

$$\int_x^b (\tilde{h}(x) - \tilde{h}(\zeta))d\lambda(\zeta) \leq 0, \quad (16)$$

$\forall x \in [a, b]$ holds, then for even $n \geq 3$, (6) is valid.

(ii) Moreover, if (6) is valid and the function $H(\cdot)$ defined in (18) is convex, then again inequality (2) holds and is called Jensen–Brunk inequality for n -convex function.

for convex function H and by our assumed conditions on functions $\tilde{h}$ and λ , this difference is non-positive by using Jensen–Steffensen inequality difference [29]. As a result, the R.H.S. of inequality (7) is non-positive and we get generalized Jensen–Steffensen inequality (2) for n -convex function. $\square$

Now, we give similar results related to Jensen–Boas inequality [30], p. 59], which is a generalization of Jensen–Steffensen inequality.

Corollary 1. Let Ψ defined in Theorem 3 be n -convex function. Also, let $\tilde{h}$ be as defined in M_1 with $a = y_0 < y_1 < \dots < y_k < \dots < y_{m-1} < y_m = b$ and $\tilde{h}$ be monotonic in each of the m intervals $((y_{k-1}, y_k))$. Then, the following results hold.

(i) If λ as defined in M_2 satisfies

Proof. We proceed with the similar idea as in the proof of Theorem 4, but under the conditions of this corollary, we employ Jensen–Brunk inequality (see [31] or [30], p. 59)) instead of Jensen–Steffensen inequality. $\square$

Remark 1. The similar result in Corollary 2 is also valid provided that the function $\tilde{h}$ is decreasing. Also, assuming that the function $\tilde{h}$ is monotonic, one can replace the conditions in Corollary 2(i) by

$$0 \leq \int_a^x |\tilde{h}(x) - \tilde{h}(\zeta)|d\lambda(\zeta) \leq \int_x^b |g(x) - \tilde{h}(\zeta)|d\lambda(\zeta). \quad (17)$$

Remark 2. It is interesting to see that by employing similar method as in Theorem 4, we can also get the generalization of classical Jensen’s inequality (2) for n -convex functions by assuming the functions $\tilde{h}$ and λ along with the respective conditions in Theorem 1.

Another important consequence of Theorem 3 can be given by setting the function $\tilde{h}$ as $\tilde{h}(\zeta) = \zeta$. This form is the generalized version of L.H.S. inequality of the Hermite–Hadamard inequality.

Corollary 3. Let $\lambda: [a, b] \rightarrow \mathbb{R}$ be a function of bounded variation such that $\lambda(a) \neq \lambda(b)$ with $[a, b] \subset [\alpha, \beta]$ and $\tilde{\zeta} = \int_a^b \zeta d\lambda(\zeta) / \int_a^b d\lambda(\zeta) \in [\alpha, \beta]$. Under the assumptions of Theorem 3, if Ψ is n -convex such that

$$R_n(\tilde{h}, s) \leq \frac{\int_a^b R_n(\tilde{h}(\zeta), s)d\lambda(\zeta)}{\int_a^b d\lambda(\zeta)}, \quad s \in [\alpha, \beta], \quad (18)$$

then we have

$$\Psi(\tilde{\zeta}) \leq \frac{\int_a^b \Psi(\zeta) d\lambda(\zeta)}{\int_a^b d\lambda(\zeta)} + \sum_{\ell=0}^{n-2} \frac{1}{\ell! (\ell+2) (\beta-\alpha)} \times \left[\Psi^{(\ell+1)}(\alpha) \left((\tilde{\zeta}-\alpha)^{(\ell+2)} - \frac{\int_a^b (\zeta-\alpha)^{(\ell+2)} d\lambda(\zeta)}{\int_a^b d\lambda(\zeta)} \right) - \Psi^{(\ell+1)}(\beta) \left((\tilde{\zeta}-\beta)^{(\ell+2)} + \frac{\int_a^b (\zeta-\beta)^{(\ell+2)} d\lambda(\zeta)}{\int_a^b d\lambda(\zeta)} \right) \right]. \quad (19)$$

If the inequality (18) holds in reverse direction, then (19) also holds reversely.

The special case of above corollary can be given in the form of following remark.

Remark 3. It is interesting to see that substituting $\lambda(\zeta) = \zeta$ gives $\int_a^b d\lambda(\zeta) = b-a$ and $\tilde{\zeta} = a+b/2$. Using these substitutions in (2) and by following remark (20), we get the L.H.S. inequality of renowned Hermite–Hadamard inequality for n -convex functions.

2.2. New Generalization of Converse of Jensen's Integral Inequality. In this section, we give the results for the

converse of Jensen's inequality to hold, giving the conditions on the real Stieltjes measure $d\lambda$, such that $\lambda(a) \neq \lambda(b)$, allowing that the measure can also be negative, but employing Montgomery identity.

To start with we need the following assumption for the results of this section:

A_3 Let $\mathbf{m}, \mathbf{M} \in [\alpha, \beta]$ ($\mathbf{m} \neq \mathbf{M}$) be such that $\mathbf{m} \leq \tilde{h}(\zeta) \leq \mathbf{M}$ for all $\zeta \in [a, b]$ where $\tilde{h}$ is defined in A_1 .

For a given function $\Psi: [\alpha, \beta] \rightarrow \mathbb{R}$, we consider the difference

$$\text{CJ}(\Psi, \tilde{h}_{[\mathbf{m}, \mathbf{M}]}; \lambda) = \frac{\int_a^b \Psi(\tilde{h}(\zeta)) d\lambda(\zeta)}{\int_a^b d\lambda(\zeta)} - \frac{\mathbf{M} - \tilde{h}}{\mathbf{M} - \mathbf{m}} \Psi(\mathbf{m}) - \frac{\tilde{h} - \mathbf{m}}{\mathbf{M} - \mathbf{m}} \Psi(\mathbf{M}), \quad (20)$$

where $\tilde{h}$ is defined in (3).

Using Montgomery identity, we obtain the following representation of the converse of Jensen's inequality.

Theorem 5. Let $\tilde{h}, \lambda$ be as defined in A_1, A_2 and let $\Psi: [\alpha, \beta] \rightarrow \mathbb{R}$ be such that for $n \geq 1$, $\Psi^{(n-1)}$ is absolutely continuous. If Ψ is n -convex such that

$$\text{CJ}(R_n(x, s), \tilde{h}_{[\mathbf{m}, \mathbf{M}]}; \lambda) \leq 0, \quad s \in [\alpha, \beta], \quad (21)$$

or

$$\frac{\int_a^b R_n(\tilde{h}(\zeta), s) d\lambda(\zeta)}{\int_a^b d\lambda(\zeta)} \leq \frac{\mathbf{M} - \tilde{h}}{\mathbf{M} - \mathbf{m}} (R_n(\mathbf{m}, s)) + \frac{\tilde{h} - \mathbf{m}}{\mathbf{M} - \mathbf{m}} (R_n(\mathbf{M}, s)), \quad s \in [\alpha, \beta], \quad (22)$$

then we get the following extension of the converse of Jensen's difference:

$$\begin{aligned} \frac{\int_a^b \Psi(\tilde{h}(\zeta)) d\lambda(\zeta)}{\int_a^b d\lambda(\zeta)} &\leq \frac{\mathbf{M} - \tilde{h}}{\mathbf{M} - \mathbf{m}} \Psi(\mathbf{m}) + \frac{\tilde{h} - \mathbf{m}}{\mathbf{M} - \mathbf{m}} \Psi(\mathbf{M}) + \sum_{\ell=0}^{n-2} \left(\frac{1}{\ell! (\ell+2) (\beta-\alpha)} \right) \\ &\times \left(\Psi^{(\ell+1)}(\alpha) \text{CJ}((x-\alpha)^{\ell+2}, \tilde{h}_{[\mathbf{m}, \mathbf{M}]}; \lambda) - \Psi^{(\ell+1)}(\beta) \text{CJ}((x-\beta)^{\ell+2}, \tilde{h}_{[\mathbf{m}, \mathbf{M}]}; \lambda) \right), \end{aligned} \quad (23)$$

where $R_n(\cdot, s)$ is defined in (5).

Proof. As $\Psi^{(n-1)}$ is absolutely continuous for $(n \geq 1)$, we can use the representation of Ψ using Montgomery identity (4) in the difference $CJ(\Psi, h_{\{m, M\}}; \lambda)$:

$$\begin{aligned} CJ(\Psi, h_{\{m, M\}}; \lambda) &= CJ\left(\frac{1}{\beta - \alpha} \int_{\alpha}^{\beta} \Psi(\zeta) d\zeta, h_{\{m, M\}}; \lambda\right) \\ &+ \sum_{\ell=0}^{n-2} \left(\frac{1}{\ell! (\ell+2) (\beta - \alpha)} \right) \Psi^{(\ell+1)}(\alpha) CJ((x - \alpha)^{\ell+2}, h_{\{m, M\}}; \lambda) \\ &- \sum_{\ell=0}^{n-2} \left(\frac{1}{\ell! (\ell+2) (\beta - \alpha)} \right) \Psi^{(\ell+1)}(\beta) CJ((x - \beta)^{\ell+2}, h_{\{m, M\}}; \lambda) + \frac{1}{(n-1)!} \int_{\alpha}^{\beta} CJ(R_n(x, s), h_{\{m, M\}}; \lambda) \Psi^{(n)}(s) ds. \end{aligned} \quad (24)$$

After simplification and following the fact that $CJ(\Psi, h_{\{m, M\}}; \lambda)$ is zero for Ψ to be constant or linear, we get the following generalized identity:

$$\begin{aligned} CJ(\Psi, h_{\{m, M\}}; \lambda) j &= \sum_{\ell=0}^{n-2} \left(\frac{1}{\ell! (\ell+2) (\beta - \alpha)} \right) \\ &\times \left(\Psi^{(\ell+1)}(\alpha) CJ((x - \alpha)^{\ell+2}, h_{\{m, M\}}; \lambda) - \Psi^{(\ell+1)}(\beta) CJ((x - \beta)^{\ell+2}, h_{\{m, M\}}; \lambda) \right) \\ &+ \frac{1}{(n-1)!} \int_{\alpha}^{\beta} CJ(R_n(x, s), h_{\{m, M\}}; \lambda) \Psi^{(n)}(s) ds. \quad (\text{CGL.1}). \end{aligned} \quad (25)$$

Now using characterizations of n -convex functions like in the proof of Theorem 3, we get (23). $\square$

The next result gives converse of Jensen's inequality for higher-order convex functions.

Theorem 6. Let Ψ defined in Theorem 5 be n -convex and h be as defined in A_3 . Then, the following results hold.

- (i) If λ is non-negative measure on $[a, b]$, then for even $n \geq 3$, (22) is valid.
- (ii) Moreover, if (22) is valid and the function $H(\cdot)$ defined in (12) is convex, then we get the following inequality for n -convex function to be valid:

$$\frac{\int_a^b \Psi(h(\zeta)) d\lambda(\zeta)}{\int_a^b d\lambda(\zeta)} \leq \frac{M - \tilde{h}}{M - m} \Psi(m) - \frac{\tilde{h} - m}{M - m} \Psi(M). \quad (26)$$

Proof. The idea of the proof is similar to that of (6), but we use converse of Jensen's inequality (see [32] or [30], p. 98). $\square$

2.3. Applications of Jensen's Integral Inequality. In this section, we give applications of Jensen's integral inequality.

Another important consequence of Theorem 3 is by setting the function h as $h(\zeta) = \zeta$ gives generalized version of L. H. S. inequality of the Hermite-Hadamard inequality.

Corollary 4. Let $\lambda: [a, b] \rightarrow \mathbb{R}$ be a function of bounded variation such that $\lambda(a) \neq \lambda(b)$ with $[a, b] \subset [\alpha, \beta]$ and $\tilde{\zeta} = \int_a^b \zeta d\lambda(\zeta) / \int_a^b d\lambda(\zeta) \in [\alpha, \beta]$. Under the assumptions of Theorem 5, if Ψ is n -convex such that

$$\frac{\int_a^b R_n(\zeta, s) d\lambda(\zeta)}{\int_a^b d\lambda(\zeta)} \leq \frac{b - \tilde{\zeta}}{b - a} (R_n(a, s)) + \frac{\tilde{\zeta} - a}{b - a} (R_n(b, s)), \quad s \in [\alpha, \beta], \quad (27)$$

then we have

$$\frac{\int_a^b \Psi(\zeta) d\lambda(\zeta)}{\int_a^b d\lambda(\zeta)} \leq \frac{b-\tilde{\zeta}}{b-a} \Psi(a) + \frac{\tilde{\zeta}-a}{b-a} \Psi(b) + \sum_{\ell=0}^{n-2} \left(\frac{1}{\ell! (\ell+2) (\beta-\alpha)} \right) \times$$

$$\left(\Psi^{(\ell+1)}(\alpha) \text{CJ}((x-\alpha)^{\ell+2}, \text{id}_{[a,b]}; \lambda) - \Psi^{(\ell+1)}(\beta) \text{CJ}((x-\beta)^{\ell+2}, \text{id}_{[a,b]}; \lambda) \right). \quad (28)$$

If the inequality (27) holds in reverse direction, then (28) also holds reversely.

The special case of above corollary can be given in the form of following remark.

Remark 4. It is interesting to see that substituting $\lambda(\zeta) = \zeta$ and by following Theorem 6, we get the R.H.S. inequality of renowned Hermite–Hadamard inequality for n -convex functions.

3. Generalization of Jensen's Discrete Inequality by Using Montgomery Identity

In this section, we give generalizations for Jensen's discrete inequality by using Montgomery identity. The proofs are similar to those of continuous case as given in previous section; therefore, we give results directly.

3.1. Generalization of Jensen's Discrete Inequality for Real Weights. In discrete case, we have that $p_{\mathcal{J}} > 0$ for all $\mathcal{J} = 1, 2, \dots, m$. Here we give generalizations of results

allowing $p_{\mathcal{J}}$ to be negative real numbers. Also, with usual notations for $p_{\mathcal{J}} x_{\mathcal{J}}$ ($\mathcal{J} = 1, 2, \dots, n$), we notate

$$\mathbf{x} = (x_1, x_2, \dots, x_m) \text{ and } \mathbf{p} = (p_1, p_2, \dots, p_m) \quad (29)$$

to be m -tuples.

$$P_v = \sum_{\mathcal{J}=1}^v p_{\mathcal{J}}, \bar{P}_v = P_m - P_{v-1} \quad (v = 1, 2, \dots, m), \quad (30)$$

and

$$\bar{x} = \frac{1}{P_m} \sum_{\mathcal{J}=1}^m p_{\mathcal{J}} x_{\mathcal{J}}. \quad (31)$$

Using Montgomery identity (4), we obtain the following representations of Jensen's discrete inequality.

Theorem 7. Let $\Psi: [\alpha, \beta] \longrightarrow \mathbb{R}$ be such that for $n \geq 1$, $\Psi^{(n-1)}$ is absolutely continuous. Also, let $x_{\mathcal{J}} \in [a, b] \subseteq [\alpha, \beta]$, $p_{\mathcal{J}} \in \mathbb{R}$ ($\mathcal{J} = 1, \dots, m$) be such that $P_m \neq 0$ and $\bar{x} \in [\alpha, \beta]$.

(i) Then, the following generalized identity holds:

$$\Psi(\bar{x}) - \frac{1}{P_m} \sum_{\mathcal{J}=1}^m p_{\mathcal{J}} \Psi(x_{\mathcal{J}}) = \sum_{\ell=2}^{n-2} \left(\frac{1}{\ell! (\ell+2) (\beta-\alpha)} \right) \times$$

$$\left\{ \Psi^{(\ell+1)}(\alpha) \left((\bar{x}-\alpha)^{\ell+2} - \frac{1}{P_m} \sum_{\mathcal{J}=1}^m p_{\mathcal{J}} (x_{\mathcal{J}}-\alpha)^{\ell+2} \right) - \Psi^{(\ell+1)}(\beta) \left((\bar{x}-\beta)^{\ell+2} - \frac{1}{P_m} \sum_{\mathcal{J}=1}^m p_{\mathcal{J}} (x_{\mathcal{J}}-\beta)^{\ell+2} \right) \right\} \quad (32)$$

$$+ \frac{1}{(n-1)!} \int_{\alpha}^{\beta} \left[R_n(\bar{x}, s) - \frac{1}{P_m} \sum_{\mathcal{J}=1}^m p_{\mathcal{J}} R_n(x_{\mathcal{J}}, s) \right] \Psi^{(n)}(s) ds, \text{ (DGI.1),}$$

where $R_n(\cdot, s)$ is defined in (5).

(ii) Moreover, if Ψ is n -convex and the inequality

$$R_n(\bar{x}, s) \leq \frac{1}{P_m} \sum_{\mathcal{J}=1}^m p_{\mathcal{J}} R_n(x_{\mathcal{J}}, s) \quad (33)$$

holds, then we have the following generalized inequality:

$$\Psi(\bar{x}) - \frac{1}{P_m} \sum_{\mathcal{J}=1}^m p_{\mathcal{J}} \Psi(x_{\mathcal{J}}) \leq \sum_{\ell=2}^{n-2} \left(\frac{1}{\ell! (\ell+2) (\beta-\alpha)} \right) \times$$

$$\left\{ \Psi^{(\ell+1)}(\alpha) \left((\bar{x}-\alpha)^{\ell+2} - \frac{1}{P_m} \sum_{\mathcal{J}=1}^m p_{\mathcal{J}} (x_{\mathcal{J}}-\alpha)^{\ell+2} \right) - \Psi^{(\ell+1)}(\beta) \left((\bar{x}-\beta)^{\ell+2} - \frac{1}{P_m} \sum_{\mathcal{J}=1}^m p_{\mathcal{J}} (x_{\mathcal{J}}-\beta)^{\ell+2} \right) \right\}. \quad (34)$$

If inequality (33) holds in reverse direction, then (34) also holds reversely.

Proof. Similar to that of Theorem 3.

In the later part of this section, we will vary our conditions on $p_{\mathcal{J}}x_{\mathcal{J}}$ ($\mathcal{J} = 1, 2, \dots, n$) to obtain generalized discrete variants of Jensen–Steffensen, Jensen’s, and Jensen–Petrovic type inequalities. We start with the following generalization of Jensen–Steffensen discrete inequality for n -convex functions. $\square$

Theorem 8. Let Ψ be as defined in Theorem 7. Also, let $\mathbf{x}$ be monotonic n -tuple, $x_{\mathcal{J}} \in [a, b] \subseteq [\alpha, \beta]$, and $\mathbf{p}$ be a real n -tuple such that

$$0 \leq P_v \leq P_m, \quad (v = 1, 2, \dots, m-1), \quad P_m > 0 \quad (35)$$

is satisfied.

- (i) If Ψ is n -convex, then for even $n \geq 3$, (33) is valid.
- (ii) Moreover, if (33) is valid and the function $H(\cdot)$ defined in (12) is convex, then we get the following generalized Jensen–Steffensen discrete inequality:

$$\Psi(\bar{x}) \leq \frac{1}{P_m} \sum_{\mathcal{J}=1}^m p_{\mathcal{J}} \Psi(x_{\mathcal{J}}). \quad (36)$$

Proof. It is interesting to see that under the assumed conditions on tuples $\mathbf{x}$ and $\mathbf{p}$, we have that $\bar{x} \in [a, b]$. For $x_1 \geq x_2 \geq \dots \geq x_m$,

$$P_m(x_1 - \bar{x}) = \sum_{\mathcal{J}=2}^m p_{\mathcal{J}}(x_1 - x_{\mathcal{J}}) = \sum_{v=2}^m (x_{v-1} - x_v)(P_m - P_{v-1}) \geq 0. \quad (37)$$

This shows that $x_1 \geq \bar{x}$. Also, $\bar{x} \geq x_m$, since we have

$$P_m(\bar{x} - x_m) = \sum_{\mathcal{J}=1}^{m-1} p_{\mathcal{J}}(x_{\mathcal{J}} - x_m) = \sum_{v=1}^{m-1} (x_v - x_{v+1})P_v \geq 0. \quad (38)$$

For further details, see the proof of Jensen–Steffensen discrete inequality ([24], p. 57). The idea of the rest of the proof is similar to that of Theorem 3, but here we employ Theorem 7 and Jensen–Steffensen discrete inequality. $\square$

Corollary 5. Let Ψ be as defined in Theorem 7 and let $x_{\mathcal{J}} \in [a, b] \subseteq [\alpha, \beta]$ with $\mathbf{p}$ being a positive n -tuple.

- (i) If Ψ is n -convex, then for even $n \geq 3$, (34) is valid.
- (ii) Moreover, if (33) is valid and the function $H(\cdot)$ defined in (12) is convex, then again we get (36) which is called Jensen’s inequality for n -convex functions.

Proof. For $p_{\mathcal{J}} > 0, x_{\mathcal{J}} \in [a, b]$ ($\mathcal{J} = 1, 2, 3, \dots, m$) ensures that $\bar{x} \in [a, b]$. So, by applying classical Jensen’s discrete

inequality (1) and idea of Theorem 8, we will get the required results. $\square$

Remark 5. Under the assumptions of Corollary 5, if we choose $P_m = 1$, then Corollary 5 (ii) gives the following inequality for n -convex functions:

$$\Psi\left(\sum_{\mathcal{J}=1}^m p_{\mathcal{J}}x_{\mathcal{J}}\right) \leq \sum_{\mathcal{J}=1}^m p_{\mathcal{J}}\Psi(x_{\mathcal{J}}). \quad (39)$$

Now we give following reverses of Jensen–Steffensen and Jensen-type inequalities.

Corollary 6. Let Ψ be as defined in Theorem 7. Also, let $\mathbf{x}$ be monotonic m -tuple, $x_{\mathcal{J}} \in [a, b] \subseteq [\alpha, \beta]$, and $\mathbf{p}$ be a real m -tuple such that there exist $m \in \{1, 2, \dots, m\}$ such that

$$0 \geq P_v, \quad \text{for } v < m \text{ and } 0 \geq \bar{P}_v, \quad \text{for } v > m, \quad (40)$$

where $P_m > 0$ and $\bar{x} \in [\alpha, \beta]$.

- (i) If Ψ is n -convex, then for even $n \geq 3$, then reverse of inequality (33) holds.
- (ii) Moreover if (33) holds reversely and the function $H(\cdot)$ defined in (12) is convex, then we get reverse of generalized Jensen–Steffensen inequality (36) for n -convex functions.

Proof. We follow the idea of Theorem 8, but according to our assumed conditions, we employ reverse of Jensen–Steffensen inequality to obtain results. $\square$

In the next corollary, we give explicit conditions on real tuple $\mathbf{p}$ such that we get reverse of classical Jensen inequality.

Corollary 7. Let Ψ be as defined in Theorem 7 and let $x_{\mathcal{J}} \in [a, b] \subseteq [\alpha, \beta]$ such that $\bar{x} \in [\alpha, \beta]$. Let $\mathbf{p}$ be a real n -tuple such that

$$0 < p_1, 0 \geq p_2, p_3, \dots, p_m, 0 < P_m \quad (41)$$

is satisfied.

- (i) If Ψ is n -convex, then for even $n \geq 3$, the reverse of inequality (33) is valid.
- (ii) Also, if reverse of (33) is valid and the function $H(\cdot)$ defined in (12) is convex, then we get reverse of (36).

Proof. We follow the idea of Theorem 8, but according to our assumed conditions, we employ reverse of Jensen inequality to obtain results. $\square$

In [33] (see also [30]), one can find the result which is equivalent to the Jensen–Steffensen and the reverse Jensen–Steffensen inequality together. It is the so-called Jensen–Petrović inequality. Here, without the proof, we give the adequate corollary which uses that result. The proof goes the same way as in the previous corollaries.

Corollary 8. Let Ψ be as defined in Theorem 7 and let $x_i \in [a, b] \subseteq [\alpha, \beta]$ be such that $x_m \geq x_{m-1}, \dots, x_2 \leq x_1$. Let $\mathbf{p}$ be a real $\mathbf{m}$ -tuple with $P_m = 1$ such that

$$0 \leq P_v, \quad \text{for } 1 \leq v < m-1 \text{ and } 0 \leq \bar{P}_v, \quad \text{for } 2 \leq v < m, \quad (42)$$

is satisfied. Then, we get the equivalent results given in Theorem 8 (i) and (ii), respectively.

Remark 6. Under the assumptions of Corollary 8, if there exist $m \in \{1, 2, \dots, n\}$ such that

$$0 \geq P_v, \quad \text{for } v < m \text{ and } 0 \geq \bar{P}_v, \quad \text{for } v > m, \quad (43)$$

and $\bar{x} \in [\alpha, \beta]$, then we get the equivalent results for reverse Jensen–Steffensen inequality given in Corollary 6 (i) and (ii), respectively.

Remark 7. It is interesting to see that the conditions on $p_{\mathcal{J}}, \mathcal{J} = 1, 2, \dots, m$ given in Corollary 8 and Remark 6 are coming from Jensen–Petrović inequality which become equivalent to conditions for $p_{\mathcal{J}}, \mathcal{J} = 1, 2, \dots, m$ for Jensen–Steffensen results given in Theorem 8 and Corollary 6, respectively, when $P_m = 1$.

Now we give results for Jensen and its reverses for $\mathbf{n}$ -tuples $\mathbf{x}$ and $\mathbf{p}$ when $\mathbf{n}$ is an odd number.

Corollary 9. Let Ψ be as defined in Theorem 7 and let $x_{\mathcal{J}} \in [a, b] \subseteq [\alpha, \beta]$ for $\mathcal{J} = 1, 2, \dots, m$ be such that $\mathbf{x}, \mathbf{p}$ be real $\mathbf{m}$ -tuples, $m = 2m+1$, $m \in \mathbb{N}$ and $\hat{x} = 1/\sum_{\mathcal{J}=1}^{2k+1} p_{\mathcal{J}} \sum_{\mathcal{J}=1}^{2k+1} p_{\mathcal{J}} x_{\mathcal{J}} \in [\alpha, \beta]$ for all $k = 1, 2, \dots, m$. If for every $k = 1, 2, \dots, m$, we have

$$(i^*) \quad p_1 > 0, p_{2k} \leq 0, p_{2k} + p_{2k+1} \leq 0, \sum_{\mathcal{J}=1}^{2k} p_{\mathcal{J}} \geq 0, \sum_{\mathcal{J}=1}^{2k+1} p_{\mathcal{J}} > 0$$

$$(ii^*) \quad x_{2k} \leq x_{2k+1}, \sum_{\mathcal{J}=1}^{2k+1} p_{\mathcal{J}} (x_{\mathcal{J}} - x_{2k+1}) \geq 0,$$

then we have the following statements to be valid.

(i) If Ψ is $\mathbf{n}$ -convex, then for even $\mathbf{n} \geq 3$, the inequality

$$R_n(\hat{x}, s) \geq \frac{1}{P_{2m+1}} \sum_{\mathcal{J}=1}^{2m+1} p_{\mathcal{J}} R_n(x_{\mathcal{J}}, s). \quad (44)$$

(ii) Also if (44) is valid and the function $H(\cdot)$ defined in (12) is convex, then we get the following generalized inequality:

$$\Psi(\hat{x}) \geq \frac{1}{P_{2m+1}} \sum_{\mathcal{J}=1}^{2m+1} p_{\mathcal{J}} \Psi(x_{\mathcal{J}}). \quad (45)$$

Proof. We employ the idea of the proofs of Theorems 7 and 8 for $\mathbf{n} = \text{odd}$ along with inequality of Vasić and Janić [34]. $\square$

Remark 8. We can also discuss the following important cases by considering the explicit conditions given in [34].

We conclude this section by giving the following important cases:

(Case 1)

Let the condition (i^*) hold and the reverse inequalities in condition (ii^*) hold. Then, again we can give inequalities (44) and (45), respectively, given in Corollary 9.

(Case 2)

If in case of conditions (i^*) and (ii^*) , the following are valid:

$$(iii^*) \quad p_1 > 0, p_{2k+1} \geq 0, p_{2k} + p_{2k+1} \geq 0, \sum_{\mathcal{J}=1}^{2k} p_{\mathcal{J}} \geq 0, \sum_{\mathcal{J}=1}^{2k+1} p_{\mathcal{J}} > 0$$

$$(iv^*) \quad x_{2k} \leq x_{2k+1}, \sum_{\mathcal{J}=1}^{2k-1} p_{\mathcal{J}} (x_{\mathcal{J}} - x_{2k}) \leq 0,$$

then we can give reverses of inequalities (44) and (45), respectively, given in Corollary 9.

(Case 3)

Finally, we can also give reverses of inequalities (44) and (45), respectively, given in Corollary 9 provided that the condition (iii^*) holds and the reverse inequalities in condition (iv^*) hold.

The result given in (Case 3) is type of generalization of inequality by Szegő [35].

3.2. Generalization of Converse Jensen's Discrete Inequality for Real Weights. In this section, we give the results for converse of Jensen's inequality in discrete case by using the Montgomery identity.

Let $x_{\mathcal{J}} \in [a, b] \subseteq [\alpha, \beta]$, $a \neq b$, $p_{\mathcal{J}} \in \mathbb{R}$ ($\mathcal{J} = 1, \dots, n$) be such that $P_m \neq 0$. Then, we have the following difference of converse of Jensen's inequality for $\Psi: [\alpha, \beta] \rightarrow \mathbb{R}$:

$$CJ_{\text{dis}}(\Psi) = \frac{1}{P_m} \sum_{\mathcal{J}=1}^m p_{\mathcal{J}} \Psi(x_{\mathcal{J}}) - \frac{b - \bar{x}}{b - a} \Psi(a) - \frac{\bar{x} - a}{b - a} \Psi(b). \quad (46)$$

Similarly, we assume the Giaccardi difference [36] given as

$$G_{\text{cardi}}(\Psi) = \sum_{\mathcal{J}=1}^m p_{\mathcal{J}} \Psi(x_{\mathcal{J}}) - A \Psi\left(\sum_{\mathcal{J}=1}^m p_{\mathcal{J}} x_{\mathcal{J}}\right) - B\left(\sum_{\mathcal{J}=1}^m p_{\mathcal{J}} - 1\right) \Psi(x_0), \quad (47)$$

where

$$A = \frac{\left(\sum_{\mathcal{J}=1}^{\mathfrak{m}} p_{\mathcal{J}}(x_{\mathcal{J}} - x_0)\right)}{\left(\sum_{\mathcal{J}=1}^{\mathfrak{m}} p_{\mathcal{J}}x_{\mathcal{J}} - x_0\right)}, B = \frac{\sum_{\mathcal{J}=1}^{\mathfrak{m}} p_{\mathcal{J}}x_{\mathcal{J}}}{\left(\sum_{\mathcal{J}=1}^{\mathfrak{m}} p_{\mathcal{J}}x_{\mathcal{J}} - x_0\right)} \text{ and } \sum_{\mathcal{J}=1}^{\mathfrak{m}} p_{\mathcal{J}}x_{\mathcal{J}} \neq x_0. \quad (48)$$

Theorem 9. Let $\Psi: [\alpha, \beta] \longrightarrow \mathbb{R}$ be such that for $\mathfrak{n} \geq 1$, $\Psi^{(\mathfrak{n}-1)}$ is absolutely continuous. Also, let $x_0, x_{\mathcal{J}} \in [a, b] \subseteq [\alpha, \beta]$, $p_{\mathcal{J}} \in \mathbb{R}$ ($\mathcal{J} = 1, \dots, \mathfrak{m}$), be such that $\sum_{\mathcal{J}=1}^{\mathfrak{m}} p_{\mathcal{J}}x_{\mathcal{J}} \neq x_0$.

(i) Then, the following generalized identity holds:

$$\begin{aligned} \text{CJ}_{\text{dis}}(\Psi) &= \sum_{\ell=0}^{\mathfrak{n}-2} \left(\frac{1}{\ell! (\ell+2) (\beta-\alpha)} \right) \left(\Psi^{(\ell+1)}(\alpha) \text{CJ}_{\text{dis}} \left((x_{\mathcal{J}} - \alpha)^{\ell+2} \right) - \Psi^{(\ell+1)}(\beta) \text{CJ}_{\text{dis}} \left((x_{\mathcal{J}} - \beta)^{\ell+2} \right) \right) \\ &\quad + \frac{1}{(\mathfrak{n}-1)!} \int_{\alpha}^{\beta} \text{CJ}_{\text{dis}}(R_{\mathfrak{n}}(x_{\mathcal{J}}, s)) \Psi^{(\mathfrak{n})}(s) ds, \text{ (DC.GI)}, \end{aligned} \quad (49)$$

where $R_{\mathfrak{n}}(\cdot, s)$ is defined in (5).

(ii) Moreover, if Ψ is $\mathfrak{n}$ -convex and the inequality

$$\text{CJ}_{\text{dis}}(R_{\mathfrak{n}}(x_{\mathcal{J}}, s)) \leq 0 \quad (50)$$

holds, then we have the following generalized inequality:

$$\text{CJ}_{\text{dis}}(\Psi) \leq \sum_{\ell=0}^{\mathfrak{n}-2} \left(\frac{1}{\ell! (\ell+2) (\beta-\alpha)} \right) \left(\Psi^{(\ell+1)}(\alpha) \text{CJ}_{\text{dis}} \left((x_{\mathcal{J}} - \alpha)^{\ell+2} \right) - \Psi^{(\ell+1)}(\beta) \text{CJ}_{\text{dis}} \left((x_{\mathcal{J}} - \beta)^{\ell+2} \right) \right). \quad (51)$$

If inequality (50) holds in reverse direction, then (51) also holds reversely.

(i) Then, the following generalized Giaccardi identity holds:

Theorem 10. Let $\Psi: [\alpha, \beta] \longrightarrow \mathbb{R}$ be such that for $\mathfrak{n} \geq 1$, $\Psi^{(\mathfrak{n}-1)}$ is absolutely continuous. Also, let $x_{\mathcal{J}} \in [a, b] \subseteq [\alpha, \beta]$, $p_{\mathcal{J}} \in \mathbb{R}$ ($\mathcal{J} = 1, \dots, \mathfrak{m}$), be such that $P_{\mathfrak{m}} \neq 0$ and $\bar{x} \in [\alpha, \beta]$.

$$\begin{aligned} G_{\text{cardi}}(\Psi) &= \sum_{\ell=0}^{\mathfrak{n}-2} \left(\frac{1}{\ell! (\ell+2) (\beta-\alpha)} \right) \left(\Psi^{(\ell+1)}(\alpha) G_{\text{cardi}} \left((x_{\mathcal{J}} - \alpha)^{\ell+2} \right) - \Psi^{(\ell+1)}(\beta) G_{\text{cardi}} \left((x_{\mathcal{J}} - \beta)^{\ell+2} \right) \right) \\ &\quad + \frac{1}{(\mathfrak{n}-1)!} \int_{\alpha}^{\beta} G_{\text{cardi}}(R_{\mathfrak{n}}(x_{\mathcal{J}}, s)) \Psi^{(\mathfrak{n})}(s) ds, \text{ (GIA.GI)}, \end{aligned} \quad (52)$$

where $R_{\mathfrak{n}}(\cdot, s)$ is defined in (5).

(ii) Moreover, if Ψ is $\mathfrak{n}$ -convex and the inequality

$$G_{\text{cardi}}(R_{\mathfrak{n}}(x_{\mathcal{J}}, s)) \leq 0 \quad (53)$$

holds, then we have the following generalized Giaccardi inequality:

$$G_{\text{cardi}}(\Psi) \leq \sum_{\ell=0}^{\mathfrak{n}-2} \left(\frac{1}{\ell! (\ell+2) (\beta-\alpha)} \right) \left(\Psi^{(\ell+1)}(\alpha) G_{\text{cardi}} \left((x_{\mathcal{J}} - \alpha)^{\ell+2} \right) - \Psi^{(\ell+1)}(\beta) G_{\text{cardi}} \left((x_{\mathcal{J}} - \beta)^{\ell+2} \right) \right). \quad (54)$$

If inequality (53) holds in reverse direction, then (54) also holds reversely.

In the later part of this section, we will vary our conditions on $p_{\mathcal{J}}x_{\mathcal{J}}$ ($\mathcal{J} = 1, 2, \dots, \mathfrak{m}$) to obtain generalized

converse discrete variants of Jensen's inequality and Giaccardi inequality for $\mathbf{n}$ -convex functions.

Theorem 11. Let Ψ be as defined in Theorem 9. Also, let $x_{\mathcal{J}} \in [a, b] \subseteq [\alpha, \beta]$ and $\mathbf{p}$ be a positive $\mathbf{m}$ -tuple.

- (i) If Ψ is $\mathbf{n}$ -convex, then for even $\mathbf{n} \geq 3$, (50) is valid.
- (ii) Moreover, if (50) is valid and the function $H(\cdot)$ defined in (12) is convex, then we get the following generalized converse of Jensen's inequality:

$$\frac{1}{P_{\mathbf{m}}} \sum_{\mathcal{J}=1}^{\mathbf{m}} p_{\mathcal{J}} \Psi(x_{\mathcal{J}}) \leq \frac{b - \bar{x}}{b - a} \Psi(a) + \frac{\bar{x} - a}{b - a} \Psi(b). \quad (55)$$

$$\sum_{\mathcal{J}=1}^{\mathbf{m}} p_{\mathcal{J}} x_{\mathcal{J}} \neq x_0 \text{ and } (x_v - x_0) \left(\sum_{\mathcal{J}=1}^{\mathbf{m}} p_{\mathcal{J}} x_{\mathcal{J}} - x_v \right) \geq 0, \quad (v = 1, \dots, \mathbf{m}). \quad (56)$$

- (i) If Ψ is $\mathbf{n}$ -convex, then for even $\mathbf{n} \geq 3$, (53) is valid.
- (ii) Moreover, if (53) is valid and the function $H(\cdot)$ defined in (12) is convex, then we get the following generalized Giaccardi inequality:

$$\sum_{\mathcal{J}=1}^{\mathbf{m}} p_{\mathcal{J}} \Psi(x_{\mathcal{J}}) \leq A \Psi \left(\sum_{\mathcal{J}=1}^{\mathbf{m}} p_{\mathcal{J}} x_{\mathcal{J}} \right) + B \left(\sum_{\mathcal{J}=1}^{\mathbf{m}} p_{\mathcal{J}} - 1 \right) \Psi(x_0), \quad (57)$$

where A and B are defined in (47).

Proof. We follow the idea of Theorem 8, but according to our assumed conditions, we employ Giaccardi inequality (see [36] or [37], p. 11) to obtain results. $\square$

3.3. Applications in Information Theory for Jensen's Discrete Inequality. Jensen's inequality plays a key role in information theory to construct lower bounds for some notable inequalities, but here we will use it to make connections between inequalities in information theory.

Let $\Psi: \mathbb{R}^+ \rightarrow \mathbb{R}^+$ be a convex function and let $p := (p_1, \dots, p_{\mathbf{m}})$ and $q := (q_1, \dots, q_{\mathbf{m}})$ be positive probability distributions; then, Ψ -divergence functional is defined (in [38]) as follows:

Proof. We follow the idea of Theorem 8, but according to our assumed conditions, we employ converse of Jensen's inequality (see [32] or [30], p. 98) to obtain results. $\square$

Finally, in this section, we give Giaccardi inequality for higher-order convex functions.

Theorem 12. Let Ψ be as defined in Theorem 9. Also, let $x_0, x_{\mathcal{J}} \in [a, b] \subseteq [\alpha, \beta]$ and $\mathbf{p}$ be a positive $\mathbf{m}$ -tuple such that

$$I_{\Psi}(\mathbf{p}, \mathbf{q}) = \sum_{\mathcal{J}=1}^{\mathbf{m}} q_{\mathcal{J}} \Psi \left(\frac{p_{\mathcal{J}}}{q_{\mathcal{J}}} \right). \quad (58)$$

Horváth et al. in [39] defined the generalized Csiszár divergence functional as follows.

Definition 2. Let I be an interval in $\mathbb{R}$ and $\Psi: I \rightarrow \mathbb{R}$ be a function. Also, let $\mathbf{p} := (p_1, \dots, p_{\mathbf{m}}) \in \mathbb{R}^{\mathbf{m}}$ and $\mathbf{q} := (q_1, \dots, q_{\mathbf{m}}) \in (0, \infty)^{\mathbf{m}}$ such that

$$\frac{p_{\mathcal{J}}}{q_{\mathcal{J}}} \in I, \quad \mathcal{J} = 1, \dots, \mathbf{m}. \quad (59)$$

Then, let

$$\tilde{I}_{\Psi}(\mathbf{p}, \mathbf{q}) = \sum_{\mathcal{J}=1}^{\mathbf{m}} q_{\mathcal{J}} \Psi \left(\frac{p_{\mathcal{J}}}{q_{\mathcal{J}}} \right). \quad (60)$$

In this section, we write Jensen's difference here that we use in upcoming results:

$$F(\mathbf{p}, x_{\mathcal{J}}, \Psi) = \Psi(\bar{x}) - \frac{1}{P_{\mathbf{m}}} \sum_{\mathcal{J}=1}^{\mathbf{m}} p_{\mathcal{J}} \Psi(x_{\mathcal{J}}). \quad (61)$$

Theorem 13. Under the assumptions of Theorem 9 (ii), let (51) hold and Ψ be $\mathbf{n}$ -convex. Also, let $\mathbf{p} := (p_1, \dots, p_{\mathbf{m}})$ in $\mathbb{R}^{\mathbf{m}}$ and $\mathbf{q} := (q_1, \dots, q_{\mathbf{m}}) \in (0, \infty)^{\mathbf{m}}$; then, we have the following results:

$$\tilde{I}_\Psi(\mathbf{p}, \mathbf{q}) \geq P_m \Psi(1) - P_m \sum_{\ell=2}^{n-2} \left(\frac{1}{\ell! (\ell+2) (\beta - \alpha)} \right) \times \left\{ \Psi^{(\ell+1)}(\alpha) \left[F\left(\mathbf{q}, \frac{p_{\mathcal{J}}}{q_{\mathcal{J}}}, (x - \alpha)^{\ell+2}\right) \right] - \Psi^{(\ell+1)}(\beta) \left[F\left(\mathbf{q}, \frac{p_{\mathcal{J}}}{q_{\mathcal{J}}}, (x - \beta)^{\ell+2}\right) \right] \right\}. \quad (62)$$

Proof. From Theorem 9 by following Jensen's difference (61), we can rearrange (34) as

$$\Psi(\bar{x}) - \frac{1}{P_m} \sum_{\mathcal{J}=1}^m p_{\mathcal{J}} \Psi(x_{\mathcal{J}}) \leq \sum_{\ell=2}^{n-2} \left(\frac{1}{\ell! (\ell+2) (\beta - \alpha)} \right) \times \left\{ \Psi^{(\ell+1)}(\alpha) \left(F(\mathbf{p}, x_{\mathcal{J}}, (x - \alpha)^{\ell+2}) \right) - \Psi^{(\ell+1)}(\beta) \left(F(\mathbf{p}, x_{\mathcal{J}}, (x - \beta)^{\ell+2}) \right) \right\}. \quad (63)$$

Now replace $p_{\mathcal{J}}$ with $q_{\mathcal{J}}$ and $x_{\mathcal{J}}$ with $p_{\mathcal{J}}/q_{\mathcal{J}}$, and we get (62). $\square$

For positive n -tuple $\mathbf{q} = (q_1, \dots, q_m)$ such that $\sum_{\mathcal{J}=1}^m q_{\mathcal{J}} = 1$, the Shannon entropy is defined by

$$S(q) = - \sum_{\mathcal{J}=1}^m q_{\mathcal{J}} \ln q_{\mathcal{J}}. \quad (64)$$

Corollary 10. Under the assumptions of Theorem 9 (ii), let (51) hold and Ψ be n -convex.

(i) If $\mathbf{q} := (q_1, \dots, q_m) \in (0, \infty)^m$, then

$$\sum_{\mathcal{J}=1}^m q_{\mathcal{J}} \ln q_{\mathcal{J}} \leq P_m \sum_{\ell=2}^{n-2} \left(\frac{1}{\ell! (\ell+2) (\beta - \alpha)} \right) \times \left\{ \frac{(-1)^{\ell+1} \ell!}{\alpha^{(\ell+1)}} F\left(q, \frac{1}{q_{\mathcal{J}}}, -\ln(\cdot)\right) - \frac{(-1)^{\ell+1} \ell!}{\beta^{(\ell+1)}} F\left(q, \frac{1}{q_{\mathcal{J}}}, -\ln(\cdot)\right) \right\}. \quad (65)$$

(ii) We can get bounds for the Shannon entropy of $\mathbf{q}$, if we choose $\mathbf{q} := (q_1, \dots, q_n)$ to be a positive probability distribution.

$$S(\mathbf{q}) \leq P_m \sum_{\ell=2}^{n-2} \left(\frac{1}{\ell! (\ell+2) (\beta - \alpha)} \right) \times \left\{ \frac{(-1)^{\ell+1} \ell!}{\alpha^{(\ell+1)}} F\left(\mathbf{q}, \frac{1}{q_{\mathcal{J}}}, -\ln(\cdot)\right) - \frac{(-1)^{\ell+1} \ell!}{\beta^{(\ell+1)}} F\left(\mathbf{q}, \frac{1}{q_{\mathcal{J}}}, -\ln(\cdot)\right) \right\}. \quad (66)$$

Proof. (i) Substituting $\Psi(x) := -\ln x$ and using $\mathbf{p} := (1, 1, \dots, 1)$ in Theorem 13, we get (65).

(ii) Since we have $\sum_{\mathcal{J}=1}^m q_{\mathcal{J}} = 1$, by multiplying -1 on both sides of (65) and taking into account (64), we get (66).

The Kullback–Leibler distance [40] between the positive probability distributions $\mathbf{p} = (p_1, \dots, p_m)$ and $\mathbf{q} = (q_1, \dots, q_m)$ is defined by

$$D(\mathbf{q} \parallel \mathbf{p}) = \sum_{\mathcal{J}=1}^m q_{\mathcal{J}} \ln \left(\frac{q_{\mathcal{J}}}{p_{\mathcal{J}}} \right). \quad (67)$$

$\square$

Corollary 11. Under the assumptions of Corollary 10,

(i) If $\mathbf{q} := (q_1, \dots, q_m), \mathbf{p} := (p_1, \dots, p_m) \in (0, \infty)^m$, then

$$\sum_{\mathcal{J}=1}^m q_{\mathcal{J}} \ln \left(\frac{q_{\mathcal{J}}}{p_{\mathcal{J}}} \right) \leq P_m \sum_{\ell=2}^{n-2} \left(\frac{1}{\ell! (\ell+2) (\beta - \alpha)} \right) \times \left\{ \frac{(-1)^{\ell+1} \ell!}{\alpha^{(\ell+1)}} F\left(q, \frac{p_{\mathcal{J}}}{q_{\mathcal{J}}}, -\ln(\cdot)\right) - \frac{(-1)^{\ell+1} \ell!}{\beta^{(\ell+1)}} F\left(q, \frac{p_{\mathcal{J}}}{q_{\mathcal{J}}}, -\ln(\cdot)\right) \right\}. \quad (68)$$

(ii) If $\mathbf{q} := (q_1, \dots, q_m), \mathbf{p} := (p_1, \dots, p_m)$ are positive probability distributions, then we have

$$D(\mathbf{q} \parallel \mathbf{p}) \leq P_m \sum_{\ell=2}^{n-2} \left(\frac{1}{\ell! (\ell+2) (\beta-\alpha)} \right) \times \left\{ \frac{(-1)^{\ell+1} \ell!}{\alpha^{(\ell+1)}} F\left(q, \frac{p_{\mathcal{J}}}{q_{\mathcal{J}}}, -\ln(\cdot)\right) - \frac{(-1)^{\ell+1} \ell!}{\beta^{\ell+1}} F\left(q, \frac{p_{\mathcal{J}}}{q_{\mathcal{J}}}, -\ln(\cdot)\right) \right\}. \quad (69)$$

Proof.

- (i) Using $\Psi(x) := -\ln x$ (which is n -convex for even n) in Theorem 13, we get (68) after simplification.
- (ii) It is a special case of (i). $\square$

3.4. Results for Zipf and Hybrid Zipf–Mandelbrot Entropy. One of the basic laws in information science is Zipf's law [41,42] which is highly applied in linguistics. Let $c \geq 0$, $d > 0$, and $N \in \{1, 2, \dots\}$; Zipf–Mandelbrot entropy can be given as

$$Z_M(H, c, d) = \frac{d}{H_{c,d}^N} \sum_{\mathcal{J}=1}^N \frac{\ln(\mathcal{J} + c)}{(\mathcal{J} + c)^d} + \ln(H_{c,d}^N), \quad (70)$$

where

$$H_{c,d}^N = \sum_{\sigma=1}^N \frac{1}{(\sigma + c)^d}. \quad (71)$$

Consider

$$q_{\mathcal{J}} = \Psi(\mathcal{J}; N, c, d) = \frac{1}{((\mathcal{J} + c)^d H_{c,d}^N)}, \quad (72)$$

where $\Psi(\mathcal{J}; m, c, d)$ is discrete probability distribution known as Zipf–Mandelbrot law. Zipf–Mandelbrot law has many application in linguistics and information sciences. Some of the recent study about Zipf–Mandelbrot law can be seen in the listed references (see [39, 43]). Now we state our results involving entropy introduced by Mandelbrot law by establishing the relationship with Shannon and relative entropies.

Theorem 14. Let $\mathbf{q}$ be Zipf–Mandelbrot law as defined in (72) with parameters $c \geq 0$, $d > 0$, and $N \in \{1, 2, \dots\}$, and we have

$$Z_M(H, c, d) = S(\mathbf{q}) \leq N \times \sum_{\ell=2}^{n-2} \left(\frac{1}{\ell! (\ell+2) (\beta-\alpha)} \right) \times \left\{ \frac{(-1)^{\ell+1} \ell!}{\alpha^{(\ell+1)}} F\left(\mathbf{q}, ((\mathcal{J} + c)^d H_{c,d}^N), -\ln(\cdot)\right) - \frac{(-1)^{\ell+1} \ell!}{\beta^{(\ell+1)}} F\left(\mathbf{q}, ((\mathcal{J} + c)^d H_{c,d}^N), -\ln(\cdot)\right) \right\}. \quad (73)$$

Proof. It is interesting to see that for $q_{\mathcal{J}}$ defined in (72), $\sum_{\mathcal{J}=1}^N q_{\mathcal{J}} = 1$. Therefore, using above $q_{\mathcal{J}}$ in Shannon entropy (64), we get Mandelbrot entropy (70):

$$S(\mathbf{q}) = - \sum_{\mathcal{J}=1}^N q_{\mathcal{J}} \ln q_{\mathcal{J}} = - \sum_{\mathcal{J}=1}^N \frac{1}{((\mathcal{J} + c)^d H_{c,d}^N)} \ln \frac{1}{((\mathcal{J} + c)^d H_{c,d}^N)} = \frac{d}{(H_{c,d}^N)} \sum_{\mathcal{J}=1}^N \frac{\ln(\mathcal{J} + c)}{(\mathcal{J} + c)^d} + \ln(H_{c,d}^N). \quad (74)$$

Finally, substituting this $q_{\mathcal{J}} = 1/((\mathcal{J} + c)^d H_{c,d}^N)$ in Corollary 10 (ii), we get the desired result. $\square$

Corollary 12. Let $\mathbf{q}$ and $\mathbf{p}$ be Zipf–Mandelbrot law with parameters $c_1, c_2 \in [0, \infty)$, $d_1, d_2 > 0$, and let

$H_{c_1, d_1}^N = \sum_{\sigma=1}^N 1/(\sigma + c_1)^{d_1}$ and $H_{c_2, d_2}^N = \sum_{\sigma=1}^N 1/(\sigma + c_2)^{d_2}$.
 Now using $q_{\mathcal{J}} = 1/(\mathcal{J} + c_1)^{d_1} H_{c_1, d_1}^N$ and $p_{\mathcal{J}} = 1/(\mathcal{J} + c_2)^{d_2} H_{c_2, d_2}^N$ in Corollary 11 (ii), the following holds:

$$\begin{aligned} D(\mathbf{q} \parallel \mathbf{p}) &= \sum_{\mathcal{J}=1}^N \frac{1}{(\mathcal{J} + c_1)^{d_1} H_{c_1, d_1}^N} \ln \left(\frac{(\mathcal{J} + c_2)^{d_2} H_{c_2, d_2}^N}{(\mathcal{J} + c_1)^{d_1} H_{c_1, d_1}^N} \right) \\ &= -Z(H, c_1, d_1) + \frac{d_2}{H_{c_1, d_1}^N} \sum_{\mathcal{J}=1}^N \frac{\ln(\mathcal{J} + c_2)}{(\mathcal{J} + c_1)^{d_1}} + \ln(H_{c_2, d_2}^N) \\ &\leq N \sum_{\ell=2}^{n-2} \left(\frac{1}{\ell! (\ell + 2) (\beta - \alpha)} \right) \times \left\{ \frac{(-1)^{\ell+1} \ell!}{\alpha^{(\ell+1)}} F \left(\mathbf{q}, \frac{(\mathcal{J} + c_1)^{d_1} H_{c_1, d_1}^N}{(\mathcal{J} + c_2)^{d_2} H_{c_2, d_2}^N}, -\ln(\cdot) \right) - \frac{(-1)^{\ell+1} \ell!}{\beta^{(\ell+1)}} F \left(\mathbf{q}, \frac{(\mathcal{J} + c_1)^{d_1} H_{c_1, d_1}^N}{(\mathcal{J} + c_2)^{d_2} H_{c_2, d_2}^N}, -\ln(\cdot) \right) \right\}. \end{aligned} \quad (75)$$

The Next Result for Hybrid Zipf–Mandelbrot Entropy. Further generalization of Zipf–Mandelbrot entropy is Hybrid

Zipf–Mandelbrot entropy. Let $N \in \{1, 2, \dots\}$, $c \geq 0$, $\omega > 0$; then, Hybrid Zipf–Mandelbrot entropy can be given as

$$\hat{Z}_M(H^*, c, d, \omega) = \frac{1}{H_{c, d, \omega}^*} \sum_{\mathcal{J}=1}^N \frac{\omega^{\mathcal{J}}}{(\mathcal{J} + c)^d} \ln \left(\frac{(\mathcal{J} + c)^d}{\omega^{\mathcal{J}}} \right) + \ln(H_{c, d, \omega}^*), \quad (76)$$

where

$$H_{c, d, \omega}^* = \sum_{\mathcal{J}=1}^N \frac{\omega^{\mathcal{J}}}{(\mathcal{J} + c)^d}. \quad (77)$$

Consider

$$q_{\mathcal{J}} = \Psi(\mathcal{J}; N, c, d, \omega) = \frac{\omega^{\mathcal{J}}}{(\mathcal{J} + c)^d H_{c, d, \omega}^*}, \quad (78)$$

which is called Hybrid Zipf–Mandelbrot law. There is a unified approach, maximization of Shannon entropy [44],

that naturally follows the path of generalization from Zipf's to Hybrid Zipf's law. Extending this idea, Jakšetić et al. in [45] presented a transition from Zipf–Mandelbrot to Hybrid Zipf–Mandelbrot law by employing maximum entropy technique with one additional constraint. It is interesting that examination of its densities provides some new insights of Lerch's transcendent.

Theorem 15. Let $\mathbf{q}$ be Hybrid Zipf–Mandelbrot law as defined in (78) with parameters $c \geq 0$, $d, \omega > 0$, and $N \in \{1, 2, \dots\}$, and we have

$$\begin{aligned} \hat{Z}_M(H^*, c, d, \omega) = S(\mathbf{q}) &\leq N \sum_{\ell=2}^{n-2} \left(\frac{1}{\ell! (\ell + 2) (\beta - \alpha)} \right) \times \\ &\left\{ \frac{(-1)^{\ell+1} \ell!}{\alpha^{(\ell+1)}} F \left(\mathbf{q}, \frac{(\mathcal{J} + c)^d H_{c, d, \omega}^*}{\omega^{\mathcal{J}}}, -\ln(\cdot) \right) - \frac{(-1)^{\ell+1} \ell!}{\beta^{(\ell+1)}} F \left(\mathbf{q}, \frac{(\mathcal{J} + c)^d H_{c, d, \omega}^*}{\omega^{\mathcal{J}}}, -\ln(\cdot) \right) \right\}. \end{aligned} \quad (79)$$

Proof. It is interesting to see that for $q_{\mathcal{F}}$ defined in (78), $\sum_{\mathcal{F}=1}^N q_{\mathcal{F}} = 1$. Therefore, using above $q_{\mathcal{F}}$ in Shannon entropy (64), we get Hybrid Zipf–Mandelbrot entropy (76):

$$\begin{aligned}
 S(\mathbf{q}) &= - \sum_{\mathcal{F}=1}^N q_{\mathcal{F}} \ln q_{\mathcal{F}} = - \sum_{\mathcal{F}=1}^N \frac{\omega^{\mathcal{F}}}{(\mathcal{F}+c)^d H_{c,d,\omega}^*} \ln \frac{\omega^{\mathcal{F}}}{(\mathcal{F}+c)^d H_{c,d,\omega}^*} \\
 &= \frac{-1}{H_{c,d,\omega}^*} \sum_{\mathcal{F}=1}^N \frac{\omega^{\mathcal{F}}}{(\mathcal{F}+c)^d} \left[\ln \left(\frac{\omega^{\mathcal{F}}}{(\mathcal{F}+c)^d} \right) + \ln \left(\frac{1}{H_{c,d,\omega}^*} \right) \right] \\
 &= \frac{1}{H_{c,d,\omega}^*} \sum_{\mathcal{F}=1}^N \frac{\omega^{\mathcal{F}}}{(\mathcal{F}+c)^d} \left[\ln \left(\frac{(\mathcal{F}+c)^d}{\omega^{\mathcal{F}}} \right) + \ln(H_{c,d,\omega}^*) \right] \\
 &= \frac{1}{H_{c,d,\omega}^*} \sum_{\mathcal{F}=1}^N \frac{\omega^{\mathcal{F}}}{(\mathcal{F}+c)^d} \ln \left(\frac{(\mathcal{F}+c)^d}{\omega^{\mathcal{F}}} \right) + \ln(H_{c,d,\omega}^*).
 \end{aligned} \tag{80}$$

Finally, substituting this $q_{\mathcal{F}} = \omega^{\mathcal{F}} / (\mathcal{F}+c)^d H_{c,d,\omega}^*$ in Corollary 10 (ii), we get the desired result. $\square$

Corollary 13. Let $\mathbf{q}$ and $\mathbf{p}$ be Hybrid Zipf–Mandelbrot law with parameters $c_1, c_2 \in [0, \infty)$, $\omega_1, \omega_2, d_1, d_2 > 0$. Now using $q_{\mathcal{F}} = \omega_1^{\mathcal{F}} / (\mathcal{F}+c_1)^{d_1} H_{c_1,d_1,\omega_1}^*$ and $p_{\mathcal{F}} = \omega_2^{\mathcal{F}} / (\mathcal{F}+c_2)^{d_2} H_{c_2,d_2,\omega_2}^*$ in Corollary 11 (ii), the following holds:

$$\begin{aligned}
 D(\mathbf{q} \parallel \mathbf{p}) &= \sum_{\mathcal{F}=1}^N \frac{\omega_1^{\mathcal{F}}}{(\mathcal{F}+c_1)^{d_1} H_{c_1,d_1,\omega_1}^*} \ln \left(\frac{\omega_1^{\mathcal{F}} (\mathcal{F}+c_2)^{d_2} H_{c_2,d_2,\omega_2}^*}{\omega_2^{\mathcal{F}} (\mathcal{F}+c_1)^{d_1} H_{c_1,d_1,\omega_1}^*} \right) \\
 &= -\hat{Z}_M(H^*, c_1, d_1, \omega_1) + \frac{1}{H_{c_1,d_1,\omega_1}^*} \sum_{\mathcal{F}=1}^N \frac{\omega_1^{\mathcal{F}}}{(\mathcal{F}+c_1)^{d_1}} \ln \left(\frac{(\mathcal{F}+c_2)^{d_2}}{\omega_2^{\mathcal{F}}} \right) + \ln(H_{c_2,d_2,\omega_2}^*) \\
 &\leq N \sum_{\ell=2}^{n-2} \left(\frac{1}{\ell! (\ell+2) (\beta-\alpha)} \right) \times \left\{ \frac{(-1)^{\ell+1} \ell!}{\alpha^{(\ell+1)}} F \left(\mathbf{q}, \frac{\omega_2^{\mathcal{F}} (\mathcal{F}+c_1)^{d_1} H_{c_1,d_1,\omega_1}^*}{\omega_1^{\mathcal{F}} (\mathcal{F}+c_2)^{d_2} H_{c_2,d_2,\omega_2}^*}, -\ln(\cdot) \right) \right. \\
 &\quad \left. - \frac{(-1)^{\ell+1} \ell!}{\beta^{(\ell+1)}} F \left(\mathbf{q}, \frac{\omega_2^{\mathcal{F}} (\mathcal{F}+c_1)^{d_1} H_{c_1,d_1,\omega_1}^*}{\omega_1^{\mathcal{F}} (\mathcal{F}+c_2)^{d_2} H_{c_2,d_2,\omega_2}^*}, -\ln(\cdot) \right) \right\}.
 \end{aligned} \tag{81}$$

Remark 9. Similarly, we can give results for Shannon entropy, Kullback–Leibler distance, Zipf–Mandelbrot entropy, and Hybrid Zipf–Mandelbrot entropy by using generalized Giaccardi inequality defined in (54) on the same steps.

4. Concluding Remarks

In this paper, we gave generalization of Jensen’s inequality as well as converse of Jensen’s inequality by using Montgomery identity. We also formulate results for other inequalities like

Jensen–Steffensen inequality, Jensen–Boas inequality, and Jensen–Brunk inequality. We can obtain Jensen–Steffensen inequality, Jensen–Boas inequality, and Jensen–Brunk inequality by changing the assumption of Jensen’s inequality. At the end, we gave applications in information theory for our obtained results, especially we gave results for Hybrid Zipf–Mandelbrot entropy for our obtained results [46].

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that there are no conflicts of interest regarding the publication of this paper.

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