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Research Article

On Omega Index and Average Degree of Graphs

Sadik Delen,[1] **Musa Demirci**,[1] **Ahmet Sinan Cevik**,[2] **and Ismail Naci Cangul**[1]

[1]*Faculty of Arts and Science, Department of Mathematics, Bursa Uludag University, 16059 Bursa, Turkey*
[2]*Faculty of Science, Department of Mathematics, Selcuk University, Konya, Turkey*

Correspondence should be addressed to Ismail Naci Cangul; cangul@uludag.edu.tr

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Average degree of a graph is defined to be a graph invariant equal to the arithmetic mean of all vertex degrees and has many applications, especially in determining the irregularity degrees of networks and social sciences. In this study, some properties of average degree have been studied. Effect of vertex deletion on this degree has been determined and a new proof of the handshaking lemma has been given. Using a recently defined graph index called *omega* index, average degree of trees, unicyclic, bicyclic, and tricyclic graphs have been given, and these have been generalized to k-cyclic graphs. Also, the effect of edge deletion has been calculated. The average degree of some derived graphs and some graph operations have been determined.

1. Introduction

Let $G = (V, E)$ be a finite, undirected, and simple graph, having $|V| = n$ vertices and $|E| = m$ edges. For a vertex $v \in V$, the number of edges of G meeting at v is denoted by $d_G(v)$ or d_v and known as the degree of v. If the set of all vertex degrees of G is

$$D = \left\{1^{(a_1)}, 2^{(a_2)}, 3^{(a_3)}, \ldots, \Delta^{(a_\Delta)}\right\}, \tag{1}$$

where a_is are nonnegative integers, then D is called the degree sequence of G. Here, Δ is the maximum vertex degree. For every graph G, there is a degree sequence. However, for a degree sequence, there may or may not exist a graph. If there is a graph, then D is said to be realizable. For every realizable degree sequence, there is at least one graph; usually, there are many.

In [1], a new graph index called Ω index was defined by

$$\Omega(D) = \sum_{i=1}^{\Delta} (i-2)a_i. \tag{2}$$

Several properties of Ω index were obtained in [1, 2].

There are some graph invariants related to the vertex degrees. The density of a graph G measures how many edges are there in the set E of edges compared to the maximum possible number of edges between the vertices of G. For a simple graph, the maximum number of edges is attained when the graph is complete. In this case, this number would be $n(n-1)/2$. So, the density of G is

$$\rho(G) = \frac{2m}{n(n-1)}. \tag{3}$$

Clearly, the density increases proportionally with the number of edges. Therefore, amongst all connected simple graphs having n vertices, trees have the lowest density and complete graph K_n has the highest density which is 1.

In this paper, we make use of Ω index to study the properties of average degree $\overline{d_G}$ of a graph G which measures how many edges has G compared to the number of vertices, that is,

$$\overline{d_G} = \frac{2m}{n}. \tag{4}$$

From the definitions, we can easily deduce the following relation between $\overline{d_G}$ and $\rho(G)$:

$$(n-1)\rho(G) = \overline{d_G}. \tag{5}$$

Also, the following relations between the Ω index of a graph G and density and average degree can be deduced as follows:

$$\rho(G) = \frac{\Omega(G) + 2n}{n(n-1)},$$
$$\overline{d_G} = \frac{\Omega(G)}{n} + 2. \quad (6)$$

If all the vertices in a graph have the same degree, say r, the graph is said r-regular. Most of the graphs are not regular and some graph indices have been defined to measure the irregularity of a graph. The Bell index of a graph G is defined by

$$B(G) = \sum_{u\in V(G)} \left(d_G(u) - \frac{2m}{n}\right)^2, \quad (7)$$

see [3], and the degree deviation index is defined by

$$S(G) = \sum_{u\in V(G)} \left|d_G(u) - \frac{2m}{n}\right|, \quad (8)$$

In [4], the degree deviation is equal to the product of the order n of G with the discrepancy of the graph. Also, the Collatz–Sinogowitz index is defined by

$$CS(G) = \lambda_1 - \frac{2m}{n}, \quad (9)$$

where λ_1 denotes the largest eigenvalue of the adjacency matrix [5]. Note that, in all these irregularity indices, the average degree of a graph is used. This makes the notion of the average degree of a graph important. In the recent paper [6], two new and structurally different irregularity indices IRA and IRB have been defined and compared with other existing irregularity measures. Naturally, this list can easily be extended to have new members of the family of irregularity indices.

The structure of this paper is planned as follows. In Section 2, two related notions, density and average degree, are studied and these two quantities are calculated for acyclic, unicyclic, bicyclic, tricyclic, and, in general, k-cyclic graphs. In Sections 3 and 4, the effects of vertex and edge deletion on average degree of a graph are formulized. In Sections 5 and 6, the average degree of some derived graphs and some binary graph operations are determined.

2. Density, Average Degree, and Cyclicness

A connected graph G having no faces is called acyclic. A graph having at least one face is named as cyclic; especially as unicyclic, bicyclic, tricyclic, etc., according to the number of faces which is $1, 2, 3$, etc., respectively. Here, using the Ω index, graphs are classified according to their cyclicness, and for each case, the density of the graph is characterized as follows:

(i) If G is acyclic, then $\rho(G) = (2/n)$
(ii) If G is unicyclic, then $\rho(G) = (2/(n-1))$
(iii) If G is bicyclic, then $\rho(G) = ((2(n+1))/(n(n-1)))$
(iv) If G is tricyclic, then $\rho(G) = ((2(n+2))/(n(n-1)))$
(v) If G is k-cyclic, for $k = 0, 1, 2, \ldots$, then $\rho(G) = ((2(n+k-1))/(n(n-1)))$

Indeed, let G be a connected k-cyclic graph. By Theorem 1 in [1], we have $k = (\Omega(G)/2) + 1$, as G is connected. Hence, $\Omega(G) = 2(k-1)$, and by Corollary 3.4 in [1], we know that $m = n + k - 1$.

The following is a similar result for an average degree. The average vertex degree of a connected k-cyclic graph is

$$\overline{d_G} = \frac{2(n+k-1)}{n}. \quad (10)$$

As a result, connected acyclic, unicyclic, bicyclic, tricyclic, etc., graphs have average degrees $2(n-1)/n$, 2, $2(n+1)/n$, $2(n+2)/n$, etc., respectively.

3. Effect of Vertex Deletion on Average Degree

Now, the effect of vertex deletion on average degree will be considered. Using the obtained formula successively, we give a result which helps to calculate the average degree of a large graph by means of the average degree of a much smaller graph. Let $G - v$ be the graph obtained by deleting the vertex v together with d_v edges incident to v. First, we have the following.

Theorem 1. *The following recurrence relation holds:*

$$n \cdot \overline{d_G} - (n-1)\overline{d}_{G-v_1} = 2d_G(v_1). \quad (11)$$

Proof. Let G be a connected simple graph of order n. Let $v_1 \in V(G)$ have degree $d_G(v_1)$ in G. $G - v_1$ is the graph obtained by deleting the vertex v_1 together with d_{v_1} edges incident to v_1. Hence, deleting v_1 from G reduces the degree of each of the d_{v_1} neighbours of v_1 by 1. So, the deletion of v_1 from G reduces the total vertex degree of G by $2d_{v_1}$. Hence,

$$\overline{d}_{G-v_1} = \frac{\sum_{u\in V(G)} d_u - 2d_{v_1}}{n-1} = \frac{n \cdot \overline{d_G} - 2d_{v_1}}{n-1}, \quad (12)$$

which gives the result. □

Applying Theorem 1 recursively, we can give another proof of the handshaking lemma by labeling the vertices $v_1, v_2, \cdots, v_n$ so that $d_{v_1} \le d_{v_2} \le d_{v_n}$.

Corollary 1. *(handshaking lemma). In every graph, the sum of vertex degrees is equal to twice the number of edges, that is,*

$$\sum_{u\in V(G)} d_u = 2m. \quad (13)$$

Proof. Applying Theorem 1 successively to delete $v_1, v_2, \ldots, v_n$ from G, respectively, we get the following equalities:

$$\begin{aligned} n\overline{d}_G - (n-1)\overline{d}_{G-v_1} &= 2d_{v_1} \\ (n-1)\overline{d}_{G-v_1} - (n-2)\overline{d}_{G-\{v_1,v_2\}} &= 2d_{v_2} \\ (n-2)\overline{d}_{G-\{v_1,v_2\}} - (n-3)\overline{d}_{G-\{v_1,v_2,v_3\}} &= 2d_{v_3} \\ &\vdots \\ (n-(n-1))\overline{d}_{G-\{v_1,v_2,\ldots,v_{n-1}\}} - (n-n)\overline{d}_{G-\{v_1,v_2,\ldots,v_n\}} &= 2d_{v_n}.j. \end{aligned} \quad (14)$$

Adding all these side by side, we obtain the result. □

So, we deleted all the vertices one by one to prove handshaking lemma. What happens if we only delete some of the vertices? We answer this question in two steps.

Theorem 2. *Let G be a connected simple graph of order n, size m, and average degree $\overline{d_G}$. If $v_1, v_2, \ldots, v_k$ are pairwise nonadjacent vertices of G, then*

$$\overline{d}_{G-\{v_1,v_2,\ldots,v_k\}} = 2\frac{m-(d_1+d_2+\cdots+d_k)}{n-k}, \quad (15)$$

where $d_i = d_{v_i}$, for $i = 1, 2, \ldots, k$.

Proof. Let $\overline{m}$ and $\overline{n}$ be the size and order of $G - \{v_1, v_2, \ldots, v_k\}$. Then, by definition, we have $\overline{d}_{G-\{v_1,v_2,\ldots,v_k\}} = 2(\overline{m}/\overline{n})$. Clearly, $\overline{n} = n - k$ as k vertices are deleted. When a vertex v_i is deleted, note that d_i incident edges are also deleted. So, if we delete pairwise nonadjacent vertices $v_1, \ldots, v_k$ from G, a total of $d_1 + d_2 + \cdots + d_k$ edges are also deleted giving that $\overline{m} = m - (d_1 + d_2 + \cdots + d_k)$. It proves the result. □

Finally, if some pairs of the deleted vertices are adjacent, then we would have the following result.

Theorem 3. *If t pairs of the vertices $v_1, v_2, \ldots, v_k$ are adjacent in G, then*

$$\overline{d}_{G-\{v_1,v_2,\ldots,v_k\}} = \frac{2m - 2(d_1+d_2+\cdots+d_k)+2t}{n-k}. \quad (16)$$

Proof. The proof is quite similar to the one of Theorem 2, except we need to deal with adjacent pairs of vertices. If v_i and v_j are two adjacent vertices in G, then the total number of neighbours of v_i is d_i and the total number of neighbours of v_j is d_j. Hence, when deleting both v_i and v_j, we also delete $d_i + d_j - 1$ edges incident to at least one of v_i or v_j. Here, -1 comes as the edge between v_i and v_j which can only be deleted once, but it is incident to both v_i and v_j. So, when $v_1, \ldots, v_k$ are deleted from G, a total of $2(d_1 + d_2 + \cdots + d_k - t)$ edges are also deleted, giving the result. □

Example 1. Let G be the graph in Figure 1, and let us delete v_1, v_2, v_4, v_5, and v_6.

Note that the average degree of G is $14/7 = 2$. The graph $G - \{v_1, v_2, v_4, v_5, v_6\}$ is in Figure 2 and has the average degree $2/2 = 1$.

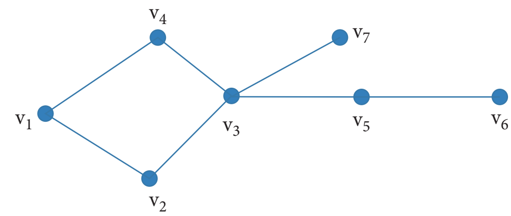

Figure 1: A graph G.

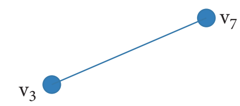

Figure 2: The graph $G - \{v_1, v_2, v_4, v_5, v_6\}$.

According to Theorem 3,

$$1 = \frac{2\cdot 7 - 2(2+2+2+2+1)+6}{7-5}, \quad (17)$$

as there are $t = 3$ pairs of deleted vertices.

4. Effect of Edge Deletion on Average Degree

We now determine the effect of edge deletion from a graph on the average vertex degree. Let G be a graph of order n and size m, and let e be an edge. The graph obtained by deleting the edge e from G is denoted by $G - e$. We have the following recurrence relation between the average degrees of G and $G - e$.

Theorem 4. *The average degree of $G - e$ is related to the average degree of G by the following recurrence relation:*

$$\overline{d_G} - \overline{d}_{G-e} = \frac{2}{n}. \quad (18)$$

Proof. As deleting an edge e will reduce the number of edges by 1 and will not change the number of vertices, we know that

$$\begin{aligned} \overline{d}_{G-e} &= \frac{\sum_{u\in V(G)} d_G(u) - 2}{n} \\ &= \frac{n\cdot\overline{d_G} - 2}{n}, \end{aligned} \quad (19)$$

giving the required result. □

5. Average Degree of Some Derived Graphs

A derived graph is a graph obtained from a given graph after some operation. In that sense, many authors consider derived graphs as graph operations. Derived graphs help to determine a property of a given graph by calculating the same property of the derived graph. In this section, we determine the average degree of some derived graphs. The derived graphs under study are the line, total, jump, and semitotal line graphs.

The line graph $L(G)$ of a graph G is constructed as follows. For each edge in the graph G, we draw a new vertex of $L(G)$ so that two edges in G, having a vertex in common, make an edge between their corresponding vertices in $L(G)$. It can be seen that the order and size of the line graph $L(G)$ are $n(L(G)) = m(G) = m$ and $m(L(G)) = ((M_1(G))/2) - m$, respectively, where $M_1(G) = \sum_{u\in V(G)} d_G(u)^2$ is the famous topological graph index called the first Zagreb index. Indeed,

$$\begin{aligned} m(L(G)) &= \frac{1}{2}\sum_{e\in E(G)} d_G(e) \\ &= \frac{1}{2}\sum_{e=uv\in E(G)} \left(d_G(u) + d_G(v) - 2\right) \\ &= \frac{1}{2}\sum_{u\in V(G)} d_G(u)^2 - \sum_{e\in E(G)} 1 \\ &= \frac{M_1(G)}{2} - m. \end{aligned} \tag{20}$$

Hence, the average degree of the line graph $L(G)$ is obtained as

$$\overline{d}_{L(G)} = \frac{M_1(G) - 2m(G)}{m(G)} = \frac{M_1(G)}{m} - 2. \tag{21}$$

The total graph $T(G)$ of a graph G, also known as the generalization of the line graph, is the graph so that the vertex set of $T(G)$ corresponds to the vertices and edges of G and two vertices are adjacent in $T(G)$ iff their corresponding elements are either adjacent or incident in G. By the definition, one can deduce the order and size of the total graph $T(G)$ as $n(T(G)) = m + n$ and $m(T(G)) = 2m + (M_1(G)/2)$, respectively. Hence, we deduce the average degree of the total graph as

$$\overline{d}_{T(G)} = \frac{4m + M_1(G)}{m+n}. \tag{22}$$

Next, we study the average degree of the jump graph. The jump graph $J(G)$ of a graph G is the complement of the line graph. That is, the jump graph $J(G)$ of a graph G is the graph defined on the edge set $E(G)$ of the graph G in which two vertices are adjacent if and only if they are not adjacent in G. The order and size of the jump graph are $n(J(G)) = m$ and $m(J(G)) = ((m(m+1))/2) - (1/2)M_1(G)$, respectively. Hence, the average degree of $J(G)$ is

$$\begin{aligned} \overline{d}_{J(G)} &= \frac{m^2 + m - M_1(G)}{m} \\ &= m + 1 - \frac{M_1(G)}{m}. \end{aligned} \tag{23}$$

Finally, we consider the semitotal line graph $T_1(G)$. Its vertex set is $V(G)\cup E(G)$ where two vertices are adjacent in $T_1(G)$ if and only if they are either adjacent or incident in G. Also, $n(T_1(G)) = m + n$ and $m(T_1(G)) = m + (M_1(G)/2)$, implying that

$$\overline{d}_{T_1(G)} = \frac{2m + M_1(G)}{m+n}. \tag{24}$$

So, it is proved.

Theorem 5. *Let G be a graph of order n and size m. Then, the average degree of the line graph $L(G)$, total graph $T(G)$, jump graph $J(G)$, and semitotal line graph $T_1(G)$ is as follows:*

$$\begin{aligned} \overline{d}_{L(G)} &= \frac{M_1(G) - 2m(G)}{m(G)} = \frac{M_1(G)}{m} - 2, \\ \overline{d}_{T(G)} &= \frac{4m + M_1(G)}{m+n}, \\ \overline{d}_{J(G)} &= m + 1 - \frac{M_1(G)}{m}, \\ \overline{d}_{T_1(G)} &= \frac{2m + M_1(G)}{m+n}. \end{aligned} \tag{25}$$

6. Average Degree of Some Graph Operations

Let $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ be two graphs such that $|E(G_1)| = m_1$ and $|E(G_2)| = m_2$. In this final section, we study the average degree of some graph operations, namely, union, join, and corona products of two graphs G_1 and G_2.

The union $G_1 \cup G_2$ of two graphs G_1 and G_2 having disjoint vertex and edge sets is obtained easily by taking union of vertex sets as its vertex set and the union of edge sets as the edge set.

Theorem 6. *The average degree of the union $G_1 \cup G_2$ of two graphs G_1 and G_2 is*

$$\overline{d}_{G_1\cup G_2} = \frac{2(m_1 + m_2)}{n_1 + n_2}. \tag{26}$$

Proof. By the definition of union operation, we can write that $|V(G_1 \cup G_2)| = n_1 + n_2$ and $|E(G_1 \vee G_2)| = m_1 + m_2$. This proves the result. □

For two graphs G_1 and G_2 with orders n_1 and n_2 and sizes m_1 and m_2, respectively, the join operation $G_1 \vee G_2$ of two graphs G_1 and G_2 having disjoint vertex sets $V(G_1)$ and $V(G_2)$ and edge sets $E(G_1)$ and $E(G_2)$ is the graph union $G_1 \cup G_2$ together with all the edges between $V(G_1)$ and $V(G_2)$.

Theorem 7. *The average degree of the join $G_1 \vee G_2$ of two graphs G_1 and G_2 is*

$$\overline{d}_{G_1\vee G_2} = \frac{2(m_1 + m_2 + n_1 n_2)}{n_1 + n_2}. \tag{27}$$

Proof. By the definition, we can deduce that $|V(G_1 \vee G_2)| = n_1 + n_2$ and $|E(G_1 \vee G_2)| = m_1 + m_2 + n_1 n_2$. Hence, the result follows. □

The corona product $G_1 \circ G_2$ of two graphs G_1 and G_2 is defined to be the graph obtained by taking one copy of G_1 (which has n_1 vertices) and n_1 copies of G_2, and then, joining the ith vertex of G_1 to every vertex in the ith copy of G_2, for $i = 1, 2, \ldots, n_1$, we obtain the following.

Theorem 8. *The average degree of the corona product $G_1 \circ G_2$ of two graphs G_1 and G_2 is*

$$\overline{d}_{G_1 \circ G_2} = \frac{2\left(m_1 + n_1\left(m_2 + n_2\right)\right)}{n_1\left(1 + n_2\right)}. \tag{28}$$

Proof. We know that $|V(G_1 \circ G_2)| = \mathrm{n}_1(1 + \mathrm{n}_2)$ and $|E(G_1 \circ G_2)| = \mathrm{m}_1 + \mathrm{n}_1(\mathrm{m}_2 + \mathrm{n}_2 \lim_{\mathrm{x} \longrightarrow \infty})$. This gives the required number. □

Data Availability

No data available.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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