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Research Article

Coupled Fixed Point Theorems for Some Type of Contraction Mappings in b-Cone and b-Theta Cone Metric Spaces

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This paper proves the existence of a unique coupled fixed point of some type of contraction mappings defined on a complete b-cone and b-theta cone metric spaces; consequently, it extends and generalizes many previous coupled fixed point theorems.

1. Introduction

In 2007, Huang and Zhang [1] introduced cone metric spaces as generalization of metric spaces by considering vector-valued metrics with values in an ordered real Banach space and hence generalized the concept of metric spaces and its completeness. They proved the existence of a unique fixed point for a contraction self-map T of cone metric space (X, d) showing that cone metric spaces provide larger categories of spaces for the fixed point theory. Recall that if $\mathbb{A}$ is a normed space, C is a cone in $\mathbb{A}$ that generates the partial ordered relation $\preceq$, and (X, d) is a cone metric space over $\mathbb{A}$, then a mapping $T\colon X \longrightarrow X$ is said to be contraction on X if and only if there is a constant $\alpha \in [0, 1)$ such that

$$d(T(x), T(y)) \preceq \alpha\, d(x, y), \quad \forall x, y \in X. \tag{1}$$

In 2008, Rezapour and Hamlbarani [2] improved some of the results in [1] by omitting the normality assumption of the cone induced the partial relation.

After that, more topological characterization of cone metric spaces linked with some fixed point theorems have been studied by many other authors (see [2–5] and the references therein).

In 2013, 2014, 2016, 2017, and then in 2020, Azam et al., citeAkbar, Xu and Radenovi [6], Huang and Radenovi [7], Sharma [8], and Sahar [9], respectively, considered cyclic and cone metric spaces over Banach algebra and b- cone metric spaces as a generalization of cone metric spaces, and they gave some further generalizations of some fixed points and proved fixed point theorems of contractive mapping in b- cone metric spaces, some of these results are proved without using the normality condition of a cone and some proved for generalized contraction multivalued mappings.

In 2020, Sahar [10] introduced the concept of theta cone metric spaces which gave larger categories of metric spaces and generalized some previous fixed points' theorems in the setting of this concept.

On the other side, in 1987, the concept of coupled fixed point was initiated by Gue and Lakshmikantham [11], in partially ordered metric spaces; after that, in 2006, Bhaskar and Lakshmikantham [12] proved existence of coupled fixed points for mappings having the mixed monotone property.

In 2009, Sabetghadam et al. [13] proved some coupled fixed point theorems for mappings satisfying different contractive conditions on complete cone metric spaces. Specifically, they proved the following.

Theorem 1 (see [13]). *Let $(X, C\ d)$ be a complete cone metric space. Suppose that the mapping $F\colon X \times X \longrightarrow X$ satisfies the following contractive condition for all $x, y, u, v \in X$:*

$$d(F(x, y), F(u, v)) \preceq k\, d(x, u) + l\, d(y, v), \tag{2}$$

where k and l are nonnegative constants with $k + l < 1$. Then, F has a unique coupled fixed point.

In 2010, Khamsi [14] gave the definition of metric type space (or b-metric space) and used this approach to proved the existence of a unique fixed point for Lipschitzian type mapping defined on a complete metric type space, and then, he noticed that, in case of normal cone, the cone metric q of the cone metric space (X, C, q) generates a metric type D: $X \times X \longrightarrow \mathbb{R}^+$ and hence a metric type space (X, D). His remarkable notice enabled him to prove the existence of unique fixed point for Lipschitzian type mapping defined on complete cone metric (X, C, q) but provided that the cone C is normal.

In 2013, Luong et al. [15] followed another direction; they proved some coincidence and coupled fixed point of two compatible mappings: F: $X \times X \longrightarrow X$ and g: $X \longrightarrow X$, F is continuous, F has the mixed g-monotone property, F and g satisfy some generalized contraction conditions, X is partially ordered set endowed with a complete cone metric and some extra conditions on convergent monotone nondecreasing and convergent monotone nonincreasing sequences in X.

In 2021, Sahar [16] generalized coupled fixed point result for some generalized contraction type of mappings of Bhaskar and Lakshmikantham [12] and of Sahar [17, 18] in theta cone metric spaces.

We have the following notations and basic definitions.

2. Preliminaries and Basic Definitions

We start with the following.

Definition 1 (see [14]). Let X be a nonempty set. Let D: $X \times X \longrightarrow \mathbb{R}^+$ be a function which satisfies the following:

(1) $D(x, y) \geq 0, D(x, y) = 0$ *if and only if* $x = y$

(2) $D(x, y) = D(y, x)$ *for every* $x, y \in X$

(3) $D(x, y) \leq K[D(x, w) + D(w, y)]$ *for some positive real number* $K > 0$

The pair (X, D) is called a metric type space, and it is called b-metric space if $K \geq 1$.

We recall some standard notations and definitions in cone metric spaces.

A subset C of a linear space $\mathbb{A}$ is said to be a cone in $\mathbb{A}$ if and only if

(1) C is nonempty closed and $C \neq \{\theta\}$, where θ is the zero (neutral element) of $\mathbb{A}$

(2) $\lambda C + \mu C \subset C$ for all nonnegative real numbers λ, μ

(3) $C \cap -C = \{\theta\}$

A cone C in a normed space $(\mathbb{A}, \|\cdot\|)$ is said to be solid if and only if it has a nonempty interior, that is, the set of all interior points of C is not empty set, $\mathrm{Int}C \neq \varnothing$.

If $(\mathbb{A}, \|\cdot\|)$ is a normed space, C is a cone in $\mathbb{A}$; then, C generates the following ordered relations:

$$\begin{aligned} u \preceq v &\Longleftrightarrow v - u \in C, \\ u < v &\Longleftrightarrow (v - u \in C \text{ and } u \neq v), \\ u \ll v &\Longleftrightarrow v - u \in \mathrm{Int}\, C. \end{aligned} \tag{3}$$

A sequence $\{z_n\}_{n \in N}$ in $\mathbb{A}$ is bounded above by $z \in \mathbb{A}$ if and only if

$$z_n \preceq z, \quad \forall n \in \mathbb{N}, \tag{4}$$

and its bounded below by $z \in \mathbb{A}$ if and only if

$$z \preceq z_n, \quad \forall n \in \mathbb{N}. \tag{5}$$

A cone C is called normal if there is a number $M > 0$ (later proved to be greater than or equal 1) such that, for all $x, y \in \mathbb{A}$,

$$\text{if } \theta \leq x \leq y, \quad \text{then } \|x\| \leq M\|y\|. \tag{6}$$

The normal constant of the normal cone C is defined to be the smallest constant M satisfying (6).

A cone C is called regular if every monotonically nonincreasing (no-decreasing) bounded above (bounded below) sequence has a limit in the norm sense of $\mathbb{A}$.

Remark 1 (see [3]). Every regular cone is normal, there are normal cones which are not regular, there are cones which are not normal, the normal constant M of any normal cone is such that $M \geq 1$, and for any real number k, $k \geq 1$, there is a cone with normal constant $M = k$.

Definition 2. Suppose that X is a nonempty set, C is a cone in a normed space $\mathbb{A}$, $r \in \mathbb{R}^+$, $r \geq 1$ (this r does not depend on the cone C itself, the cone may not be normal), and q is a function; q: $X \times X \longrightarrow C$ satisfies the following:

(1) $\theta \ll q(u, v)$, $\forall u, v \in X$

(2) $q(u, v) = \theta \Leftrightarrow u = v$

(3) $q(u, v) = q(v, u)$, $\quad \forall u, v \in X$

(4) $q(u, v) \preceq r[q(u, w) + q(w, v)]$, $\quad \forall u, v, w \in X$

Then, (X, C, q) is defined to be a b-cone metric space over C. In particular, if $r = 1$, then (X, C, q) is cone metric space.

A sequence $\{v_n\}_{n \in \mathbb{N}}$ in a cone (b- cone) metric space (X, C, q) (with C solid) is Cauchy if and only if for every $z \in \mathbb{A}$ with $\theta \ll z$ there is $n_0 \in \mathbb{N}$ such that $\mathbf{q}(v_n, v_m) \ll z$ for all $n, m \geq n_0$.

A sequence $\{v_n\}_{n \in \mathbb{N}}$ in a cone (b- cone) metric space (X, C, q) (with C solid) is convergent sequence if and only if there is v such that, for every $z \in \mathbb{A}$ with $\theta \ll z$, there is $n_0 \in \mathbb{N}$ such that $q(v_n, v) \ll z$ for all $n \geq n_0$.

A cone (b- cone) metric space (X, C, q) is complete whenever every Cauchy sequence in (X, C, q) converges to an element belonging to (X, C, q).

The following is a remarkable and an excitable notice given by Khamsi [14].

Remark 2. If (X, C, q) is a cone metric space, where C is normal cone with normal constant $M \geq 1$, then the composite function $D(x, y) = (\|\cdot\| o q)(x, y) = \|q(x, y)\|$, D: $X \times X \longrightarrow \mathbb{R}^+$ is a b-metric on X. Indeed, for all $x, y, w \in X$, we have

(1) $D(x,y) \geq 0$ and $D(x,y) = 0$ if and only if $\|q(x,y)\| = 0$, if and only if $q(x,y) = \theta$, and if and only if $x = y$

(2) $D(x,y) = \|q(x,y)\| = \|q(y,x)\| = D(y,x)$

(3) Since $q(x,y) \preceq q(x,w) + q(w,y)$, then $\|q(x,y)\| \leq M[\|q(x,w) + q(w,y)\|]$; hence, $D(x,y) = \|q(x,y)\| \leq M[\|q(x,w) + q(w,y)\|] \leq M[\|q(x,w)\| + \|q(w,y)\|] = M[D(x,w) + D(w,y)]$

That is, the cone metric (X, C, q) for normal cone C generates a b-metric space (X, D).

Next, we have the following.

Definition 3 (see [10]). Let be an ordered linear space, where $\preceq$ is an ordered relation induced by some cone $C \subset \mathbb{A}$ and Θ: $C \times C \longrightarrow C$ be a continuous mapping with respect to each variable, and we denote

$$\operatorname{Im}(\Theta) = \{t : t \in C \text{ such that } \exists u_0, v_0 \in \mathbb{A}, \Theta(u_0, v_0) = t\}. \tag{7}$$

Then, Θ is said to be an ordered-action mapping on $\mathbb{A}$ if and only if it satisfies the following conditions:

(1) $\Theta(\theta, \theta) = \theta$ and $\Theta(u,v) = \Theta(v,u)$, for every $u, v \in C$

(2) $\Theta(u,v) < \Theta(w,t)$ if either $\begin{cases} u < w \text{ and } v \preceq t \\ \text{or} \\ u \preceq w \text{ and } v < t \end{cases}$

(3) For every $u \in \operatorname{Im}(\Theta)$ and every $\theta \preceq v \preceq u$, there is $\theta \preceq w \preceq u$ such that $\Theta(v,w) = u$

(4) $\Theta(u,\theta) \preceq u$, for every $u \in C/\{\theta\}$

In addition, the concept of Θ-cone-metric space is as follows.

Definition 4 (see [10]). Let $(\mathbb{A}, \preceq)$ be an ordered normed space, where $\preceq$ is an ordered relation induced by some cone $C \subset \mathbb{A}$ and Θ be an ordered-action mapping on $\mathbb{A}$. If X is a nonempty set, then a function d_Θ: $X \times X \longrightarrow C$ is said to be Θ-cone-metric on X if and only if d_Θ satisfies the following conditions:

(1) $d_\Theta(x,y) = \theta \Leftrightarrow x = y$

(2) $d_\Theta(x,y) = d_\Theta(y,x)$, $\forall x, y \in X$

(3) $d_\Theta(x,y) \preceq r\,\Theta(d_\Theta(x,z), d_\Theta(z,y))$, $\forall x, y, z \in X$, where r is real number such that $r \geq 1$

The triple (X, C, d_Θ) is defined to be a b-Θ-cone-metric space.

Remark 3. The constant r does not depend on the cone C in general whether it is normal cone or not.

If $r = 1$, then (X, C, d_Θ) is Θ-cone-metric space, meaning that the class of all Θ-cone-metric spaces is included in the class of all b-Θ-cone-metric spaces.

If $\Theta(u,v) = u + v$, then (X, C, d_Θ) is b- cone-metric space, meaning that the class of all cone metric spaces is included in the class of all theta cone metric spaces.

A sequence $\{u_n\}_{n\in\mathbb{N}}$ in (X, C, d_Θ) converges to u whenever, for each $z \in \operatorname{Im}(\theta)$ with $\theta < z$, there is $n_0 \in \mathbb{N}$ such that $d_\Theta(u_n, u) < z$ for all $n \geq n_0$. We instead write $u_n \xrightarrow[n\to\infty]{d_\Theta} u$.

A sequence $\{u_n\}_{n\in\mathbb{N}}$ in (X, C, d_Θ) is Cauchy whenever, for each $z \in \operatorname{Im}(\theta)$ with $\theta < z$, there is $n_0 \in \mathbb{N}$ such that $d_\Theta(u_n, u_m) < z$ for all $n, m \geq n_0$.

A Θ cone (b- cone) metric space (X, C, d_Θ) is complete whenever every Cauchy sequence in (X, C, d_Θ) converges to an element belonging to (X, C, d_Θ).

On the other side, we have the following.

Definition 5. An element $(x,y) \in X \times X$ is said to be a coupled fixed point of the mapping F: $X \times X \longrightarrow X$ if and only if $F(x,y) = x$ and $F(y,x) = y$.

In this paper, we consider the corresponding definition for contraction type of mappings on complete b-cone metric spaces and generalize the coupled fixed point theorem of Sabetghadam et al. (Theorem 1) in this setting. On the other side, we consider the concept of b-theta-cone metrics and also prove the existence of unique coupled fixed point of some contraction type of mappings that gives another generalization of some previous coupled fixed point theorems.

Since the class of all b-cone metric spaces is including the class of cone metric spaces as supported with the example below, the results of this paper are real generalizations of some previous results. Moreover and in particular, some results of [15] can be extended to the case of theta cone metric spaces and do not affect the validity of this paper.

3. Main Results

Let p be any real number, $p > 0$, and $C_p := \{\{\lambda_n\}_{n\in\mathbb{N}}: \{\lambda_n\}_{n\in\mathbb{N}} \in l_p, \lambda_n \geq 0 \forall n \in \mathbb{N}\}$. Then, C_p is a normal cone in the Banach space l_p with normal constant $M = 1$, where $\|\{\lambda_n\}_{n\in\mathbb{N}}\|_p = \sqrt[p]{\sum_{n\in\mathbb{N}} |\lambda_n|^p}$ for every $\{\lambda_n\}_{n\in\mathbb{N}} \in l_p$, if $\{\lambda_n\}_{n\in\mathbb{N}} \preceq \{\mu_n\}_{n\in\mathbb{N}}$, then $\{\mu_n - \lambda_n\}_{n\in\mathbb{N}} \in C_p$, meaning that $\mu_n - \lambda_n \geq 0$, $\lambda_n \leq \mu_n$ for every $n \in \mathbb{N}$ and therefore $\sqrt[p]{\sum_{n\in\mathbb{N}} |\lambda_n|^p} \leq \sqrt[p]{\sum_{n\in\mathbb{N}} |\mu_n|^p}$, equivalently, $\|\{\lambda_n\}_{n\in\mathbb{N}}\|_p \leq \|\{\mu_n\}_{n\in\mathbb{N}}\|_p$.

The class of cone metric spaces is larger than the class of metric spaces. Indeed, we have the following.

From any metric space (X, d), we define infinitely many cone metric spaces in such a way that if (X, C, d^*) is one of these cone metric spaces, then we have

$$d(u,v) = \|d^*(u,v)\|, \quad \forall u, v \in X. \tag{8}$$

Indeed, let (X, d) be a metric space, p be a given real number, and $\lambda = \{\lambda_n\}_{n\in\mathbb{N}} \in l_p$. Define

$$d^*(u,v) = \left\{\frac{d(u,v)|\lambda_n|}{\|\lambda\|_p}\right\}_{n\in\mathbb{N}}, \quad \forall u, v \in X. \tag{9}$$

Clearly, d^*: $X \times X \longrightarrow C_p$ is cone metric on X and $d(u,v) = \|d^*(u,v)\|_p$ for every $u, v \in X$.

Specifically and in particular, the following is cone metric on X and $d(u,v) = \|d^*(u,v)\|_p$ for every $u, v \in X$:

$$d^*(u,v) = \left\{\frac{d(u,v)}{2^{(n/p)}}\right\}_{n\in\mathbb{N}}, \quad \forall u,v \in X. \tag{10}$$

Every cone metric space is b-cone metric space with $r = 1$ and the converse is not true, meaning that the class of all b-cone metric spaces is larger than the class of all cone metric spaces. Indeed, we have the following.

Let $M_m(l_p)$ be the Banach space of all $m \times m$ matrices, and the entries of each matrix in $M_m(l_p)$ are elements of l_p, the usual linear structure of addition and scalar multiplication:

$$\begin{aligned} A &= \left[a^{ij}\right]_{1\le i,j\le m} \in M_m(l_p) \Longrightarrow a^{ij} = \left\{a_n^{ij}\right\}_{n\in\mathbb{N}} \in l_p, \quad \forall 1 \le i, j \le m, \\ \|A\|_\infty &= \left\|\left[a^{ij}\right]_{1\le i,j\le m}\right\|_\infty = \max_{i=1}^m \sum_{j=1}^m \left\|a_{ij}\right\|_p. \end{aligned} \tag{11}$$

Let $C^* := \left\{[a^{ij}]_{1\le i,j\le m} : a^{ij} \in C_p, \forall 1 \le i, j \le m\right\}$. Then, C^* is a normal cone in $\mathbb{A} = M_m(l_p)$ with normal constant $M = 1$.

Let us now give an example of b-cone metric space which is not cone metric space.

Let $X = M_m(l_p)$, using the fact that the cone is normal with normal constant $M = 1$, and Khamsi notices that every cone metric p on X generates a metric $D(x,y) = \|p(x,y)\|$ on X, $D: X \times X \longrightarrow \mathbb{R}^+$.

Differently, we have the following. Let p be a real number, $0 < p < 1$, and q, $q: X \times X \longrightarrow C^*$ be defined by

$$\begin{aligned} q(A,B) &= q\left(\left[a^{ij}\right]_{1\le i,j\le m}, \left[b^{ij}\right]_{1\le i,j\le m}\right) \\ &:= \left[\left\{\left\|a^{ij} - b^{ij}\right\|_p \delta_{jk}\right\}_{k\in\mathbb{N}}\right]_{1\le i,j\le m}. \end{aligned} \tag{12}$$

Then, (X, C^*, q) is b-cone metric space with $r = 2^{(1/p)}$ which is not a cone metric space. Indeed, let $A, B, C \in X$ with $A = [a^{ij}]_{1\le i,j\le m}$, $B = [b^{ij}]_{1\le i,j\le m}$, and $C = [c^{ij}]_{1\le i,j\le m}$, and we have

$$\begin{aligned} q(A,B) &= q\left(\left[a^{ij}\right]_{1\le i,j\le m}, \left[b^{ij}\right]_{1\le i,j\le m}\right) \\ &= \left[\left\{\left\|a^{ij} - b^{ij}\right\|_p \delta_{jk}\right\}_{k\in\mathbb{N}}\right]_{1\le i,j\le m} = \left[\left\{\left\|\left(a^{ij} - c^{ij}t + nqhc^{ij} - b^{ij}\right)\right\|_p \delta_{jk}\right\}_{k\in\mathbb{N}}\right]_{1\le i,j\le m} \\ &\prec \left[\left\{2^{(1/p)}\left(\left\|a^{ij} - c^{ij}\right\|_p + \left\|c^{ij} - b^{ij}\right\|_p\right)\delta_{jk}\right\}_{k\in\mathbb{N}}\right]_{1\le i,j\le m} \\ &= 2^{(1/p)}\left[\left\{\left\|a^{ij} - c^{ij}\right\|_p \delta_{jk}\right\}_{k\in\mathbb{N}} + \left\{\left\|c^{ij} - b^{ij}\right\|_p \delta_{jk}\right\}_{k\in\mathbb{N}}\right]_{1\le i,j\le m} \\ &= 2^{(1/p)}\left(\left[\left\{\left\|a^{ij} - c^{ij}\right\|_p \delta_{jk}\right\}_{k\in\mathbb{N}}\right]_{1\le i,j\le m} + \left[\left\{\left\|c^{ij} - b^{ij}\right\|_p \delta_{jk}\right\}_{k\in\mathbb{N}}\right]_{1\le i,j\le m}\right) \\ &= 2^{(1/p)}(q(A,C) + q(C,B)). \end{aligned} \tag{13}$$

Here, the constant $r = 2^{(1/p)}$ is completely different from the normal constant of the cone which is $M = 1$ in addition that the range of q is a subset of the given cone not subset of positive real numbers $\mathbb{R}^+$. That is why the results of this paper do not interfere, but it provides generalized results of [14].

We have the following results, where cones are not necessarily normal but solid.

Theorem 2. *Let (X, C, q) be a complete b- cone metric space. Suppose that the mapping $T: X \times X \longrightarrow X$ satisfies the following contractive condition:*

$$q(T(x,y), T(u,v)) \preceq kq(x,u) + lq(y,v), \quad \forall x, y, u, v \in X, \tag{14}$$

where k and l are nonnegative constants with $r(k+l) < 1$. Then, T has a unique coupled fixed point.

Proof. Select u_0, v_0 in X and set $u_1 = T(u_0, v_0), v_1 = T(v_0, u_0), \ldots, u_{n+1} = T(u_n, v_n), v_{n+1} = T(v_n, u_n)$. Then, by (14), we have

$$\begin{aligned} q(u_{n+1}, u_n) &= q(T(u_n, v_n), T(u_{n-1}, v_{n-1})) \\ &\preceq kq(u_n, u_{n-1}) + lq(v_n, v_{n-1}). \end{aligned} \tag{15}$$

Similarly,

$$\begin{aligned} q(v_{n+1}, v_n) &= q(T(v_n, u_n), T(v_{n-1}, u_{n-1})) \\ &\preceq kq(v_n, v_{n-1}) + lq(u_n, u_{n-1}). \end{aligned} \tag{16}$$

Let $z_n = q(v_n, v_{n-1}) + q(u_n, u_{n-1})$. Then, we have

$$\begin{aligned} z_{n+1} &= q(u_{n+1}, u_n) + q(v_{n+1}, v_n) \\ &\preceq kq(u_n, u_{n-1}) + lq(v_n, v_{n-1}) + kq(v_n, v_{n-1}) + lq(u_n, u_{n-1}) \\ &= (k+l)[q(u_n, u_{n-1}) + q(v_n, v_{n-1})] = (k+l)z_n. \end{aligned} \tag{17}$$

For each $n \in \mathbb{N}$, we have

$$\theta \preceq z_n \preceq (k+l)z_{n-1} \preceq (k+l)^2 z_{n-2} \preceq, \ldots, \preceq (k+l)^n z_0. \tag{18}$$

If $z_0 = \theta$, then (x_0, y_0) is coupled fixed point of T; therefore, we continue by letting $\theta < z_0$. Now, let $n, m \in \mathbb{N}$, $n \leq m$. Then,

$$\begin{aligned} q(u_n, u_m) &\preceq r[q(u_n, u_{n+1}) + q(u_{n+1}, u_m)] \\ &\preceq r[q(u_n, u_{n+1}) + r[q(u_{n+1}, u_{n+2}) + q(u_{n+2}, u_m)]] \\ &\preceq r[q(u_n, u_{n+1}) + r[q(u_{n+1}, u_{n+2}) + r[q(u_{n+2}, u_{n+3}) + q(u_{n+3}, u_m)]]] \\ &= rq(u_n, u_{n+1}) + r^2 q(u_{n+1}, u_{n+2}) + r^3 q(u_{n+2}, u_{n+3}) + \cdots\cdots \\ &\quad + r^{m-n} q(u_{m-1}, u_m). \end{aligned} \tag{19}$$

Similarly,

$$q(v_n, v_m) \preceq rq(v_n, v_{n+1}) + r^2 q(v_{n+1}, v_{n+2}) + r^3 q(v_{n+2}, v_{n+3}) + \cdots\cdots + r^{m-n} q(v_{m-1}, v_m). \tag{20}$$

Adding, we obtain

$$\begin{aligned} q(u_n, u_m) + q(v_n, v_m) &\preceq r[q(u_n, u_{n+1}) + q(v_n, v_{n+1})] + \\ &+ r^2[q(u_{n+1}, u_{n+2}) + q(v_{n+1}, v_{n+2})] + \\ &+ r^3[q(u_{n+2}, u_{n+3}) + q(v_{n+2}, v_{n+3})] + \cdots + r^{m-n}[q(u_{m-1}, u_m) + q(v_{m-1}, v_m)]. \end{aligned} \tag{21}$$

This shows that

$$q(u_n, u_m) + q(v_n, v_m) \preceq rz_{n+1} + r^2 z_{n+2} + r^3 z_{n+3} + \cdots + r^{m-n} z_m. \tag{22}$$

Using the two inequalities (18) and (22), we have

$$\begin{aligned} q(u_n, u_m) + q(v_n, v_m) &\preceq r(k+l)^{n+1} z_0 + r^2 (k+l)^{n+2} z_0 \\ &\quad + \cdots + r^{m-n} (k+l)^{n+(m-n)} z_0 \\ &= (k+l)^n \left[r(k+l) + r^2 (k+l)^2 + r^3 (k+l)^3 + \cdots + r^{m-n} (k+l)^{m-n} \right] z_0 \\ &= (k+l)^n \left[[r(k+l)] + [r(k+l)]^2 + [r(k+l)]^3 + \cdots + [r(k+l)]^{m-n} \right] z_0 \\ &\preceq (k+l)^n \left[\frac{r(k+l)}{1 - [r(k+l)]} \right] z_0. \end{aligned} \tag{23}$$

Let $\theta \ll z$, that is, $z \in \mathrm{Int}(C)$. Then, there is a neighborhood of z with some radius $\delta > 0$ say $z + N_\delta(\theta)$ such that $z + N_\delta(\theta) \subset C$, and for this $\delta > 0$, there is a natural number n_0 such that $(k+l)^n < \delta[(1 - [r(k+l)]/\|z_0\| r(k+l))]$ for

every $n \geq n_0$; hence, $[(\|z_0\| r(k+l)/1 - [r(k+l)])](k+l)^n < \delta$ for every $n \geq n_0$; consequently, $[\|(r(k+l)/1 - [r(k+l)])\|(k+l)^n z_0] < \delta$ for every $n \geq n_0$. This proves that $\pm(r(k+l)/1 - [r(k+l)])(k+l)^n z_0 \in N_\delta(\theta)$ for every $n \geq n_0$. Therefore, $z \pm (r(k+l)/1 - [r(k+l)])(k+l)^n z_0 \in z + N_\delta(\theta) \subset C$; hence, $z \pm (r(k+l)/1 - [r(k+l)])(k+l)^n z_0 \in \operatorname{Int}(C)$ for every $n \geq n_0$:

$$\frac{r(k+l)}{1-[r(k+l)]}(k+l)^n z_0 \ll z, \quad \forall n \geq n_0. \tag{24}$$

Using (24), there is a sequence of neighborhoods $\{N_{\delta_n}(\theta)\}_{n \geq n_0}$ such that $[z - (r(k+l)/1 - [r(k+l)])](k+l)^n z_0 + N_{\delta_n}(\theta) \subset C$, using (23) gives

$$\begin{aligned}&\left[(k+l)^n\left[\frac{r(k+l)}{1-[r(k+l)]}\right]z_0 - q(u_n,u_m) + q(v_n,v_m)\right]\\&+\left[z - \frac{r(k+l)}{1-[r(k+l)]}\right](k+l)^n z_0 + N_{\delta_n}(\theta)\\&\subset C + C \subset C, \quad \forall n \geq n_0.\end{aligned} \tag{25}$$

This means

$$[z - q(u_n,u_m) + q(v_n,v_m)] + N_{\delta_n}(\theta) \subset C, \quad \forall n \geq n_0. \tag{26}$$

Consequently, $[z - q(u_n,u_m) + q(v_n,v_m)] \in \operatorname{Int}(C)$ for every $n \geq n_0$, and we have

$$q(u_n,u_m) + q(v_n,v_m) \ll z, \quad \forall n \geq n_0. \tag{27}$$

Since $n \leq m$, $q(u_n,u_m) \preceq q(u_n,u_m) + q(v_n,v_m)$, and $q(v_n,v_m) \preceq q(u_n,u_m) + q(v_n,v_m)$, we similarly conclude that

$$q(u_n,u_m) \ll z, \quad \forall n,m \geq n_0, \tag{28}$$

$$q(v_n,v_m) \ll z, \quad \forall n,m \geq n_0. \tag{29}$$

The two inequalities (28) and (29) prove that the two sequences $\{u_n\}_{n\in\mathbb{N}}$ and $\{v_n\}_{n\in\mathbb{N}}$ are Cauchy sequences in (X,C,q). Since (X,C,q) is complete b- cone metric space, there are two limits. Let $f \geq 1$, we have $\theta \ll [(1/r[k+l+1]f)]z$. Then,

$$\begin{aligned}&\forall \theta \ll z \exists n_1 \in \mathbb{N} \text{ such that } q(u_n,w_0) \ll \left[\frac{1}{r[k+l+1]f}\right]z, \quad \forall n \geq n_1,\\&\forall \theta \ll z \exists n_2 \in \mathbb{N} \text{ such that } q(v_n,z_0) \ll \left[\frac{1}{r[k+l+1]f}\right]z, \quad \forall n \geq n_2.\end{aligned} \tag{30}$$

Take $N_0 = \max\{n_1,n_2\}$, and we have

$$\begin{aligned}q(T(w_0,z_0),w_0) &\preceq r\left[q\left(T(w_0,z_0),u_{N_0+1}\right) + q\left(u_{N_0+1},w_0\right)\right]\\&= r\left[q\left(T(w_0,z_0),T\left(u_{N_0},v_{N_0}\right)\right) + q\left(u_{N_0+1},w_0\right)\right]\\&= r\left[kq\left(w_0,u_{N_0}\right) + lq\left(z_0,v_{N_0}\right)\right] + rq\left(u_{N_0+1},w_0\right)\\&= r\left[k\left[\frac{1}{r[k+l+1]f}\right]z + l\left[\frac{1}{r[k+l+1]f}\right]z\right]\\&\quad + r\left[\frac{1}{r[k+l+1]f}\right]z\\&= r[k+l+1]\left[\frac{1}{r[k+l+1]f}\right]z = \left[\frac{1}{f}\right]z.\end{aligned} \tag{31}$$

Since $f \geq 1$ is an arbitrary number, we have $q(T(w_0,z_0),w_0) = \theta$, and hence, $T(w_0,z_0) = w_0$, and similarly, $T(z_0,w_0) = z_0$, meaning that (w_0,z_0) is a coupled fixed point of T. Finally, we show that such a coupled fixed point is unique. Contrarily, suppose that (u_1,v_1) is another coupled fixed point. Then, we have

$$\begin{aligned}q(w_0,u_1) &= q(T(w_0,z_0),T(u_1,v_1)) \preceq kq(w_0,u_1) + lq(z_0,v_1),\\q(z_0,v_1) &= q(T(z_0,w_0),T(v_1,u_1)) \preceq kq(z_0,v_1) + lq(w_0,u_1).\end{aligned} \tag{32}$$

Adding, we obtain

$$q(w_0,u_1) + q(z_0,v_1) \preceq [k+l][q(w_0,u_1) + q(z_0,v_1)]. \tag{33}$$

Since $[k+l] < 1$, we see that $q(w_0,u_1) + q(z_0,v_1) = \theta$; hence, $q(w_0,u_1) = \theta$ and $q(z_0,v_1) = \theta$ meaning that $(w_0,z_0) = (u_1,v_1)$ and the proof of the theorem is completed.

The following corollary proves Theorem 1 of Sabetghadam et al. □

Corollary 1. *Let (X,C,q) be a complete cone metric space. Suppose that the mapping $T\colon X \times X \longrightarrow X$ satisfies the following contractive condition for all $x,y,u,v \in X$:*

$$q(T(x,y),T(u,v)) \preceq kq(x,u) + lq(y,v), \tag{34}$$

where k and l are nonnegative constants with $k+l<1$. Then, T has a unique coupled fixed point.

Proof. Using Theorem 2 with $r = 1$ completes the proof.

We also have the following. □

Corollary 2. *Let (X,C,q) be a complete cone metric space. Suppose that the mapping $T\colon X \times X \longrightarrow X$ satisfies the following contractive condition for all $x,y,u,v \in X$:*

$$q(T(x,y),T(u,v)) \preceq t[q(x,u)+q(y,v)], \tag{35}$$

where t is a nonnegative constant with $t < (1/2)$. Then, T has a unique coupled fixed point.

Proof. Using Corollary 1 with $l = k < (1/2)$, we obtain the following. □

Corollary 3. *Let (X, C, q) be a complete b- cone metric space. Suppose that the mapping T: $X \times X \longrightarrow X$ satisfies the following contractive condition for all $x, y, u, v \in X$:*

$$q(T(x,y),T(u,v)) \preceq t[q(x,u)+q(y,v)], \tag{36}$$

where t is nonnegative constant with $rt < (1/2)$. Then, T has a unique coupled fixed point.

Proof. Using Theorem 2 with $k = l$ completes the proof. □

Theorem 3. *Let (X, C, q) be a complete b-cone metric space. Suppose that the mapping T: $X \times X \longrightarrow X$ satisfies the following contractive condition for all $x, y, u, v \in X$:*

$$q(T(x,y),T(u,v)) \preceq aq(x,u) + bq(T(x,y),x) + cq(T(u,v),u), \quad \forall x,y,u,v \in X, \tag{37}$$

where $a, b, c \in \mathbb{R}$ are nonnegative constants with $r[a+b+c] < 1$. Then, T has a unique coupled fixed point.

Proof. Construct the two sequences $\{u_n\}_{n\in\mathbb{N}}$ and $\{v_n\}_{n\in\mathbb{N}}$ as in Theorem 2. Then, by (37), we have

$$\begin{aligned} q(u_{n+1},u_n) &= q(T(u_n,v_n),T(u_{n-1},v_{n-1})) \\ &\preceq aq(u_n,u_{n-1}) + bq(T(u_n,v_n),u_n) + cq(T(u_{n-1},v_{n-1}),u_{n-1}) \\ &\preceq aq(u_n,u_{n-1}) + bq(u_{n+1},u_n) + cq(u_n,u_{n-1}) \\ &\preceq [a+c]q(u_n,u_{n-1}) + bq(u_{n+1},u_n). \end{aligned} \tag{38}$$

Consequently,

$$q(u_{n+1},u_n) \preceq \left[\frac{a+c}{1-b}\right] q(u_n,u_{n-1}). \tag{39}$$

Similarly,

$$q(v_{n+1},v_n) \preceq \left[\frac{a+c}{1-b}\right] q(v_n,v_{n-1}). \tag{40}$$

We have $r[(a+c/1-b)] < 1$; then, the two sequences $\{u_n\}_{n\in\mathbb{N}}$ and $\{v_n\}_{n\in\mathbb{N}}$ are Cauchy sequences, and the rest of the proof is similar to the proof of Theorem 2. □

Corollary 4. *Let (X, C, q) be a complete b- cone metric space. Suppose that the mapping T: $X \times X \longrightarrow X$ satisfies the following contractive condition for all $x, y, u, v \in X$:*

$$\begin{aligned} &q(T(x,y),T(u,v)) \preceq t[q(T(x,y),x) + q(T(u,v),u)], \\ &\quad \forall x,y,u,v \in X, \end{aligned} \tag{41}$$

where $t \in \mathbb{R}$ is nonnegative constants with $rt < (1/2)$. Then, T has a unique coupled fixed point.

The following is a coupled fixed point theorem of contraction type of mappings in b-Θ-cone metric spaces.

Theorem 4. *Let (X, C, d_Θ) be a complete b-Θ-cone metric space. Suppose that the mapping T: $X \times X \longrightarrow X$ satisfies the following contractive condition for all $x, y, u, v \in X$:*

$$d_\Theta(T(x,y),T(u,v)) \preceq kd_\Theta(x,u) + ld_\Theta(y,v), \tag{42}$$

where k and l are nonnegative constants with $k + l < 1$. Then, T has a unique coupled fixed point.

Proof. Construct the two sequences $\{u_n\}_{n\in\mathbb{N}}$ and $\{v_n\}_{n\in\mathbb{N}}$ in X and the sequence $\{z_n\}_{n\in\mathbb{N}}$ in C as in Theorem 2. Then, by (42), we have

$$\begin{aligned} &d_\Theta(u_{n+1},u_n) \preceq kd_\Theta(u_n,u_{n-1}) + ld_\Theta(v_n,v_{n-1}) \\ &\prec (k+l)[d_\Theta(u_n,u_{n-1}) + d_\Theta(v_n,v_{n-1})], \\ &\quad d_\Theta(v_{n+1},v_n) \preceq kd_\Theta(v_n,v_{n-1}) + ld_\Theta(u_n,u_{n-1}) \\ &\preceq (k+l)[d_\Theta(u_n,u_{n-1}) + d_\Theta(v_n,v_{n-1})], \\ &\quad \theta \preceq z_n \preceq (k+l)z_{n-1} \preceq (k+l)^2 z_{n-2} \preceq \cdots \preceq (k+l)^n z_0. \end{aligned} \tag{43}$$

The last inequalities give

$$d_\Theta(u_{n+1},u_n) \preceq (k+l)z_n \preceq (k+l)^{n+1} z_0, \tag{44}$$

$$d_\Theta(v_{n+1},v_n) \preceq (k+l)z_n \preceq (k+l)^{n+1} z_0. \tag{45}$$

If $z_0 = \theta$, then (x_0, y_0) is coupled fixed of T; therefore, we continue by letting $\theta < z_0$. Now, let $\theta \ll z$, $z \in \mathrm{Int}(C)$. Then, there is a neighborhood of z with some radius $\delta > 0$ say $z + N_\delta(\theta)$ such that $z + N_\delta(\theta) \subset C$, and for this $\delta > 0$, there is a natural number n_0 such that $(k+l)^{n+1} < \delta[(1/\|z_0\|)]$ for every $n \geq n_0$; hence, $\|z_0\|(k+l)^{n+1} < \delta$ for every $n \geq n_0$; consequently, $\|(k+l)^{n+1} z_0\| < \delta$ for every $n \geq n_0$. This proves that $\pm(k+l)^{n+1} z_0 \in N_\delta(\theta)$ for every $n \geq n_0$. Therefore,

$$(k+l)^{n+1} z_0 \ll z, \quad \forall n \geq n_0. \tag{46}$$

Using (46) with both (44) and (45), we have

$$d_\Theta(u_{n+1},u_n) \ll z, \quad \forall n \geq n_0, \tag{47}$$

$$d_\Theta(v_{n+1},v_n) \ll z, \quad \forall n \geq n_0. \tag{48}$$

Now, we are going to show that the two sequences $\{u_n\}_{n\in\mathbb{N}}$ and $\{v_n\}_{n\in\mathbb{N}}$ are Cauchy sequences; for this reason, suppose that one of them is not Cauchy, say $\{u_n\}_{n\in\mathbb{N}}$. Then,

there exists $\overline{z} \in \mathrm{Im}(\Theta)$, $\theta < \overline{z}$ and sequences of natural numbers $\{i_n\}_{n\in N}$ and $\{j_n\}_{n\in N}$ such that for any $i_n > j_n > n$,

$$\begin{aligned} \overline{z} &\preceq d_\Theta\left(u_{i_n}, u_{j_n}\right), \\ d_\Theta\left(u_{i_n-1}, u_{j_n}\right) &< \overline{z}. \end{aligned} \tag{49}$$

Let $f \in \mathbb{R}^+$ be such that $r\Theta(f\overline{z}, \overline{z}) < \overline{z}$; using (47), for such $\overline{z}$, there is $n_0 \in \mathbb{N}$ such that $d_\Theta(u_{n+1}, u_n) \ll f\overline{z}$ for every $n \geq n_0$; in particular, we have $d_\Theta(u_{i_n}, u_{i_n-1}) \ll f\overline{z}$ for every $i_n \geq n_0$; hence, we have the following contradiction:

$$\begin{aligned} \overline{z} &\preceq d_\Theta\left(u_{i_n}, u_{j_n}\right) \preceq r\Theta\left(d_\Theta\left(u_{i_n}, u_{i_n-1}\right), d_\Theta\left(u_{i_n-1}, u_{j_n}\right)\right) \\ &< r\Theta\left(d_\Theta\left(u_{i_n}, u_{i_n-1}\right), \overline{z}\right) \preceq r\Theta(f\overline{z}, \overline{z}) < \overline{z}. \end{aligned} \tag{50}$$

Hence, the two sequences are Cauchy sequences. Since (X, C, d_Θ) is complete, there are two limits for both of them say u_0 and v_0, respectively:

$$\begin{aligned} u_n &\longrightarrow_{n\longrightarrow\infty}^{d_\Theta} u_0, \\ v_n &\longrightarrow_{n\longrightarrow\infty}^{d_\Theta} v_0. \end{aligned} \tag{51}$$

We show that (u_0, v_0) is coupled fixed point of T. Using the properties of Θ, we see that

$$\begin{aligned} d_\Theta(T(u_0, v_0), u_0) &\preceq r\Theta(d_\Theta(T(u_0, v_0), u_n), d_\Theta(u_n, u_0)) \\ &\preceq r\Theta(d_\Theta(T(u_0, v_0), T(u_{n-1}, v_{n-1})), d_\Theta(u_n, u_0)) \\ &\preceq r\Theta((kd_\Theta(u_0, u_{n-1}) + ld_\Theta(v_0, v_{n-1})), d_\Theta(u_n, u_0)) \\ &\longrightarrow_{n\longrightarrow\infty} r\Theta((k\theta + l\theta), \theta) = r\Theta(\theta, \theta) = r\theta = \theta. \end{aligned} \tag{52}$$

Consequently, $d_\Theta(T(u_0, v_0), u_0) = \theta$, meaning that $T(u_0, v_0) = u_0$; similarly, $T(v_0, u_0) = v_0$. The uniqueness of such a coupled fixed point is as before; clearly, the coupled fixed point is unique. □

Corollary 5. *Let (X, C, d_Θ) be a complete theta cone metric space. Suppose that the mapping T: $X \times X \longrightarrow X$ satisfies the following contractive condition for all $x, y, u, v \in X$:*

$$d_\Theta(T(x, y), T(u, v)) \preceq kd_\Theta(x, u) + ld_\Theta(y, v), \tag{53}$$

where k and l are nonnegative constants with $k + l < 1$. Then, T has a unique coupled fixed point.

As in Theorem 3, we have the following.

Corollary 6. *Let (X, C, d_Θ) be a complete Θ-cone metric space. Suppose that the mapping T: $X \times X \longrightarrow X$ satisfies the following contractive condition for all $x, y, u, v \in X$:*

$$\begin{aligned} d_\Theta(T(x, y), T(u, v)) &\preceq ad_\Theta(x, u) + bd_\Theta(T(x, y), x) \\ &+ cd_\Theta(T(u, v), u), \quad \forall x, y, u, v \in X, \end{aligned} \tag{54}$$

where $a, b, c \in \mathbb{R}$ are nonnegative constants with $a + b + c < 1$. Then, T has a unique coupled fixed point.

Corollary 7. *Let (X, C, d_Θ) be a complete b- Θ-cone metric space. Suppose that the mapping T: $X \times X \longrightarrow X$ satisfies the following contractive condition for all $x, y, u, v \in X$:*

$$\begin{aligned} q(T(x, y), T(u, v)) &\preceq t\left[d_\Theta(T(x, y), x) + d_\Theta(T(u, v), u)\right], \\ &\forall x, y, u, v \in X, \end{aligned} \tag{55}$$

where $t \in \mathbb{R}$ is nonnegative constants with $t < (1/2)$. Then, T has a unique coupled fixed point.

4. Conclusion

The results of this paper prove the existence of a unique coupled fixed point of some well-known type contraction of mappings but are defined on complete b-cone and b-theta cone metric spaces; consequently, it extends and generalizes some previous coupled fixed point theorems.

Data Availability

No data were used to support this study.

Conflicts of Interest

The author has no conflicts of interest.

Authors' Contributions

The sole author contributed 100% to the article. The author read and approved the final manuscript.

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