Research Article

Stability and Convergence Analysis for Set-Valued Extended Generalized Nonlinear Mixed Variational Inequality Problems and Generalized Resolvent Dynamical Systems

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Received 8 February 2021; Accepted 11 June 2021; Published 22 July 2021

Academic Editor: Efthymios G. Tsionas

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In this paper, we study a set-valued extended generalized nonlinear mixed variational inequality problem and its generalized resolvent dynamical system. A three-step iterative algorithm is constructed for solving set-valued extended generalized nonlinear variational inequality problem. Convergence and stability analysis are also discussed. We have shown the globally exponential convergence of generalized resolvent dynamical system to a unique solution of set-valued extended generalized nonlinear mixed variational inequality problem. In support of our main result, we provide a numerical example with convergence graphs and computation tables. For illustration, a comparison of our three-step iterative algorithm with Ishikawa-type algorithm and Mann-type algorithm is shown.

1. Introduction

Variational inequality theory was introduced by Hartmann and Stampacchia [1] in 1966 as a tool for the study of partial differential equations with applications principally drawn from mechanics. Variational inclusions are the generalized forms of variational inequalities and they have wide range of applications in industry, mathematical finance, and economics and in several branches of applied sciences; see [2–6] and the references therein.

We would like to emphasize that the projection method cannot be applied for solving the mixed variational inequalities involving the nonlinear term. To overcome this drawback, it is assumed that the nonlinear term involved in the general mixed variational inequalities is a proper, convex, and lower-semicontinuous functional. It is well known that the subdifferential of a proper, convex, and lower-semicontinuous functional is a maximal monotone operator. This characterization enables researchers to define the resolvent operator associated with the maximal monotone operator; see, for example, [7–9] and the references therein. The resolvent operator technique is used to establish the equivalence between the variational inequalities and fixed-point problems; see, for example, [10–12] and the references therein.

In recent years, considerable interest has been shown in developing various extensions and generalizations of variational inequalities and variational inclusions, both for their own sake and for their applications; see, for example, [13–16] and the references therein. There are significant developments of these problems related to set-valued operators, nonconvex optimization, iterative methods, and structural analysis. Glowinski and Tallec [17] suggested and analyzed some three-step splitting methods for solving variational inequality problems by using the Lagrange multipliers technique. They showed that three-step splitting methods
are numerically more efficient as compared with one-step and two-step splitting methods. They studied the convergence of these splitting methods under the assumption that the underlying operator is monotone and Lipschitz continuous. For the applications and convergence analysis of projection-based splitting methods, we refer to [18–20]. Moreover, one can find the important aspects of generalized variational inclusions in [21–26].

In this paper, we introduce and study a set-valued extended generalized nonlinear mixed variational inequality problem including many existing problems studied by several authors. A three-step iterative algorithm is constructed for solving our problem, which is more general than Ishikawa-type algorithm and Mann-type algorithm. Existence of solution and convergence of the iterative sequences as well as stability analysis are shown. We also study generalized resolvent dynamical system associated with set-valued extended generalized nonlinear mixed variational inequality problem. Using Matlab R2019a program, we construct a numerical example for illustration.

2. Preliminaries

Let $\mathcal{H}$ be a real Hilbert space whose inner product and norm are denoted by $\langle \cdot, \cdot \rangle$ and $\| \cdot \|$, respectively. Let $CB(\mathcal{H})$ be the family of all nonempty bounded closed subsets of $\mathcal{H}$. Let $\overline{D}(\cdot, \cdot)$ be the Hausdörff metric on $CB(\mathcal{H})$ defined by

$$\overline{D}(A, B) = \max \left\{ \sup_{x \in A} d(x, B), \sup_{y \in B} d(A, y) \right\}, \quad \forall A, B \in CB(\mathcal{H}),$$

where $d(x, B) = \inf_{y \in B} d(x, y)$ and $d(A, y) = \inf_{x \in A} d(x, y)$. The following definitions are required to achieve the goal.

Definition 1. Let $g: \mathcal{H} \to \mathcal{H}$ and $T: \mathcal{H} \to \mathcal{H}$ be the single-valued mappings. Then, we have the following.

(i) $g$ is said to be Lipschitz continuous, if there exists a constant $\lambda_g > 0$ such that

$$\| g(x) - g(y) \| \leq \lambda_g \| x - y \|, \quad \forall x, y \in \mathcal{H}. \quad (2)$$

(ii) $g$ is said to be strongly monotone, if there exists a constant $\xi > 0$ such that

$$\langle g(x) - g(y), x - y \rangle \geq \xi \| x - y \|^2, \quad \forall x, y \in \mathcal{H}. \quad (3)$$

(iii) $T$ is said to be relaxed Lipschitz continuous, if there exists a constant $\alpha > 0$ such that

$$\langle T(x) - T(y), x - y \rangle \leq -\alpha \| x - y \|^2, \quad \forall x, y \in \mathcal{H}. \quad (4)$$

(iv) $T$ is said to be strongly monotone with respect to $g$, if there exists a constant $\lambda_T > 0$ such that

$$\langle T(g(x)) - T(g(y)), g(x) - g(y) \rangle \geq \lambda_T \| x - y \|^2, \quad \forall x, y \in \mathcal{H}. \quad (5)$$

Definition 2. (see [27, 28]) Let $A: \mathcal{H} \to CB(\mathcal{H})$ be a set-valued mapping and let $T: \mathcal{H} \to \mathcal{H}$ be a single-valued mapping. Then, $A$ is said to be

(i) $\overline{D}$-Lipschitz continuous if there exists a constant $\delta_A > 0$ such that

$$\overline{D}(A(x), A(y)) \leq \delta_A \| x - y \|, \quad \forall x, y \in \mathcal{H}. \quad (6)$$

(ii) Strongly monotone with respect to $T$ if there exists a constant $\tau > 0$ such that

$$\langle T(u_1) - T(u_2), x - y \rangle \geq \tau \| x - y \|^2, \quad \forall u_1 \in A(x), u_2 \in A(y). \quad (7)$$

(iii) Relaxed Lipschitz continuous with respect to $T$ if there exists a constant $\beta > 0$ such that

$$\langle T(u_1) - T(u_2), x - y \rangle \leq -\beta \| x - y \|^2, \quad \forall u_1 \in A(x), u_2 \in A(y). \quad (8)$$

(iv) Relaxed monotone with respect to $T$ if there exists a constant $\kappa > 0$ such that

$$\langle T(u_1) - T(u_2), x - y \rangle \geq -\kappa \| x - y \|^2, \quad \forall u_1 \in A(x), u_2 \in A(y). \quad (9)$$
Definition 3. (see [27, 28]) Let $A: \mathcal{H} \rightarrow CB(\mathcal{H})$ be a set-valued mapping. For all $x, y \in \mathcal{H}$, the mapping $\eta(\cdot, \cdot): \mathcal{H} \times \mathcal{H} \rightarrow \mathcal{H}$ is called

(i) Lipschitz continuous in the first argument with respect to $A$ if there exists a constant $\lambda_n > 0$ such that

$$\|\eta(u_1, \cdot) - \eta(u_2, \cdot)\| \leq \lambda_n\|u_1 - u_2\|, \quad \forall u_1 \in A(x), u_2 \in A(y).$$

(ii) Lipschitz continuous in the second argument with respect to $A$ if there exists a constant $\lambda_n > 0$ such that

$$\|\eta(\cdot, u_1) - \eta(\cdot, u_2)\| \leq \lambda_n\|u_1 - u_2\|, \quad \forall u_1 \in A(x), u_2 \in A(y).$$

(iii) Relaxed Lipschitz continuous in the first argument with respect to $A$ if there exists a constant $r_1 > 0$ such that

$$\langle \eta(u_1, \cdot) - \eta(u_2, \cdot), x - y \rangle \leq -r_1\|x - y\|^2, \quad \forall u_1 \in A(x), u_2 \in A(y).$$

(iv) Relaxed Lipschitz continuous in the second argument with respect to $A$ if there exists a constant $r_2 > 0$ such that

$$\langle \eta(\cdot, u_1) - \eta(\cdot, u_2), x - y \rangle \leq -r_2\|x - y\|^2, \quad \forall u_1 \in A(x), u_2 \in A(y).$$

Similarly, we can define Lipschitz continuity and relaxed Lipschitz continuity of $\eta$ in the third argument with respect to $A$.

Definition 4 (see [29]). Let $S, T: \mathcal{H} \rightarrow \mathcal{H}$ be a single-valued mapping, $x_0 \in \mathcal{H}$, and

$$x_{n+1} = S(T, x_n)$$

defines an iterative sequence which yields a sequence of points $\{x_n\}$ in $\mathcal{H}$. Suppose that $F(T) = \{p \in \mathcal{H}: Tp \neq p\} \neq \emptyset$ and $\{x_n\}$ converges to a fixed point $x^* \in T$. Let $\{u_n\} \subset \mathcal{H}$ and

$$\lambda_n = ||u_{n+1} - S(T, u_n)||.$$  

$$\lim_{n \rightarrow \infty} \lambda_n = 0$$ implies that $u_n \rightarrow x^*$, and consequently the iterative sequence $\{x_n\}$ is said to be $T$-stable or stable with respect to $T$.

Definition 5. A set-valued mapping $M: \mathcal{H} \rightarrow CB(\mathcal{H})$ is said to be monotone if, for any $x, y \in \mathcal{H}$,

$$\langle u_1 - u_2, x - y \rangle \geq 0, \quad \forall u_1 \in M(x), u_2 \in M(y).$$

Definition 6. Let $\psi: \mathcal{H} \rightarrow \mathbb{R} \cup \{\infty\}$ be a proper functional. A vector $\xi^* \in \mathcal{H}$ is called a subgradient of $\psi$ at $x \in \text{dom} \psi$ if

$$\langle \xi^*, y - x \rangle \leq \psi(y) - \psi(x), \quad \forall y \in \mathcal{H}.$$  

Each $\psi$ can be associated with the following map $\partial \psi$, called subdifferential of $\psi$ at $x$, defined by

$$\partial \psi(x) = \{\xi^* \in H: \langle \xi^*, y - x \rangle \leq \psi(y) - \psi(x), \quad \forall y \in \mathcal{H}\}.$$  

Definition 7 (see [30]). If $M$ is a maximal monotone mapping, then the resolvent operator associated with $M$ of parameter $\omega$ is defined as

$$\mathcal{R}_\omega M(x) = (I + \omega M)^{-1}(x), \quad \forall x \in \mathcal{H}.$$  

A monotone set-valued mapping $M$ is called maximal if its graph, $\text{Gr}(M) = \{(x, y) \in \mathcal{H} \times \mathcal{H}: y \in M(x)\}$, is not properly contained in the graph of any other monotone mapping. It is well defined that $M$ is a maximal monotone mapping if and only if $(I + \omega M)(\mathcal{H}) = \mathcal{H}$, for all $\omega > 0$, where $I$ denotes the identity mapping on $\mathcal{H}$.
It is well known that the resolvent operator $\mathcal{R}_w^\omega$ is single-valued and nonexpansive.

If $\psi$ is a proper, convex, and lower-semicontinuous functional, then its subdifferential $\partial \psi$ is a maximal monotone operator. Thus, we can define the resolvent operator associated with the subdifferential $\partial \psi$ of parameter $\omega$ as

$$
\mathcal{R}_w^\omega (x) = (1 + \omega \partial \psi)^{-1} (x), \quad \forall x \in \mathcal{H}.
$$

(20)

The resolvent operator $\mathcal{R}_w^\omega$ has the following useful characterization.

Lemma 1 (see [30]). For any $z \in \mathcal{H}$, $x \in \mathcal{H}$ satisfies the inequality

$$
\langle x-z, y-x \rangle \geq \omega \psi (x) - \omega \psi (y), \quad \forall y \in \mathcal{H},
$$

(21)

if and only if $x = \mathcal{R}_w^\omega (z)$, where $\mathcal{R}_w^\omega$ is the resolvent operator associated with $\partial \psi$ of parameter $\omega > 0$.

Lemma 2 (see [31]). Let $\{x_n\}$ be a nonnegative real sequence and let $\{\xi_n\}$ be a real sequence in $[0, 1]$ such that $\sum_{n=0}^{\infty} \xi_n = \infty$. If there exists a positive integer $m$ such that

$$
\langle S(a) - (T(b) - \omega \eta(c, d, e)), y - g(x) \rangle \geq \omega \psi (g(x), x) - \omega \psi (y, x), \quad \forall y \in \mathcal{H}.
$$

(23)

Problem (23) is equivalent to finding $x \in \mathcal{H}$, $a \in A(x)$, $b \in B(x)$, $c \in C(x)$, $d \in D(x)$, and $e \in E(x)$ such that

$$
0 \in S(a) - (T(b) - \omega \eta(c, d, e)) + \omega \partial \psi (g(x), x).
$$

(24)

Problem (24) is called the set-valued extended generalized nonlinear mixed variational inclusion problem.

3.1. Special Cases

(i) If $C, D, E, \eta \equiv 0$ and $\psi(\cdot, x) = \psi(\cdot)$, then problem (23) reduces to the following problem which is to find $x \in \mathcal{H}$, $a \in A(x)$, and $b \in B(x)$ such that

$$
\langle S(a) - T(b), y - g(x) \rangle \geq \psi(g(x), x) - \psi(y), \quad \forall y \in \mathcal{H}.
$$

(25)

Problem (25) was introduced and studied by Huang [27].

(ii) If $\eta(c, d, e) \equiv \eta(c, d)$, $A, B, E, S, T \equiv 0$, and $\psi(\cdot, x) = \psi(\cdot)$, then problem (23) reduces to the following problem which is to find $x \in \mathcal{H}$, $c \in C(x)$, and $d \in D(x)$ such that

$$
\langle \eta(c, d), y - g(x) \rangle \geq \psi(g(x)) - \psi(y), \quad \forall y \in \mathcal{H}.
$$

(26)

Problem (26) was introduced and studied by Noor [32].

(iii) If $A, B, E, S, T \equiv 0$, $\eta(c, d, e) \equiv \eta(c, d)$, $\psi(\cdot, x) = \psi(\cdot)$, and $g = I$ (the identity mapping), then problem (23) reduces to the following problem which is to find $x \in \mathcal{H}$, $c \in C(x)$, and $d \in D(x)$ such that

$$
\langle \eta(c, d), y - g(x) \rangle \geq 0, \quad \forall y \in \mathcal{H}.
$$

(29)

Problem (29) was introduced and studied by Noor [34]. The following lemma ensures the equivalence between the set-valued extended generalized nonlinear mixed variational inequality problem (23) and a fixed-point problem.

Lemma 3. Let the mappings $S, T, g, \eta, A, B, C, D, E$ be the same as in problem (23). Then $(x, a, b, c, d, e)$ is a solution of SEGNMVI (23) if and only if $(x, a, b, c, d, e)$ satisfies the following relation:

$$
g(x) = \mathcal{R}_w^\omega [g(x) - \{S(a) - T(b) - \omega \eta(c, d, e)\}],
$$

(30)
where \( x \in \mathcal{H}, a \in A(x), b \in B(x), c \in C(x), d \in D(x), \) and \( e \in E(x); \psi, \omega > 0 \) and \( \mathcal{R}_w^{\psi,(\cdot,x)} \) is the resolvent operator associated with \( \partial \psi(\cdot,x) \) of parameter \( \omega \).

**Proof.** Assume that \( x \in \mathcal{H}, a \in A(x), b \in B(x), c \in C(x), d \in D(x), \) and \( e \in E(x) \) satisfy relation (23); that is,
\[
g(x) = \mathcal{R}_w^{\psi,(\cdot,x)} [g(x) - \{S(a) - T(b) - \omega \eta(c,d,e)\}].
\]  
(31)

Since \( \mathcal{R}_w^{\psi,(\cdot,x)} = (I + \omega \partial \psi(\cdot,x))^{-1} \), the above inequality holds if and only if
\[
g(x) - \{S(a) - T(b) - \omega \eta(c,d,e)\} \in g(x) + \omega \partial \psi(g(x), x).
\]  
(32)

By using the definition of subdifferential of \( \psi(\cdot,x) \), the above relations hold if and only if

\[
\langle T(b) - \omega \eta(c,d,e), y - g(x) \rangle \leq \omega \psi(y, x) - \omega \psi(g(x), x), \quad \forall y \in \mathcal{H}.
\]  
(33)

Hence, we have

\[
\langle S(a) - (T(b) - \omega \eta(c,d,e)), y - g(x) \rangle \geq \omega \psi(g(x), x) - \omega \psi(y, x), \quad \forall y \in \mathcal{H},
\]  
(34)

that is, \( (x, a, b, c, d, e) \) is a solution of SEGNNVI (23).

The following theorem guarantees the existence of a unique solution of SEGNNVI (23). \( \square \)

**Theorem 1.** Let \( S, T, g, \mathcal{H} \to \mathcal{H} \) and \( \eta, \mathcal{H} \times \mathcal{H} \times \mathcal{H} \to \mathcal{H} \) be the single-valued mappings such that \( S \) is \( \lambda_\Sigma \)-Lipschitz continuous, \( T \) is \( \lambda_T \)-Lipschitz continuous and \( \mu_T \)-strongly monotone, \( g \) is \( \lambda_g \)-Lipschitz continuous and \( \mu_g \)-strongly monotone such that \( g(\mathcal{H}) = \mathcal{H} \), and \( \eta \) is Lipschitz continuous in the first argument with respect to \( C \) with constant \( \lambda_\eta \). Lipschitz continuous in the second argument with respect to \( D \) with constant \( \lambda_D \), and Lipschitz continuous in the third argument with respect to \( E \) with constant \( \lambda_E \), respectively. Let \( A, B, C, D, E : \mathcal{H} \to CB(\mathcal{H}) \) be the \( D \)-Lipschitz continuous mappings with constants \( \delta_A, \delta_B, \delta_C, \delta_D, \) and \( \delta_E \), respectively, \( A \) is \( \mu_\Sigma \)-strongly monotone with respect to \( g \), \( B \) is \( \mu_T \)-strongly monotone with respect to \( T \), \( \eta \) is relaxed Lipschitz continuous with respect to \( C \) in the first argument with constant \( \lambda_\eta \), \( \eta \) is relaxed Lipschitz continuous with respect to \( D \) in the second argument with constant \( \lambda_D \), and \( \eta \) is relaxed Lipschitz continuous with respect to \( E \) in the third argument with constant \( \lambda_E \), respectively. Let \( \psi : \mathcal{H} \times \mathcal{H} \to \mathbb{R} \cup \{+\infty\} \) be such that, for each fixed \( x \in \mathcal{H}, \psi(\cdot,x) \) is a proper convex lower-semicontinuous functional on \( \mathcal{H} \) satisfying \( g(\mathcal{H}) \cap \text{dom}(\partial \psi(\cdot,x)) + \Omega \), where \( \partial \psi(\cdot,x) \) is a subdifferential of \( \psi(\cdot,x) \). Suppose that there exist constants \( \omega > 0 \) and \( \xi > 0 \) such that, for each \( z \in \mathcal{H} \),
\[
\left\| \mathcal{R}_w^{\psi,(\cdot,x)}(z) - \mathcal{R}_w^{\psi,(\cdot,y)}(z) \right\| \leq \xi \|x - y\|,
\]  
(35)

and the following conditions are satisfied:

\[
\omega - \frac{t_2}{t_1} < \frac{\sqrt{t_1^2 - t_1 t_2}}{t_1} ,
\]  
(36)

Then, the SEGNNVI (23) admits a unique solution \( (x^*, u^*, v^*, w^*, e^*) \), where \( x^* \in \mathcal{H}, a^* \in A(x^*), b^* \in B(x^*), \) \( c^* \in C(x^*), d^* \in D(x^*), \) and \( e^* \in E(x^*) \).

**Proof.** By Lemma 3, it is sufficient to prove that there exist \( x^* \in \mathcal{H}, a^* \in A(x^*), b^* \in B(x^*), \) \( c^* \in C(x^*), d^* \in D(x^*), \) and \( e^* \in E(x^*) \) satisfying (30). Define a set-valued mapping:
Suppose that 

\[ G(x) = \bigcup_{a \in A(x), b \in B(x), c \in C(x), d \in D(x), e \in E(x)} x - g(x) + R^{\psi, \xi}_w [g(x) - \{S(a) - T(b) - \omega \eta (c, d, e)]], \quad \forall x \in . \]  

(37)

\[ \varphi_i = x_i - g(x_i) + R^{\psi, \xi}_w [g(x_i) - \{S(a_i) - (T(b_i) - \omega \eta (c_i, d_i, e_i)]]. \]

(38)

We show that mapping \( G \) is a contraction mapping. Suppose that \( i \in \{1, 2\} \), and for any \( x_i \in \mathcal{X} \) and \( \varphi_i \in G(x_i) \) there exist \( a_i \in A(x_i), b_i \in B(x_i), c_i \in C(x_i), d_i \in D(x_i), \) and \( e_i \in E(x_i) \) such that

\[
\| \varphi_1 - \varphi_2 \| \leq \| x_1 - x_2 - (g(x_1) - g(x_2)) \| + R^{\psi, \xi}_w (x_1) g(x_1) - S(a_1) - T(b_1) \\
- \omega \eta (c_1, d_1, e_1) - R^{\psi, \xi}_w (x_2) g(x_2) - S(a_2) - T(b_2) \\
- \omega \eta (c_2, d_2, e_2) \\
\leq \| x_1 - x_2 - (g(x_1) - g(x_2)) \| + R^{\psi, \xi}_w (x_1) g(x_1) - S(a_1) - T(b_1) \\
- \omega \eta (c_1, d_1, e_1) - R^{\psi, \xi}_w (x_2) g(x_2) - S(a_2) - (T(b_2)) \\
- \omega \eta (c_2, d_2, e_2) + R^{\psi, \xi}_w (x_2) g(x_2) - S(a_2) - (T(b_2)) \\
- \omega \eta (c_2, d_2, e_2) - R^{\psi, \xi}_w (x_2) g(x_2) - S(a_2) - T(b_2) \\
- \omega \eta (c_2, d_2, e_2) \\
\leq \| x_1 - x_2 - (g(x_1) - g(x_2)) \| + g(x_1) - S(a_1) - T(b_1) \\
- \omega \eta (c_1, d_1, e_1) - [g(x_2) - \{S(a_2) - (T(b_2) - \omega \eta (c_2, d_2, e_2))] \\
+ \xi \| x_1 - x_2 \| \\
\leq \| x_1 - x_2 - (g(x_1) - g(x_2)) \| + \| g(x_1) - g(x_2) - (S(a_1) - S(a_2)) \| \\
+ \| x_1 - x_2 - (T(b_1) - T(b_2)) \| + (x_1 - x_2) + \omega \eta (c_1, d_1, e_1) \\
- \eta (c_2, d_2, e_2) + \xi \| x_1 - x_2 \| .
\]

(39)

It follows from \( \mu \)–strong monotonicity and \( \lambda_g \)–Lipschitz continuity of \( g \) that

\[
\| x_1 - x_2 - (g(x_1) - g(x_2)) \|^2 = \| x_1 - x_2 \|^2 - 2 \langle g(x_1) - g(x_2), x_1 - x_2 \rangle \\
+ \| g(x_1) - g(x_2) \|^2 \\
\leq \| x_1 - x_2 \|^2 - 2 \mu_g \| x_1 - x_2 \|^2 + \lambda_g \| x_1 - x_2 \|^2 \\
\leq (1 - 2 \mu_g + \lambda_g^2) \| x_1 - x_2 \|^2,
\]

(40)

that is,
By using the \(\mu_{B}\)-strong monotonicity of \(B\) with respect to \(T\) and \(D\)-Lipschitz continuity of \(B\) with constant \(\delta_{B}\), we have

\[
\|x_{1} - x_{2} - (g(x_{1}) - g(x_{2}))\| \leq \sqrt{(1 - 2\mu_{g} + \lambda_{g}^{2})} \|x_{1} - x_{2}\|. 
\] (41)

\[
\|x_{1} - x_{2} - (T(b_{1}) - T(b_{2}))\|^{2} = \|x_{1} - x_{2}\|^{2} - 2\langle T(b_{1}) - T(b_{2}), x_{1} - x_{2}\rangle \\
+ \|T(b_{1}) - S(b_{2})\|^{2} \\
\leq \|x_{1} - x_{2}\|^{2} - 2\mu_{B}\|x_{1} - x_{2}\|^{2} + \lambda_{g}^{2}\delta_{B}\|x_{1} - x_{2}\|^{2} \\
\leq (1 - 2\mu_{B} + \lambda_{g}^{2}\delta_{B})\|x_{1} - x_{2}\|^{2}, 
\] (42)

that is,

\[
\|x_{1} - x_{2} - (T(b_{1}) - T(b_{2}))\| \leq \sqrt{(1 - 2\mu_{B} + \lambda_{g}^{2}\delta_{B})}\|x_{1} - x_{2}\|. 
\] (43)

By using the \(\mu_{A}\)-strong monotonicity of \(A\) with respect to \(S\), \(D\)-Lipschitz continuity of \(A\) with constant \(\delta_{A}\), and \(\lambda_{g}\)-Lipschitz continuity of \(g\), respectively, we evaluate

\[
\|\langle x_{1} - x_{2}\rangle + \omega(\eta(c_{1}, d_{1}, e_{1}) - \eta(c_{2}, d_{2}, e_{2}))\|^{2} \\
= \|x_{1} - x_{2}\|^{2} + 2\omega\|\eta(c_{1}, d_{1}, e_{1}) - \eta(c_{2}, d_{2}, e_{2}), x_{1} - x_{2}\rangle \\
+ \omega^{2}\|\eta(c_{1}, d_{1}, e_{1}) - \eta(c_{2}, d_{2}, e_{2})\|^{2} \\
= \|x_{1} - x_{2}\|^{2} + 2\omega\|\eta(c_{1}, d_{1}, e_{1}) - \eta(c_{2}, d_{1}, e_{1}), x_{1} - x_{2}\rangle \\
+ \langle \eta(c_{2}, d_{1}, e_{1}) - \eta(c_{2}, d_{2}, e_{1}), x_{1} - x_{2}\rangle + \langle \eta(c_{2}, d_{2}, e_{1}) - \eta(c_{2}, d_{2}, e_{2}), x_{1} - x_{2}\rangle + \omega^{2}\|\eta(c_{1}, d_{1}, e_{1}) - \eta(c_{2}, d_{2}, e_{2})\|^{2} \\
\leq \|x_{1} - x_{2}\|^{2} - 2(r_{1} + r_{2} + r_{3})\|x_{1} - x_{2}\|^{2} \\
+ \omega^{2}\|\eta(c_{1}, d_{1}, e_{1}) - \eta(c_{2}, d_{2}, e_{2})\|^{2}. 
\] (45)

Since mapping \(\eta\) is Lipschitz continuous in the first argument with respect to \(C\) with constant \(\lambda_{c}\), Lipschitz continuous in the second argument with respect to \(D\) with constant \(\lambda_{d}\), Lipschitz continuous in the third argument with respect to \(E\) with constant \(\lambda_{e}\), \(D\)-Lipschitz continuous of \(C, D, E\) with constants \(\delta_{C}, \delta_{D}, \text{ and } \delta_{E}\), respectively, we have

\[
\|\eta(c_{1}, d_{1}, e_{1}) - \eta(c_{2}, d_{2}, e_{2})\| \leq \|\eta(c_{1}, d_{1}, e_{1}) - \eta(c_{2}, d_{1}, e_{1})\| + \|\eta(c_{2}, d_{1}, e_{1}) - \eta(c_{2}, d_{2}, e_{1})\| + \|\eta(c_{2}, d_{2}, e_{1}) - \eta(c_{2}, d_{2}, e_{2})\| \\
\leq \lambda_{c}\|c_{1} - c_{2}\| + \lambda_{d}\|d_{1} - d_{2}\| + \lambda_{e}\|e_{1} - e_{2}\| \\
\leq \lambda_{c}\|D(C(x_{1}), C(x_{2}))\| + \lambda_{d}\|D(D(x_{1}), D(x_{2}))\| \\
+ \lambda_{e}\|D(E(x_{1}), E(x_{2}))\| \\
\leq \lambda_{c}\delta_{C}\|x_{1} - x_{2}\| + \lambda_{d}\delta_{D}\|x_{1} - x_{2}\| \\
+ \lambda_{e}\delta_{E}\|x_{1} - x_{2}\| \\
= (\lambda_{c}\delta_{C} + \lambda_{d}\delta_{D} + \lambda_{e}\delta_{E})\|x_{1} - x_{2}\|. 
\] (46)
Combining (45) and (46), we obtain

\[
\|x_1 - x_2\| + \omega(\eta, d_1, e_1) - \eta(\epsilon, d_2, e_2) \leq \sqrt{1 - 2\omega(r_1 + r_2 + r_3) + \omega^2(\lambda_1, \delta_C + \lambda_2, \delta_D + \lambda_3, \delta_E)^2}\|x_1 - x_2\|.
\] (47)

From (39)–(47), we have

\[
\|\varphi_1 - \varphi_2\| \leq \sqrt{1 - 2\mu_g + \lambda_2^2}\|x_1 - x_2\|.
\] (48)

which implies that

\[
\|\varphi_1 - \varphi_2\| \leq \Theta(\varphi)\|x_1 - x_2\|,
\] (49)

where

\[
\Theta(\varphi) = \theta + \sqrt{1 - 2\mu_g + \lambda_2^2} + \sqrt{\lambda_1^2 + \lambda_2^2}\varphi.
\] (50)

By condition (36), it is clear that 0 ≤ \Theta(\varphi) < 1. Since \( \varphi_1 \in G(x_1) \) and \( \varphi_2 \in G(x_2) \) are arbitrary, we obtain

\[
\hat{D}(G(x_1), G(x_2)) \leq \Theta(\varphi)\|x_1 - x_2\|, \quad \forall x_1, x_2 \in \mathcal{H}.
\] (51)

By Theorem 1 of Siddique and Ansari [35], it follows that \( G \) has a fixed point \( x^* \in \mathcal{H} \) such that \( a^* \in A(x^*), b^* \in B(x^*), c^* \in C(x^*), d^* \in D(x^*), \) and \( e^* \in E(x^*) \); that is, \( G(x^*) = x^* \). From (30), we conclude that

\[
g(x^*) = \mathcal{D}(\mathcal{H}, \mathcal{X})[g(x^*) - \mathcal{S}(a^*) - (T(b^*) - \omega)(c^*, d^*, e^*)].
\] (52)

Thus, by Lemma 1, \( x^* \in \mathcal{H} \) such that \( a^* \in A(x^*), b^* \in B(x^*), c^* \in C(x^*), d^* \in D(x^*), \) and \( e^* \in E(x^*) \) is a solution of SEGNMVI (23).

4. Convergence and Stability Analysis

We establish a three-step iterative algorithm for solving SEGNMVI (23) and discuss the convergence and stability analysis.

Algorithm 1. Let the mappings \( S, T, g, \eta, A, B, C, D, E \) be the same as in problem (23). For any \( x_0 \in \mathcal{H}, a_0 \in A(x_0), b_0 \in B(x_0), c_0 \in C(x_0), d_0 \in D(x_0), \) and \( e_0 \in D(x_0), \) as \( g(H) = H, \) compute the iterative sequences \( \{x_n\}, \{a_n\}, \{b_n\}, \{c_n\}, \{d_n\}, \) and \( \{e_n\} \) by the following iterative process:

\[
\begin{align*}
x_{n+1} &= (1 - \gamma_n)x_n + \gamma_n\left[ y_n - g(y_n) + \mathcal{D}(\mathcal{H}, \mathcal{X})[g(y_n) - \mathcal{S}(\tilde{a}_n) - (T(\tilde{b}_n) - \omega)(\tilde{c}_n, \tilde{d}_n, \tilde{e}_n)]\right] + \delta_n, \\
y_n &= (1 - \sigma_n)x_n + \sigma_n\left[ z_n - g(z_n) + \mathcal{D}(\mathcal{H}, \mathcal{X})[g(z_n) - \mathcal{S}(\tilde{a}_n) - (T(\tilde{b}_n) - \omega)(\tilde{c}_n, \tilde{d}_n, \tilde{e}_n)]\right] + \eta_n, \\
z_n &= (1 - \varsigma_n)x_n + \varsigma_n\left[ x_n - g(x_n) + \mathcal{D}(\mathcal{H}, \mathcal{X})[g(x_n) - \mathcal{S}(\tilde{a}_n) - (T(\tilde{b}_n) - \omega)(\tilde{c}_n, \tilde{d}_n, \tilde{e}_n)]\right] + \zeta_n.
\end{align*}
\] (53)
\[
\varphi_n = \|u_n1 + (1 - y_n)u_n + y_n \left( s_n - g(s_n) + R^{\rho}_w(\cdot, s_n) \left( g(s_n) - S(a_n) \right) \right) + \omega(T(b_n) - \omega(\varphi_n, d_n, e_n)) \| + y_n|d_n|
\]
\[
s_n = (1 - a_n)u_n + \sigma_a u_n - T(b_n) + R^{\rho}_w(\cdot, s_n) \left( g(s_n) - S(a_n) \right) \left( g(s_n) - S(a_n) \right) - \omega(T(b_n) - \omega(\varphi_n, d_n, e_n)) + \omega(\varphi_n, d_n, e_n) + \omega(\varphi_n, d_n, e_n)
\]
\[
z_n = (1 - \zeta_n)u_n + \sigma_a u_n - g(u_n) + R^{\rho}_w(\cdot, s_n) \left( g(u_n) - S(a_n) \right) \left( g(u_n) - S(a_n) \right) - \omega(T(b_n) - \omega(\varphi_n, d_n, e_n)) + \omega(\varphi_n, d_n, e_n) + \omega(\varphi_n, d_n, e_n)
\]
(54)

where \( a_n \in A(u_n), b_n \in B(u_n), c_n \in C(u_n), d_n \in D(u_n), e_n \in E(u_n), a_n' \in A(s_n), b_n' \in B(s_n), c_n' \in C(s_n), d_n' \in D(s_n), e_n' \in E(s_n), a_n'' \in A(t_n), b_n'' \in B(t_n), c_n'' \in C(t_n), d_n'' \in D(t_n), \) and \( e_n'' \in E(t_n) \) can be chosen arbitrarily. Let \( \{l_n\}, \{p_n\}, \) and \( \{q_n\} \) be the three sequences in \( \mathcal{H} \) to take into account the possible inexact computation and the sequences \( y_n, \sigma_n, \) and \( c_n \) satisfy the following conditions: \( 0 \leq y_n, \sigma_n, c_n \leq 1, n \geq 0, \) and \( \sum_{n=0}^{\infty} y_n \) diverges.

**Remark 1.** If \( c_n = 0, \forall n \geq 0, \) then Algorithm 1 becomes Ishikawa-type iterative algorithm. If \( b_n, c_n = 0, \forall n \geq 0, \) then Algorithm 1 becomes Mann-type iterative algorithm. Also, we remark that, for suitable choices of operators involved in Algorithm 1, we can easily obtain many algorithms studied by several authors; see, for example, [27, 33, 34].

**Theorem 2.** Suppose that Theorem 1 holds. Additionally, if \( \lim_{n \to \infty} \|l_n\| = 0, \lim_{n \to \infty} \|p_n\| = 0, \) and \( \lim_{n \to \infty} \|q_n\| = 0, \) then

(i) The sequences \( \{x_n\}, \{a_n\}, \{b_n\}, \{c_n\}, \{d_n\}, \) and \( \{e_n\} \) generated by Algorithm 1 converge strongly to \( x^*, a^*, b^*, c^*, d^*, \) and \( e^* \), respectively, and \( (x^*, a^*, b^*, c^*, d^*, e^*) \) is the solution of SEGNMVI (23)

(ii) Furthermore, if \( 0 < \varepsilon < y_n^* \) then \( \lim_{n \to \infty} x_n = x^* \) if and only if \( \lim_{n \to \infty} y_n = 0, \) where \( y_n \) is given in (54); that is, the sequences \( \{x_n\}, \{a_n\}, \{b_n\}, \{c_n\}, \{d_n\}, \) and \( \{e_n\} \) generated by (53) are \( R^{\rho}(x, \cdot, \cdot, \cdot) \)-stable.

**Proof.** (i) Suppose that \( (x^*, a^*, b^*, c^*, d^*, e^*) \) is a unique solution of SEGNMVI (23). Then, we have

\[
x_n = (1 - y_n)x_n + y_n \left( g(x_n) + R^{\rho}_w(\cdot, x_n) \left( g(x_n) - S(a_n) - T(b_n) \right) - \omega(\varphi_n, d_n, e_n) \right)
\]

\[
= (1 - y_n)x_n + y_n \left( g(x_n) + R^{\rho}_w(\cdot, x_n) \left( g(x_n) - S(a_n) - T(b_n) \right) - \omega(\varphi_n, d_n, e_n) \right)
\]

\[
= (1 - \zeta_n)x_n + \zeta_n \left( g(x_n) + R^{\rho}_w(\cdot, x_n) \left( g(x_n) - S(a_n) - T(b_n) \right) - \omega(\varphi_n, d_n, e_n) \right)
\]

Using Algorithm 1 and (49), we have

\[
\|x_n1 - x^*\| = \left\| (1 - y_n)x_n + y_n \left( g(x_n) + R^{\rho}_w(\cdot, x_n) \left( g(x_n) - S(a_n) - T(b_n) \right) - \omega(\varphi_n, d_n, e_n) \right) \right\|
\]

\[
\leq (1 - y_n)\|x_n - x^*\| + y_n \left\| g(x_n) + R^{\rho}_w(\cdot, x_n) \left( g(x_n) - S(a_n) - T(b_n) \right) - \omega(\varphi_n, d_n, e_n) \right\|
\]

\[
\leq (1 - y_n)\|x_n - x^*\| + y_n \left\| g(x_n) + R^{\rho}_w(\cdot, x_n) \left( g(x_n) - S(a_n) - T(b_n) \right) - \omega(\varphi_n, d_n, e_n) \right\|
\]

where

\[
\Theta(\varphi) = \varphi + \sqrt{1 - 2\omega(r_1 + r_2 + r_3) + \omega^2 (\lambda_1 \delta C + \lambda_2 \delta D + \lambda_3 \delta E)^2}
\]

\[
\varphi = \sqrt{(1 - 2\mu_g + \lambda_2 \delta B) + \sqrt{(1 - 2\mu_g + \lambda_2 \delta B)^2 + \lambda_2 \delta B + \lambda_3 \delta A}} + \xi.
\]

Using the same argument as for (56), we estimate
\[ y_n - x^* \| \leq \left\| (1 - \sigma_n) x_n + \sigma_n \left( z_n - g(z_n) + R_n(x_n) \right) - S(a_n) \right\| \]

Using (58) and (60),

\[ \| y_n - x^* \| \leq \| x_n - x^* \| + \sigma_n \| q_n \|. \]  

Since \( 1 - \sigma_n (1 - \Theta (\ast)) < 1 \), we have

\[ \| z_n - x^* \| \leq \| x_n - x^* \| + \sigma_n \| q_n \|. \]  

Using (58) and (60),

\[ \| y_n - x^* \| \leq \| x_n - x^* \| + \sigma_n \Theta (\ast) \| q_n \| + \sigma_n \| p_n \|. \]  

Since \( 1 - \sigma_n 1 - \Theta (\ast) < 1 \), we have

\[ \| y_n - x^* \| \leq \| x_n - x^* \| + \sigma_n \Theta (\ast) \| q_n \| + \sigma_n \| p_n \|. \]  

Using (56) and (62),

\[ \left\| x_{n+1} - x^* \right\| \leq \left( 1 - \gamma_n \right) \left\| x_n - x^* \right\| + \gamma_n \Theta (\ast) \left\| x_n - x^* \right\| + \sigma_n \| q_n \| + \| p_n \| \]

Using (58) and (60),

\[ \| y_n - x^* \| \leq \left( 1 - \sigma_n \right) \left\| x_n - x^* \right\| + \sigma_n \Theta (\ast) \left\| x_n - x^* \right\| + \sigma_n \| q_n \| + \sigma_n \| p_n \|. \]  

Using (56) and (62),

\[ \left\| x_{n+1} - x^* \right\| \leq \left( 1 - \gamma_n \right) \left\| x_n - x^* \right\| + \gamma_n \Theta (\ast) \left\| x_n - x^* \right\| + \sigma_n \| q_n \| + \| p_n \| \]

Using (56) and (62),

\[ \| y_n - x^* \| \leq \left( 1 - \sigma_n \right) \left\| x_n - x^* \right\| + \sigma_n \Theta (\ast) \left\| x_n - x^* \right\| + \sigma_n \| q_n \| + \sigma_n \| p_n \|. \]  

Using (58) and (60),

\[ \| y_n - x^* \| \leq \left( 1 - \sigma_n \right) \left\| x_n - x^* \right\| + \sigma_n \Theta (\ast) \left\| x_n - x^* \right\| + \sigma_n \| q_n \| + \sigma_n \| p_n \|. \]  

Using (56) and (62),

\[ \left\| x_{n+1} - x^* \right\| \leq \left( 1 - \gamma_n \right) \left\| x_n - x^* \right\| + \gamma_n \Theta (\ast) \left\| x_n - x^* \right\| + \sigma_n \| q_n \| + \| p_n \| \]

Using (56) and (62),

\[ \| y_n - x^* \| \leq \left( 1 - \sigma_n \right) \left\| x_n - x^* \right\| + \sigma_n \Theta (\ast) \left\| x_n - x^* \right\| + \sigma_n \| q_n \| + \sigma_n \| p_n \|. \]  

Using (56) and (62),

\[ \left\| x_{n+1} - x^* \right\| \leq \left( 1 - \gamma_n \right) \left\| x_n - x^* \right\| + \gamma_n \Theta (\ast) \left\| x_n - x^* \right\| + \sigma_n \| q_n \| + \| p_n \| \]

Using (56) and (62),

\[ \| y_n - x^* \| \leq \left( 1 - \sigma_n \right) \left\| x_n - x^* \right\| + \sigma_n \Theta (\ast) \left\| x_n - x^* \right\| + \sigma_n \| q_n \| + \sigma_n \| p_n \|. \]  

Using (56) and (62),

\[ \left\| x_{n+1} - x^* \right\| \leq \left( 1 - \gamma_n \right) \left\| x_n - x^* \right\| + \gamma_n \Theta (\ast) \left\| x_n - x^* \right\| + \sigma_n \| q_n \| + \| p_n \| \]

Using (56) and (62),

\[ \| y_n - x^* \| \leq \left( 1 - \sigma_n \right) \left\| x_n - x^* \right\| + \sigma_n \Theta (\ast) \left\| x_n - x^* \right\| + \sigma_n \| q_n \| + \sigma_n \| p_n \|. \]  

Using (56) and (62),

\[ \left\| x_{n+1} - x^* \right\| \leq \left( 1 - \gamma_n \right) \left\| x_n - x^* \right\| + \gamma_n \Theta (\ast) \left\| x_n - x^* \right\| + \sigma_n \| q_n \| + \| p_n \| \]

Using (56) and (62),

\[ \| y_n - x^* \| \leq \left( 1 - \sigma_n \right) \left\| x_n - x^* \right\| + \sigma_n \Theta (\ast) \left\| x_n - x^* \right\| + \sigma_n \| q_n \| + \sigma_n \| p_n \|. \]  

Using (56) and (62),

\[ \left\| x_{n+1} - x^* \right\| \leq \left( 1 - \gamma_n \right) \left\| x_n - x^* \right\| + \gamma_n \Theta (\ast) \left\| x_n - x^* \right\| + \sigma_n \| q_n \| + \| p_n \| \]

Using (56) and (62),

\[ \| y_n - x^* \| \leq \left( 1 - \sigma_n \right) \left\| x_n - x^* \right\| + \sigma_n \Theta (\ast) \left\| x_n - x^* \right\| + \sigma_n \| q_n \| + \sigma_n \| p_n \|. \]  

Using (56) and (62),

\[ \left\| x_{n+1} - x^* \right\| \leq \left( 1 - \gamma_n \right) \left\| x_n - x^* \right\| + \gamma_n \Theta (\ast) \left\| x_n - x^* \right\| + \sigma_n \| q_n \| + \| p_n \| \]
Let $\Gamma_n = (\Theta(\Phi))^2 \sigma_n c_n \|q_n\| + \Theta(\Phi) \sigma_n |p_n| + \gamma_n J_n^I$, $1 - \Theta(\Phi))$, $\Delta_n = \|x_n - x^*\|$, and $Y_n = \gamma_n (1 - \Theta(\Phi))$; then, (63) can be rewritten as

$$\Delta_{n+1} \leq (1 - Y_n) \Delta_n + Y_n \Gamma_n. \quad (64)$$

By Lemma 2 and the assumptions $\lim_{n \to \infty} \|p_n\| = \lim_{n \to \infty} \|p_n\| = \lim_{n \to \infty} |q_n| = 0$, we conclude that $\Delta_n \to 0$, as $n \to \infty$, and so we obtain $x_n \to x$, as $n \to \infty$. Since $A$ is $D$-Lipschitz continuous with constant $\delta_A$, we have

$$|a_n - a^*| \leq \tilde{D}(A(x_n), A(x^*)) \leq \delta_A \|x_n - x^*\| \to 0, \quad \text{and} \quad a_n \to a^*, \text{as} \quad n \to \infty. \quad (65)$$

Hence, $a^* \in A(x^*)$. Similarly, we can show that $b^* \in B(x^*)$, $c^* \in C(x^*)$, $d^* \in D(x^*)$, and $e \in E(x^*)$. By Lemma 3, we conclude that $(x^*, a^*, b^*, c^*, d^*, e^*)$ is the solution of SEGNMVI (23).

**Proof: (ii).** Let $G(x^*) = x^* - g(x^*) + A^\Phi(x^*) g(x^*) = S(a^*) - (T(b^*) - \omega \eta (c^*, d^*, e^*))$. Using Algorithm 1, Lemma 2, and Lemma 3, we obtain

$$\|u_{n+1} - x^*\| \leq \|u_{n+1} \| - \|[(1 - \gamma_n) u_n + \gamma_n G(t_n)] + \gamma_n J_n^I\|$$

$$\leq \|u_{n+1} \| - \|[(1 - \gamma_n) u_n + \gamma_n G(t_n)] + \gamma_n J_n^I\|$$

$$\leq \|u_{n+1} \| - \|[(1 - \gamma_n) u_n + \gamma_n G(t_n)] + \gamma_n J_n^I\|$$

$$\leq \|u_{n+1} \| - \|[(1 - \gamma_n) u_n + \gamma_n G(t_n)] + \gamma_n J_n^I\|$$

Using (54), we have

$$\|t_n - x^*\| \leq \|[(1 - \sigma_n) u_n + \sigma_n G(s_n)] + \sigma_n p_n\|$$

$$\leq (1 - \sigma_n) \|u_n - x^*\| + \sigma_n \|G(s_n) - G(x^*)\| + \sigma_n \|p_n\|$$

Again, using (54), we have

$$\|s_n - x^*\| \leq \|[(1 - \sigma_n) u_n + \sigma_n G(s_n)] + \sigma_n p_n\|$$

$$\leq (1 - \sigma_n) \|u_n - x^*\| + \sigma_n \|G(s_n) - G(x^*)\| + \sigma_n \|p_n\|$$

$$\leq (1 - \sigma_n) \|u_n - x^*\| + \sigma_n \|G(s_n) - G(x^*)\| + \sigma_n \|p_n\|$$

$$\leq (1 - \sigma_n) \|u_n - x^*\| + \sigma_n \|G(s_n) - G(x^*)\| + \sigma_n \|p_n\|.$$
\[ \|s_n - x^*\| \leq \|(1 - \xi)u_n + \xi G(u_n) + \xi q_n\| - \|(1 - \xi)x^* + \xi G(x^*)\| \\
\leq (1 - \xi)\|u_n - x^*\| + \xi\|G(u_n) - G(x^*)\| + \xi q_n \]
\leq (1 - \xi)\|u_n - x^*\| + \xi\|\theta(x)\|u_n - x^*\| + \xi q_n \]
\leq (1 - \xi)(1 - \theta(x))\|u_n - x^*\| + \xi\|q_n\| \\
\leq \|u_n - x^*\| + \xi\|q_n\|. \tag{69} \]

Using (58) and (60),
\[ \|s_n - x^*\| \leq (1 - \sigma_n)\|u_n - x^*\| + \sigma_n\|\theta(x)\|\|u_n - x^*\| + \sigma_n\|q_n\| \\
= (1 - \sigma_n(1 - \theta(x)))\|u_n - x^*\| + \sigma_n\|\theta(x)\|q_n\| + \sigma_n\|p_n\| \tag{70} \]
\[ \leq \|u_n - x^*\| + \sigma_n\|\theta(x)\|q_n\| + \sigma_n\|p_n\|. \]

Using (68) and (70),
\[ \|u_{n+1} - x^*\| \leq \|u_{n+1} - [(1 - \gamma_n)u_n + \gamma_n F(t_n)]\| + (1 - \gamma_n)\|u_n - x^*\| \\
+ \gamma_n\|\theta(x)\|\|u_n - x^*\| + \sigma_n\|\theta(x)\|\|q_n\| + \sigma_n\|p_n\| + \gamma_n\|p_n\| \]
\[ \leq \varphi_n + (1 - \gamma_n(1 - \theta(x)))\|u_n - x^*\| + (\theta(x))^2 \gamma_n\sigma_n\|q_n\| \\
+ \theta(x)\gamma_n\sigma_n\|p_n\| + \gamma_n\|p_n\| \tag{71} \]
\[ \leq \varphi_n + (1 - \gamma_n(1 - \theta(x)))\|u_n - x^*\| \\
+ \gamma_n(1 - \theta(x)) \left( \frac{(\theta(x))^2 \gamma_n\sigma_n\|q_n\| + \theta(x)\sigma_n\|p_n\| + \gamma_n\|p_n\|}{1 - \theta(x)} \right). \]

As \(0 < \epsilon \leq \gamma_n\), (70) becomes
\[ \|u_{n+1} - x^*\| \leq (1 - \gamma_n(1 - \theta(x)))\|u_n - x^*\| + \gamma_n(1 - \theta(x)) \]
\[ \times \left( \frac{\varphi_n + \theta(x)^2 \gamma_n\sigma_n\|q_n\| + \theta(x)\sigma_n\|p_n\| + \gamma_n\|p_n\|}{\epsilon(1 - \theta(x))} \right). \tag{72} \]

Assume that \(\lim_{n \to \infty} \varphi_n = 0\); hence, \(\lim_{n \to \infty} \|u_n - x^*\| = \lim_{n \to \infty} \|\theta(x)\|q_n\| = 0\). Conversely, assume that \(\lim_{n \to \infty} \|u_n - x^*\| = \lim_{n \to \infty} \|\theta(x)\|q_n\| = 0\), such that \(a^* \in A(x^*), b^* \in B(x^*), c^* \in C(x^*), d^* \in D(x^*), \) and \(e \in E(x^*)\), where \(\lim_{n \to \infty} \|u_n\| = \lim_{n \to \infty} \|\theta(x)\|q_n\| = \lim_{n \to \infty} \|\theta(x)\|q_n\| = 0\). From (55) and as \(\lim_{n \to \infty} \|l_n\| = \lim_{n \to \infty} \|\theta(x)\|q_n\| = 0\), we have...
\[ \varphi_n = \left\| u_{n+1} - \left[ (1 - \gamma_n)u_n + y_n G(t_n) + y_n l_n \right] \right\| \]
\leq \left\| u_{n+1} - x^* \right\| + \left\| \left( (1 - \gamma_n)u_n + y_n G(t_n) + y_n l_n \right) - x^* \right\| 
\leq \left\| u_{n+1} - x^* \right\| + \left\| \left( (1 - \gamma_n)u_n + y_n G(t_n) + y_n l_n \right) - \left[ (1 - \gamma_n)x^* + y_n G(x^*) \right] \right\| 
\leq \left\| u_{n+1} - x^* \right\| + \left(1 - \gamma_n \right) \left\| u_n - x^* \right\| + y_n \left\| G(t_n) - G(x^*) \right\| + y_n \left\| l_n \right\| 
\leq \left(1 - \gamma_n \right) \left\| u_n - x^* \right\| + (1 - \gamma_n) (1 - \Theta(\eta)) \left\| u_n - x^* \right\| + \Theta(\eta) y_n \sigma_\eta \left\| p_n \right\| + y_n \left\| l_n \right\| . \]

which implies that
\[ \lim_{n \to \infty} \varphi_n = 0. \] (74)

Hence, the sequences \( \{u_n\}, \{s_n\}, \{c_n\}, \{d_n\}, \) and \( \{e_n\} \) generated by Algorithm 1 are \( \mathcal{A}_6^{\psi}(x^*) \)-stable.

If we take \( S, T, A, B, E \equiv 0, \) \( \eta(c, d, \epsilon) \equiv \eta(c, d), \) and \( \psi(\cdot, x) = \psi(\cdot) \) in SEGNMVI (23), we obtain the following corollary which ensures the solvability and stability of problem (26).

\[ \boxempty \]

**Corollary 1.** Let \( g: \mathcal{H} \to \mathcal{H} \) and \( \eta: \mathcal{H} \times \mathcal{H} \to \mathcal{H} \) be the single-valued mappings such that \( g \) is \( \lambda_2 \)-Lipschitz continuous and \( \mu_2 \)-strongly monotone and \( g(\mathcal{H}) = \mathcal{H}; \) \( \eta \) is Lipschitz continuous in the first argument with respect to \( C \) with constant \( \lambda_\eta \) and Lipschitz continuous in the second argument with respect to \( D \) with constant \( \lambda_\eta, \) respectively. Let \( C, D: \mathcal{H} \to CB(\mathcal{H}) \) be the \( \tilde{D} \)-Lipschitz continuous mappings with constants \( \delta_\tilde{C} \) and \( \delta_\tilde{D} \), respectively; \( \eta \) is relaxed Lipschitz continuous with respect to \( C \) in the first argument with constant \( r_1 \), and \( \eta \) is relaxed Lipschitz continuous with respect to \( D \) in the second argument with constant \( r_2 \), respectively. Let \( \psi: \mathcal{H} \to [0, \infty) \) be a proper convex lower-semicontinuous functional on satisfying \( \varphi(\mathcal{H}) \cap \text{dom}(\psi) \neq \emptyset, \) where \( \varphi \) is a subdifferential of \( \psi. \)

Let \( x_0 \in C(x_0) \) and \( d_0 \in D(x_0), \) and the following iterative sequences \( \{x_n\}, \{c_n\}, \) and \( \{d_n\} \) are obtained by the following iterative scheme:

\[ \begin{align*}
x_{n+1} &= (1 - \gamma_n)x_n + y_n (g(y_n) + \mathcal{A}_\psi^\psi (g(y_n) - \omega y (\tilde{a}_n, d_n))) + y_n l_n, \\
y_n &= (1 - \sigma_n)x_n + \sigma_n \left[ g(z_n) - g(z_n) + \mathcal{A}_\psi^\psi (z_n - \omega y (\tilde{a}_n, d_n)) \right] + \sigma_n p_n, \\
z_n &= (1 - \zeta_n)x_n + \zeta_n \left[ x_n - g(x_n) + \mathcal{A}_\psi^\psi (g(x_n) - \omega y (c_n, d_n)) \right] + \zeta_n q_n, \end{align*} \] (76)

where \( c_n \in C(x_n), \) \( d_n \in D(x_n), \tilde{a}_n \in C(y_n), \tilde{a}_n \in D(y_n), \) \( \tilde{a}_n \in C(z_n), \) and \( \tilde{a}_n \in D(z_n) \) can be chosen arbitrarily.

Let \( \{u_n\} \) be any sequence in \( \mathcal{H} \) and define \( \{\varphi_n\} \) by

\[ \begin{align*}
\varphi_n &= \left\| u_{n+1} - \left[ (1 - \gamma_n)u_n + y_n \left[ s_n - g(s_n) + \mathcal{A}_\psi^\psi (g(s_n) - \omega y (c_n, d_n)) \right] + y_n l_n \right] \right\|, \\
s_n &= (1 - \sigma_n)u_n + \sigma_n \left[ t_n - g(t_n) + \mathcal{A}_\psi^\psi (g(t_n) - \omega y (c_n, d_n)) \right] + \sigma_n p_n, \\
z_n &= (1 - \zeta_n)u_n + \zeta_n \left[ u_n - u_n + \mathcal{A}_\psi^\psi (g(u_n) - \omega y (c_n, d_n)) \right] + \zeta_n q_n, \end{align*} \] (77)
where \( c_n \in C(u_n), d_n \in D(u_n), c'_n \in C(s_n), d'_n \in D(s_n),
\), and \( \gamma_n, d_n, \) and \( \varsigma_n \) satisfy the following conditions: \( 0 \leq \gamma_n, \sigma_n, \varsigma_n \leq 1, \forall n \geq 0, \) and \( \Sigma_{n=0}^{\infty} \gamma_n \) diverges. If \( \lim_{n \to \infty} \| p_n \| = 0, \)
\( \lim_{n \to \infty} \| q_n \| = 0, \) and \( \lim_{n \to \infty} \| q_n \| = 0, \) then

(i) The sequences \( \{ x_n \}, \{ c_n \}, \) and \( \{ d_n \} \) converge strongly to \( x^*, c^*, \) and \( d^* \), respectively, and \( \{ x^*, c^*, d^* \} \) is a solution of problem (26)

(ii) Furthermore, if \( 0 < \varepsilon < \gamma_n, \) then \( \lim_{n \to \infty} \| u_n - x^* \| = 0, \) and only if \( \lim_{n \to \infty} \| q_n \| = 0, \) where \( q_n \) is given by (77); that is, the sequences \( \{ x_n \}, \{ c_n \}, \) and \( \{ d_n \} \) generated by (76) are \( \mathcal{R}^{(\nu)} \)-stable:

\[
\begin{align*}
\left\| \omega - \left( \frac{\mu + \lambda t_1 \delta}{1 - \lambda t_1 \delta} (1 - t_1) \right) \right\| & < \sqrt{\frac{(\mu \lambda - (1 - t_1) \lambda t_1 \delta)}{\left( \lambda \delta - \lambda t_1 \delta \right)^2}} t_1 (2 - t_1), \\
\delta \lambda t_1 \delta & < \mu + t_1 \lambda t_1 \delta, \lambda t_1 \delta > \lambda t_1 \delta,
\end{align*}
\]

\[
\omega \lambda t_1 \delta < (1 - t_1), t_1 = 2 \sqrt{2 - 2 \delta + \lambda t_1 \delta}, t_1 < 1.
\]

For any \( x_0 \in \mathcal{H}, a_0 \in A(x_0), \) and \( b_0 \in B(x_0), \) as \( g(\mathcal{H}) = \mathcal{H}, \) we can obtain the iterative sequences \( \{ x_n \}, \{ a_n \}, \) and \( \{ b_n \} \) by the following iterative scheme:

\[
x_{n+1} = (1 - \gamma_n) x_n + \gamma_n g(y_n - g(y_n) + \mathcal{R}^{(\nu)} \left[ g(y_n) - \omega \left( S(\alpha_n) - T(\beta_n) \right) \right]) + \gamma_n u_n,
\]

\[
y_n = (1 - \sigma_n) x_n + \sigma_n g(z_n - g(z_n) + \mathcal{R}^{(\nu)} \left[ g(z_n) - \omega \left( S(\alpha_n) - T(\beta_n) \right) \right]) + \sigma_n \phi_n,
\]

\[
z_n = (1 - \varsigma_n) x_n + \varsigma_n g(x_n - g(x_n) + \mathcal{R}^{(\nu)} \left[ g(x_n) - \omega \left( S(\alpha_n) - T(\beta_n) \right) \right]) + \varsigma_n \eta_n,
\]

where \( a_n \in A(x_n), b_n \in B(x_n), \alpha_n \in A(y_n), \beta_n \in B(y_n), \alpha_n \in A(z_n), \) and \( \beta_n \in B(z_n) \) can be chosen arbitrarily.

Let \( \{ u_n \} \) be any sequence in \( \mathcal{H} \) and define \( \{ \phi_n \} \) by

\[
\phi_n = \| u_{n+1} - \left[ (1 - \gamma_n) u_n + \gamma_n g(z_n - g(z_n) + \mathcal{R}^{(\nu)} \left[ g(z_n) - \omega \left( S(\alpha_n) - T(\beta_n) \right) \right]) + \gamma_n \phi_n \right\|,
\]

\[
s_n = (1 - \sigma_n) u_n + \sigma_n g(t_n - g(t_n) + \mathcal{R}^{(\nu)} \left[ g(t_n) - \omega \left( S(\alpha_n) - T(\beta_n) \right) \right]) + \sigma_n \phi_n,
\]

\[
z_n = (1 - \varsigma_n) u_n + \varsigma_n g(u_n - g(u_n) + \mathcal{R}^{(\nu)} \left[ g(u_n) - \omega \left( S(\alpha_n) - T(\beta_n) \right) \right]) + \varsigma_n \eta_n,
\]

where \( a_n \in A(u_n), b_n \in B(u_n), a'_n \in A(s_n), b'_n \in B(s_n), a''_n \in A(t_n), \) and \( b''_n \in B(t_n) \) can be chosen arbitrarily. Let \( \{ u_n \}, \)
\( \{ \phi_n \}, \) and \( \{ \eta_n \} \) be the three sequences in \( \mathcal{H} \) to take into account the possible inexact computation and the sequences \( \gamma_n, \sigma_n, \) and \( \varsigma_n \) satisfy the following conditions:

\[
0 \leq \gamma_n, \sigma_n, \varsigma_n \leq 1, \forall n \geq 0, \text{ and } \Sigma_{n=0}^{\infty} \gamma_n \text{ diverges. If } \lim_{n \to \infty} \| u_n \| = 0, \lim_{n \to \infty} \| \phi_n \| = 0, \text{ and } \lim_{n \to \infty} \| \eta_n \| = 0, \text{ then}
\]

(i) The sequences \( \{ x_n \}, \{ a_n \}, \) and \( \{ b_n \} \) generated by the suggested Algorithm 1 converge strongly to \( x^*, a^*, \)

If we take \( C, D, E, \eta \equiv 0 \) and \( \psi(.x) = \psi(.) \) in SEGMNVI (23), we obtain the following corollary to study stability and convergence analysis of problem (25).

**Corollary 2.** Let \( S, T, g : \mathcal{H} \to \mathcal{H} \) be the single-valued mappings such that \( S \) is \( \lambda_S \)-Lipschitz continuous, \( T \) is \( \lambda_T \)-Lipschitz continuous and \( \mu_T \)-strongly monotone, \( g \) is \( \lambda_g \)-Lipschitz continuous and \( \mu_g \)-strongly monotone, and \( g(\mathcal{H}) = \mathcal{H}. \) Let \( A, B : \mathcal{H} \to CB(\mathcal{H}) \) be the \( \mathcal{D} \)-Lipschitz continuous mappings with constants \( \delta_A \) and \( \delta_B, \) respectively, \( A \) is \( \mu_A \)-strongly monotone with respect to \( S, \) and \( B \) is \( \mu_B \)-strongly monotone with respect to \( T, \) respectively. Let \( \psi : \mathcal{H} \to \mathbb{R} \cup \{ + \infty \} \) be a proper convex lower-semi-continuous functional on \( \mathcal{H} \) satisfying \( g(\mathcal{H}) \cap \text{dom}(\partial \psi) \neq \emptyset, \) where \( \partial \psi \) is a subdifferential of \( \psi. \) Suppose that there exists a constant \( \omega > 0 \) and the following conditions are satisfied:

\[
\omega^2 (\lambda_t - \lambda_g)^2 - (\lambda_t^2 \lambda_g^2 - \lambda_t^2 \lambda_g^2) t_1 (2 - t_1) < 0,
\]

\[
\omega^2 (\lambda_t - \lambda_g)^2 < 1 - \omega \lambda_t \lambda_g,
\]

\[
t_1 = 2 \sqrt{1 - 2 \omega \lambda_t + \omega^2},
\]

\[t_1 < 1.
\]
and $b^*$, respectively, and $(x^*, a^*, b^*)$ is the solution of SEGNMVI (23)

(ii) Furthermore, if $0 < \epsilon < \gamma_0$, then

$$\lim_{n \to \infty} \epsilon_n = x^*$$

if and only if $\lim_{n \to \infty} \Theta_n = 0$, where $\Theta_n$ is given in (80); that is, the sequences $\{x_n\}$, $\{a_n\}$, and $\{b_n\}$ generated by (68) are $\Delta^\alpha_{w^*}$-stable.

5. Generalized Resolvent Dynamical System

In this section, we consider the dynamical system technique to study the existence of unique solution of SEGNMVI (23). Using Lemma 3, we suggest and analyze the following generalized resolvent dynamical system associated with SEGNMVI (23):

$$\frac{dx}{dt} = \omega \left( \Delta^\alpha_{w}(x) \left[ \{S(a) - T(b) - \omega \eta(c, d, e) \} - g(x) \right] \right), \quad x(t_0) = x_0 \in \mathcal{H},$$

where $x \in \mathcal{H}$, $a \in A(x)$, $b \in B(x)$, $c \in C(x)$, $d \in D(x)$, $e \in E(x)$, and $\omega > 0$ is a parameter. System (81) is called generalized resolvent dynamical system associated with the set-valued extended generalized nonlinear mixed variational inequality problem (23).

**Definition 8.** (see [38]) The dynamical system is said to converge to the solution set $\Omega^*$ of problem (23), if, irrespective of the initial point, the trajectory of the dynamical system satisfies

$$\lim_{t \to \infty} \text{dist}(x(t), \Omega^*) = 0,$$

(82)

where $\text{dist}(x(t), \Omega^*) = \inf_{y \in \Omega^*} \|x - y\|$.

If the solution set $\Omega^*$ has a unique solution $x^* \in \mathcal{H}$ with $a^* \in A(x^*)$, $b^* \in B(x^*)$, $c^* \in C(x^*)$, $d^* \in D(x^*)$, and $e^* \in E(x^*)$, then (81) implies that $\lim_{t \to \infty} x(t) = x^*$.

**Definition 9** (see [38]). The dynamical system is said to be globally exponentially stable with degree $\theta$ at $x^*$, if, irrespective of the initial point, the trajectory of the dynamical system satisfies

$$\|x(t) - x^*\| \leq c_0 \|x(t) - x^*\| \exp(-\theta(t - t_0)), \quad \forall t \geq t_0,$$

(83)

where $c_0$ and $\theta$ are positive constants independent of the initial point.

**Lemma 4** (see [38]). Let $\tilde{x}$ and $\tilde{y}$ be real-valued nonnegative continuous functions with domain $\{t; t \geq t_0\}$ and let

$$F(x) = \omega \left( \Delta^\alpha_{w}(x) \left[ \{S(a) - T(b) - \omega \eta(c, d, e) \} - g(x) \right] \right), \quad \forall x \in \mathcal{H}.$$

Then, for all $x, x' \in \mathcal{H}$, such that $a \in A(x), b \in B(x), c \in C(x), d \in D(x), e \in E(x)$,

$$a' \in A(x'), b' \in B(x'), c' \in C(x'), d' \in D(x'), e \in E(x),$$

and using the arguments as for (39), we have
\[ \|F(x) - F(x')\| \leq \omega \sqrt{1 - 2\mu_g + \lambda_g^2} \|x - x'\|, \]  
\[ \|x - x' - g(x) - g(x')\| \leq \sqrt{1 - 2\mu_g + \lambda_g^2} \|x - x'\|, \]  
\[ \|x - x' - T(b) - T(b')\| \leq \sqrt{1 - 2\mu_B + \lambda_B^2\delta_B^2} \|x - x'\|, \]  
\[ \|g(x) - g(x') - S(a) - S(a')\| \leq \sqrt{1 - 2\omega(r_1 + r_2 + r_3) + \omega^2(\lambda_{\eta_1}\delta_C + \lambda_{\eta_2}\delta_D + \lambda_{\eta_3}\delta_E)^2} \|x - x'\|. \]

Using the same arguments as for (41), (43), (44), and (47), we have

\[ \|F(x) - F(x')\| \leq \omega (1 + \Theta(\xi)) \|x - x'\|, \]  

where

\[ \Theta(\xi) = \sqrt{1 - 2\omega(r_1 + r_2 + r_3) + \omega^2(\lambda_{\eta_1}\delta_C + \lambda_{\eta_2}\delta_D + \lambda_{\eta_3}\delta_E)^2} + \sqrt{1 - 2\mu_B + \lambda_B^2\delta_B^2} + \sqrt{1 - 2\mu_g + \lambda_g^2} + \lambda_g^2\delta_g^2 + \xi. \]

Hence, mapping \( F \) is locally Lipschitz continuous in \( \mathcal{H} \). Therefore, for each \( x_0 \in \mathcal{H} \), there exists a unique and continuous solution \( x(t) \) such that \( a(t) \in A(x(t)), \ b(t) \in B(x(t)), \ c(t) \in C(x(t)), \ d(t) \in D(x(t)), \) and \( e(t) \in E(x(t)) \) of generalized resolvent dynamical system (81) defined in the interval \( t_0 \leq t < T \) with the initial condition \( x(t_0) = x_0 \). Let \( [t_0, T) \) be its maximal interval of existence. Now, we show that \( T = \infty \). For any \( x \) such that \( a \in A(x), \ b \in B(x), \ c \in C(x), \ d \in D(x), \) and \( e \in E(x) \), we have
\[ \|F(x)\| \leq \omega \left\| R^{s(x)}_w [g(x) - [S(a) - (T(b) - \omega \eta(c, d, e))] - g(x)] \right\| \\
\leq \omega \left\| R^{s(x)}_w [g(x) - [S(a) - (T(b) - \omega \eta(c, d, e))] - g(x^*)] \right\| \\
+ \omega \|g(x) - g(x^*)\| \\
\leq \omega \left\| R^{s(x)}_w [g(x) - [S(a) - (T(b) - \omega \eta(c, d, e))] - g(x^*)] \right\| \\
+ \|x - x^*\| + \|g(x) - g(x^*)\| \leq \omega \|g(x) - [S(a) - (T(b) - \omega \eta(c, d, e))]\| \\
- [g(x) - [S(a) - (T(b') - \omega \eta(c', d', e'))]] \\
+ (1 + \epsilon)\|x - x^*\| + \|g(x) - g(x^*)\| \leq \omega (1 + \Theta(\epsilon))\|x - x^*\| \\
\leq \omega (1 + \Theta(\epsilon))\|x\| + \omega (1 + \Theta(\epsilon))\|x^*\|. \]

and then
\[
\|x(t)\| \leq \|x_0\| + \int_{t_0}^{t} \|F(x(s))\| ds \\
\leq \left(\|x_0\| + k_1(t - t_0)\right) + k_2 \int_{t_0}^{t} \|x(s)\| ds, \tag{96}
\]
where \( k_1 = \omega (1 + \Theta(\epsilon))\|x^*\| \) and \( k_2 = \omega (1 + \Theta(\epsilon)) \). Using Lemma 4, we have
\[
\|x(t)\| \leq \left(\|x_0\| + k_1(t - t_0)\right)e^{k_1(t - t_0)}, \quad \forall t \in [t_0, T]. \tag{97}
\]

Hence, the solution is bounded for \( t \in [t_0, T] \), if \( T \) is finite, and thus, \( T = \infty \).

By using the technique of Xia and Wang [36, 37], we show that the trajectory of solution of generalized resolvent dynamical system (81) converges to a unique solution of the set-valued extended generalized nonlinear mixed variational inequality problem (23).

**Theorem 4.** Suppose that Theorem 1 holds. Additionally, if \( 1 - t_1 < \mu - \xi < (1 - t_2)(1 + t_2) \), where \( t_1 \) and \( t_2 \) are the same as in (36), then the generalized resolvent dynamical system (81) converges globally uniquely to the unique solution of the set-valued extended generalized nonlinear mixed variational inequality problem (23).

**Proof.** By Theorem 1, there exists a unique solution \( x^* \in \mathcal{H} \) such that \( a^* \in A(x^*), b^* \in B(x^*) \), \( c^* \in C(x^*), d^* \in D(x^*) \), and \( e^* \in E(x^*) \) for problem (23). By applying Lemma 3, we have \( g(x^*) = R^{s(x^*)}_w [g(x^*) - [S(a^*) - (T(b^*) - \omega \eta(c^*, d^*, e^*))]] \). By Theorem 4, there exists a unique solution of system (81), that is, \( x(t) \), such that \( a(t) \in A(x(t)), b(t) \in B(x(t)), c(t) \in C(x(t)), d(t) \in D(x(t)), \) and \( e(t) \in E(x(t)) \) for any fixed \( x_0 \in \mathcal{H} \). Let \( x(t) = x(t, t_0; x_0) \) be the solution of (81) with \( x(t_0) = x_0 \). Now, we consider the Lyapunov function \( L \) defined on \( \mathcal{H} \) by
\[
L(x) = \|x - x^*\|^2, \quad \forall x \in \mathcal{H}. \tag{98}
\]

From (81), (90)–(92), and (98) and using \( \mu_g \)-strong monotonicity of \( g \), we have
\[
\frac{dL}{dt} = \frac{dL}{d\frac{dx}{dt}} = 2\langle x(t) - x^*, \frac{dx}{dt} \rangle \\
= 2\omega \langle x(t) - x^*, R^{s(x)}_w [g(x) - [S(a) - (Tb - \omega \eta(c, d, e))] - g(x)] \rangle \\
= -2\omega \langle x(t) - x^*, g(x) - g(x^*) \rangle \\
+ 2\omega \langle x(t) - x^*, R^{s(x)}_w [g(x) - [S(a) - (T(b) - \omega \eta(c, d, e))] - g(x^*)] \rangle \\
\leq -2\omega \mu_g \|x(t) - x^*\|^2 + 2\omega \|x(t) - x^*\| \left\| R^{s(x)}_w [g(x) - [S(a) - (T(b) - \omega \eta(c, d, e))] - g(x^*)] \right\| \\
- \omega \eta(c, d, e) - R^{s(x)}_w [g(x^*) - [S(a^*) - (T(b^*) - \omega \eta(c^*, d^*, e^*))]] \\
\leq -2\omega \mu_g - \xi - \sqrt{(1 - 2\mu_g + \lambda^2 B^2)} + \sqrt{(\lambda^2 A^2 - 2\mu_A + \lambda^2 A^2)} \\
+ \sqrt{1 - 2\omega (r_1 + r_2 + r_3) + \omega^2 (\lambda c + \lambda d + \lambda \delta_3)^2} \|x(t) - x^*\|^2, \tag{99}
\]
that is,
\[
\frac{d}{dt}\|x - x^*\|^2 \leq -2\omega(\mu_g - \Phi)\|x(t) - x^*\|^2, \quad (100)
\]
where
\[
\Phi = \sqrt{1 - 2\mu_g + \lambda_1^2\delta_b^2} + \sqrt{\lambda_2^2 - 2\mu_A + \lambda_3^2\delta_a^2} + \lambda_4\delta_c + \lambda_5\delta_d + \lambda_6\delta_e^2).
\]
(101)

Therefore, we have
\[
\|x - x^*\|^2 \leq \|x_0 - x^*\|e^{-\omega(\mu_g - \Phi)(t - t_0)}, \quad (102)
\]

The condition in (35) and the fact that 1 - t_1 < \mu_g - \xi < (1 - t_2)(1 + t_2) guarantee that \mu_g - \Phi > 0. Hence, the trajectory of the solution of generalized resolvent dynamical system (81) converges globally exponentially to the unique solution of the set-valued extended generalized nonlinear mixed variational inequality problem (23). □

6. Numerical Example

In support of Theorem 2, we provide a numerical example with convergence graphs and computation tables. We compare our three-step iterative Algorithm 1 with Ishikawa-type and Mann-type algorithms. The convergence graphs and the computation tables are provided for the sequences generated by Algorithm 1.

Example 1. Let \( R = \mathbb{R} \) with the usual inner product and norm. Let \( S, T, g: \mathbb{R} \rightarrow \mathbb{R} \), \( \eta: \mathbb{R} \times \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R} \), and \( A, B, C, D, E: \mathbb{R} \rightarrow \mathbb{R} \) be the mappings defined by
\[
S(x) = \frac{x}{2} + 1, \quad T(x) = 255x - 254, \quad g(x) = \frac{x}{5}, \quad \eta(x, y, z) = x + y + z, \quad A(x) = \left\{ \frac{x}{2} \right\},
\]
(103)
\[
B(x) = \left\{ 1 - \frac{x}{3} \right\}, \quad C(x) = \left\{ \frac{x + 1}{2} \right\}, \quad D(x) = \left\{ \frac{1}{x + 1} \right\}, \quad E(x) = \left\{ \frac{1}{x} \right\}, \quad \forall x \in \mathbb{R}.
\]

It is easy to see that \( S \) is 3/4-Lipschitz continuous mapping, \( T \) is 256-Lipschitz continuous and 254-strongly monotone mapping, \( g \) is 1/2-Lipschitz continuous and 1/4-strongly monotone mapping, \( \eta \) is Lipschitz continuous in first, second, and third arguments with respect to \( C, D, \) and \( E \) with constants (3/4), (1/5), (1/10), respectively, \( \eta \) is relaxed Lipschitz continuous in first, second, and third arguments with respect to \( C, D, \) and \( E \) with constants (3/5), (1/6), (1/10), respectively, and \( A, B, C, D, \) and \( E \) are \( D \)-Lipschitz continuous mappings with constants (3/4), (1/2), (3/4), (1/6), and (1/10), respectively.

Let \( \psi: \mathbb{R} \rightarrow \mathbb{R} \) be a functional satisfying \( g(R) \cap \text{dom}(\partial \psi(\cdot, x)) \neq \emptyset \), let \( \partial \psi: \mathbb{R} \rightarrow 2^\mathbb{R} \) be the subdifferential of \( \psi \), and \( J_\omega^{\partial \psi(x)}(x) \) is the resolvent operator associated with the subdifferential \( \partial \psi \) such that
\[
\psi(g(x), x) = 25g(x)^2 + x^2, \quad \forall x \in \mathbb{R}. \quad (104)
\]

Then,
\[
J_\omega^{\partial \psi(x)}(x) = \left\{ \frac{x}{1 + 4\omega} \right\}, \quad \forall x \in \mathbb{R}. \quad (105)
\]

For \( \omega = 1 \), mapping \( G \) defined in (37) has a fixed point \( x^* = 0 \in \mathcal{H} \) such that \( 0 \in A(0), 1 \in B(0), -1 \in C(0), -1 \in D(0), \) and \( 0 \in E(0) \) which is a solution of SEGNMVI (23). We verify the conditions of Algorithm 1.

Let \( y_n = (1/n), \sigma_n = (n + 1/2n^2 + n), \varphi_n = (n + 2 + 5n^2 + 2n), l_n = (n^2 + 1/3n^2 + n^2), p_n = (1/n + 1), \) and \( q_n = (1/n^2) \). It is easy to check that the sequences \{\( y_n \), \( \sigma_n \), \( \varphi_n \), \( l_n \), \( p_n \), \( q_n \)\} satisfy the conditions \( 0 \leq \psi_n \sigma_n \varphi_n \leq 1 \), and \( \Sigma_{n=0}^{\infty} y_n = \infty \) given in Algorithm 1. For \( \omega = 1 \), we obtain the sequences \{\( x_n \), \( y_n \), \( z_n \)\} generated by Algorithm 1 as
\[
z_n = \left( \frac{5n^2 - 2}{5n^2 + 2} \right) x_n + \frac{2}{(5n^2 + 2)} \left( \frac{6}{5} x_n - \frac{1685}{100} x_n \right)
\]
\[
+ \frac{2(n + 1)}{n^2(5n + 2)}
\]
\[
y_n = \left( \frac{2n^2 - 1}{2n^2 + n} \right) x_n + \frac{n + 1}{2n^2 + n} \left( \frac{6}{5} y_n - \frac{1685}{100} y_n \right)
\]
\[
+ \left( \frac{n + 1}{2n^2 + n} \right) \left( \frac{1}{n^2 + 1} \right)
\]
\[
x_{n+1} = \left( \frac{n - 1}{n} \right) x_n + \frac{1}{n} \left( \frac{6}{5} y_n - \frac{1685}{100} y_n \right) + \frac{n^2 + 1}{3n(5n + 1)}
\]
(106)

All codes are written in MATLAB version R2019a, for different choices of initial values \( x_0 = 3, 5, -3, \) and \( -5 \), which ensures that the sequence \{\( x_n \)\} converges to \( x^* = 0 \).

It is shown in Figure 1 that the sequence \{\( x_n \)\} converges to 0. In Table 1, comparing different initial values \{\( x_n \)\} and for various number of iterations, it is obtained that the sequence \{\( x_n \)\} converges to 0.
Remark 2. Consider the mappings defined in Example 1; we compare Algorithm 1 with the Ishikawa-type algorithm and Mann-type algorithm. By taking $\sigma_n = 0, \forall n \geq 0$, Algorithm 1 becomes Ishikawa-type Algorithm; that is, we can compute the sequences $x_n$ and $y_n$ by the following Ishikawa-type algorithm:

$$y_n = \left(\frac{2n^2 - 1}{2n^2 + n}\right)x_n - \frac{1565}{100} \left(\frac{n + 1}{2n^2 + n}\right)y_n$$

$$+ \left(\frac{n + 1}{2n^2 + n}\right) \left(\frac{1}{n^2 + 1}\right)$$

$$x_{n+1} = \left(\frac{n - 1}{n}\right)x_n + \frac{1}{n} \left(\frac{6}{5}y_n - \frac{1685}{100}y_n\right) + \frac{n^2 + 1}{3n^2(3n + 1)}$$

(107)

By taking $\sigma_n = 0, \forall n \geq 0$, Algorithm 1 becomes Mann-type algorithm; we can compute the sequence $\{x_n\}$ by the following Mann-type algorithm:

$$x_{n+1} = \left(\frac{n - 1}{n}\right)x_n - \left(\frac{1565}{100n}\right)x_n + \frac{n^2 + 1}{3n^2(3n + 1)}$$

(108)

The iterative schemes will suspend when the stopping criteria $\|x_{n+1} - x_n\| \leq 10^{-6}$ are fulfilled. The comparison of Algorithm 1 with Ishikawa-type algorithm and Mann-type algorithm is given in Figure 2 and Table 2, for initial value $x_0 = 5$.

We conclude that our algorithm is fast, efficient, and stable, and it takes average of 16–20 iterations to converge. Thus, algorithm 1 has a faster rate of convergence than Ishikawa-type algorithm and Mann-type algorithm.

Table 1: The values of $x_n$ with initial values $x_0 = 3, 5, -3, \text{ and } -5$.

<table>
<thead>
<tr>
<th>No. of iterations</th>
<th>For $x_0 = 3x_n$</th>
<th>For $x_0 = 5x_n$</th>
<th>For $x_0 = -3x_n$</th>
<th>For $x_0 = -5x_n$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$n = 1$</td>
<td>1.440314</td>
<td>2.885622</td>
<td>-1.440044</td>
<td>-2.400163</td>
</tr>
<tr>
<td>$n = 2$</td>
<td>0.691572</td>
<td>1.665419</td>
<td>-0.691171</td>
<td>-1.152086</td>
</tr>
<tr>
<td>$n = 3$</td>
<td>0.332131</td>
<td>0.961242</td>
<td>-0.331668</td>
<td>-1.152086</td>
</tr>
<tr>
<td>$n = 4$</td>
<td>0.159578</td>
<td>0.554863</td>
<td>-0.159085</td>
<td>-0.265306</td>
</tr>
<tr>
<td>$n = 5$</td>
<td>0.076742</td>
<td>0.320343</td>
<td>-0.076235</td>
<td>-0.127227</td>
</tr>
<tr>
<td>$n = 6$</td>
<td>0.036976</td>
<td>0.185001</td>
<td>-0.036462</td>
<td>-0.060941</td>
</tr>
<tr>
<td>$n = 7$</td>
<td>0.017885</td>
<td>0.106896</td>
<td>-0.017368</td>
<td>-0.029120</td>
</tr>
<tr>
<td>$n = 8$</td>
<td>0.008721</td>
<td>0.061822</td>
<td>-0.008202</td>
<td>-0.013844</td>
</tr>
<tr>
<td>$n = 9$</td>
<td>0.004322</td>
<td>0.035810</td>
<td>-0.003802</td>
<td>-0.006510</td>
</tr>
<tr>
<td>$n = 10$</td>
<td>0.002210</td>
<td>0.020798</td>
<td>-0.00169</td>
<td>-0.006510</td>
</tr>
<tr>
<td>$n = 11$</td>
<td>0.001986</td>
<td>0.012135</td>
<td>-0.001690</td>
<td>-0.001300</td>
</tr>
<tr>
<td>$n = 12$</td>
<td>0.000709</td>
<td>0.004250</td>
<td>-0.000489</td>
<td>-0.000489</td>
</tr>
<tr>
<td>$n = 13$</td>
<td>0.000026</td>
<td>0.001624</td>
<td>-0.000019</td>
<td>-0.000189</td>
</tr>
<tr>
<td>$n = 14$</td>
<td>0</td>
<td>0.001070</td>
<td>-0.000044</td>
<td>-0.000025</td>
</tr>
<tr>
<td>$n = 15$</td>
<td>0</td>
<td>0.000313</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>$n = 20$</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
</tbody>
</table>

Figure 1: The convergence of $x_n$ with initial values $x_0 = 3, 5, -3, \text{ and } -5$. 

Table 1: The values of $x_n$ with initial values $x_0 = 3, 5, -3, \text{ and } -5$. 

![Graph showing the convergence of $x_n$ with initial values $x_0 = 3, 5, -3, \text{ and } -5$.](image-url)
7. Conclusion

The purpose of this paper is to introduce and study a set-valued extended generalized nonlinear mixed variational inequality problem and a generalized resolvent dynamical system. A three-step iterative algorithm is defined to obtain the solution of set-valued extended generalized nonlinear mixed variational inequality problem and it is shown through a numerical example that the rate of convergence of our three-step algorithm is faster than Ishikawa-type algorithm and Mann-type algorithm. It is shown that the trajectory of the solution of generalized resolvent dynamical system converges globally exponentially to a unique solution of set-valued extended generalized nonlinear mixed variational inequality problem. We remark that there is still a wide scope of studying globally asymptotic stability of generalized resolvent dynamical system and one can explore many applications of our results in pure and applied sciences.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

Authors’ Contributions

All authors contributed equally and significantly in writing this article and have read and approved the final manuscript.

Acknowledgments

The research of Ching-Feng Wen was funded by Ministry of Science and Technology, Taiwan, under Grant no. 109-2115-M-037-001.

References


