Research Article
On a Sum Involving the Sum-of-Divisors Function

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Let \( \sigma(n) \) be the sum of all divisors of \( n \) and let \([t]\) be the integral part of \( t \). In this paper, we shall prove that

\[
\sum_{n \leq x} \sigma \left( [x/n] \right) \leq \pi^2/6 x \log x + O(x (\log x)^{2/3} (\log_2 x)^{4/3})
\]

for \( x \to \infty \), and that the error term of this asymptotic formula is \( \Omega(x) \).

1. Introduction

As usual, denote by \( \varphi(n) \) the Euler function and by \([t]\) the integral part of real \( t \), respectively. Recently, Bordellès et al. [1] studied the asymptotic behaviour of the quantity

\[
S_\varphi(x) = \sum_{n \leq x} \varphi \left( \left\lfloor \frac{x}{n} \right\rfloor \right),
\]

for \( x \to \infty \). By exponential sum technique, they proved

\[
\left( \frac{2629}{4009} \cdot \frac{6}{\pi^2} + o(1) \right) x \log x \leq S_\varphi(x)
\]

and conjectured that

\[
S_\varphi(x) \sim \frac{6}{\pi^2} x \log x, \quad \text{as } x \to \infty.
\]

Very recently, Wu [2] improved (2) and Zhai [3] resolved conjecture (3) by showing

\[
S_\varphi(x) = \frac{6}{\pi^2} x \log x + O(x (\log x)^{2/3} (\log_2 x)^{4/3}),
\]

and also proved that the error term in (4) is \( \Omega(x) \), where \( \log_2 \) denotes the iterated logarithm. Some related works can be found in [4, 5]. Since the sum-of-divisors function \( \sigma(n) = \sum_{d \mid n} d \) has similar properties as the Euler function \( \varphi(n) \) in many cases, it seems natural and interesting to consider its analogy of (3).

Our result is as follows.

Theorem 1

(i) For \( x \to \infty \), we have

\[
S_\sigma(x) = \sum_{n \leq x} \sigma \left( \left\lfloor \frac{x}{n} \right\rfloor \right) = \frac{\pi^2}{6} x \log x
\]

\[
+ O(x (\log x)^{2/3} (\log_2 x)^{4/3}).
\]

(ii) Let \( E(x) \) be the error term in (5). Then, for \( x \to \infty \), we have

\[
E(x) = \Omega(x), \quad \text{i.e. } \limsup_{x \to \infty} \frac{|E(x)|}{x} > 0.
\]

Let \( \mu(n) \) be the Möbius function and define \( \text{id}(n) = n \) and \( 1(n) = 1 \) for all integers \( n \geq 1 \). Then, \( \varphi = \text{id} \ast \mu \) and \( \sigma = \text{id} \ast 1 \). In Zhai’s approach proving (4), the inequality
\[ \sum_{n \leq x} \mu(n) \ll x \exp\left(-c \sqrt{\log x}\right), \quad (x \geq 1), \]  

plays a key role, where \( c > 0 \) is a positive constant. Clearly, such a bound is not true for 1. By refining Zhai’s approach, we shall prove our result.

## 2. Preliminary Lemmas

As in [3], we need some bounds on exponential sums of the type \( \sum_{n \leq N} e(T/n) \) where \( N < N' \leq 2N \). For large values of \( N \), Zhai used the theory of exponent pair, and for smaller ones the Vinogradov method. Both estimates are contained in the following general theorem of Karatsuba [6, Theorem 1], which will be a key tool for proving Theorem 1.

**Lemma 1.** Let \( k \geq 2 \) and \( M \) and \( P \) be integers, \( P \) being positive. Let \( f \in \mathbb{G}^{k+1}([M, M + P]; \mathbb{R}) \). Suppose that there exist positive absolute constants \( c_0, c_1, c_2, c_3, c_4 \), and \( c_5 \) such that \( c_5 < 1, c_1 < 1, \) and \( c_2 + c_3 < c_4 \) an integer \( r \) such that \( c_4 k \leq r \leq k; \) and distinct numbers \( s_j \geq 2 \) \( (j = 1, \ldots, r) \) not exceeding \( k \), such that for \( M \leq t \leq M + P \) the following inequalities are satisfied:

- (i) \( |f^{(k+1)}(t)/(k+1)!| \leq P^{-c_4(k+1)} \).
- (ii) \( P^{-c_3 s_j} \leq |f^{(s_j)}(t)/s_j!| \leq P^{-c_4 s_j}, \) \( (j = 1, \ldots, r) \).

Then, for each positive integer \( P_1 \) not exceeding \( P \), we have

\[ \left| e(f(m)) \right| \leq AP^{1-(c/k^2)}, \]

where \( e(t) := e^{2\pi it} \) and \( A > 0, c > 0 \) are absolute constants.

The next two lemmas are essentially a special case of [7, Lemmas 2.5 and 2.6] with \( a = 1 \). The only difference is that the ranges of \( T \) and \( N \) here are slightly larger than those of [7, Lemmas 2.5 and 2.6] \( (T \geq N^2 \) in place of \( T \geq N^{(3/2)} \) and \( N \leq x^{(3/2)} \) in place of \( N \leq x^{(1/2)} \), respectively). Although the proof is completely similar, for the convenience of readers, we still reproduce a proof here.

**Lemma 2.** Let \( e^{100} \leq N < N' \leq 2N \) and \( T \geq N^{(3/2)} \). Then, there exists an absolute positive constant \( c_t \) such that

\[ \sum_{T \leq n < N'} e\left(\frac{T}{n}\right) \ll N \exp\left(-c_t \frac{\log^3 N}{\log^2 T}\right), \]

where the implied constant is absolute.

**Proof.** We apply Lemma 1 to \( f(t) = (T/t) \) with \( M = N, P = N, P_1 = N' - N \). For this, we choose and take the \( s_j \) to be all integers \( s \) such that

\[ \frac{\log(T/n)}{\log N} \leq s \leq 4 \frac{\log(T/n)}{\log N}. \]

Obviously the number \( r \) of \( s_j \) is between \( c_1 k \) and \( k \). Next we shall verify that \( f(t) \) satisfies the conditions (i) and (ii) of Lemma 1 with the parameters chosen above.

For \( N \leq t \leq 2N \), we have

\[ \left| \frac{f^{(k+1)}(t)}{(k+1)!} \right| = Tt^{-k-2} \leq T N^{-k-2} = N^{-\eta_1}, \]

where

\[ \eta_1 := k + 1 - \frac{\log(T/n)}{\log N} \geq k + 1 - \frac{1}{100} k \geq \frac{99}{100} (k + 1) = c_1 (k + 1). \]

Similarly for \( N \leq t \leq 2N \), we find the inequality

\[ |f^{(s_j)}(t)/s_j!| \leq N^{-\eta_2}, \]

where

\[ \eta_2 := k - \frac{\log(T/n)}{\log N} \geq k - \frac{3}{4}s_j \geq c_3 s_j. \]

For the lower bound of (ii), we have

\[ \frac{f^{(s_j)}(t)}{s_j!} = Tt^{-s_j} \geq T (2N)^{-s_j} = N^{-\eta_3}, \]

where

\[ \eta_3 := s_j - \frac{\log(T/n)}{\log N} \geq s_j - \frac{2}{100} (s_j + 1) \leq \frac{87}{100} s_j = c_2 s_j. \]

From Lemma 1, there exist two positive constants \( c \) and \( A \) such that
Let $\psi(t) = t - [t] - (1/2)$. Let $c_5$ be the constant defined by Lemma 2 and $c_6 = (8/9)^2 c_5$, $c^* = ((3/5)c_6)^{-(1/3)}$. Then, we have
\[
\sum_{N \leq n < N'} \frac{1}{n} \psi\left(\frac{x}{n}\right) \ll c_5 (\log N)^3/(\log x)^2 \cdot \frac{\psi\left(\frac{x}{n}\right)}{\psi\left(\frac{x}{n-1}\right)},
\]
uniformly for $x \geq 10$, $\exp\{c^* (\log x)^{(2/3)}\} \leq N \leq x^{(2/3)}$ and $N < N' \leq 2N$.

Proof. By invoking a classical result on $\psi(t)$ (see 8, page 39) we can write, for any $H \geq 1$,
\[
\sum_{N \leq n < N'} \psi\left(\frac{x}{n}\right) \ll N H^{-1} + \sum_{1 \leq h \leq H} h^{-1} \sum_{N \leq n < N'} e\left(\frac{hx}{n}\right).
\]

An application of Lemma 2 with $T = h x \geq x \geq N^{(3/2)}$ yields
\[
\sum_{N \leq n < N'} \psi\left(\frac{x}{n}\right) \ll N \left( H^{-1} + e^{-c_5 (\log N)^3/\log (Hx)} \log H \right).
\]

Taking $H = \exp\{\log N\}^3/(\log x)^2 \leq x^{(8/27)}$, we easily deduce that
\[
\sum_{N \leq n < N'} \psi\left(\frac{x}{n}\right) \ll N \left( e^{-c_5 (\log N)^3/\log (Hx)} + e^{-c_5 (\log N)^3/\log (\log x)^2} \right).
\]

The first term can be absorbed by the second, since $c_5$ can be chosen small enough to ensure that $c_5 < 1$ and since $\exp\{c^* (\log x)^{(2/3)}\} \leq N'$ implies $\log N^3/\log x^2 \geq c^*$. Hence,
\[
\sum_{N \leq n < N'} \psi\left(\frac{x}{n}\right) \ll N e^{-c_5 (\log N)^3/\log (\log x)^2} \frac{(\log N)^3}{\log x^2},
\]
and an Abel summation produces the required result. \(\square\)

Lemma 4. Let $2 \leq z_1 < z_2 \leq x$ and $F_x(t) = (1/t)\psi(x/t)$. Denote by $V_{F_x}[z_1, z_2]$ the total variation of $F_x$ on $[z_1, z_2]$. Then,
\[
V_{F_x}[z_1, z_2] \ll \frac{x}{z_1} + \frac{1}{z_1},
\]
where the implied constant is absolute.

Proof. If $z_1 = t_0 < t_1 < \cdots < t_n = z_2$ is a partition of the interval $[z_1, z_2]$, then
\[
\sum_{k=1}^{n} \left| F_x(t_k) - F_x(t_{k-1}) \right| \ll \sum_{k=1}^{n} \left| \frac{1}{t_k} \psi\left(\frac{x}{t_k}\right) - \frac{1}{t_{k-1}} \psi\left(\frac{x}{t_{k-1}}\right) \right|.
\]

Since $|\psi(t)| \leq 1$ for all $t$, we have
\[
\sum_{k=1}^{n} \left| \frac{1}{t_k} \psi\left(\frac{x}{t_k}\right) - \frac{1}{t_{k-1}} \psi\left(\frac{x}{t_{k-1}}\right) \right| \ll \sum_{k=1}^{n} \left| \frac{1}{z_k} \psi\left(\frac{x}{z_k}\right) - \frac{1}{z_{k-1}} \psi\left(\frac{x}{z_{k-1}}\right) \right|.
\]

Inserting these two bounds into (24), we obtain the required result. \(\square\)

3. Proof of Theorem 1

3.1. A Formula on the Mean Value of $\sigma(n)$

Lemma 5

(i) For $x \geq 2$ and $1 \leq z \leq x^{(1/3)}$, we have
\[
\sum_{n \leq x} \sigma(n) = \frac{n}{12} x^2 - x \left( \frac{z - [z]^2 + [z]}{2z} + O\left(\frac{x}{z}\right) \right) - \Delta(x, z),
\]
where
\[
\Delta(x, z) = \sum_{d \mid (x/z)} \frac{x}{d} \psi\left(\frac{x}{d}\right).
\]

(ii) For $x \rightarrow \infty$, we have
\[
\sum_{n \leq x} \sigma(n) = \frac{n^2}{12} x^2 + O(x \log x).
\]

Proof. Using $\sigma(n) = \sum_{d \mid n} m$, the hyperbole principle of Dirichlet allows us to write
\[
\sum_{n \leq x} \sigma(n) = \sum_{d \mid n} m = S_1 + S_2 - S_3,
\]
where
\[ S_1 := \sum_{d \leq (x,z)} \sum_{m \leq (x,z)} \frac{m}{m}, \]
\[ S_2 := \sum_{m \leq z} \sum_{d \leq (x,m)} \frac{m}{m}, \]
\[ S_3 := \sum_{d \leq (x,z)} \sum_{m \leq z} m. \quad (31) \]

Firstly we have
\[ S_2 = \sum_{m \leq z} \left[ \frac{m}{m} \right] x[z] + O(z^2), \quad (32) \]
\[ S_3 = \left[ \frac{x}{z} \right] \left[ \frac{x}{z} + 1 \right] \frac{1}{2} = \frac{x}{z} \left[ \frac{x}{z} + 1 \right] + O(z^2). \quad (33) \]

Secondly we can write
\[ S_1 = \frac{1}{2} \sum_{d \leq (x,z)} \frac{x^2}{d} - \psi \left( \frac{x}{d} \right) = \frac{1}{2} \sum_{d \leq (x,z)} \left[ \frac{x^2}{d^2} - 2 \frac{x}{d} \psi \left( \frac{x}{d} \right) + \psi \left( \frac{x}{d} \right)^2 - 1 \right] \]
\[ = \frac{\pi^2}{12} x^2 - \frac{x}{2} x z - \Delta(x, z) + O(x/z), \quad (34) \]
where \( \Delta(x, z) \) is as in (28). Inserting (32), (33), and (34) into (30) and using \( z^2 \leq (x/z) \), we get (27).

Taking \( z = 1 \) in (27) and noticing that
\[ \sum_{d \leq x} \frac{1}{d^2} = \frac{\pi^2}{6} + O(1/x), \]
\[ \sum_{d \leq x} \frac{x}{d} \psi \left( \frac{x}{d} \right) \ll x \log x, \quad (35) \]
we obtain the required bound. This completes the proof. \( \square \)

3.2. Estimates of Error Terms

**Lemma 6.** Let \( N_0 := \exp \left\{ (6/c_6) (\log x) \right\} \), \( x \geq 10 \) and \( 2 \leq z \leq \sqrt{N_0} \), we have
\[ \left| \sum_{N_0 < z < \sqrt{N_0}} \Delta \left( \frac{x}{n}, z \right) \right| \ll \sum_{N_0 < z < \sqrt{N_0}} \Delta \left( \frac{x}{n} - 1, z \right) \ll \left| \frac{1}{\log x} + \frac{\log x}{z} \right|. \quad (36) \]

**Proof.** Denote by \( \Delta_1(x, z) \) and \( \Delta_2(x, z) \) two sums on the left-hand side of (36), respectively. By (28) of Lemma 5, we can write
\[ \Delta_1(x, z) = x \sum_{N_0 < z < \sqrt{N_0}} \sum_{d \leq \min \left\{ N_0, (x, z) \right\}} \frac{1}{d} \psi \left( \frac{x}{d} \right) \]
\[ = x \sum_{d \leq (x, z)} \frac{1}{d^2} \sum_{N_0 < z < \min \left\{ \sqrt{N_0}, (x, z) \right\}} \frac{1}{n} \psi \left( \frac{x}{n} \right) \]
\[ = x \Delta_1^1(x, z) + x \Delta_1^2(x, z), \quad (37) \]
where
\[ \Delta_1^1(x, z) := \sum_{d \leq (x, z)} \frac{1}{d} \sum_{N_0 < z < \min \left\{ \sqrt{N_0}, (x, z) \right\}} \frac{1}{n} \psi \left( \frac{x}{n} \right), \]
\[ \Delta_1^2(x, z) := \sum_{d \leq (x, z)} \frac{1}{d} \sum_{N_0 < z < \min \left\{ \sqrt{N_0}, (x, z) \right\}} \frac{1}{n} \psi \left( \frac{x}{n} \right), \quad (38) \]

For \( 0 \leq k \leq \left( \log (x/d)^{(2/3)} / N_0 \right) \log 2 \), let \( N_k := 2^k N_0 \) and define
\[ \mathfrak{G}_k(d) := \sum_{N_k < z < 2N_k} \frac{1}{n} \psi \left( \frac{x}{n} \right). \quad (39) \]

Noticing that \( N_0 \leq N_k \leq (x/d)^{(2/3)} \), we can apply Lemma 3 to derive that
\[ \mathfrak{G}_k(d) \ll e^{-\theta \left( \frac{\log N_k}{\log (x/d)^{(2/3)}} \right)^2}, \quad (40) \]
with \( \theta(t) = c_6 t - \log t \). It is clear that \( \theta(t) \) is increasing on \( [c_6, \infty) \). On the other hand, for \( k \geq 0 \) and \( d \geq 1 \), we have
\[ (\log N_k)^{3/2} / (\log (x/d))^{3/2} \geq (\log N_0)^{3/2} / (\log x)^{3/2} = (6/c_6) \log_2 x. \quad (41) \]

Thus,
\[ \theta \left( \frac{(\log N_k)^{3/2}}{(\log (x/d))^{3/2}} \right) \geq \theta \left( \frac{6}{c_6} \log_2 x \right) \]
\[ = 6 \log_2 x - \log \left( \frac{6}{c_6} \log_2 x \right) \geq 5 \log_2 x, \quad (42) \]
which implies that \( \mathfrak{G}_k(d) \ll (\log x)^{-5} \). Inserting this into the expression of \( \Delta_1^1(x, z) \), we get
\[ \Delta_1(x, z) \ll \sum_{d \leq (x, z)} \frac{1}{d} \sum_{N_k < z < (x/d)^{(2/3)}} \mathfrak{G}_k(d) \ll (\log x)^{-3}. \quad (43) \]

Next we bound \( \Delta_1^2(x, z) \). Let \( F(t) \) be a function of bounded variation on \( [n, n+1] \) for each integer \( n \) and let \( V_F [n, n+1] \) be the total variation of \( F \) on \( [n, n+1] \). Integrating by parts, we have
\[
\int_{n}^{n+1} \left( t - n - \frac{1}{2} \right) dF(t) = \frac{1}{2} (F(n + 1) + F(n)) - \int_{n}^{n+1} F(t) \, dt. \tag{44}
\]

From this, we can derive that

\[
\frac{1}{2} (F(n + 1) + F(n)) = \int_{n}^{n+1} F(t) \, dt + O(V_F [n, n + 1]),
\tag{45}
\]

for \( n \geq 1 \). Summing over \( n \), we find that

\[
\sum_{N_1 < n \leq N_2} F(n) = \int_{N_1}^{N_2} F(t) \, dt + \frac{1}{2} (F(N_1) + F(N_2)) + O(V_F [N_1, N_2]).
\tag{46}
\]

We apply this formula to

\[
F_{(x/d)}(t) = \frac{1}{t} \psi \left( \frac{(x/d)}{t} \right),
\]

\[
N_1 = \left\lfloor \frac{(x/d)^{(2/3)}}{\sqrt{z}} \right\rfloor,
\]

\[
N_2 = \left\lfloor \min \left\{ \frac{x}{\sigma (dz)}, \frac{x}{(dz)^{(d/3)}} \right\} \right\rfloor.
\tag{47}
\]

According to Lemma 4, we have

\[
V_F [N_1, N_2] \ll (x/d)^{-(1/3)},
\]

and thus by putting \( u = (x/d)/t \), we obtain, with the notation \( x_{d,1} = \max (\sqrt{x/d}, tz) \) and \( x_{d,2} = (x/d)^{(1/3)} \),

\[
\sum_{(x/d)^{(2/3)} < n \leq \min \{ \sqrt{x/d}, (x/dz) \}} \frac{1}{n} \psi \left( \frac{x}{dn} \right) = \int_{x_{d,1}}^{x_{d,2}} \psi \left( \frac{(x/d)}{u} \right) \, du + O \left( \frac{x}{d} \right)^{-(1/3)}
\ll z^{-1} + \left( \frac{x}{d} \right)^{-(1/3)}
\ll z^{-1},
\tag{48}
\]

where we have used the fact that \( z \leq \sqrt{N_0} \) and \( d \leq (x/(N_0z)) \Rightarrow z \leq (x/d)^{(1/3)} \) and the bound

\[
\int_{x_{d,1}}^{x_{d,2}} \psi \left( \frac{(x/d)}{u} \right) \frac{du}{u} = \int_{x_{d,1}}^{x_{d,2}} \psi (t) \frac{dt}{t} - \frac{1}{x_{d,2}^{(2/3)}} \int_{x_{d,1}}^{x_{d,2}} \psi (t) \frac{dt}{t}
\ll x_{d,1}^{-1} + (x_{d,2})^{-(2/3)} \ll z^{-1} + (x/d)^{-(2/3)} \ll z^{-1}.
\tag{49}
\]

Using (48), a simple partial integration allows us to derive that

\[
\Delta_1^t (x, z) \ll z^{-1} \sum_{d \leq x} \{N_{1,z} (N_{1,z})^{-1} \ll z^{-1} \log x. \tag{50}
\]

Combining (43) and (50), it follows that

\[
|\Delta_1 (x, z)| \ll x (\log x)^{-3} + xz^{-1} \log x. \tag{51}
\]

Similarly, we can prove the same bound for \( |\Delta_2 (x, z)| \). This completes the proof. \( \Box \)

3.3. End of the Proof of Theorem 1. Let \( c_x \) be the constant given in Lemma 3 and \( N_0 := \exp \left( (6/c_x) (\log x)^{1/3} \right) \). Let \( z \in [2, \sqrt{N_0}] \) be a parameter to be chosen later.

Putting \( d = [x/n] \), we have \( (x/n) - 1 < d \leq (x/n) \) and \( x/(d + 1) < n \leq (x/d) \). We have, with the convention \( \sigma (0) = 0 \),

\[
S_\sigma (x) = \sum_{d \leq x} \frac{\sigma (d)}{(x/d)(\log d)} - \sum_{n \leq x} \sigma (d - 1),
\]

\[
= \sum_{d \leq x} \sigma (d) - \sum_{d \leq x, d \leq 2} \sigma (d - 1) \tag{52}
\]

By the hyperbole principle of Dirichlet, we can write

\[
S_\sigma (x) = S_1 (x, \sigma) + S_2 (x, \sigma) - S_3 (x, \sigma), \tag{53}
\]

where

\[
S_1 (x, \sigma) = \sum_{d \leq \sqrt{x}, d \leq x} \sigma (d) - \sigma (d - 1),
\]

\[
S_2 (x, \sigma) = \sum_{n \leq x} \sigma (d) - \sigma (d - 1), \tag{54}
\]

\[
S_3 (x, \sigma) = \sum_{d \leq \sqrt{x}, d \leq \sqrt{x}} \sigma (d) - \sigma (d - 1).
\]

With the help of the bound \( \sigma (n) \ll n \log n \), we can derive that

\[
S_3 (x, \sigma) = \left[ \sqrt{x} \right] \sigma \left( \left[ \sqrt{x} \right] \right) \ll x \log x. \tag{55}
\]

For evaluating \( S_1 (x, \sigma) \), we write

\[
S_1 (x, \sigma) = \sum_{d \leq \sqrt{x}} \sigma (d) - \sigma (d - 1) \left[ \frac{x}{d} \right]
\ll x \sum_{d \leq \sqrt{x}} \sigma (d) - \sigma (d - 1) \cdot \left[ \frac{x}{d} \right] + O \left( \sum_{d \leq \sqrt{x}} \sigma (d) - \sigma (d - 1) \right). \tag{56}
\]

With the help of Lemma 5 (ii), a simple partial integration gives us
\[ \sum_{d \leq \sqrt{x}} \frac{\sigma(d) - \sigma(d - 1)}{d} = \sum_{d \leq \sqrt{x}} \frac{\sigma(d)}{d} \]

\[ = \sum_{d \leq \sqrt{x}} \frac{\sigma(d)}{d^2} - \sum_{d \leq \sqrt{x}} \frac{\sigma(d)}{d} \]

\[ = \left( \sum_{d \leq \sqrt{x}} t^{-2} \left( \frac{12}{\pi} t^2 + O(t \log t) \right) \right) + O(1) \]

\[ = \frac{\pi^2}{12} \log x + O(1), \]

\[ \sum_{d \leq \sqrt{x}} |\sigma(d) - \sigma(d - 1)| \]

\[ \leq \sum_{d \leq \sqrt{x}} \sigma(d) \ll x. \]  \hspace{1cm} (57)

Inserting these estimates into (56), we find that

\[ S_1(x, \sigma) = \frac{\pi^2}{12} x \log x + O(x). \]  \hspace{1cm} (58)

Finally, we evaluate \( S_2(x, \sigma) \). For this, we write

\[ S_2(x, \sigma) = S'_2(x, \sigma) + S''_2(x, \sigma), \]  \hspace{1cm} (59)

where

\[ S'_2(x, \sigma) = \sum_{n \leq \sqrt{x}} \left( \frac{\pi^2}{12} \cdot \frac{x}{n} - \Delta \left( \frac{x}{n}, z \right) \right) + O \left( \frac{x}{n} \right) - \Delta_1(x, z) + \Delta_2(x, z), \]

\[ \sum_{d \leq \sqrt{x}} \sigma(d) - \sum_{d \leq \sqrt{x}} \sigma(d - 1) = S_1(p) - S_\sigma(p - 1) \]

\[ \geq \frac{\pi^2}{6} (\log p - \log (p - 1)) + E(p) - E(p - 1) \]

\[ \geq E(p) - E(p - 1) \geq -2E^* (p), \]  \hspace{1cm} (66)

where \( E^*(p) = \max\{|E(p)|, |E(p - 1)|\} \). On the other hand, we have

\[ \sum_{d \mid p} (\sigma(d) - \sigma(d - 1)) = \sigma(p) - \sigma(p - 1) + 1 \]

\[ \leq p + 1 - \left( p - 1 + \frac{1}{2} (p - 1) + 2 + 1 \right) + 1 \leq \frac{1}{4} p. \]  \hspace{1cm} (67)

Thus, \( E^*(p) \geq (1/8)p \) for all odd primes.

**Data Availability**

No data were used to support this study.
Conflicts of Interest

The authors declare that they have no conflicts of interest.

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