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Research Article

On a Sum Involving the Sum-of-Divisors Function

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Let $\sigma(n)$ be the sum of all divisors of n and let $[t]$ be the integral part of t. In this paper, we shall prove that $\sum_{n\le x}\sigma([x/n]) = (\pi^2/6)x\log x + O(x(\log x)^{(2/3)}(\log_2 x)^{(4/3)})$ for $x \longrightarrow \infty$, and that the error term of this asymptotic formula is $\Omega(x)$.

1. Introduction

As usual, denote by $\varphi(n)$ the Euler function and by $[t]$ the integral part of real t, respectively. Recently, Bordellès et al. [1] studied the asymptotic behaviour of the quantity

$$S_\varphi(x) := \sum_{n\le x}\varphi\left(\left[\frac{x}{n}\right]\right), \tag{1}$$

for $x \longrightarrow \infty$. By exponential sum technique, they proved that

$$\begin{aligned}\left(\frac{2629}{4009}\cdot\frac{6}{\pi^2}+o(1)\right)x\log x &\le S_\varphi(x)\\ &\le\left(\frac{2629}{4009}\cdot\frac{6}{\pi^2}+\frac{1380}{4009}+o(1)\right)x\log x,\end{aligned} \tag{2}$$

and conjectured that

$$S_\varphi(x)\sim\frac{6}{\pi^2}x\log x,\quad \text{as } x\longrightarrow\infty. \tag{3}$$

Very recently, Wu [2] improved (2) and Zhai [3] resolved conjecture (3) by showing

$$S_\varphi(x)=\frac{6}{\pi^2}x\log x+O\left(x(\log x)^{(2/3)}(\log_2 x)^{(1/3)}\right), \tag{4}$$

and also proved that the error term in (4) is $\Omega(x)$, where $\log_2$ denotes the iterated logarithm. Some related works can be found in [4, 5]. Since the sum-of-divisors function $\sigma(n) := \sum_{d|n} d$ has similar properties as the Euler function $\varphi(n)$ in many cases, it seems natural and interesting to consider its analogy of (3).

Our result is as follows.

Theorem 1

(i) For $x \longrightarrow \infty$, we have

$$\begin{aligned}S_\sigma(x) := \sum_{n\le x}\sigma\left(\left[\frac{x}{n}\right]\right) &= \frac{\pi^2}{6}x\log x\\ &+O\left(x(\log x)^{(2/3)}(\log_2 x)^{(4/3)}\right).\end{aligned} \tag{5}$$

(ii) Let $E(x)$ be the error term in (5). Then, for $x \longrightarrow \infty$, we have

$$E(x)=\Omega(x),\quad \text{i.e. } \limsup_{x\longrightarrow\infty}\frac{|E(x)|}{x}>0. \tag{6}$$

Let $\mu(n)$ be the Möbius function and define $\mathrm{id}(n)=n$ and $1(n)=1$ for all integers $n\ge 1$. Then, $\varphi=\mathrm{id}*\mu$ and $\sigma=\mathrm{id}*1$. In Zhai's approach proving (4), the inequality

$$\sum_{n\le x}\mu(n)\ll x\exp\left\{-c\sqrt{\log x}\right\},\quad (x\ge 1), \tag{7}$$

plays a key role, where $c>0$ is a positive constant. Clearly, such a bound is not true for 1. By refining Zhai's approach, we shall prove our result.

2. Preliminary Lemmas

As in [3], we need some bounds on exponential sums of the type $\sum_{N\le n<N'} e(T/n)$ where $N<N'\le 2N$. For large values of N, Zhai used the theory of exponent pair, and for smaller ones the Vinogradov method. Both estimates are contained in the following general theorem of Karatsuba [6, Theorem 1], which will be a key tool for proving Theorem 1.

Lemma 1. *Let $k\ge 2$ and M and P be integers, P being positive. Let $f\in\mathscr{C}^{k+1}([M,M+P];\mathbb{R})$. Suppose that there exist positive absolute constants c_0,c_1,c_2,c_3, and c_4 such that $c_0<1$, $c_1<1$, and $c_2+c_4<c_1$; an integer r such that $c_0k\le r\le k$; and distinct numbers $s_j\ge 2\,(j=1,\dots,r)$ not exceeding k, such that for $M\le t\le M+P$ the following inequalities are satisfied:*

(i) $|f^{(k+1)}(t)/(k+1)!|\le P^{-c_1(k+1)}$.

(ii) $P^{-c_2s_j}\le|f^{(s_j)}(t)/s_j!|\le P^{-c_3s_j}$, $(j=1,\dots,r)$.

Then, for each positive integer P_1 not exceeding P, we have

$$\left|\sum_{M\le m\le M+P_1-1} e(f(m))\right|\le AP^{1-(c/k^2)}, \tag{8}$$

where $e(t):=e^{2\pi it}$ and $A>0$, $c>0$ are absolute constants.

The next two lemmas are essentially a special case of [7, Lemmas 2.5 and 2.6] with $a=1$. The only difference is that the ranges of T and N here are slightly larger than those of [7, Lemmas 2.5 and 2.6] ($T\ge N^2$ in place of $T\ge N^{(3/2)}$ and $N\le x^{(2/3)}$ in place of $N\le x^{(1/2)}$, respectively). Although the proof is completely similar, for the convenience of readers, we still reproduce a proof here.

Lemma 2. *Let $e^{100}\le N<N'\le 2N$ and $T\ge N^{(3/2)}$. Then, there exists an absolute positive constant c_5 such that*

$$\sum_{N\le n<N'} e\left(\frac{T}{n}\right)\ll N\exp\left\{-\left(\frac{c_5\log^3 N}{\log^2 T}\right)\right\}, \tag{9}$$

where the implied constant is absolute.

Proof. We apply Lemma 1 to $f(t):=(T/t)$ with $M=N$, $P=N$, $P_1=N'-N$. For this, we choose

$$\begin{aligned} c_0&=\frac{1}{100},\\ c_1&=\frac{99}{100},\\ c_2&=\frac{87}{100},\\ c_3&=\frac{3}{4},\\ c_4&=\frac{1}{100},\\ k&=\left[100\frac{\log(T/n)}{\log N}\right], \end{aligned} \tag{10}$$

and take the s_j to be all integers s such that

$$4\frac{\log(T/n)}{\log N}\le s\le 5\frac{\log(T/n)}{\log N}. \tag{11}$$

Obviously the number r of s_j is between c_0k and k. Next we shall verify that $f(t)$ satisfies the conditions (i) and (ii) of Lemma 1 with the parameters chosen above.

For $N\le t\le 2N$, we have

$$\left|\frac{f^{(k+1)}(t)}{(k+1)!}\right|=Tt^{-k-2}\le TN^{-k-2}=N^{-\eta_1}, \tag{12}$$

where

$$\eta_1:=k+1-\frac{\log(T/n)}{\log N}\ge k+1-\frac{1}{100}k\ge\frac{99}{100}(k+1)=c_1(k+1). \tag{13}$$

Similarly for $N\le t\le 2N$, we find the inequality $|f^{(s_j)}(t)/s_j!|\le N^{-\eta_3}$, where

$$\eta_3:=s_j-\frac{\log(T/n)}{\log N}\ge\frac{3}{4}s_j=c_3s_j. \tag{14}$$

For the lower bound of (ii), we have

$$\left|\frac{f^{(s_j)}(t)}{s_j!}\right|=Tt^{-1-s_j}\ge T(2N)^{-1-s_j}=N^{-\eta_2}, \tag{15}$$

where

$$\begin{aligned}\eta_2&:=s_j-\frac{\log(T/n)}{\log N}+\frac{\log 2}{\log N}(s_j+1)\\ &\le\frac{4}{5}s_j+\frac{\log 2}{100}(s_j+1)\le\frac{87}{100}s_j=c_2s_j.\end{aligned} \tag{16}$$

From Lemma 1, there exist two positive constants c and A such that

$$\left|\sum_{N\le n<N'} e\left(\frac{T}{n}\right)\right| \le AN^{1-(c/k^2)} \le AN\ \exp\left\{-\frac{c_5\log^3 N}{\log^2(T/n)}\right\}, \tag{17}$$

with $c_5 := 10^{-4}c$. This completes the proof of Lemma 2. □

Lemma 3. *Define $\psi(t) := t-[t]-(1/2)$. Let c_5 be the constant defined by Lemma 2 and $c_6 := (8/9)^2 c_5$, $c^* := ((3/5)c_6)^{-(1/3)}$. Then, we have*

$$\sum_{N\le n<N'} \frac{1}{n}\psi\left(\frac{x}{n}\right) \ll e^{-c_6(\log N)^3/(\log x)^2}\frac{(\log N)^3}{(\log x)^2}, \tag{18}$$

uniformly for $x\ge 10$, $\exp\left\{c^*(\log x)^{(2/3)}\right\} \le N \le x^{(2/3)}$ and $N<N'\le 2N$.

Proof. By invoking a classical result on $\psi(t)$ (see 8, page 39]) we can write, for any $H\ge 1$,

$$\sum_{N\le n<N'} \psi\left(\frac{x}{n}\right) \ll NH^{-1} + \sum_{1\le h\le H} h^{-1}\left|\sum_{N\le n<N'} e\left(\frac{hx}{n}\right)\right|. \tag{19}$$

An application of Lemma 2 with $T = hx \ge x \ge N^{(3/2)}$ yields

$$\sum_{N\le n<N'} \psi\left(\frac{x}{n}\right) \ll N\left(H^{-1} + e^{-c_5(\log N)^3/\log^2(Hx)}\log H\right). \tag{20}$$

Taking $H = \exp\left\{(\log N)^3/(\log x)^2\right\} \le x^{(8/27)}$, we easily deduce that

$$\sum_{N\le n<N'} \psi\left(\frac{x}{n}\right) \ll N\left(e^{-(\log N)^3/(\log x)^2} + e^{-c_6(\log N)^3/(\log x)^2}\frac{(\log N)^3}{(\log x)^2}\right). \tag{21}$$

The first term can be absorbed by the second, since c_5 can be chosen small enough to ensure that $c_6<1$ and since $\exp\left\{c^*(\log x)^{(2/3)}\right\} \le N$ implies $(\log N)^3/(\log x)^2 \ge c^*$. Hence,

$$\sum_{N\le n<N'} \psi\left(\frac{x}{n}\right) \ll Ne^{-c_6(\log N)^3/(\log x)^2}\frac{(\log N)^3}{(\log x)^2}, \tag{22}$$

and an Abel summation produces the required result. □

Lemma 4. *Let $2\le z_1<z_2\le x$ and $F_x(t) := (1/t)\psi(x/t)$. Denote by $V_{F_x}[z_1,z_2]$ the total variation of F_x on $[z_1,z_2]$. Then,*

$$V_{F_x}[z_1,z_2] \ll \frac{x}{z_1^2} + \frac{1}{z_1}, \tag{23}$$

where the implied constant is absolute.

Proof. If $z_1 = t_0 < t_1 < \cdots < t_n = z_2$ is a partition of the interval $[z_1,z_2]$, then

$$\begin{aligned}\sum_{k=1}^{n}\left|F_x(t_k)-F_x(t_{k-1})\right| &= \sum_{k=1}^{n}\left|\frac{1}{t_k}\psi\left(\frac{x}{t_k}\right)-\frac{1}{t_{k-1}}\psi\left(\frac{x}{t_{k-1}}\right)\right| \\ &\le \sum_{k=1}^{n}\left|\left(\frac{1}{t_k}-\frac{1}{t_{k-1}}\right)\psi\left(\frac{x}{t_k}\right)\right| + \sum_{k=1}^{n}\frac{1}{t_{k-1}}\left|\psi\left(\frac{x}{t_k}\right)-\psi\left(\frac{x}{t_{k-1}}\right)\right|.\end{aligned} \tag{24}$$

Since $|\psi(t)|\le 1$ for all t, we have

$$\sum_{k=1}^{n}\left|\left(\frac{1}{t_k}-\frac{1}{t_{k-1}}\right)\psi\left(\frac{x}{t_k}\right)\right| \le \frac{1}{z_1}-\frac{1}{z_2} < \frac{1}{z_1}. \tag{25}$$

On the other hand, since $\psi(u)$ is of period 1, we have

$$\begin{aligned}&\sum_{k=1}^{n}\frac{1}{t_{k-1}}\left|\psi\left(\frac{x}{t_k}\right)-\psi\left(\frac{x}{t_{k-1}}\right)\right| \\ &\qquad \le \frac{1}{z_1}\left(\frac{x}{z_1}+1\right)V_\psi[0,1] \le 2\left(\frac{x}{z_1^2}+\frac{1}{z_1}\right).\end{aligned} \tag{26}$$

Inserting these two bounds into (24), we obtain the required result. □

3. Proof of Theorem 1

3.1. A Formula on the Mean Value of $\sigma(n)$

Lemma 5

(i) For $x\ge 2$ and $1\le z\le x^{(1/3)}$, we have

$$\sum_{n\le x}\sigma(n) = \frac{\pi^2}{12}x^2 - x\frac{(z-[z])^2+[z]}{2z} + O\left(\frac{x}{z}\right) - \Delta(x,z), \tag{27}$$

where

$$\Delta(x,z) := \sum_{d\le(x/z)}\frac{x}{d}\psi\left(\frac{x}{d}\right). \tag{28}$$

(ii) For $x\longrightarrow\infty$, we have

$$\sum_{n\le x}\sigma(n) = \frac{\pi^2}{12}x^2 + O(x\log x). \tag{29}$$

Proof. Using $\sigma(n) = \sum_{dm=n} m$, the hyperbole principle of Dirichlet allows us to write

$$\sum_{n\le x}\sigma(n) = \sum_{dm\le x} m = S_1+S_2-S_3, \tag{30}$$

where

$$S_1 := \sum_{d\le(x/z)}\sum_{m\le(x/d)} m,$$
$$S_2 := \sum_{m\le z}\sum_{d\le(x/m)} m, \tag{31}$$
$$S_3 := \sum_{d\le(x/z)}\sum_{m\le z} m.$$

Firstly we have

$$S_2 = \sum_{m\le z} m\left[\frac{x}{m}\right] = x[z] + O\left(z^2\right), \tag{32}$$

$$S_3 = \left[\frac{x}{z}\right]\frac{[z]([z]+1)}{2} = x\frac{[z]([z]+1)}{2z} + O\left(z^2\right). \tag{33}$$

Secondly we can write

$$\begin{aligned} S_1 &= \frac{1}{2}\sum_{d\le(x/z)}\left(\frac{x}{d}-\psi\left(\frac{x}{d}\right)-\frac{1}{2}\right)\left(\frac{x}{d}-\psi\left(\frac{x}{d}\right)+\frac{1}{2}\right) \\ &= \frac{1}{2}\sum_{d\le(x/z)}\left(\frac{x^2}{d^2}-2\frac{x}{d}\psi\left(\frac{x}{d}\right)+\psi\left(\frac{x}{d}\right)^2-\frac{1}{4}\right) \\ &= \frac{\pi^2}{12}x^2-\frac{1}{2}xz-\Delta(x,z)+O(x/z), \end{aligned} \tag{34}$$

where $\Delta(x,z)$ is as in (28). Inserting (32), (33), and (34) into (30) and using $z^2\le(x/z)$, we get (27).

Taking $z=1$ in (27) and noticing that

$$\sum_{d\le x}\frac{1}{d^2} = \frac{\pi^2}{6}+O\left(\frac{1}{x}\right),$$
$$\left|\sum_{d\le x}\frac{x}{d}\psi\left(\frac{x}{d}\right)\right| \ll x\log x, \tag{35}$$

we obtain the required bound. This completes the proof. □

3.2. Estimates of Error Terms

Lemma 6. *Let* $N_0 := \exp\left\{(6/c_6)(\log x)^{(2/3)}(\log_2 x)^{(1/3)}\right\}$, *where* c_6 *is given as in Lemma 3. Let* $\Delta(x,z)$ *be defined by (28). Then, for* $x\ge 10$ *and* $2\le z\le\sqrt{N_0}$, *we have*

$$\begin{aligned} &\left|\sum_{N_0<n\le\sqrt{x}}\Delta\left(\frac{x}{n},z\right)\right| + \left|\sum_{N_0<n\le\sqrt{x}}\Delta\left(\frac{x}{n}-1,z\right)\right| \\ &\ll x\left(\frac{1}{(\log x)^3}+\frac{\log x}{z}\right). \end{aligned} \tag{36}$$

Proof. Denote by $\Delta_1(x,z)$ and $\Delta_2(x,z)$ two sums on the left-hand side of (36), respectively. By (28) of Lemma 5, we can write

$$\begin{aligned} \Delta_1(x,z) &= x\sum_{N_0<n\le\sqrt{x}}\sum_{d\le(x/(nz))}\frac{1}{dn}\psi\left(\frac{x}{dn}\right) \\ &= x\sum_{d\le(x/(N_0z))}\frac{1}{d}\sum_{N_0<n\le\min(\sqrt{x},x/(dz))}\frac{1}{n}\psi\left(\frac{x}{dn}\right) \\ &= x\Delta_1^{\dagger}(x,z)+x\Delta_1^{\sharp}(x,z), \end{aligned} \tag{37}$$

where

$$\Delta_1^{\dagger}(x,z) := \sum_{d\le(x/(N_0z))}\frac{1}{d}\sum_{N_0<n\le(x/d)^{(2/3)}}\frac{1}{n}\psi\left(\frac{x}{dn}\right),$$
$$\Delta_1^{\sharp}(x,z) := \sum_{d\le(x/(N_0z))}\frac{1}{d}\sum_{(x/d)^{(2/3)}<n\le\min\{\sqrt{x},(x/dz)\}}\frac{1}{n}\psi\left(\frac{x}{dn}\right). \tag{38}$$

For $0\le k\le(\log((x/d)^{(2/3)}/N_0))/\log 2$, let $N_k := 2^kN_0$ and define

$$\mathfrak{S}_k(d) := \sum_{N_k<n\le 2N_k}\frac{1}{n}\psi\left(\frac{x}{dn}\right). \tag{39}$$

Noticing that $N_0\le N_k\le(x/d)^{(2/3)}$, we can apply Lemma 3 to derive that

$$\mathfrak{S}_k(d)\ll e^{-\vartheta\left((\log N_k)^3/(\log(x/d))^2\right)}, \tag{40}$$

with $\vartheta(t) := c_6t-\log t$. It is clear that $\vartheta(t)$ is increasing on $[c_6,\infty)$. On the other hand, for $k\ge 0$ and $d\ge 1$, we have

$$(\log N_k)^3/(\log(x/d))^2 \ge (\log N_0)^3/(\log x)^2 = (6/c_6)\log_2 x. \tag{41}$$

Thus,

$$\begin{aligned} \vartheta\left(\frac{(\log N_k)^3}{(\log(x/d))^2}\right) &\ge \vartheta\left(\left(\frac{6}{c_6}\right)\log_2 x\right) \\ &= 6\log_2 x-\log\left(\left(\frac{6}{c_6}\right)\log_2 x\right)\ge 5\log_2 x, \end{aligned} \tag{42}$$

which implies that $\mathfrak{S}_k(d)\ll(\log x)^{-5}$. Inserting this into the expression of $\Delta_1^{\dagger}(x,z)$, we get

$$\Delta_1^{\dagger}(x,z)\ll\sum_{d\le(x/(N_0z))}\frac{1}{d}\sum_{2^kN_0\le(x/d)^{(2/3)}}\left|\mathfrak{S}_k(d)\right|\ll(\log x)^{-3}. \tag{43}$$

Next we bound $\Delta_1^{\sharp}(x,z)$. Let $F(t)$ be a function of bounded variation on $[n,n+1]$ for each integer n and let $V_F[n,n+1]$ be the total variation of F on $[n,n+1]$. Integrating by parts, we have

$$\int_n^{n+1}\left(t-n-\frac{1}{2}\right)\mathrm{d}F(t)=\frac{1}{2}(F(n+1)+F(n))-\int_n^{n+1}F(t)\mathrm{d}t. \tag{44}$$

From this, we can derive that

$$\frac{1}{2}(F(n+1)+F(n))=\int_n^{n+1}F(t)\mathrm{d}t+O\left(V_F[n,n+1]\right), \tag{45}$$

for $n\geq 1$. Summing over n, we find that

$$\sum_{N_1<n\leq N_2}F(n)=\int_{N_1}^{N_2}F(t)\mathrm{d}t+\frac{1}{2}\left(F(N_1)+F(N_2)\right)+O\left(V_F[N_1,N_2]\right). \tag{46}$$

We apply this formula to

$$F_{(x/d)}(t)=\frac{1}{t}\psi\left(\frac{(x/d)}{t}\right),$$
$$N_1=\left[(x/d)^{(2/3)}\right], \tag{47}$$
$$N_2=\left[\min\left\{\sqrt{x},\frac{x}{(\mathrm{d}z)}\right\}\right].$$

According to Lemma 4, we have $V_{F_{(x/d)}}[N_1,N_2]\ll(x/d)^{-(1/3)}$, and thus by putting $u=(x/d)/t$, we obtain, with the notation $x_{d,1}=\max(\sqrt{x}/d,tz)$ and $x_{d,2}=(x/d)^{(1/3)}$,

$$\sum_{(x/d)^{(2/3)}<n\leq\min\left\{\sqrt{x},(x/(dz))\right\}}\frac{1}{n}\psi\left(\frac{x}{\mathrm{d}n}\right)=\int_{x_{d,1}}^{x_{d,2}}\frac{\psi(u)}{u}\mathrm{d}u+O\left(\left(\frac{x}{d}\right)^{-(1/3)}\right)$$
$$\ll z^{-1}+\left(\frac{x}{d}\right)^{-(1/3)}\ll z^{-1}, \tag{48}$$

where we have used the fact that $z\leq\sqrt{N_0}$ and $d\leq(x/(N_0z))\Rightarrow z\leq(x/d)^{(1/3)}$ and the bound

$$\int_{x_{d,1}}^{x_{d,2}}\frac{\psi(u)}{u}\mathrm{d}u=\int_{x_{d,1}}^{x_{d,2}}\left(\int_{x_{d,1}}^{u}\psi(t)\mathrm{d}t\right)\frac{\mathrm{d}u}{u^2}-\frac{1}{x_{d,2}^{(2/3)}}\int_{x_{d,1}}^{x_{d,2}}\psi(t)\mathrm{d}t$$
$$\ll x_{d,1}^{-1}+\left(x_{d,2}\right)^{-(2/3)}\ll z^{-1}+(x/d)^{-(2/3)}\ll z^{-1}. \tag{49}$$

Using (48), a simple partial integration allows us to derive that

$$\Delta_1^{\sharp}(x,z)\ll z^{-1}\sum_{d\leq x/(N_0z)}d^{-1}\ll z^{-1}\log x. \tag{50}$$

Combining (43) and (50), it follows that

$$\left|\Delta_1(x,z)\right|\ll x(\log x)^{-3}+xz^{-1}\log x. \tag{51}$$

Similarly, we can prove the same bound for $|\Delta_2(x,z)|$. This completes the proof. □

3.3. End of the Proof of Theorem 1. Let c_6 be the constant given as in Lemma 3 and $N_0:=\exp\{(6/c_6)\ (\log x)^{(2/3)}(\log_2 x)^{(1/3)}\}$. Let $z\in[2,\sqrt{N_0}]$ be a parameter to be chosen later.

Putting $d=[x/n]$, we have $(x/n)-1<d\leq(x/n)$ and $x/(d+1)<n\leq(x/d)$. We have, with the convention $\sigma(0)=0$,

$$\begin{aligned}S_\sigma(x)&=\sum_{d\leq x}\sigma(d)\sum_{(x/(d+1))<n\leq(x/d)}1\\&=\sum_{dn\leq x}\sigma(d)-\sum_{dn\leq x,d\geq 2}\sigma(d-1)\\&=\sum_{dn\leq x}(\sigma(d)-\sigma(d-1)).\end{aligned} \tag{52}$$

By the hyperbole principle of Dirichlet, we can write

$$S_\sigma(x)=S_1(x,\sigma)+S_2(x,\sigma)-S_3(x,\sigma), \tag{53}$$

where

$$\begin{aligned}S_1(x,\sigma)&:=\sum_{d\leq\sqrt{x},\mathrm{d}n\leq x}(\sigma(d)-\sigma(d-1)),\\S_2(x,\sigma)&:=\sum_{n\leq\sqrt{x},\mathrm{d}n\leq x}(\sigma(d)-\sigma(d-1)),\\S_3(x,\sigma)&:=\sum_{d\leq\sqrt{x},n\leq\sqrt{x}}(\sigma(d)-\sigma(d-1)).\end{aligned} \tag{54}$$

With the help of the bound $\sigma(n)\ll n\log_2 n$, we can derive that

$$S_3(x,\sigma)=[\sqrt{x}]\sigma([\sqrt{x}])\ll x\log_2 x. \tag{55}$$

For evaluating $S_1(x,\sigma)$, we write

$$\begin{aligned}S_1(x,\sigma)&=\sum_{d\leq\sqrt{x}}(\sigma(d)-\sigma(d-1))\left[\frac{x}{d}\right]\\&=x\sum_{d\leq\sqrt{x}}\frac{\sigma(d)-\sigma(d-1)}{d}+O\left(\sum_{d\leq\sqrt{x}}|\sigma(d)-\sigma(d-1)|\right).\end{aligned} \tag{56}$$

With the help of Lemma 5 (ii), a simple partial integration gives us

$$\begin{aligned}\sum_{d\le\sqrt{x}}\frac{\sigma(d)-\sigma(d-1)}{d} &= \sum_{d\le\sqrt{x}}\frac{\sigma(d)}{d(d+1)}\\ &= \sum_{d\le\sqrt{x}}\frac{\sigma(d)}{d^2}-\sum_{d\le\sqrt{x}}\frac{\sigma(d)}{d^2(d+1)}\\ &= \int_{1-}^{\sqrt{x}} t^{-2}\mathrm{d}\left(\frac{\pi^2}{12}t^2+O(t\log t)\right)+O(1)\\ &= \frac{\pi^2}{12}\log x+O(1),\\ &\sum_{d\le\sqrt{x}}|\sigma(d)-\sigma(d-1)|\\ &\ll \sum_{d\le\sqrt{x}}\sigma(d)\ll x.\end{aligned} \tag{57}$$

Inserting these estimates into (56), we find that

$$S_1(x,\sigma)=\frac{\pi^2}{12}x\log x+O(x). \tag{58}$$

Finally, we evaluate $S_2(x,\sigma)$. For this, we write

$$S_2(x,\sigma)=S_2^{\dagger}(x,\sigma)+S_2^{\sharp}(x,\sigma), \tag{59}$$

where

$$\begin{aligned}S_2^{\dagger}(x,\sigma) &:= \sum_{n\le N_0,\mathrm{d}n\le x}(\sigma(d)-\sigma(d-1)),\\ S_2^{\sharp}(x,\sigma) &:= \sum_{N_0<n\le\sqrt{x},\mathrm{d}n\le x}(\sigma(d)-\sigma(d-1)).\end{aligned} \tag{60}$$

By the bound that $\sigma(n)\ll n\log_2 n$, we have

$$\begin{aligned}S_2^{\dagger}(x,\sigma) &= \sum_{n\le N_0}\sigma\left(\left[\frac{x}{n}\right]\right)\ll\sum_{n\le N_0}\left(\frac{x}{n}\right)\log_2 x\\ &\ll x(\log x)^{(2/3)}(\log_2 x)^{(4/3)}.\end{aligned} \tag{61}$$

On the other hand, (27) of Lemma 5 allows us to derive that

$$\begin{aligned}\sum_{d\le x}\sigma(d)-\sum_{d\le x-1}\sigma(d) &= \frac{\pi^2}{12}\left(x^2-(x-1)^2\right)-\Delta(x,z)\\ &\quad+\Delta(x-1,z)+O\left(\frac{x}{z}\right)\\ &= \frac{\pi^2}{6}x-\Delta(x,z)+\Delta(x-1,z)+O\left(\frac{x}{z}\right),\end{aligned} \tag{62}$$

where $\Delta(x,z)$ is given by (28). Thus,

$$\begin{aligned}S_2^{\sharp}(x,\sigma) &= \sum_{N_0<n\le\sqrt{x}}\left(\frac{\pi^2}{6}\cdot\frac{x}{n}-\Delta\left(\frac{x}{n},z\right)+\Delta\left(\frac{x}{n}-1,z\right)+O\left(\frac{x}{nz}\right)\right)\\ &= \frac{\pi^2}{12}x\log x+O\left(x(\log x)^{(2/3)}(\log_2 x)^{(1/3)}+xz^{-1}\log x\right)-\Delta_1(x,z)+\Delta_2(x,z),\end{aligned} \tag{63}$$

where

$$\begin{aligned}\Delta_1(x,z) &:= \sum_{N_0<n\le\sqrt{x}}\Delta\left(\frac{x}{n},z\right)\ll x(\log x)^{-3}+xz^{-1}\log x,\\ \Delta_2(x,z) &:= \sum_{N_0<n\le\sqrt{x}}\Delta\left(\frac{x}{n}-1,z\right)\ll x(\log x)^{-3}+xz^{-1}\log x,\end{aligned} \tag{64}$$

thanks to Lemma 6. Inserting these estimates into (59), we find that

$$S_2(x,\sigma)=\frac{\pi^2}{12}x\log x+O\left(x(\log x)^{(2/3)}(\log_2 x)^{(4/3)}+xz^{-1}\log x\right), \tag{65}$$

Now (5) follows from (53), (55), (58), and (66) with the choice of $z=(\log x)^{(1/3)}$.

3.4. Proof of Theorem 1. (ii) For any odd prime p, (52) allows us to write

$$\begin{aligned}\sum_{d\mid p}(\sigma(d)-\sigma(d-1)) &= S_\sigma(p)-S_\sigma(p-1)\\ &= \frac{\pi^2}{6}(\log p-\log(p-1))+E(p)-E(p-1)\\ &\ge E(p)-E(p-1)\ge -2E^*(p),\end{aligned} \tag{66}$$

where $E^*(p):=\max\{|E(p)|,|E(p-1)|\}$. On the other hand, we have

$$\begin{aligned}\sum_{d\mid p}(\sigma(d)-\sigma(d-1)) &= \sigma(p)-\sigma(p-1)+1\\ &\le p+1-\left(p-1+\frac{1}{2}(p-1)+2+1\right)+1\le -\frac{1}{4}p.\end{aligned} \tag{67}$$

Thus, $E^*(p)\ge(1/8)p$ for all odd primes.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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