Hindawi
Journal of Mathematics
Volume 2021, Article ID 5577203, 14 pages
https://doi.org/10.1155/2021/5577203

Research Article

Inequalities for Riemann–Liouville Fractional Integrals of Strongly (s, m)-Convex Functions

Fuzhen Zhang[1] **Ghulam Farid**[2] **and Saira Bano Akbar**[3]

[1]*Foundation Department of Jiangsu College of Safety Technology, Xuzhou 221000, China*
[2]*Department of Mathematics, COMSATS University Islamabad, Attock Campus, Attock, Pakistan*
[3]*Department of Mathematics, COMSATS University Islamabad, Lahore Campus, Lahore, Pakistan*

Correspondence should be addressed to Fuzhen Zhang; shenzhuren1@sina.com and Ghulam Farid; faridphdsms@hotmail.com

Received 18 January 2021; Revised 7 February 2021; Accepted 9 February 2021; Published 12 March 2021

Academic Editor: Ahmet Ocak Akdemir

The results of this paper provide two Hadamard-type inequalities for strongly (s, m)-convex functions via Riemann–Liouville fractional integrals and error estimations of well-known fractional Hadamard inequalities. Their special cases are given and connected with the results of some published papers.

1. Introduction

The most prominent inequality for convex functions is the well-known Hadamard inequality stated in the following.

Theorem 1 (see [1]). *Let $f: I \subset \mathbb{R} \longrightarrow \mathbb{R}$ be a convex function on the interval I, where $x, y \in I$ with $x < y$. Then, the following inequality holds:*

$$f\left(\frac{x+y}{2}\right) \leq \frac{1}{y-x} \int_x^y f(u)\mathrm{d}u \leq \frac{f(x)+f(y)}{2}. \quad (1)$$

Convex functions are extended, generalized, and refined in different ways to define new types of convex functions. For instance, s-convex, m-convex, (s, m)-convex, strongly convex, and strongly (s, m)-convex functions are extensions of convex functions. The aim of this paper is to establish integral inequalities by using the class of strongly (s, m)-convex functions. We give definitions of (s, m)-convex and strongly (s, m)-convex functions as follows.

Definition 1 (see [2]). A function $f: [0, \infty) \longrightarrow \mathbb{R}$ is called (s, m)-convex in the second sense, if the following inequality holds:

$$f(tx + m(1-t)y) \leq t^s f(x) + m(1-t)^s f(y), \quad (2)$$

for every $x, y \in [0, \infty)$, $t \in [0, 1]$ and $[s, m) \in (0, 1] \times [0, 1)$.

Definition 2 (see [3]). A function $f: [0, \infty) \longrightarrow \mathbb{R}$ is called strongly (s, m)-convex in the second sense with modulus $C \geq 0$, if the following inequality holds:

$$\begin{aligned} f(tx + m(1-t)y) \leq{}& t^s f(x) + m(1-t)^s f(y) \\ &- Cmt(1-t)(y-x)^2, \end{aligned} \quad (3)$$

for every $x, y \in [0, \infty)$, $t \in [0, 1]$ and $[s, m) \in (0, 1] \times [0, 1)$.

By setting $(s, m) = (s, 1)$, $(s, m) = (1, m)$, and $(s, m) = (1, 1)$ in (2), we get s-convex [4], m-convex [5], and convex functions, respectively, while by setting $(s, m) = (s, 1)$, $(s, m) = (1, m)$, and $(s, m) = (1, 1)$ in (3), we get strongly s-convex [6], strongly m-convex [7], and strongly convex [6] functions, respectively.

Next we give definition of Riemann–Liouville fractional integrals $J_{x^+}^{\alpha} f$ and $J_{y^-}^{\alpha} f$ which are utilized to get the desired results of this paper.

Definition 3 (see [8]). Let $f \in L_1[x, y]$. Then, Riemann–Liouville fractional integral operators of order $\alpha > 0$ are given by

$$J_{x^+}^{\alpha} f(u) = \frac{1}{\Gamma(\alpha)} \int_x^u (u-t)^{\alpha-1} f(t)\mathrm{d}t, \quad u > x,$$
$$J_{y^-}^{\alpha} f(u) = \frac{1}{\Gamma(\alpha)} \int_u^y (t-u)^{\alpha-1} f(t)\mathrm{d}t, \quad u < y, \tag{4}$$

where $\Gamma(\alpha) = \int_0^{\infty} e^{-t} t^{\alpha-1}\mathrm{d}t$ is the gamma function and $J_{x^+}^0 f(u) = J_{y^-}^0 f(u) = f(u)$.

The following special functions are also involved in the findings of this paper.

Definition 4. The beta function, also referred to as first type of Euler integral, is defined by

$$\beta(\alpha, s) = \int_0^1 t^{\alpha-1}(1-t)^{s-1}\mathrm{d}t, \tag{5}$$

where $\mathrm{Re}(\alpha), \mathrm{Re}(s) > 0$.

Close association of the beta function to the gamma function is an important factor of the beta function

$$\beta(\alpha, s) = \frac{\Gamma(\alpha)\Gamma(s)}{\Gamma(\alpha+s)}. \tag{6}$$

The beta function is symmetric, i.e., $\beta(\alpha, s) = \beta(s, \alpha)$. A generalization of the beta function, called the incomplete beta function, is defined by

$$\beta(b; \alpha, s) = \int_0^b t^{\alpha-1}(1-t)^{s-1}\mathrm{d}t, \tag{7}$$

where $\mathrm{Re}(\alpha), \mathrm{Re}(s) > 0$ with $0 < b < 1$. The incomplete beta function $\beta(b; \alpha, s)$ weakens to the ordinary $\beta(\alpha, s)$ (beta function) by setting $b = 1$.

In [8], the Hadamard inequality is studied for Riemann–Liouville fractional integrals which is stated in the following theorem.

Theorem 2. *Let $f\colon [x, y] \longrightarrow \mathbb{R}$ be a positive function with $0 \le x < y$ and $f \in L_1[x, y]$. If f is convex function on $[x, y]$, then the following inequality for fractional integrals holds:*

$$f\left(\frac{x+y}{2}\right) \le \frac{\Gamma(\alpha+1)}{2(y-x)^{\alpha}}\left[\left(J_{x^+}^{\alpha} f\right)(y) + \left(J_{y^-}^{\alpha} f\right)(x)\right] \le \frac{f(x)+f(y)}{2}, \tag{8}$$

with $\alpha > 0$.

The inequality stated in the aforementioned theorem motivates the researchers to work in this direction by establishing other kinds of inequalities for Riemann–Liouville fractional integrals. In the past decade, several classical inequalities have been extended via different kinds of fractional integral operators. The Hadamard inequality is one of the most studied inequalities for fractional integral operators. For some recent work, we refer the readers to [3, 8–17].

This paper is organized as follows. In Section 2, two versions of the Hadamard inequality for strongly (s,m)-convex functions via Riemann–Liouville fractional integrals are given. Their connection with the well-known results is established in the form of corollaries and remarks. In Section 3, the error estimations of Hadamard inequalities for Riemann–Liouville fractional integrals are obtained by using differentiable strongly (s,m)-convex functions.

2. Main Results

Theorem 3. *Let $f \in L_1[x, y]$ be a positive function with $0 \le x < y$. If f is strongly (s,m)-convex function on $[x, my]$ with modulus $C \ge 0$, $m \ne 0, 0 < s \le 1$, then the following fractional integral inequality holds:*

$$2^{s-1} f\left(\frac{x+my}{2}\right) + \frac{2^{s-3}Cm\alpha}{\alpha+2}\left((x-y)^2 + \frac{2(my-(x/m))^2}{\alpha(\alpha+1)} + \frac{2(x-y)(my-(x/m))}{(\alpha+1)}\right)$$
$$\le \frac{\Gamma(\alpha+1)}{2(my-x)^{\alpha}}\left[J_{x^+}^{\alpha} f(my) + m^{\alpha+1} J_{y^-}^{\alpha} f\left(\frac{x}{m}\right)\right] \le \frac{\alpha(f(x)+mf(y))}{2(\alpha+s)} \tag{9}$$
$$+ \frac{m\alpha\beta(\alpha, s+1)\left(f(y)+mf\left(x/m^2\right)\right)}{2} - \frac{Cm\alpha\left((y-x)^2 + m\left(y-\left(x/m^2\right)\right)^2\right)}{2(\alpha+1)(\alpha+2)},$$

with $\alpha > 0$.

Proof. Since f is strongly (s,m)-convex function, for $u, v \in [x, y]$, we have

$$f\left(\frac{u+mv}{2}\right) \le \frac{f(u)+mf(v)}{2^s} - \frac{Cm}{4}|u-v|^2. \tag{10}$$

By setting $u = xt + m(1-t)y$ and $v = yt + (1-t)(x/m)$, we have

$$f\left(\frac{x+my}{2}\right) \le \frac{1}{2^s} f(xt + m(1-t)y) + \frac{m}{2^s} f\left(yt + (1-t)\frac{x}{m}\right) - \frac{Cm}{4}\left|t(x-y) + (1-t)\left(my - \frac{x}{m}\right)\right|^2. \tag{11}$$

By multiplying inequality (11) with $t^{\alpha-1}$ on both sides and then integrating over the interval $[0,1]$, we get

$$f\left(\frac{x+my}{2}\right)\int_0^1 t^{\alpha-1}\mathrm{d}t \leq \frac{1}{2^s}\int_0^1 f(xt+m(1-t)y)t^{\alpha-1}\mathrm{d}t + \frac{m}{2^s}\int_0^1 f\left(yt+(1-t)\frac{x}{m}\right)t^{\alpha-1}\mathrm{d}t - \frac{Cm}{4}\int_0^1 \left|t(x-y)+(1-t)\left(my-\frac{x}{m}\right)\right|^2 t^{\alpha-1}\mathrm{d}t. \tag{12}$$

By change of variables, we will get

$$\frac{1}{\alpha}f\left(\frac{x+my}{2}\right) \leq \frac{\Gamma(\alpha)}{2^s(my-x)^\alpha}\left[\frac{1}{\Gamma(\alpha)}\int_{my}^{x}(my-u)^{\alpha-1}f(u)\mathrm{d}u + \frac{m^{\alpha+1}}{\Gamma(\alpha)}\cdot\int_{(x/m)}^{y}\left(v-\frac{x}{m}\right)^{\alpha-1}f(v)\mathrm{d}v\right] - \frac{Cm}{4}\left(\frac{(x-y)^2}{\alpha+2}+\frac{2(my-(x/m))^2}{\alpha(\alpha+1)(\alpha+2)}+\frac{2(x-y)(my-(x/m))}{(\alpha+1)(\alpha+2)}\right). \tag{13}$$

Further, the above inequality takes the following form:

$$2^{s-1}f\left(\frac{x+my}{2}\right) \leq \frac{\Gamma(\alpha+1)}{2(my-x)^\alpha}\left[J_{x^+}^{\alpha}f(my)+m^{\alpha+1}J_{y^-}^{\alpha}f\left(\frac{x}{m}\right)\right] - \frac{2^{s-1}Cm\alpha}{4}\left(\frac{(x-y)^2}{\alpha+2}+\frac{2(my-(x/m))^2}{\alpha(\alpha+1)(\alpha+2)}+\frac{2(x-y)(my-(x/m))}{(\alpha+1)(\alpha+2)}\right). \tag{14}$$

From the definition of strongly (s, m)-convex function with modulus C, for $t \in [0,1]$, we have the following inequality:

$$f(tx+m(1-t)y)+mf\left(yt+(1-t)\frac{x}{m}\right) \leq t^s(f(x)+mf(y))+m(1-t)^s\left(f(y)+mf\left(\frac{x}{m^2}\right)\right) - Cmt(1-t)\left((y-x)^2+m\left(y-\frac{x}{m^2}\right)^2\right). \tag{15}$$

By multiplying inequality (15) with $t^{\alpha-1}$ on both sides and then integrating over the interval $[0,1]$, we get

$$\int_0^1 f(tx+m(1-t)y)t^{\alpha-1}\mathrm{d}t + m\int_0^1 f\left(yt+(1-t)\frac{x}{m}\right)t^{\alpha-1}\mathrm{d}t \leq (f(x)+mf(y))\int_0^1 t^{s+\alpha-1}\mathrm{d}t + m\left(f(y)+mf\left(\frac{x}{m^2}\right)\right)\int_0^1 t^{\alpha-1}(1-t)^s\mathrm{d}t - Cm\left((y-x)^2+m\left(y-\frac{x}{m^2}\right)^2\right)\int_0^1 t^\alpha(1-t)\mathrm{d}t. \tag{16}$$

By change of variables, we will get

$$\frac{\Gamma(\alpha)}{(my-x)^{\alpha}}\left[\frac{1}{\Gamma(\alpha)}\int_{my}^{x}(my-u)^{\alpha-1}f(u)du+\frac{m^{\alpha+1}}{\Gamma(\alpha)}\int_{(x/m)}^{y}\left(v-\frac{x}{m}\right)^{\alpha-1}f(v)dv\right]$$
$$\leq\frac{f(x)+mf(y)}{\alpha+s}+m\left(f(y)+mf\left(\frac{x}{m^2}\right)\right)\beta(s+1,\alpha)-\frac{Cm\left((y-x)^2+m\left(y-\left(x/m^2\right)\right)^2\right)}{(\alpha+1)(\alpha+2)}. \tag{17}$$

Further, the above inequality takes the following form:

$$\frac{\Gamma(\alpha+1)}{2(my-x)^{\alpha}}\left[J_{x^+}^{\alpha}f(my)+m^{\alpha+1}J_{y^-}^{\alpha}f\left(\frac{x}{m}\right)\right]$$
$$\leq\frac{\alpha(f(x)+mf(y))}{2(\alpha+s)}+\frac{m\alpha}{2}\left(f(y)+mf\left(\frac{x}{m^2}\right)\right)\beta(s+1,\alpha)$$
$$-\frac{Cm\alpha\left((y-x)^2+m\left(y-\left(x/m^2\right)\right)^2\right)}{2(\alpha+1)(\alpha+2)}. \tag{18}$$

From inequalities (14) and (18), one can get inequality (9). □

Remark 1

(i) For $s=1$ in (9), we have the result for strongly m-convex function [18].

(ii) For $m=1$ and $s=1$ in (9), we have the result for strongly convex function.

(iii) For $m=1$, $s=1$, and $C=0$, we get [[16], Theorem 2]

(iv) For $m=1$, $s=1$, $\alpha=1$, and $C=0$, we get the classical Hadamard inequality.

(v) For $m=1$ and $C=0$, we get [[17], Theorem 3].

Corollary 1. *For $m=1$, we have the result for Riemann–Liouville fractional integrals of strongly s-convex functions:*

$$2^{s-1}f\left(\frac{x+y}{2}\right)+\frac{2^{s-1}C\alpha(y-x)^2\left(\alpha^2-\alpha+2\right)}{4(\alpha+1)(\alpha+2)}$$
$$\leq\frac{\Gamma(\alpha+1)}{2(y-x)^{\alpha}}\left[J_{x^+}^{\alpha}f(y)+J_{y^-}^{\alpha}f(x)\right]$$
$$\leq\frac{f(x)+f(y)}{2}\left(\frac{\alpha}{\alpha+s}+\frac{\Gamma(\alpha+1)\Gamma(s+1)}{\Gamma(\alpha+s+1)}\right)-\frac{C\alpha(y-x)^2}{(\alpha+1)(\alpha+2)}. \tag{19}$$

Corollary 2. *For $\alpha=1$ and $m=1$, the following inequality holds for strongly s-convex function:*

$$2^{s-1}f\left(\frac{x+y}{2}\right)+\frac{2^{s-1}C(y-x)^2}{12}$$
$$\leq\frac{1}{y-x}\int_{x}^{y}f(u)du\leq\frac{f(x)+f(y)}{s+1}-\frac{C(y-x)^2}{6}. \tag{20}$$

In the next theorem, we give another version of the Hadamard inequality.

Theorem 4. *Under the assumptions of Theorem 3, the following fractional integral inequality holds:*

$$2^{s-1}f\left(\frac{x+my}{2}\right)$$
$$+\frac{Cm\alpha}{2^{4-s}}\left[\frac{(x-y)^2}{2(\alpha+2)}+\frac{(my-(x/m))^2\left(\alpha^2+5\alpha+8\right)}{2\alpha(\alpha+1)(\alpha+2)}+\frac{(x-y)(my-(x/m))(\alpha+3)}{(\alpha+1)(\alpha+2)}\right]$$
$$\leq\frac{2^{\alpha-1}\Gamma(\alpha+1)}{(my-x)^{\alpha}}\left[\left(J_{((x+my)/2)^+}^{\alpha}f\right)(ym)+m^{\alpha+1}\left(J_{((x+ym)/2m)^-}^{\alpha}f\right)\left(\frac{x}{m}\right)\right]$$
$$\leq\frac{\alpha(f(x)+mf(y))}{2^{s+1}(\alpha+s)}+2^{\alpha-1}m\alpha\left(f(y)+mf\left(\frac{x}{m^2}\right)\right)\beta\left(\frac{1}{2};s+1,\alpha\right)$$
$$-\frac{Cm\alpha\left((y-x)^2+m\left(y-\left(x/m^2\right)\right)^2\right)(\alpha+3)}{8(\alpha+1)(\alpha+2)}, \tag{21}$$

with $\alpha > 0$.

Proof. Let $t \in [0,1]$. Using strong (s,m)-convexity of function f for $u = x(t/2) + m((2-t)/2)y$ and $v = ((2-t)/2)(x/m) + y(t/2)$ in inequality (10), we have

$$f\left(\frac{x+my}{2}\right) \le \frac{1}{2^s} f\left(x\frac{t}{2} + m\left(\frac{2-t}{2}\right)y\right) + \frac{m}{2^s} f\left(\left(\frac{2-t}{2}\right)\frac{x}{m} + y\frac{t}{2}\right) - \frac{Cm}{4}\left|\frac{t}{2}(x-y) + \frac{2-t}{2}\left(my - \frac{x}{m}\right)\right|^2. \tag{22}$$

By multiplying (22) with $t^{\alpha-1}$ on both sides and making integration over $[0,1]$, we get

$$f\left(\frac{x+my}{2}\right)\int_0^1 t^{\alpha-1}\mathrm{d}t \le \frac{1}{2^s}\int_0^1 f\left(x\frac{t}{2} + m\left(\frac{2-t}{2}\right)y\right)t^{\alpha-1}\mathrm{d}t + \frac{m}{2^s}\int_0^1 f\left(\left(\frac{2-t}{2}\right)\frac{x}{m} + y\frac{t}{2}\right)t^{\alpha-1}\mathrm{d}t - \frac{Cm}{4}\int_0^1\left|\frac{t}{2}(x-y) + \frac{2-t}{2}\left(my - \frac{x}{m}\right)\right|^2 t^{\alpha-1}\mathrm{d}t. \tag{23}$$

By using change of variables and computing the last integral, from (23), we get

$$\frac{2^s}{\alpha} f\left(\frac{x+my}{2}\right) \le \frac{2^\alpha\Gamma(\alpha)}{(my-x)^\alpha}\left[\frac{1}{\Gamma(\alpha)}\int_{my}^{((x+my)/2)}(my-u)^{\alpha-1}f(u)\mathrm{d}u + \frac{m^{\alpha+1}}{\Gamma(\alpha)}\int_{(x/m)}^{((ym+x)/2m)}\left(v-\frac{x}{m}\right)^{\alpha-1}f(v)\mathrm{d}v\right] - \frac{2^s Cm}{4}\left[\frac{(x-y)^2}{4(\alpha+2)} + \frac{(my-(x/m))^2(\alpha^2+5\alpha+8)}{4\alpha(\alpha+1)(\alpha+2)} + \frac{(x-y)(my-(x/m))(\alpha+3)}{2(\alpha+1)(\alpha+2)}\right]. \tag{24}$$

Further, it takes the following form:

$$2^{s-1} f\left(\frac{x+my}{2}\right) \le \frac{2^{\alpha-1}\Gamma(\alpha+1)}{(my-x)^\alpha}\left[\left(J^\alpha_{((x+my)/2)^+}f\right)(ym) + m^{\alpha+1}\left(J^\alpha_{((ym+x)/2m)^-}f\right)\left(\frac{x}{m}\right)\right] - \frac{2^{s-1}Cm\alpha}{4}\left[\frac{(x-y)^2}{4(\alpha+2)} + \frac{(my-(x/m))^2(\alpha^2+5\alpha+8)}{4\alpha(\alpha+1)(\alpha+2)} + \frac{(x-y)(my-(x/m))(\alpha+3)}{2(\alpha+1)(\alpha+2)}\right]. \tag{25}$$

The first inequality of (21) can be seen in (25). Now we prove the second inequality of (21). Since f is strongly (s,m)-convex function and $t \in [0,1]$, we have the following inequality:

$$f\left(x\frac{t}{2}+m\left(\frac{2-t}{2}\right)y\right)+mf\left(\left(\frac{2-t}{2}\right)\frac{x}{m}+y\frac{t}{2}\right)\leq\left(\frac{t}{2}\right)^{s}(f(x)+mf(b))$$
$$+m\left(\frac{2-t}{2}\right)^{s}\left(f(y)+mf\left(\frac{x}{m^{2}}\right)\right)-\frac{Cmt(2-t)}{4}\left[(y-x)^{2}+m\left(y-\frac{x}{m^{2}}\right)^{2}\right]. \tag{26}$$

By multiplying inequality (26) with $t^{\alpha-1}$ on both sides and making integration over $[0,1]$, we get

$$\int_{0}^{1}f\left(x\frac{t}{2}+m\left(\frac{2-t}{2}\right)y\right)t^{\alpha-1}\mathrm{d}t+m\int_{0}^{1}f\left(\left(\frac{2-t}{2}\right)\frac{x}{m}+y\frac{t}{2}\right)t^{\alpha-1}\mathrm{d}t$$
$$\leq\frac{1}{2^{s}}(f(x)+mf(y))\int_{0}^{1}t^{s+\alpha-1}\mathrm{d}t+\frac{m}{2^{s}}\left(f(y)+mf\left(\frac{x}{m^{2}}\right)\right)\int_{0}^{1}(2-t)^{s}t^{\alpha-1}\mathrm{d}t \tag{27}$$
$$-\frac{Cm}{4}\left[(y-x)^{2}+m\left(y-\frac{x}{m^{2}}\right)\right]\int_{0}^{1}t^{\alpha}(2-t)\mathrm{d}t.$$

By using change of variables and computing the last integral, from (27), we get

$$\frac{2^{\alpha}\Gamma(\alpha)}{(my-x)^{\alpha}}\left[\frac{1}{\Gamma(\alpha)}\int_{my}^{((x+my)/2)}(my-u)^{\alpha-1}f(u)\mathrm{d}u+\frac{m^{\alpha+1}}{\Gamma(\alpha)}\int_{(x/m)}^{((my+x)/2m)}\left(v-\frac{x}{m}\right)^{\alpha-1}f(v)\mathrm{d}v\right]$$
$$\leq\frac{f(x)+mf(y)}{2^{s}(\alpha+s)}+2^{\alpha}m\left(f(y)+mf\left(\frac{x}{m^{2}}\right)\right)\beta\left(\frac{1}{2}:s+1,\alpha\right) \tag{28}$$
$$-\frac{Cm\left((y-x)^{2}+m\left(y-\left(x/m^{2}\right)\right)^{2}\right)(\alpha+3)}{4(\alpha+1)(\alpha+2)}.$$

Further, it takes the following form:

$$\frac{2^{\alpha-1}\Gamma(\alpha+1)}{(my-x)^{\alpha}}\left[\left(J^{\alpha}_{((x+my)/2)^{+}}f\right)(ym)+m^{\alpha+1}\left(J^{\alpha}_{((x+ym)/2m)^{-}}f\right)\left(\frac{x}{m}\right)\right]$$
$$\leq\frac{\alpha(f(x)+mf(y))}{2^{s+1}(\alpha+s)}+2^{\alpha-1}m\alpha\left(f(y)+mf\left(\frac{x}{m^{2}}\right)\right)\beta\left(\frac{1}{2}:s+1,\alpha\right) \tag{29}$$
$$-\frac{Cm\alpha\left((y-x)^{2}+m\left(y-\left(x/m^{2}\right)\right)^{2}\right)(\alpha+3)}{8(\alpha+1)(\alpha+2)}.$$

From inequalities (25) and (29), we have inequality (21). □

Remark 2

(i) For $s = 1$ in (21), we get the result for strongly m-convex function [18].

(ii) For $m = 1, s = 1$, and $C = 0$, we get [[16], Theorem 2]

(iii) For $m = 1$, $s = 1$, $\alpha = 1$, and $C = 0$, we get the classical Hadamard inequality.

Corollary 3. *For $m = 1$ and $s = 1$ in (21), we have the result for Riemann–Liouville fractional integrals of strongly convex function:*

$$\begin{aligned} & f\left(\frac{x+y}{2}\right) + \frac{C(y-x)^2}{2(\alpha+1)(\alpha+2)} \\ & \leq \frac{2^{\alpha-1}\Gamma(\alpha+1)}{(y-x)^{\alpha}}\left[\left(J^{\alpha}_{((x+y)/2)^+} f\right)(y) + \left(J^{\alpha}_{((x+y)/2)^-} f\right)(x)\right] \\ & \leq \frac{f(x)+f(y)}{2} - \frac{C\alpha(y-x)^2(\alpha+3)}{4(\alpha+1)(\alpha+2)}. \end{aligned} \tag{30}$$

Corollary 4. *For $m = 1$ in (21), we get the result for Riemann–Liouville fractional integrals of strongly s-convex function:*

$$\begin{aligned} & 2^{s-1} f\left(\frac{x+y}{2}\right) \\ & + \frac{2^s C(x-y)^2}{4(\alpha+1)(\alpha+2)} \leq \frac{2^{\alpha-1}\Gamma(\alpha+1)}{(y-x)^{\alpha}}\left[\left(J^{\alpha}_{((x+y)/2)^+} f\right)(y) + \left(J^{\alpha}_{((x+y)/2)^-} f\right)(x)\right] \\ & \leq \alpha(f(x)+f(y))\left(\frac{1}{2^{s+1}(\alpha+s)} + 2^{\alpha-1}\beta\left(\frac{1}{2}; s+1, \alpha\right)\right) - \frac{C\alpha(y-x)^2(\alpha+3)}{4(\alpha+1)(\alpha+2)}. \end{aligned} \tag{31}$$

Corollary 5. *For $m = 1$ and $C = 0$ in (21), we get the result for Riemann–Liouville fractional integrals of s-convex function:*

$$\begin{aligned} 2^{s-1} f\left(\frac{x+y}{2}\right) & \leq \frac{2^{\alpha-1}\Gamma(\alpha+1)}{(y-x)^{\alpha}}\left[\left(J^{\alpha}_{((x+y)/2)^+} f\right)(y) + \left(J^{\alpha}_{((x+y)/2)^-} f\right)(x)\right] \\ & \leq \alpha(f(x)+f(y))\left(\frac{1}{2^{s+1}(\alpha+s)} + 2^{\alpha-1}\beta\left(\frac{1}{2}; s+1, \alpha\right)\right). \end{aligned} \tag{32}$$

Corollary 6. *For $m = 1$ and $\alpha = 1$ in (1), we have the Hadamard inequality for strongly s-convex function:*

$$\begin{aligned} & 2^{s-1} f\left(\frac{x+y}{2}\right) + \frac{2^s C(x-y)^2}{24} \leq \frac{1}{y-x}\int_x^y f(u)\mathrm{d}u \\ & \leq \frac{f(x)+f(y)}{s+1} - \frac{C(y-x)^2}{6}. \end{aligned} \tag{33}$$

3. Error Estimations of Riemann–Liouville Fractional Integral Inequalities

The following two lemmas are very useful to obtain the results of this section.

Lemma 1 (see [8]). *Let $f\colon [x, y] \longrightarrow \mathbb{R}$ be a differentiable mapping on (x, y) with $x < y$. If $f' \in Ł[x, y]$, then the following fractional integral equality holds:*

$$\frac{f(x)+f(y)}{2}-\frac{\Gamma(\alpha+1)}{2(y-x)^{\alpha}}\left[\left(J_{x^+}^{\alpha}f\right)(y)+\left(J_{y^-}^{\alpha}f\right)(x)\right]$$
$$=\frac{y-x}{2}\int_0^1\left[(1-t)^{\alpha}-t^{\alpha}\right]f'(tx+(1-t)y)\mathrm{d}t. \tag{34}$$

Lemma 2 (see [10]). *Let $f\colon [x,y]\longrightarrow\mathbb{R}$ be a differentiable mapping on (x,y) with $x<y$. If $f'\in[x,my], m\in(0,1]$, then the following equality for fractional integrals holds:*

$$\frac{2^{\alpha-1}\Gamma(\alpha+1)}{(my-x)^{\alpha}}\left[\left(J_{((x+my)/2)^+}^{\alpha}f\right)(my)+m^{\alpha+1}\left(J_{((x+my)/2m)^-}^{\alpha}f\right)\left(\frac{x}{m}\right)\right]$$
$$-\frac{1}{2}\left[f\left(\frac{x+my}{2}\right)+mf\left(\frac{x+my}{2m}\right)\right]| \tag{35}$$
$$=\frac{mb-a}{4}\left[\int_0^1 t^{\alpha}f'\left(x\frac{t}{2}+m\left(\frac{2-t}{2}\right)y\right)\mathrm{d}t+\int_0^1 t^{\alpha}f'\left(y\frac{t}{2}+\left(\frac{2-t}{2}\right)\frac{x}{m}\right)\mathrm{d}t\right].$$

Theorem 5. *Let $f\colon [x,y]\longrightarrow\mathbb{R}$ be a differentiable mapping on (x,y) with $x<y$. If $|f'|$ is a strongly (s,m)-convex function on $[x,my]$ with modulus $C\geq 0$, $m\neq 0$, and $0<s\leq 1$, then the following fractional integral inequality holds:*

$$\left|\frac{f(x)+f(y)}{2}-\frac{\Gamma(\alpha+1)}{2(y-x)^{\alpha}}\left[\left(J_{x^+}^{\alpha}f\right)(y)+\left(J_{y^-}^{\alpha}f\right)(x)\right]\right|$$
$$\leq\frac{y-x}{2}\left[\left|f'(x)\right|\left(\beta\left(\frac{1}{2};\alpha+1,s+1\right)-\beta\left(\frac{1}{2};s+1,\alpha+1\right)+\frac{1-(1/2)^{\alpha+s}}{\alpha+s+1}\right)\right.$$
$$+m\left|f'\left(\frac{y}{m}\right)\right|\left(\beta\left(\frac{1}{2};\alpha+1,s+1\right)-\beta\left(\frac{1}{2};s+1,\alpha+1\right)+\frac{1-(1/2)^{\alpha+s}}{\alpha+s+1}\right) \tag{36}$$
$$\left.-\frac{2Cm((y/m)-x)^2}{(\alpha+2)(\alpha+3)}\left(1-\frac{\alpha+4}{2^{\alpha+2}}\right)\right],$$

with $\alpha>0$.

Proof. Since $|f'|$ is strongly (s,m)-convex function on $[x,y]$, for $t\in[0,1]$, we have

$$\left|f'(tx+(1-t)y)\right|\leq t^s\left|f'(x)\right|+m(1-t)^s\left|f'\left(\frac{y}{m}\right)\right|-Cmt(1-t)\left(\frac{y}{m}-x\right)^2. \tag{37}$$

By using Lemma 1 and (37), we have

$$
\begin{aligned}
&\left|\frac{f(x)+f(y)}{2}-\frac{\Gamma(\alpha+1)}{2(y-x)^{\alpha}}\left[\left(J_{x^{+}}^{\alpha} f\right)(y)+\left(J_{y^{-}}^{\alpha} f\right)(x)\right]\right| \\
&\leq \frac{y-x}{2} \int_{0}^{1}\left|(1-t)^{\alpha}-t^{\alpha}\right|\left|f'\left(tx+m(1-t)\frac{y}{m}\right)\right| \mathrm{d}t \\
&\leq \frac{y-x}{2} \int_{0}^{1}\left|(1-t)^{\alpha}-t^{\alpha}\right|\left(t^{s}\left|f'(x)\right|+m(1-t)^{s}\left|f'\left(\frac{y}{m}\right)\right|-Cmt(1-t)\left(\frac{y}{m}-x\right)^{2}\right) \mathrm{d}t \\
&\leq \frac{y-x}{2}\left[\int_{0}^{(1/2)}\left((1-t)^{\alpha}-t^{\alpha}\right)\left(t^{s}\left|f'(x)\right|+m(1-t)^{s}\left|f'\left(\frac{y}{m}\right)\right|-Cmt(1-t)\left(\frac{y}{m}-x\right)^{2}\right) \mathrm{d}t\right. \\
&\left.+\int_{(1/2)}^{1}\left(t^{\alpha}-(1-t)^{\alpha}\right)\left(t^{s}\left|f'(x)\right|+m(1-t)^{s}\left|f'\left(\frac{y}{m}\right)\right|-Cmt(1-t)\left(\frac{y}{m}-x\right)^{2}\right) \mathrm{d}t\right] \\
&\leq \frac{y-x}{2}\left[\left|f'(x)\right|\left(\beta\left(\frac{1}{2};\alpha+1,s+1\right)-\beta\left(\frac{1}{2};s+1,\alpha+1\right)+\frac{1-(1/2)^{\alpha+s}}{\alpha+s+1}\right)+m\left|f'\left(\frac{y}{m}\right)\right|\right. \\
&\left.\cdot\left(\beta\left(\frac{1}{2};\alpha+1,s+1\right)-\beta\left(\frac{1}{2};s+1,\alpha+1\right)+\frac{1-(1/2)^{\alpha+s}}{\alpha+s+1}\right)-\frac{2Cm((y/m)-x)^{2}}{(\alpha+2)(\alpha+3)}\left(1-\frac{\alpha+4}{2^{\alpha+2}}\right)\right].
\end{aligned}
\tag{38}
$$

After simplifying the last inequality of (38), we get (36). □

Remark 3

(i) By setting $C=0$ in inequality (36), one can get result for (s,m)-convex function.

(ii) By setting $s=1$ in inequality (36), we get [[18], Theorem 8].

Corollary 7. *By taking $m=1$ in (36), we have the result for Riemann–Liouville fractional integrals of strongly s-convex function:*

$$
\begin{aligned}
&\left|\frac{f(x)+f(y)}{2}-\frac{\Gamma(\alpha+1)}{2(y-x)^{\alpha}}\left[\left(J_{x^{+}}^{\alpha} f\right)(y)+\left(J_{y^{-}}^{\alpha} f\right)(x)\right]\right| \\
&\leq \frac{y-x}{2}\left[\left|f'(x)\right|\left(\beta\left(\frac{1}{2};\alpha+1,s+1\right)-\beta\left(\frac{1}{2};s+1,\alpha+1\right)+\frac{1-(1/2)^{\alpha+s}}{\alpha+s+1}\right)\right. \\
&+\left|f'(y)\right|\left(\beta\left(\frac{1}{2};\alpha+1,s+1\right)-\beta\left(\frac{1}{2};s+1,\alpha+1\right)+\frac{1-(1/2)^{\alpha+s}}{\alpha+s+1}\right) \\
&\left.-\frac{2C(y-x)^{2}}{(\alpha+2)(\alpha+3)}\left(1-\frac{\alpha+4}{2^{\alpha+2}}\right)\right].
\end{aligned}
\tag{39}
$$

Corollary 8. *By taking $m=1$ and $s=1$ in inequality (36), we have the result for Riemann–Liouville fractional integrals of strongly convex function:*

$$
\begin{aligned}
&\left|\frac{f(x)+f(y)}{2}-\frac{\Gamma(\alpha+1)}{2(y-x)^{\alpha}}\left[\left(J_{x^{+}}^{\alpha} f\right)(y)+\left(J_{y^{-}}^{\alpha} f\right)(x)\right]\right| \\
&\leq \frac{y-x}{2}\left[\frac{1-(1/2)^{\alpha}}{(\alpha+1)}\left(\left|f'(x)\right|+\left|f'(y)\right|\right)-\frac{2C(y-x)^{2}}{(\alpha+2)(\alpha+3)}\left(1-\frac{\alpha+4}{2^{\alpha+2}}\right)\right].
\end{aligned}
\tag{40}
$$

Corollary 9. *By taking $m = s = 1$ and $\alpha = 1$ in inequality (36), we get the following inequality:*

$$\left|\frac{f(x)+f(y)}{2} - \frac{1}{y-x}\int_x^y f(u)\mathrm{d}u\right| \leq \frac{y-x}{8}\left(\left|f'(x)\right| + \left|f'(y)\right|\right) - \frac{C(y-x)^3}{32}. \tag{41}$$

Inequality (41) provides the refinement of [[19], Theorem 2.2].

Theorem 6. *Let $f\colon [x, y] \longrightarrow \mathbb{R}$ be a differentiable mapping on (x, y) with $x < y$. If $|f'|^q$ is strongly (s, m)-convex on $[x, my]$ with modulus $C \geq 0$, $(s, m) \in (0, 1]^2$ for $q \geq 1$, then the following inequality for fractional integrals holds:*

$$\begin{aligned}
&\left|\frac{2^{\alpha-1}\Gamma(\alpha+1)}{(my-x)^{\alpha}}\left[\left(J^{\alpha}_{((x+my)/2)^{+}} f\right)(my) + m^{\alpha+1}\left(J^{\alpha}_{((x+my)/2m)^{-}} f\right)\left(\frac{x}{m}\right)\right]\right. \\
&\left.-\frac{1}{2}\left[f\left(\frac{x+my}{2}\right) + mf\left(\frac{x+my}{2m}\right)\right]\right| \leq \frac{my-x}{4(\alpha+1)^{(1/p)}} \\
&\cdot\left[\left(\frac{\left|f'(x)\right|^q}{2^s(\alpha+s+1)} + 2^{\alpha+1}m\left|f'(y)\right|^q\beta\left(\frac{1}{2}; s+1, \alpha+1\right) - \frac{Cm(y-x)^2(\alpha+4)}{4(\alpha+2)(\alpha+3)}\right)^{(1/q)}\right. \\
&\left.+\left(\frac{\left|f'(y)\right|^q}{2^s(\alpha+s+1)} + 2^{\alpha+1}m\left|f'\left(\frac{x}{m^2}\right)\right|^q\beta\left(\frac{1}{2}; s+1, \alpha+1\right) - \frac{Cm\left((x/m^2)-y\right)^2(\alpha+4)}{4(\alpha+2)(\alpha+3)}\right)^{(1/q)}\right].
\end{aligned} \tag{42}$$

Proof. By applying Lemma 2 and strong (s, m)-convexity of $|f'|$, we have

$$\begin{aligned}
&\left|\frac{2^{\alpha-1}\Gamma(\alpha+1)}{(my-x)^{\alpha}}\left[\left(J^{\alpha}_{((x+my)/2)^{+}} f\right)(my) + m^{\alpha+1}\left(J^{\alpha}_{((x+my)/2m)^{-}} f\right)\left(\frac{x}{m}\right)\right] - \frac{1}{2}\left[f\left(\frac{x+my}{2}\right)\right.\right. \\
&\left.\left.+mf\left(\frac{x+my}{2m}\right)\right]\right| \leq \frac{my-x}{4}\left[\int_0^1\left|t^{\alpha}f'\left(\frac{t}{2}x + m\left(\frac{2-t}{2}\right)y\right)\right|\mathrm{d}t\right. \\
&\left.+\int_0^1\left|t^{\alpha}f'\left(\left(\frac{2-t}{2}\right)\frac{x}{m} + \frac{t}{2}y\right)\right|\mathrm{d}t\right] \leq \frac{my-x}{4}\left[\left(\frac{\left|f'(x)\right| + \left|f'(y)\right|}{2^s}\right)\int_0^1 t^{\alpha+s}\mathrm{d}t\right. \\
&+\frac{m}{2^s}\left(\left|f'(y)\right| + \left|f'\left(\frac{x}{m^2}\right)\right|\right)\int_0^1 t^{\alpha}(2-t)^s\mathrm{d}t - \frac{Cm}{4}\left((y-x)^2 + \left(\frac{x}{m^2} - y\right)^2\right) \\
&\left.\cdot\int_0^1 t^{\alpha+1}(2-t)\mathrm{d}t\right] = \frac{my-x}{4}\left[\left(\frac{\left|f'(x)\right| + \left|f'(y)\right|}{2^s(\alpha+s+1)}\right) + 2^{\alpha+1}m\left(\left|f'(y)\right| + \left|f'\left(\frac{x}{m^2}\right)\right|\right)\right. \\
&\left.\cdot\beta\left(\frac{1}{2}; s+1, \alpha+1\right) - \frac{Cm(\alpha+4)}{4(\alpha+2)(\alpha+3)}\left((y-x)^2 + \left(\frac{x}{m^2} - y\right)^2\right)\right].
\end{aligned} \tag{43}$$

Now, for strong (s,m)-convexity of $|f'|^q$, $q>1$, using power mean inequality, we get

$$\begin{aligned}
&\left|\frac{2^{\alpha-1}\Gamma(\alpha+1)}{(my-x)^{\alpha}}\left[\left(J^{\alpha}_{((x+my)/2)^{+}}f\right)(my)+m^{\alpha+1}\left(J^{\alpha}_{((x+my)/2m)^{-}}f\right)\left(\frac{x}{m}\right)\right]-\frac{1}{2}\left[f\left(\frac{x+my}{2}\right)\right.\right.\\
&\left.\left.+mf\left(\frac{x+my}{2m}\right)\right]\right|\leq\frac{my-x}{4}\left(\int_0^1 t^{\alpha}\mathrm{d}t\right)^{1-(1/q)}\left[\left(\int_0^1 t^{\alpha}\left|f'\left(\frac{t}{2}x+m\left(\frac{2-t}{2}\right)y\right)\right|^q\mathrm{d}t\right)^{(1/q)}\right.\\
&\left.+\left(\int_0^1 t^{\alpha}\left|f'\left(\left(\frac{2-t}{2}\right)\frac{x}{m}+\frac{t}{2}y\right)\right|^q\mathrm{d}t\right)^{(1/q)}\right]\leq\frac{my-x}{4(\alpha+1)^{(1/p)}}\\
&\left(\frac{|f(x)|^q}{2^s}\int_0^1 t^{\alpha+s}\mathrm{d}t+\frac{m|f'(y)|^q}{2^s}\int_0^1 t^{\alpha}(2-t)^s\mathrm{d}t-\frac{Cm(y-x)^2}{4}\int_0^1 t^{\alpha+1}(2-t)\mathrm{d}t\right)^{(1/q)}\\
&+\left(\frac{|f'(y)|^q}{2^s}\int_0^1 t^{\alpha+s}\mathrm{d}t+\frac{m}{2^s}\left|f'\left(\frac{x}{m^2}\right)\right|^q\int_0^1 t^{\alpha}(2-t)^s\mathrm{d}t-\frac{Cm\left(\left(x/m^2\right)-y\right)^2}{4}\int_0^1 t^{\alpha+1}(2-t)\mathrm{d}t\right)^{(1/q)}\\
&\leq\frac{my-x}{4(\alpha+1)^{(1/p)}}\left[\left(\frac{|f'(x)|^q}{2^s(\alpha+s+1)}+2^{\alpha+1}m|f'(y)|^q\beta\left(\frac{1}{2};s+1,\alpha+1\right)-\frac{Cm(y-x)^2(\alpha+4)}{4(\alpha+2)(\alpha+3)}\right)^{(1/q)}\right.\\
&\left.+\left(\frac{|f'(y)|^q}{2^s(\alpha+s+1)}+2^{\alpha+1}m\left|f'\left(\frac{x}{m^2}\right)\right|^q\beta\left(\frac{1}{2};s+1,\alpha+1\right)-\frac{Cm\left(\left(x/m^2\right)-y\right)^2(\beta+4)}{4(\beta+2)(\beta+3)}\right)^{(1/q)}\right].
\end{aligned}\tag{44}$$

Hence, we have inequality (42). □

Remark 4

(i) For $s=1$ in inequality (42), we have the result for strongly m-convex function [18].

(ii) For $s=1, m=1$, and $C=0$ in inequality (42), we get [[16], Theorem 5].

(iii) For $s=1, m=1, C=0$, and $\alpha=1$ in inequality (42), we get the inequality proved by Kirmaci in [20].

Corollary 10. *For $s=1$ and $m=1$ in inequality (42), we have the result for Riemann–Liouville fractional integrals of strongly convex function:*

$$\begin{aligned}
&\left|\frac{2^{\alpha-1}\Gamma(\alpha+1)}{(y-x)^{\alpha}}\left[\left(J^{\alpha}_{((x+y)/2)^{+}}f\right)(y)+\left(J^{\alpha}_{((x+y)/2)^{-}}f\right)(x)\right]-f\left(\frac{x+y}{2}\right)\right|\leq\frac{y-x}{4(\alpha+1)(2\alpha+4)^{(1/q)}}\\
&\cdot\left[\left(\left|f'(x)\right|^q(\alpha+1)+\left|f'(y)\right|^q(\alpha+3)-\frac{C(y-x)^2(\alpha+1)(\alpha+4)}{2(\alpha+3)}\right)^{(1/q)}\right.\\
&\left.+\left(\left|f'(y)\right|^q(\alpha+1)+\left|f'(x)\right|^q(\alpha+3)-\frac{C(y-x)^2(\alpha+1)(\alpha+4)}{2(\alpha+3)}\right)^{(1/q)}\right].
\end{aligned}\tag{45}$$

Corollary 11. *For $m = 1$ in inequality (42), we have the result for Riemann–Liouville fractional integrals of strongly s-convex function:*

$$\left|\frac{2^{\alpha-1}\Gamma(\alpha+1)}{(y-x)^{\alpha}}\left[\left(J^{\alpha}_{((x+y)/2)^{+}}f\right)(y)+\left(J^{\alpha}_{(x+y)/2^{-}}f\right)(x)\right]-f\left(\frac{x+y}{2}\right)\right|$$
$$\leq\frac{y-x}{4(\alpha+1)^{(1/p)}}\left[\left(\frac{|f'(x)|^{q}}{2^{s}(\alpha+s+1)}+2^{\alpha+1}|f'(y)|^{q}\beta\left(\frac{1}{2};s+1,\alpha+1\right)-\frac{C(y-x)^{2}(\alpha+4)}{4(\alpha+2)(\alpha+3)}\right)^{(1/q)}\right. \tag{46}$$
$$\left.+\left(\frac{|f''(y)|^{q}}{2^{s}(\alpha+s+1)}+2^{\alpha+1}|f'(x)|^{q}\beta\left(\frac{1}{2};s+1,\alpha+1\right)-\frac{C(y-x)^{2}(\alpha+4)}{4(\alpha+2)(\alpha+3)}\right)^{(1/q)}\right].$$

Theorem 7. *Let $f\colon [x,y]\longrightarrow\mathbb{R}$ be a differentiable mapping on (x,y) with $x<y$. If $|f'|^{q}$ is strongly (s,m)-convex function on $[x,my]$ with modulus $C\geq 0$, $(s,m)\in(0,1]^{2}$ for $q>1$, then the following fractional integral inequality holds:*

$$\left|\frac{2^{\alpha-1}\Gamma(\alpha+1)}{(my-x)^{\alpha}}\left[\left(J^{\alpha}_{((x+my)/2)^{+}}f\right)(my)+m^{\alpha+1}\left(J^{\alpha}_{((x+my)/2m)^{-}}f\right)\left(\frac{x}{m}\right)\right]-\frac{1}{2}\left[f\left(\frac{x+my}{2}\right)\right.\right.$$
$$\left.\left.+mf\left(\frac{x+my}{2m}\right)\right|\leq\frac{(my-x)\left(2^{s}(s+1)\right)^{(1/p)-1}}{4(\alpha p+1)^{(1/p)}}\left[\left(|f'(x)|^{q}+m\left(2^{s+1}-1\right)|f'(y)|^{q}\right.\right.$$
$$\left.-\frac{2^{s}Cm(s+1)(y-x)^{2}}{6}\right)^{(1/q)}+\left(m\left(2^{s+1}-1\right)\left|f'\left(\frac{x}{m^{2}}\right)\right|^{q}+|f'(y)|^{q}\right. \tag{47}$$
$$\left.\left.-\frac{2^{s}Cm(s+1)\left((x/m^{2})-y\right)^{2}}{6}\right)^{(1/q)}\right]\leq\frac{(my-x)\left(2^{s}(s+1)\right)^{(1/p)-1}}{4(\alpha p+1)^{(1/p)}}\left[|f'(x)|+|f'(y)|\right.$$
$$\left.+m\left(2^{s+1}-1\right)\left(\left|f'\left(\frac{x}{m^{2}}\right)\right|+|f'(y)|\right)-\frac{2^{s}Cm(s+1)}{6}\left((y-x)^{2}+\left(\frac{x}{m^{2}}-y\right)^{2}\right)\right],$$

where $\alpha>0$.

Proof. By applying Lemma 2 and then using Hölder inequality and strong (s,m)-convexity of $|f'|^{q}$, we get

$$\left|\frac{2^{\alpha-1}\Gamma(\alpha+1)}{(my-x)^{\alpha}}\left[\left(J^{\alpha}_{((x+my)/2)^{+}}f\right)(my)+m^{\alpha+1}\left(J^{\alpha}_{((x+my)/2m)^{-}}f\right)\left(\frac{x}{m}\right)\right]-\frac{1}{2}\left[f\left(\frac{x+my}{2}\right)\right.\right.$$
$$\left.\left.+mf\left(\frac{x+my}{2m}\right)\right]\right|\leq\frac{my-x}{4}\left(\int_{0}^{1}t^{p\alpha}dt\right)^{1/p}\left[\left(\int_{0}^{1}\left|f'\left(x\frac{t}{2}+m\left(\frac{2-t}{2}\right)y\right)\right|^{q}dt\right)^{1/q}\right.$$
$$\left.+\left(\int_{0}^{1}\left|f'\left(y\frac{t}{2}+\left(\frac{2-t}{2}\right)\frac{x}{m}\right)\right|^{q}dt\right)^{1/q}\right]\leq\frac{my-x}{4}\left(\frac{1}{\alpha p+1}\right)^{1/p}\left[\left(|f'(x)|^{q}\int_{0}^{1}\left(\frac{t}{2}\right)^{s}dt\right.\right.$$
$$+m|f'(y)|^{q}\int_{0}^{1}\left(\frac{2-t}{2}\right)^{s}dt-\frac{Cm(y-x)^{2}}{4}\int_{0}^{1}t(2-t)dt\Big)^{1/q}+\left(|f'(y)|^{q}\int_{0}^{1}\left(\frac{t}{2}\right)^{s}dt\right.$$

$$+m\left|f'\left(\frac{x}{m^2}\right)\right|^q\int_0^1\left(\frac{2-t}{2}\right)^s dt-\frac{Cm((x/m^2)-y)^2}{4}\int_0^1 t(2-t)dt\Bigg)^{1/q}\Bigg]=\frac{(my-x)(s+1)^{(1/p)-1}}{2^{2-s((1/p)-1)}(\alpha p+1)^{(1/p)}}$$
$$\left[\left(\left|f'(x)\right|^q+m\left|f'(y)\right|^q\left(2^{s+1}-1\right)-\frac{2^sCm(s+1)(y-x)^2}{6}\right)^{1/q}+\left(m\left|f'\left(\frac{x}{m^2}\right)\right|^q\left(2^{s+1}-1\right)\right.\right.$$
$$\left.\left.+\left|f'(y)\right|^q-\frac{2^sCm(s+1)((x/m^2)-y)^2}{6}\right)^{1/q}\right]\leq\frac{(my-x)\left(2^s(s+1)\right)^{(1/p)-1}}{4(\alpha p+1)^{(1/p)}}\left[\left|f'(x)\right|+\left|f'(y)\right|\right.$$
$$\left.+m\left(2^{s+1}-1\right)\left(\left|f'(y)\right|+\left|f'\left(\frac{x}{m^2}\right)\right|\right)-\frac{2^sCm(s+1)}{6}\left((y-x)^2+\left(\frac{x}{m^2}-y\right)^2\right)\right]. \tag{48}$$

We have used $A^q+B^q\leq(A+B)^q$, for $A\geq 0, B\geq 0$. This completes the proof. □

Remark 5

(i) For $s=1$ in inequality (47), we get [[18], Theorem 10].

(ii) For $s=1$ and $C=0$ in inequality (47), we get [[10], Theorem 2.7].

(iii) For $s=1, m=1$, and $C=0$ in inequality (47), we get [[16], Theorem 6].

Corollary 12. *For $\alpha=1$ and $m=1$, we have the result for s-convex function:*

$$\left|\frac{1}{y-x}\int_x^y f(u)du-f\left(\frac{x+y}{2}\right)\right|\leq\frac{(y-x)\left(2^s(s+1)\right)^{(1/p)-1}}{4(p+1)^{(1/p)}}$$
$$\cdot\left[\left(\left|f'(x)\right|^q+\left(2^{s+1}-1\right)\left|f'(y)\right|^q-\frac{2^sC(s+1)(y-x)^2}{6}\right)^{1/q}\right.$$
$$\left.+\left(\left(2^{s+1}-1\right)\left|f'(x)\right|^q+\left|f'(y)\right|^q-\frac{2^sC(s+1)(y-x)^2}{6}\right)^{1/q}\right] \tag{49}$$
$$\leq\frac{(y-x)\left(2^s(s+1)\right)^{(1/p)-1}}{4(p+1)^{(1/p)}}\left[2^{s+1}\left(\left|f'(x)\right|+\left|f'(y)\right|\right)-\frac{2^sC(s+1)}{3}(y-x)^2\right].$$

Corollary 13. *For $\alpha=1$ and $m=q=1$, we have*

$$\left|\frac{1}{y-x}\int_x^y f(u)du-f\left(\frac{x+y}{2}\right)\right|\leq\frac{y-x}{4(s+1)}\left[2\left(\left|f'(x)\right|+\left|f'(y)\right|\right)-\frac{C(s+1)}{3}(y-x)^2\right]. \tag{50}$$

Corollary 14. *For $\alpha=1$ and $m=q=s=1$, we have*

$$\left|\frac{1}{y-x}\int_x^y f(u)du-f\left(\frac{x+y}{2}\right)\right|\leq\frac{y-x}{4}\left[\left(\left|f'(x)\right|+\left|f'(y)\right|\right)-\frac{C}{3}(y-x)^2\right]. \tag{51}$$

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

References

[1] S. S. Dragomir and C. E. M. Pearce, *Selected Topics on Hermite-Hadamard Inequalities and Applications, RGMIA Monographs*, Victoria University, Footscray, Australia, 2000.

[2] J. Park, "Some Hadamard's type inequalities for co-ordinated (s,m)-convex mappings in the second sense," *Far East Journal of Mathematical Sciences*, vol. 51, pp. 205–216, 2011.

[3] M. Bracamonte, J. Gimenez, and M. Vivas-Cortez, "Hermite-hadamard-fejer type inequalities for strongly (s, m)-Convex functions with modulus c, in second sense$(s,m)C$-convex functions with modulus in the second sense," *Applied*

Mathematics & Information Sciences, vol. 10, no. 6, pp. 2045–2053, 2016.

[4] H. Hudzik and L. Maligranda, "Some remarks on s-convex functions," *Aequationes Mathematicae*, vol. 48, pp. 100–111, 1994.

[5] G. Toader, "Some generalizations of the convexity," in *Proceedings of the Colloquium on Approximation and Optimization*, pp. 329–338, University Cluj-Napoca, Trento, Italy, October 1985.

[6] B. T. Polyak, "Existence theorems and convergence of minimizing sequences in extremum problems with restrictions," *Soviet Mathematics Doklady*, vol. 7, pp. 72–75, 1966.

[7] T. Lara, N. Merentes, R. Quintero, and E. Rosales, "On strongly m-convex functions," *Mathematica Aeterna*, vol. 5, no. 3, pp. 521–535, 2015.

[8] M. Z. Sarikaya, E. Set, H. Yaldiz, and N. Basak, "Hermite-Hadamard's inequalities for fractional integrals and related fractional inequalities," *Mathematical and Computer Modelling*, vol. 57, no. 9-10, pp. 2403–2407, 2013.

[9] S. I. Butt, A. O. Akdemir, J. Nasir, and F. Jarad, "Some Hermite-Jensen-Mercer like inequalities for convex functions through a certain generalized fractional integrals and related results," *Miskolc Mathematical Notes*, vol. 21, no. 2, pp. 689–715, 2020.

[10] G. Farid, A. U. Rehman, and B. Tariq, "On Hadamard-type inequalities for -convex functions via Riemann-Liouville fractional integrals," *Studia Universitatis Babes-Bolyai Mathematica*, vol. 62, no. 2, pp. 141–150, 2017.

[11] M. Gurbuz and M. E. Ozdemir, "On some inequalities for product of different kinds of convex functions," *Turkish Journal of Science*, vol. 5, no. 1, pp. 23–27, 2020.

[12] S. M. Kang, G. Abbas, G. Farid, and W. Nazeer, "A generalized Fejér–Hadamard inequality for harmonically convex functions via generalized fractional integral operator and related results," *Mathematics*, vol. 6, no. 7, p. 122, 2018.

[13] S. M. Kang, G. Farid, W. Nazeer, and S. Naqvi, "A version of the Hadamard inequality for Caputo fractional derivatives and related results," *Journal of Computational Analysis and Applications*, vol. 27, no. 6, pp. 962–972, 2019.

[14] S. Mehmood and G. Farid, "Fractional integrals inequalities for exponentially m-convex functions," *Open Journal of Mathematical Sciences*, vol. 4, pp. 78–85, 2020.

[15] X. Qiang, G. Farid, J. Pečarić, and S. B. Akbar, "Generalized fractional integral inequalities for exponentially (s, m)-convex functions," *Journal of Inequalities and Applications*, vol. 70, p. 2020, 2020.

[16] M. Z. Sarikaya and H. Yildirim, "On Hermite-Hadamard type inequalities for Riemann-Liouville fractional integrals," *Miskolc Mathematical Notes*, vol. 17, no. 2, pp. 1049–1059, 2017.

[17] E. Set, M. Z. Sarikaya, M. E. Ozdemir, and H. Yildirim, "The Hermite-Hadamarad's inequality for some convex functions via fractional integrals and related results," *JAMSI*, vol. 10, no. 2, 2014.

[18] Y. C. Kwun, G. Farid, S. B. Akbar, and S. M. Kang, "Riemann-Liouville fractional version of Hadamard inequality for strongly m-convex functions," submitted.

[19] S. S. Dragomir and R. P. Agarwal, "Two inequalities for differentiable mappings and applications to special means of real numbers and to trapezoidal formula," *Applied Mathematics Letters*, vol. 11, no. 5, pp. 91–95, 1998.

[20] U. S. Kirmaci, "Inequalities for differentiable mappings and applications to special means of real numbers to midpoint formula," *Applied Mathematics and Computation*, vol. 147, no. 1, pp. 137–146, 2004.