Research Article

A Criterion for the Existence of the Unique Periodic Solution of One-Dimensional Periodic Differential Equation

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In this paper, we discuss one-dimensional differential equation with ω -period. By using the fixed point theory, the existence of a periodic solution is obtained; by using the second Lyapunov method, the uniqueness and stability of the periodic solution are obtained.

1. Introduction

Consider the first-order nonautonomous differential equation:

$$\frac{dx}{dt} = f(t, x), \quad (1)$$

where $t, x \in \mathbb{R}$ and $f(t, x)$ is continuous and is an ω -periodic function on t .

As we all know, the first-order differential equation (1) is widely used to establish mathematical models in many fields, such as physics, biology, economy, and medicine. Because the nonautonomous differential equations have important practical applications, it has been a research subject of many scientists for a long time. The main problems studied are the existence, stability, and number of periodic solutions (see [1–8]).

In [9], the authors gave several methods to study periodic solutions of one-dimensional ω -periodic differential equations, including Poincaré's map, the stability of zero solution and its multiplicity, normal form, and averaging method. They summarized and improved some existing results and obtained some new conclusions as well.

As we all know, the existence, uniqueness, and stability of periodic solutions of differential equations have always been an important research hotspot in the field of differential equations (see [10–20]). However, the above literatures are

basically about the study of periodic solutions of specific equations, rather than the study of general differential equations.

In [21], the existence of a periodic solution for Abel's differential equation is obtained first by using the fixed-point theorem. By constructing the Lyapunov function, the uniqueness, and stability of the periodic solution of the equation are obtained.

Stimulated by the works of [21], in this paper, we consider more general differential equation (1). By using the fixed point theorem and constructing Lyapunov function, we give a new criterion for the existence, uniqueness, and stability of the periodic solutions of the equation (1). These results generalize some related results in some literatures.

2. Some Lemmas and Abbreviations

Lemma 1 (see [22]). *Consider the equation:*

$$\frac{dx}{dt} = a(t)x + b(t), \quad (2)$$

where $a(t)$ and $b(t)$ are ω -periodic continuous functions on $\mathbb{R}$. If $\int_0^\omega a(t)dt \neq 0$, then (2) has a unique ω -periodic continuous solution $\eta(t)$, $\text{mod}(\eta(t)) \subseteq \text{mod}(a(t), b(t))$, and $\eta(t)$ can be written as

$$\eta(t) = \begin{cases} \int_{-\infty}^t e^{\int_s^t a(\mu) d\mu} b(s) ds, & \int_0^\omega a(t) dt < 0, \\ -\int_t^{+\infty} e^{\int_s^t a(\mu) d\mu} b(s) ds, & \int_0^\omega a(t) dt > 0. \end{cases} \quad (3)$$

Lemma 2 (see [23]). Assume that an ω -periodic sequence $\{f_n(t)\}$ is convergent uniformly on any compact set of $\mathbb{R}$, $f(t)$ is an ω -periodic function, and $\text{mod}(f_n) \subseteq \text{mod}(f)$ ($n = 1, 2, \dots$); then, $\{f_n(t)\}$ is convergent uniformly on $\mathbb{R}$.

Lemma 3 (see [24]). Assume $\mathbb{V}$ is a metric space and $\mathbb{C}$ is a convex closed set of $\mathbb{V}$; its boundary is $\partial\mathbb{C}$ if $T: \mathbb{V} \rightarrow \mathbb{V}$ is a continuous compact mapping, such that $T(\partial\mathbb{C}) \subseteq \mathbb{C}$, then T has at least a fixed point on $\mathbb{C}$.

For convenience, assume that $f(t)$ is an ω -periodic continuous function on $\mathbb{R}$, we denote

$$\begin{aligned} f_M &= \sup_{t \in [0, \omega]} f(t), \\ f_L &= \inf_{t \in [0, \omega]} f(t). \end{aligned} \quad (4)$$

3. Main Results

In this section, firstly, we use the mean value theorem to transform equation (1) into an equivalent equation. Then, we use the fixed point theorem to get the existence of periodic solution of equation (1). Finally, we use Lyapunov function method to obtain the uniqueness and stability of the periodic solution.

Theorem 1. Consider equation (1), $f: \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$ is continuous, $f(t + \omega, x) = f(t, x)$, and f'_x exists and is a continuous function, and $r(t)$ is an ω -periodic continuous function on $\mathbb{R}$. Assume that the following conditions hold:

$$\begin{aligned} (H_1) f[t, r(t)] &= 0, \\ (H_2) f'_x(t, x) &\leq 0 \quad (\equiv 0), \end{aligned} \quad (5)$$

then (1) equation (1) has a unique ω -periodic continuous solution $\Phi(t)$ and (2) $\Phi(t)$ is globally attractive.

Proof. (1) By (H_1) and differential mean value theorem, equation (1) is transformed into the equation:

$$\begin{aligned} \frac{dx}{dt} &= f(t, x) \\ &= f(t, x) - f[t, r(t)] \\ &= f'_x[t, r(t) + \theta_1(x)(x - r(t))](x - r(t)), \\ &\quad (0 < \theta_1(x) < 1). \end{aligned} \quad (6)$$

Noting that $f: \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$ is continuous, $f(t + \omega, x) = f(t, x)$, f'_x exists and is a continuous

function, and from (6), it follows that $f'_x[t, r(t) + \theta_1(x)(x - r(t))]$ is continuous and

$$\begin{aligned} &f'_x[t + \omega, r(t + \omega) + \theta_1(x)(x - r(t + \omega))] \\ &= f'_x[t, r(t) + \theta_1(x)(x - r(t))]. \end{aligned} \quad (7)$$

Defining

$$\mathbb{S} = \{\psi(t) \in C(\mathbb{R}, \mathbb{R}) | \psi(t + \omega) = \psi(t)\}, \quad (8)$$

$\forall \varphi(t), \psi(t) \in \mathbb{S}$, the distance is defined as follows:

$$\rho(\varphi, \psi) = \sup_{t \in [0, \omega]} |\varphi(t) - \psi(t)|, \quad (9)$$

and thus, $(\mathbb{S}, \rho)$ is a complete metric space.

Take a convex closed set $\mathbb{B}$ of $\mathbb{S}$ as follows:

$$\mathbb{B} = \{\psi(t) \in \mathbb{S} | r_L \leq \psi(t) \leq r_M, \text{mod}(\psi) \subseteq \text{mod}(f)\}, \quad (10)$$

$\forall \psi(t) \in \mathbb{B}$, and consider the equation:

$$\begin{aligned} \frac{dx}{dt} &= f'_x[t, r(t) + \theta_1(\psi(t))(\psi(t) - r(t))](x - r(t)) \\ &= f'_x[t, r(t) + \theta_1(\psi(t))(\psi(t) - r(t))]x \\ &\quad - f'_x[t, r(t) + \theta_1(\psi(t))(\psi(t) - r(t))]r(t) \\ &= f'_{x_\psi}(t)x - f'_{x_\psi}(t)r(t), \end{aligned} \quad (11)$$

where

$$f'_x[t, r(t) + \theta_1(\psi(t))(\psi(t) - r(t))] \triangleq f'_{x_\psi}(t). \quad (12)$$

It follows from (7), (10), and (12) that $f'_{x_\psi}(t)$ is an ω -periodic continuous function on $t \in \mathbb{R}$; thus, there is a positive number δ such that

$$|f'_x[t, r(t) + \theta_1(\psi(t))(\psi(t) - r(t))]| \leq \delta. \quad (13)$$

It follows from (H_2) that

$$\int_0^\omega f'_{x_\psi}(t) dt < 0. \quad (14)$$

Since $f'_{x_\psi}(t)$ and $r(t)$ are ω -periodic continuous functions, it follows

$$f'_{x_\psi}(t)r(t) \quad (15)$$

is an ω -periodic continuous function. It follows from (14) and Lemma 1 that equation (11) has a unique ω -periodic continuous solution as follows:

$$\eta(t) = - \int_{-\infty}^t e^{\int_s^t f'_{x_\psi}(\mu) d\mu} f'_{x_\psi}(s)r(s) ds, \quad (16)$$

$$\text{mod}(\eta) \subseteq \text{mod}\left(f'_{x_\psi}(t), f'_{x_\psi}(t)r(t)\right). \quad (17)$$

It follows from (6), (10), and (11) that

$$\begin{aligned} \text{mod}\left(f'_{x_\psi}(t)\right) &\subseteq \text{mod}(f), \\ \text{mod}\left(f'_{x_\psi}(t)r(t)\right) &\subseteq \text{mod}(f), \end{aligned} \quad (18)$$

and hence, we have

$$\text{mod}(\eta) \subseteq \text{mod}(f). \quad (19)$$

It follows from (10), (14), and (16) that

$$\begin{aligned} \eta(t) &\geq -r_L \int_{-\infty}^t e^{\int_s^t f'_{x_\psi}(\mu) d\mu} f'_{x_\psi}(s) ds \\ &= r_L \int_{-\infty}^t e^{\int_s^t f'_{x_\psi}(\mu) d\mu} d\left(\int_s^t f'_{x_\psi}(\mu) d\mu\right) \\ &= r_L \left[e^{\int_s^t f'_{x_\psi}(\mu) d\mu} \right]_{-\infty}^t \\ &= r_L \left[1 - e^{\int_{-\infty}^t f'_{x_\psi}(\mu) d\mu} \right] (-\infty < t < +\infty) \\ &= r_L, \\ \eta(t) &\leq -r_M \int_{-\infty}^t e^{\int_s^t f'_{x_\psi}(\mu) d\mu} f'_{x_\psi}(s) ds \\ &= r_M \int_{-\infty}^t e^{\int_s^t f'_{x_\psi}(\mu) d\mu} d\left(\int_s^t f'_{x_\psi}(s) d\mu\right) \\ &= r_M \left[e^{\int_s^t f'_{x_\psi}(\mu) d\mu} \right]_{-\infty}^t \\ &= r_M \left[1 - e^{\int_{-\infty}^t f'_{x_\psi}(\mu) d\mu} \right] (-\infty < t < +\infty) \\ &= r_M, \end{aligned} \quad (20)$$

and therefore, $\eta(t) \in \mathbb{B}$.

Define a mapping as follows:

$$(T\psi)(t) = - \int_{-\infty}^t e^{\int_s^t f'_{x_\psi}(\mu) d\mu} f'_{x_\psi}(s) r(s) ds, \quad (21)$$

and thus, if given any $\psi(t) \in \mathbb{B}$, then $(T\psi)(t) \in \mathbb{B}$; hence, $T: \mathbb{B} \longrightarrow \mathbb{B}$.

Now, we prove that the mapping T is a compact mapping.

Consider any sequence $\{\psi_n(t)\} \subseteq \mathbb{B} (n = 1, 2, \dots)$; then, it follows

$$r_L \leq \psi_n(t) \leq r_M, \text{mod}(\psi_n) \subseteq \text{mod}(f), \quad (n = 1, 2, \dots). \quad (22)$$

On the other hand, $(T\psi_n)(t) = x_{\psi_n}(t)$ satisfies

$$\begin{aligned} \frac{dx_{\psi_n}(t)}{dt} &= f'_x[t, r(t) + \theta_2(\psi_n(t))(\psi_n(t) - r(t))]x_{\psi_n}(t) \\ &\quad - f'_x[t, r(t) + \theta_2(\psi_n(t))(\psi_n(t) - r(t))]r(t) \\ s &= f'_{x_{\psi_n}}(t)x_{\psi_n}(t) - f'_{x_{\psi_n}}(t)r(t), \end{aligned} \quad (23)$$

where

$$f'_x[t, r(t) + \theta_2(\psi_n(t))(\psi_n(t) - r(t))] \triangleq f'_{x_{\psi_n}}(t). \quad (24)$$

Since $f: \mathbb{R} \times \mathbb{R} \longrightarrow \mathbb{R}$ is continuous, $f(t + \omega, x) = f(t, x)$, f'_x exists and is a continuous function, and from (23) and $\{\psi_n(t)\} \subseteq \mathbb{B} (n = 1, 2, \dots)$, it follows that $f'_{x_{\psi_n}}(t)$ is an ω -periodic continuous function on $t \in \mathbb{R}$.

It follows from (10), (13), (22), and (24) that

$$|f'_x[t, r(t) + \theta_2(\psi_n(t))(\psi_n(t) - r(t))]| \leq \delta. \quad (25)$$

Thus, we have

$$\left| \frac{dx_{\psi_n}(t)}{dt} \right| \leq 2|f'_{x_{\psi_n}}(t)|r|_M \leq 2\delta|r|_M, \quad (26)$$

$$\text{mod}(x_{\psi_n}(t)) \subseteq \text{mod}(f). \quad (27)$$

Hence, $\{dx_{\psi_n}(t)/dt\}$ is uniformly bounded; therefore, $\{x_{\psi_n}(t)\}$ is uniformly bounded and equi-continuous on $\mathbb{R}$. By the theorem of Ascoli-Arzelà, for any sequence $\{x_{\psi_n}(t)\} \subseteq \mathbb{B}$, there exists a subsequence (also denoted by $\{x_{\psi_n}(t)\}$) such that $\{x_{\psi_n}(t)\}$ is convergent uniformly on any compact set of $\mathbb{R}$. By (27), combined with Lemma 2, $\{x_{\psi_n}(t)\}$ is convergent uniformly on $\mathbb{R}$, that is to say, T is relatively compact on $\mathbb{B}$.

Next, we prove that T is a continuous mapping.

Assume $\{\psi_n(t)\} \subseteq \mathbb{B}$, $\psi(t) \in \mathbb{B}$, and

$$\psi_n(t) \longrightarrow \psi(t), \quad (n \longrightarrow \infty). \quad (28)$$

Since $f(t, x)$ has first-order continuous partial derivative on x , it follows from (28) that

$$\begin{aligned} &f'_x[t, r(t) + \theta_2(\psi_n(t))(\psi_n(t) - r(t))] \\ &\longrightarrow f'_x[t, r(t) + \theta_1(\psi(t))(\psi(t) - r(t))], \end{aligned} \quad (29)$$

$(n \longrightarrow \infty),$

that is,

$$f'_{x_{\psi_n}}(t) \longrightarrow f'_{x_\psi}(t), \quad (n \longrightarrow \infty). \quad (30)$$

It follows from (21) that

$$\begin{aligned}
& |(T\psi_n)(t) - (T\psi)(t)| \\
&= \left| \int_{-\infty}^t e^{\int_s^t f'_{x_{\psi_n}}(\mu) d\mu} f'_{x_{\psi_n}}(s) r(s) ds - \int_{-\infty}^t e^{\int_s^t f'_{x_{\psi}}(\mu) d\mu} f'_{x_{\psi}}(s) r(s) ds \right| \\
&= \left| \int_{-\infty}^t e^{\int_s^t f'_{x_{\psi_n}}(\mu) d\mu} (f'_{x_{\psi_n}}(s) - f'_{x_{\psi}}(s)) r(s) ds \right. \\
&\quad \left. + \int_{-\infty}^t \left(e^{\int_s^t f'_{x_{\psi_n}}(\mu) d\mu} - e^{\int_s^t f'_{x_{\psi}}(\mu) d\mu} \right) f'_{x_{\psi}}(s) r(s) ds \right| \\
&= \left| \int_{-\infty}^0 e^{\int_{s+t}^t f'_{x_{\psi_n}}(\mu) d\mu} (f'_{x_{\psi_n}}(t+s) - f'_{x_{\psi}}(t+s)) r(t+s) ds \right. \\
&\quad \left. + \int_{-\infty}^0 \left(e^{\int_{s+t}^t f'_{x_{\psi_n}}(\mu) d\mu} - e^{\int_{s+t}^t f'_{x_{\psi}}(\mu) d\mu} \right) f'_{x_{\psi}}(t+s) r(t+s) ds \right| \\
&= \left| \int_{-\infty}^0 e^{\int_{s+t}^t f'_{x_{\psi_n}}(\mu) d\mu} (f'_{x_{\psi_n}}(t+s) - f'_{x_{\psi}}(t+s)) r(t+s) ds \right. \\
&\quad \left. + \int_{-\infty}^0 e^{\xi} \left(\int_{s+t}^t (f'_{x_{\psi_n}}(\mu) - f'_{x_{\psi}}(\mu)) d\mu \right) f'_{x_{\psi}}(t+s) r(t+s) ds \right| \\
&\leq \left(\int_{-\infty}^0 e^{\int_{s+t}^t f'_{x_{\psi_n}}(\mu) d\mu} |r(t+s)| ds + \int_{-\infty}^0 e^{\xi} \left(\int_{s+t}^t d\mu \right) |f'_{x_{\psi}}(s+t)| |r(s+t)| ds \right) \rho(f'_{x_{\psi_n}}, f'_{x_{\psi}}) \\
&= \left(\int_{-\infty}^{-T_0} e^{\int_{t+s}^t f'_{x_{\psi_n}}(\mu) d\mu} |r(t+s)| ds + \int_{-T_0}^0 e^{\int_{t+s}^t f'_{x_{\psi_n}}(\mu) d\mu} |r(t+s)| ds \right. \\
&\quad \left. + \int_{-\infty}^{-T_0} e^{\xi} \left(\int_{t+s}^t d\mu \right) |f'_{x_{\psi}}(s+t)| |r(s+t)| ds \right. \\
&\quad \left. + \int_{-T_0}^0 e^{\xi} \left(\int_{t+s}^t d\mu \right) |f'_{x_{\psi}}(s+t)| |r(s+t)| ds \right) \rho(f'_{x_{\psi_n}}, f'_{x_{\psi}}),
\end{aligned} \tag{31}$$

where ξ is between $\int_{s+t}^t f'_{x_{\psi_n}}(\mu) d\mu$ and $\int_{s+t}^t f'_{x_{\psi}}(\mu) d\mu$. Since $f'_{x_{\psi_n}}(t)$ and $f'_{x_{\psi}}(t)$ are ω -periodic continuous functions on $t \in \mathbb{R}$, it follows

$$\begin{aligned}
\lim_{s \rightarrow -\infty} \frac{1}{-s} \int_s^0 f'_{x_{\psi_n}}(\mu) d\mu &= \frac{1}{\omega} \int_0^\omega f'_{x_{\psi_n}}(\mu) d\mu \triangleq -\lambda_1, \\
\lim_{s \rightarrow -\infty} \frac{1}{-s} \int_s^0 f'_{x_{\psi}}(\mu) d\mu &= \frac{1}{\omega} \int_0^\omega f'_{x_{\psi}}(\mu) d\mu \triangleq -\lambda_2.
\end{aligned} \tag{32}$$

Thus, there is a $T_0 > 0$ (also, assume it is T_0 in the above) when $s \leq -T_0$, such that

$$\begin{aligned}
\left| \frac{-1}{s} \int_{t+s}^t f'_{x_{\psi_n}}(\mu) d\mu + \lambda_1 \right| &< \frac{\lambda_1}{2}, \\
\left| \frac{-1}{s} \int_{t+s}^t f'_{x_{\psi}}(\mu) d\mu + \lambda_2 \right| &< \frac{\lambda_2}{2},
\end{aligned} \tag{33}$$

that is,

$$\begin{aligned}
\int_{t+s}^t f'_{x_{\psi_n}}(\mu) d\mu &< \frac{\lambda_1 s}{2}, \\
\int_{t+s}^t f'_{x_{\psi}}(\mu) d\mu &< \frac{\lambda_2 s}{2}.
\end{aligned} \tag{34}$$

Set

$$\frac{\lambda_0 s}{2} = \max\left(\frac{\lambda_1 s}{2}, \frac{\lambda_2 s}{2}\right). \tag{35}$$

Hence, we have

$$\begin{aligned}
& |(T\psi_n)(t) - (T\psi)(t)| \\
&\leq \left(\int_{-\infty}^{-T_0} e^{(\lambda_0 s/2)} |r|_M ds + T_0 e^{\delta T_0} |r|_M - \int_{-\infty}^{-T_0} e^{(\lambda_0 s/2)} s \delta |r|_M ds + T_0^2 e^{\delta T_0} |r|_M \right) \rho(f'_{x_{\psi_n}}, f'_{x_{\psi}}) \\
&= |r|_M \left(\frac{2}{\lambda_0} e^{-(\lambda_0 T_0/2)} + T_0 e^{\delta T_0} + \frac{4}{\lambda_0^2} e^{-(\lambda_0 T_0/2)} \delta + T_0^2 e^{\delta T_0} \delta \right) \rho(f'_{x_{\psi_n}}, f'_{x_{\psi}}).
\end{aligned} \tag{36}$$

It follows from (30) and above inequality that

$$(T\psi_n)(t) \longrightarrow (T\psi)(t), \quad (n \longrightarrow \infty). \quad (37)$$

Therefore, T is continuous. By (21), easy to see, $T(\partial\mathbb{B}) \subseteq \mathbb{B}$. According to Lemma 3, T has at least a fixed point on $\mathbb{B}$, the fixed point is the ω -periodic continuous solution $\Phi(t)$ of equation (1), and

$$r_L \leq \Phi(t) \leq r_M. \quad (38)$$

(2) Construct a Lyapunov function as follows:

$$V(t, x(t)) = (x(t) - \Phi(t))^2, \quad (39)$$

where $x(t)$ is the unique solution with initial value $x(t_0) = x_0$ of equation (1) and $\Phi(t)$ is the periodic solution of equation (1).

Differentiating both sides of (39) along the solution of equation (1), we have

$$\begin{aligned} \frac{dV(t, x(t))}{dt} &= 2(x(t) - \Phi(t)) \left[\frac{dx(t)}{dt} - \frac{d\Phi(t)}{dt} \right] \\ &= 2(x(t) - \Phi(t)) [f(t, x(t)) - f(t, \Phi(t))] \\ &= 2(x(t) - \Phi(t)) [f'_x(t, \zeta)(x(t) - \Phi(t))] \\ &= 2f'_x(t, \zeta)(x(t) - \Phi(t))^2, \end{aligned} \quad (40)$$

where ζ is between $x(t)$ and $\Phi(t)$.

By (H_2) , it follows

$$|x(t) - \Phi(t)| \longrightarrow 0, \quad (t \longrightarrow +\infty). \quad (41)$$

Thus, $x(t)$ cannot be a periodic solution of equation (1), and it is proved that equation (1) has a unique ω -periodic continuous solution $\Phi(t)$ which is globally attractive.

This is the end of the proof of Theorem 1. $\square$

Theorem 2. Consider equation (1), $f: \mathbb{R} \times \mathbb{R} \longrightarrow \mathbb{R}$ is continuous and $f(t + \omega, x) = f(t, x)$, and f'_x exists and is a continuous function; $r(t)$ is an ω -periodic continuous function on $\mathbb{R}$. Assume that the following conditions hold:

$$\begin{aligned} (H_1) f[t, r(t)] &= 0, \\ (H_2) f'_x(t, x) &\geq 0 \quad (\equiv 0). \end{aligned} \quad (42)$$

Then, (1) equation (1) has a unique ω -periodic continuous solution $\Phi(t)$ and (2) $\Phi(t)$ is unstable.

Proof.

(1) By (H_1) and differential mean value theorem, equation (1) is transformed into the equation

$$\begin{aligned} \frac{dx}{dt} &= f(t, x) = f(t, x) - f[t, r(t)] \\ &= f'_x[t, r(t) + \theta_1(x)(x - r(t))](x - r(t)), \\ &\quad (0 < \theta_1(x) < 1). \end{aligned} \quad (43)$$

Noting that $f: \mathbb{R} \times \mathbb{R} \longrightarrow \mathbb{R}$ is continuous, $f(t + \omega, x) = f(t, x)$, f'_x exists and is a continuous function, and from (43), it follows that $f'_x[t, r(t) + \theta_1(x)(x - r(t))]$ is continuous and

$$\begin{aligned} f'_x[t + \omega, r(t + \omega) + \theta_1(x)(x - r(t + \omega))] \\ = f'_x[t, r(t) + \theta_1(x)(x - r(t))]. \end{aligned} \quad (44)$$

Defining

$$\mathbb{S} = \{\psi(t) \in C(\mathbb{R}, \mathbb{R}) | \psi(t + \omega) = \psi(t)\}, \quad (45)$$

$\forall \varphi(t), \psi(t) \in \mathbb{S}$, the distance is defined as follows:

$$\rho(\varphi, \psi) = \sup_{t \in [0, \omega]} |\varphi(t) - \psi(t)|. \quad (46)$$

Thus, $(\mathbb{S}, \rho)$ is a complete metric space.

Take a convex closed set $\mathbb{B}$ of $\mathbb{S}$ as follows:

$$\mathbb{B} = \{\psi(t) \in \mathbb{S} | r_L \leq \psi(t) \leq r_M, \text{mod}(\psi) \subseteq \text{mod}(f)\}, \quad (47)$$

$\forall \psi(t) \in \mathbb{B}$, and consider the equation

$$\begin{aligned} \frac{dx}{dt} &= f'_x[t, r(t) + \theta_1(\psi(t))(\psi(t) - r(t))](x - r(t)) \\ &= f'_x[t, r(t) + \theta_1(\psi(t))(\psi - r(t))]x - f'_x[t, r(t) \\ &\quad + \theta_1(\psi(t))(\psi - r(t))]r(t) \\ &= f'_{x_\psi}(t)x - f'_{x_\psi}(t)r(t), \end{aligned} \quad (48)$$

where

$$f'_x[t, r(t) + \theta_1(\psi(t))(\psi - r(t))] \triangleq f'_{x_\psi}(t). \quad (49)$$

It follows from (44), (47), and (49) that $f'_{x_\psi}(t)$ is an ω -periodic continuous function on $t \in \mathbb{R}$; thus, there is a positive number δ such that

$$|f'_x[t, r(t) + \theta_1(\psi(t))(\psi(t) - r(t))]| \leq \delta. \quad (50)$$

It follows from (H_2) that

$$\int_0^\omega f'_{x_\psi}(t) dt > 0. \quad (51)$$

Since $f'_{x_\psi}(t)$ and $r(t)$ are ω -periodic continuous functions, it follows

$$f'_{x_\psi}(t)r(t) \quad (52)$$

is an ω -periodic continuous function. It follows from (51) and Lemma 1 that equation (48) has a unique ω -periodic continuous solution as follows:

$$\eta(t) = \int_t^{+\infty} e^{\int_s^t f'_{x_\psi}(\mu) d\mu} f'_{x_\psi}(s) r(s) ds, \quad (53)$$

$$\text{mod}(\eta) \subseteq \text{mod}\left(f'_{x_\psi}(t), f'_{x_\psi}(t) r(t)\right). \quad (54)$$

It follows from (43), (47), and (48) that

$$\text{mod}\left(f'_{x_\psi}(t)\right) \subseteq \text{mod}(f), \quad (55)$$

$$\text{mod}\left(f'_{x_\psi}(t) r(t)\right) \subseteq \text{mod}(f).$$

Hence, we have

$$\text{mod}(\eta) \subseteq \text{mod}(f). \quad (56)$$

It follows from (47), (51), and (53) that

$$\begin{aligned} \eta(t) &\geq r_L \int_t^{+\infty} e^{\int_s^t f'_{x_\psi}(\mu) d\mu} f'_{x_\psi}(s) ds \\ &= -r_L \int_t^{+\infty} e^{\int_s^t f'_{x_\psi}(\mu) d\mu} d\left(\int_s^t f'_{x_\psi}(\mu) d\mu\right) \\ &= -r_L \left[e^{\int_s^t f'_{x_\psi}(\mu) d\mu} \right]_t^{+\infty} \\ &= -r_L \left[e^{\int_{+\infty}^t f'_{x_\psi}(\mu) d\mu} - 1 \right], \quad (-\infty < t < +\infty) \\ &= r_L, \end{aligned}$$

$$\begin{aligned} \eta(t) &\leq r_M \int_t^{+\infty} e^{\int_s^t f'_{x_\psi}(\mu) d\mu} f'_{x_\psi}(s) ds \\ &= -r_M \int_t^{+\infty} e^{\int_s^t f'_{x_\psi}(\mu) d\mu} d\left(\int_s^t f'_{x_\psi}(\mu) d\mu\right) \\ &= -r_M \left[e^{\int_s^t f'_{x_\psi}(\mu) d\mu} \right]_t^{+\infty} \\ &= -r_M \left[e^{\int_{+\infty}^t f'_{x_\psi}(\mu) d\mu} - 1 \right], \quad (-\infty < t < +\infty) \\ &= r_M. \end{aligned} \quad (57)$$

Therefore, $\eta(t) \in \mathbb{B}$.

Define a mapping as follows:

$$(T\psi)(t) = \int_t^{+\infty} e^{\int_s^t f'_{x_\psi}(\mu) d\mu} f'_{x_\psi}(s) r(s) ds. \quad (58)$$

Thus, if given any $\psi(t) \in \mathbb{B}$, then $(T\psi)(t) \in \mathbb{B}$, and hence, $T: \mathbb{B} \rightarrow \mathbb{B}$.

Now, we prove that the mapping T is a compact mapping.

Consider any sequence $\{\psi_n(t)\} \subseteq \mathbb{B}$ ($n = 1, 2, \dots$), and then, it follows

$$r_L \leq \psi_n(t) \leq r_M, \text{mod}(\psi_n) \subseteq \text{mod}(f), \quad (n = 1, 2, \dots). \quad (59)$$

On the other hand, $(T\psi_n)(t) = x_{\psi_n}(t)$ satisfies

$$\begin{aligned} \frac{dx_{\psi_n}(t)}{dt} &= f'_x[t, r(t) + \theta_2(\psi_n(t))(\psi_n(t) - r(t))]x_{\psi_n}(t) - f'_x[t, r(t) + \theta_2(\psi_n(t))(\psi_n(t) - r(t))]r(t) \\ &= f'_{x_{\psi_n}}(t)x_{\psi_n}(t) - f'_{x_{\psi_n}}(t)r(t), \end{aligned} \quad (60)$$

where

$$f'_x[t, r(t) + \theta_2(\psi_n(t))(\psi_n(t) - r(t))] \triangleq f'_{x_{\psi_n}}(t). \quad (61)$$

Since $f: \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$ is continuous, $f(t + \omega, x) = f(t, x)$, f'_x exists and is a continuous function, and from (60) and

$\{\psi_n(t)\} \subseteq \mathbb{B}$ ($n = 1, 2, \dots$), it follows that $f'_{x_{\psi_n}}(t)$ is an ω -periodic continuous function on $t \in \mathbb{R}$.

It follows from (47), (50), (59), and (61) that

$$|f'_x[t, r(t) + \theta_2(\psi_n(t))(\psi_n(t) - r(t))]| \leq \delta. \quad (62)$$

Thus, we have

$$\left| \frac{dx_{\psi_n}(t)}{dt} \right| \leq 2 \|f'_{x_{\psi_n}}(t)\| r(t) \leq 2\delta \|r\|_M, \quad (63)$$

$$\text{mod}(x_{\psi_n}(t)) \subseteq \text{mod}(f). \quad (64)$$

Hence, $\{dx_{\psi_n}(t)/dt\}$ is uniformly bounded, and therefore, $\{x_{\psi_n}(t)\}$ is uniformly bounded and equicontinuous on $\mathbb{R}$. By the theorem of Ascoli-Arzelà, for any sequence $\{x_{\psi_n}(t)\} \subseteq \mathbb{B}$, there exists a subsequence (also denoted by $\{x_{\psi_n}(t)\}$) such that $\{x_{\psi_n}(t)\}$ is convergent uniformly on any compact set of $\mathbb{R}$. By (64), combined with Lemma 2, $\{x_{\psi_n}(t)\}$ is convergent uniformly on $\mathbb{R}$, that is to say, T is relatively compact on $\mathbb{B}$.

Next, we prove that T is a continuous mapping.

Assume $\{\psi_n(t)\} \subseteq \mathbb{B}$, $\psi(t) \in \mathbb{B}$,

$$\psi_n(t) \longrightarrow \psi(t), \quad (n \longrightarrow \infty). \quad (65)$$

Since $f(t, x)$ has first-order continuous partial derivative on x , we have

$$\begin{aligned} & f'_x[t, r(t) + \theta_2(\psi_n(t))(\psi_n(t) - r(t))] \\ & \longrightarrow f'_x[t, r(t) + \theta_1(\psi(t))(\psi(t) - r(t))], \end{aligned} \quad (66)$$

($n \longrightarrow \infty$),

that is,

$$f'_{x_{\psi_n}}(t) \longrightarrow f'_{x_{\psi}}(t), \quad (n \longrightarrow \infty). \quad (67)$$

It follows from (58) that

$$\begin{aligned} & |(T\psi_n)(t) - (T\psi)(t)| \\ &= \left| \int_t^{+\infty} e^{\int_s^t f'_{x_{\psi_n}}(\mu) d\mu} f'_{x_{\psi_n}}(s) r(s) ds - \int_t^{+\infty} e^{\int_s^t f'_{x_{\psi}}(\mu) d\mu} f'_{x_{\psi}}(s) r(s) ds \right| \\ &= \left| \int_t^{+\infty} e^{\int_s^t f'_{x_{\psi_n}}(\mu) d\mu} (f'_{x_{\psi_n}}(s) - f'_{x_{\psi}}(s)) r(s) ds \right. \\ & \quad \left. + \int_t^{+\infty} \left(e^{\int_s^t f'_{x_{\psi_n}}(\mu) d\mu} - e^{\int_s^t f'_{x_{\psi}}(\mu) d\mu} \right) f'_{x_{\psi}}(s) r(s) ds \right| \\ &= \left| \int_0^{+\infty} e^{\int_{s+t}^t f'_{x_{\psi_n}}(\mu) d\mu} (f'_{x_{\psi_n}}(t+s) - f'_{x_{\psi}}(t+s)) r(t+s) ds \right. \\ & \quad \left. + \int_0^{+\infty} \left(e^{\int_{s+t}^t f'_{x_{\psi_n}}(\mu) d\mu} - e^{\int_{s+t}^t f'_{x_{\psi}}(\mu) d\mu} \right) f'_{x_{\psi}}(t+s) r(t+s) ds \right| \\ &= \left| \int_0^{+\infty} e^{\int_{s+t}^t f'_{x_{\psi_n}}(\mu) d\mu} (f'_{x_{\psi_n}}(t+s) - f'_{x_{\psi}}(t+s)) r(t+s) ds \right. \\ & \quad \left. + \int_0^{+\infty} e^{\xi} \left(\int_{s+t}^t (f'_{x_{\psi_n}}(\mu) - f'_{x_{\psi}}(\mu)) d\mu \right) f'_{x_{\psi}}(t+s) r(t+s) ds \right| \\ &\leq \left(\int_0^{+\infty} e^{\int_{s+t}^t f'_{x_{\psi_n}}(\mu) d\mu} |r(t+s)| ds \right. \\ & \quad \left. + \int_0^{+\infty} e^{\xi} \left| \int_{s+t}^t d\mu \right| |f'_{x_{\psi}}(s+t)| \|r(s+t)\| ds \right) \rho(f'_{x_{\psi_n}}, f'_{x_{\psi}}) \\ &= \left(\int_0^{T_0} e^{\int_{t+s}^t f'_{x_{\psi_n}}(\mu) d\mu} |r(t+s)| ds + \int_{T_0}^{+\infty} e^{\int_{t+s}^t f'_{x_{\psi_n}}(\mu) d\mu} |r(t+s)| ds \right. \\ & \quad \left. + \int_0^{T_0} e^{\xi} \left| \int_{t+s}^t d\mu \right| |f'_{x_{\psi}}(s+t)| \|r(s+t)\| ds \right. \\ & \quad \left. + \int_{T_0}^{+\infty} e^{\xi} \left| \int_{t+s}^t d\mu \right| |f'_{x_{\psi}}(s+t)| \|r(s+t)\| ds \right) \rho(f'_{x_{\psi_n}}, f'_{x_{\psi}}), \end{aligned} \quad (68)$$

where ξ is between $\int_{s+t}^t f'_{x_{\psi_n}}(\mu) d\mu$ and $\int_{s+t}^t f'_{x_{\psi}}(\mu) d\mu$. Since $f'_{x_{\psi_n}}(\mu)$ and $f'_{x_{\psi}}(\mu)$ are ω -periodic continuous functions on $\mathbb{R}$, it follows

$$\lim_{s \longrightarrow +\infty} \frac{1}{s} \int_0^s f'_{x_{\psi_n}}(\mu) dt = \frac{1}{\omega} \int_0^\omega f'_{x_{\psi_n}}(\mu) dt \triangleq \lambda_1, \quad (69)$$

$$\lim_{s \longrightarrow +\infty} \frac{1}{s} \int_0^s f'_{x_{\psi}}(\mu) dt = \frac{1}{\omega} \int_0^\omega f'_{x_{\psi}}(\mu) dt \triangleq \lambda_2,$$

and thus, there is a $T_0 > 0$ (also, assume it is T_0 in the above), when $s \geq T_0$, such that

$$\left| \frac{-1}{s} \int_{t+s}^t f'_{x_{\psi_n}}(\mu) dt - \lambda_1 \right| < \frac{\lambda_1}{2}, \quad (70)$$

$$\left| \frac{-1}{s} \int_{t+s}^t f'_{x_{\psi}}(\mu) dt - \lambda_2 \right| < \frac{\lambda_2}{2},$$

that is,

$$\begin{aligned} \int_{t+s}^t f'_{x_{\psi_n}}(\mu) d\mu &< -\frac{\lambda_1 s}{2}, \\ \int_{t+s}^t f'_{x_{\psi}}(\mu) d\mu &< -\frac{\lambda_2 s}{2}. \end{aligned} \quad (71)$$

Set

$$\frac{\lambda_0 s}{2} = \max\left(-\frac{\lambda_1 s}{2}, -\frac{\lambda_2 s}{2}\right), \quad (72)$$

and hence, we have

$$\begin{aligned} & |(T\psi_n)(t) - (T\psi)(t)| \\ & \leq \left(T_0 e^{\delta T_0} |r|_M + \int_{T_0}^{+\infty} e^{-(\lambda_0 s/2)} |r|_M ds + T_0^2 e^{\delta T_0} \delta |r|_M \right. \\ & \quad \left. + \int_{T_0}^{+\infty} e^{-(\lambda_0 s/2)} s \delta |r|_M ds \right) \rho(f'_{x_{\psi_n}}, f'_{x_{\psi}}) \\ & = |r|_M \left(T_0 e^{\delta T_0} + \frac{2}{\lambda_0} e^{-(\lambda_0 T_0/2)} + T_0^2 e^{\delta T_0} \delta + \frac{4}{\lambda_0^2} e^{-(\lambda_0 T_0/2)} \delta \right) \rho(f'_{x_{\psi_n}}, f'_{x_{\psi}}). \end{aligned} \quad (73)$$

It follows from (67) and the above inequality that

$$(T\psi_n)(t) \longrightarrow (T\psi)(t), \quad (n \longrightarrow \infty). \quad (74)$$

Therefore, T is continuous. By (58), easy to see, $T(\partial\mathbb{B}) \subseteq \mathbb{B}$. According to Lemma 3, T has at least a fixed point on $\mathbb{B}$, the fixed point is the ω -periodic continuous solution $\Phi(t)$ of equation (1), and

$$r_L \leq \Phi(t) \leq r_M. \quad (75)$$

(2) Construct a Lyapunov function as follows:

$$V(t, x(t)) = (x(t) - \Phi(t))^2, \quad (76)$$

where $x(t)$ is the unique solution with initial value $x(t_0) = x_0$ of equation (1) and $\Phi(t)$ is the periodic solution of equation (1).

Differentiating both sides of (76) along the solution of equation (1), we have

$$\begin{aligned} \frac{dV(t, x(t))}{dt} &= 2(x(t) - \Phi(t)) \left[\frac{dx(t)}{dt} - \frac{d\Phi(t)}{dt} \right] \\ &= 2(x(t) - \Phi(t)) [f(t, x(t)) - f(t, \Phi(t))] \\ &= 2(x(t) - \Phi(t)) [f'_x(t, \zeta)(x(t) - \Phi(t))] \\ &= 2f'_x(t, \zeta)(x(t) - \Phi(t))^2, \end{aligned} \quad (77)$$

where ζ is between $x(t)$ and $\Phi(t)$.

By (H_2) , it follows

$$|x(t) - \Phi(t)| \longrightarrow +\infty, \quad (t \longrightarrow +\infty), \quad (78)$$

and thus, $x(t)$ cannot be a periodic solution of equation (1), and it is proved that equation (1) has a unique ω -periodic continuous solution $\Phi(t)$ which is unstable.

This is the end of the proof of Theorem 2. $\square$

4. Examples

The following examples show the feasibility of our main results.

Example 1. Consider the equation

$$\frac{dx}{dt} = -e^x + e^{\sin t}, \quad (79)$$

where

$$\begin{aligned} f(t, x) &= -e^x + e^{\sin t}, \\ f(t, \sin t) &= 0, \\ f'_x(t, x) &= -e^x \leq 0 \quad (\equiv 0), \end{aligned} \quad (80)$$

and hence, equation (79) satisfies all the conditions of Theorem 1. It follows from Theorem 1 that equation (79) has a unique 2π -periodic solution $r(t)$ which is globally attractive.

Clearly, according to the solution curve of equation (79), if given any initial value $x(0) = x_0$ (e.g., $x(0) = 0.2$), then the solution curve of equation (79) does tend to the curve of the periodic solution $r(t)$ (see Figure 1).

Example 2. Consider the equation

$$\frac{dx}{dt} = x^5 + x^3 - \cos^5 t - \cos^3 t, \quad (81)$$

where

FIGURE 1: The curve of the solution of equation (79) with initial value $x(0) = 0.2$. FIGURE 2: The curve of the solution of equation (81) with initial value $x(0) = 0.6$.

$$\begin{aligned} f(t, x) &= x^5 + x^3 - \cos^5 t - \cos^3 t, \\ f(t, \cos t) &= 0, \\ f'_x(t, x) &= 5x^4 + 3x^2 \geq 0 \quad (\equiv 0). \end{aligned} \quad (82)$$

Hence, equation (81) satisfies all the conditions of Theorem 2. It follows from Theorem 2 that equation (81) has a unique 2π -periodic solution $r(t)$ which is unstable.

Clearly, according to the solution curve of equation (81), if given any initial value $x(0) = x_0$ (e.g., $x(0) = 0.6$), then the solution curve of equation (81) arrives at enough large $(-\infty)$ at some time t^* (see Figure 2).

5. Conclusion

In this paper, a simple criterion for the existence, uniqueness, and stability of the periodic solution of one-dimensional periodic differential equations is given. This criterion can be applied to many one-dimensional periodic differential equations.

Data Availability

No data were used to support the findings of the study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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