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Research Article

On a New Criterion for the Solvability of Non-Simple Finite Groups and m-Abelian Solvability

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This paper is devoted to introduce a sufficient condition for the solvability of finite groups. Also, it presents the concepts of m-abelian and m-cyclic solvability as new generalizations of solvability and polycyclicity, respectively. These new generalizations show a connection between prime powers of elements in a finite group G and its solvability.

1. Introduction

An important problem in the theory of groups came to light after Galois' work [4]. This problem is concerned with determining whether a group G is solvable or not. According to the literature, many conditions and criteria were introduced to deal with this problem. Feit and Thompson had proved that each finite group of odd order is solvable (see [4, 5]).

Arad and Ward had proved Hall's Conjecture about solvability in [6].

In [7], Dolfi et al. introduced the following criterion.

G is solvable if, for all conjugacy classes C and D of G consisting of elements of prime power order, there exist $x \in C$ and $y \in D$such that x and y generate a solvable group.

Through this paper, we use the concept of (m-power closed group) [3] to introduce a sufficient condition that implies the solvability of a finite group.

On the other hand, we define two new solvability systems (m-abelian/m-cyclic solvability) based on m-power closed groups. These notions will provide us an interesting connection between prime divisors of the order $|G|$ of a finite group G and its solvability. In fact, m-abelian solvability turns out to be equivalent to the classical solvability whenever m is a prime divisor of $|G|$. We recall some basic definitions and theorems.

All groups throughout this paper are considered finite.

Definition 1 (see [1])

(a) Let G be a group and m be a fixed integer. We say that G is of exponent type m if for any $x, y \in G$, there exists $z \in G$ such that $x^m y^m = z^m$.

(b) Let G be a group and H▸G, and we say that H is m-normal in G if for any $x, y \in G$, there exists $z \in G$ such that $z^m x^m y^m \in H$. If this is the case, we denote H $\blacktriangleright_m$G.

Theorem 1 (see [1]). *Let G be a group and m be a fixed integer; then,*

(a) G is of exponent type m if and only if $G_m = \{g^m; g \in G\}$ is a subgroup of G.

(b) The homomorphic image of a group of exponent type m is also of exponent type m.

Theorem 2 (see [1]). *Suppose that G is a finite group of exponent type m, where m is a prime divisor of $|G|$. Then, G is solvable if and only if G_m is solvable.*

Definition 2 (see [1]). Suppose that G is a finite group of exponent type m. We say G is m-abelian if G_m is abelian.

Theorem 3. *Let G be an m-abelian group. Then, the homomorphic image of G is also m-abelian.*

Proof. It is clear by Theorem 1 and the fact that the homomorphic image of an abelian group is abelian. □

Definition 3. Let G be a group. The m-th commutator of x, $y\in$ G is defined as

$$[x, y]_m = x^{-m} y^{-m} x^m y^m. \tag{1}$$

The m-derived subgroup of G denoted by $(G)_m'$ is defined to be the subgroup generated by all m-th commutators of G.

Lemma 1. *Let G be a group of exponent type m. Then,*

(a) $(G)_m' \blacktriangleright G$.

(b) G is m-abelian if and only if $(G)_m' = \{e\}$.

(c) For $H \blacktriangleright_m G$, *G/H is m-abelian if and only if* $(G)_m' \leq H$.

Proof

(a) Let φ be a homomorphism on G. $\forall\, z\in (G)_m'$, we have

(i) $z = \prod_{i=1}^{i=n} x_i^{-m} y_i^{-m} x_i^m y_i^m$; $x_i, y_i \in G$, so $\varphi(G) = \prod_{i=1}^{i=n} [\varphi(x_i)]^{-m} [\varphi(y_i)]^{-m} [\varphi(x_i)]^m [\varphi(y_i)]^m$, and hence $\varphi(z) \in (G)_m'$ and $(G)_m'$ is fully invariant and then normal.

(b) G_m is abelian if and only if $(G)_m' = (G)_m' = \{e\}$.

(c) It is easy and clear.

□

Lemma 2. *The direct product of two m-abelian groups is m-abelian again.*

Proof. Let G, H be two m-abelian groups; then, $G \times H$ is a group of exponent type m and $(G \times H)_m = G_m \times H_m$. Since the direct product of abelian groups is abelian, $(G \times H)_m$ is abelian. Thus, G×H is m-abelian. □

Theorem 4. *Let G be an m-abelian group with m being a prime divisor of* $|G|$. *Then, G is solvable.*

Proof. The result holds directly from Theorem 2. □

2. m-Abelian Solvable Groups

Definition 4

(a) Let $\{e\} = H_0 \leq H_1 \leq \cdots \leq H_n = G$ be a subnormal series of a group G.

(i) We say that it is an m-abelian solvable series if H_i/H_{i-1} is m-abelian for every $1 \leq i \leq$ n.

(b) We say that G is m-abelian solvable if it has an m-abelian solvable series.

Lemma 3. *Let G be a group, and we have*

(a) If G is (m-abelian), then it is (m-abelian solvable).

(b) If G is solvable, then G is (m-abelian solvable) for each integer m.

(c) The homomorphic image of (m-abelian solvable) group is (m-abelian solvable).

Proof

(a) If G is (m-abelian), then it has an (m-abelian solvable) series $\{e\} \leq G$.

(b) If G is solvable, then it has a subnormal series $\{e\} = H_0 \leq H_1 \leq \cdots \leq H_n = G$ with abelian factors. Since every abelian group is (m-abelian) for each integer m, G must be (m-abelian solvable).

(c) Let φ: $G \longrightarrow K$ be a group homomorphism; first of all we will prove that $\varphi((G)_m') = [\varphi(G)]_m'$. $\forall h \in \varphi((G)_m')$, $h = \varphi(\prod_{i=1}^{i=n} x_i^{-m} y_i^{-m} x_i^m y_i^m) = \prod_{i=1}^{i=n} [\varphi(x_i)]^{-m} [\varphi(y_i)]^{-m} [\varphi(x_i)]^{m} [\varphi(y_i)]^m \in [\varphi(G)]_m'$, and thus $\varphi((G)_m') \leq [\varphi(G)]_m'$. The other inclusion can be proved by the same.

Suppose that G is (m-abelian solvable); then, it has an (m-abelian solvable) series $\{e\} = H_0 \leq H_1 \leq \cdots \leq H_n = G$. We have $H_{i-1} \blacktriangleright_m H_i$ and $(H_i)_m' \leq H_{i-1}$; this implies that $\varphi[(H_i)_m'] = [\varphi(H_i)]_m' \leq \varphi(H_{i-1})$. It is easy to show that$\varphi(H_{i-1}) \blacktriangleright_m \varphi(H_i)$, and thus we obtain an (m-abelian solvable) series $\{e\} \leq \varphi(H_0) \leq \varphi(H_1) \leq \cdots \leq \varphi(H_n) = \varphi(G)$, and hence G is (m-abelian solvable). □

Theorem 5. *Let G be a group, and we have*

(a) If $H \blacktriangleright_m G$ *and H is solvable with* $(G)_m' \leq H$ *H, then G is (m-abelian solvable).*

(b) If there is a positive integer k such $G^{(k)}$ *is (m-abelian), then G is (m-abelian solvable).*

(c) If $H \blacktriangleright G$ *and H, G/H are (m-abelian solvable), then G is (m-abelian solvable).*

Proof

(a) Let $\{e\} = H_0 \leq H_1 \leq \cdots \leq H_n = H$ be the solvable series of H; we have that $\{e\} = H_0 \leq H_1 \leq \cdots \leq H_n = H \leq G$ is (m-abelian solvable) series of G, and our proof is complete.

(b) It is easy to see that $\{e\} \leq G^{(k)} \leq G^{(k-1)} \leq \cdots \leq G^0 = G$ is an (m-abelian solvable) series of G.

(c) Suppose that H, G/H are (m-abelian solvable), and we have the following two (m-abelian solvable) series: $\{e\} = H_0 \leq H_1 \leq \cdots \leq H_n = H$ and $\{H\} = \overline{K_0} \leq \overline{K_1} \leq \cdots \leq \overline{K_m} = G/H$. For each $0 \leq j \leq$ m we can find a subgroup $K_j \leq G$ such that $K_0 = H$

, $K_m = G, K_j \blacktriangleright K_{j+1}, \overline{K_j} = K_j/H$; by isomorphism theorem, we obtain $\overline{K_j}/\overline{K_{j-1}} \cong (K_j/H)/(K_{j-1}/H) \cong K_j/K_{j-1}$, and thus K_j/K_{j-1} is (m-abelian), $1 \le j \le m$, and this implies that the series $\{e\} = H_0 \le H_1 \le \cdots \le H_n = H \le K_0 \le K_1 \le \cdots \le K_m = G$ is (m-abelian solvable). □

Theorem 6

(a) The direct product of two (m-abelian solvable) groups is (m-abelian solvable).

(b) The direct product of finite number of (m-abelian solvable) groups is (m-abelian solvable).

Proof

(a) Let G, H be two (m-abelian solvable) groups, and we have the following (m-abelian solvable) series: $\{e_1\} = H_0 \le H_1 \le \cdots \le H_n = H$ and $\{e_2\} = K_0 \le K_1 \le \cdots \le K_m = G$; without affecting the generality, we can assume that n ≥ m; let the series $(*)$ be $H_0 \times K_0 \le H_1 \times K_1 \le \cdots \le H_m \times K_m \le H_{m+1} \times K_m \le \cdots \le H_n \times K_m = H \times G$; since $H_i \times K_j/H_{i-1} \times K_{j-1} \cong (H_i/H_{i-1}) \times (K_j/K_{j-1})$ for $1 \le i \le n, 1 \le j \le m$ and $(H_i \times K_m/H_{i-1} \times K_m) \cong (H_i/H_{i-1}) \times (K_m/K_m)$; $m + 1 \le i \le n$, each factor of series $(*)$ is (m-abelian), and hence $H \times G$ is (m-abelian solvable).

(b) It holds directly by an easy induction. □

Theorem 7. *Let G be a finite (m-abelian solvable) group where prime m divides its order; then, G is solvable.*

Proof. There is an (m-abelian solvable) series $\{e\} = H_0 \le H_1 \le \cdots \le H_n = G$, and we have that $H_1/H_0 \cong H_1$ is (m-abelian), so H_1 is solvable and H_2/H_1 is (m-abelian); thus, it is solvable, and H_2 is solvable. By the same argument, we find that G is solvable. □

Example 1. Consider the finite group $G = D_4$; we have $Z(G)$ as a normal subgroup of order 2, and hence $G/Z(G)$ is of order 4.

$Z(G)$, $G/Z(G)$ are 2-abelian groups since they are abelian; this implies that G is 2-abelian solvable, and then it is solvable.

3. (m-Cyclic Solvable) Groups

Definition 5

(a) Let G be an (m-group); it is (m-cyclic) if G_m is cyclic.

(b) Let G be a group with a subnormal series $\{e\} = H_0 \le H_1 \le \cdots \le H_n = G$, and we say that it is (m-cyclic solvable) series if H_i/H_{i-1} is (m-cyclic) for each $1 \le i \le$ n.

(c) We say that G is (m-cyclic solvable) if it has an (m-cyclic solvable) series.

Lemma 4. *Let G be a group, and we have*

(a) If G is cyclic, then it is (m-cyclic) for each integer m.

(b) If G is an (m-cyclic) group, then the homomorphic image of G is (m-cyclic).

(c) If G is an (m-cyclic) group with a prime m/|G|, then G is solvable.

Proof

(a) A subgroup of cyclic group is cyclic, so it is clear.

(b) It is known that the homomorphic image of any cyclic group is cyclic and by Theorem 1, the proof is complete.

(c) It holds easily, since each m-cyclic group is m-abelian group. □

Theorem 8. *Let G be a group, and we have*

(a) If G is (m-cyclic), then it is (m-cyclic solvable).

(b) If G is polycyclic, then G is (m-cyclic solvable) for each integer m.

(c) The homomorphic image of any (m-cyclic solvable) group is (m-cyclic solvable).

Proof

(a) If G is (m-cyclic), then it has an (m-cyclic solvable) series $\{e\} \le G$.

(b) If G is polycyclic, then it has a subnormal series $\{e\} = H_0 \le H_1 \le \cdots \le H_n = G$ with cyclic factors. Since every cyclic group is (m-cyclic) for each integer m, G must be (m-cyclic solvable).

(c) Assume that G is (m-cyclic solvable); then, it has an (m-cyclic solvable) series $\{e\} = H_0 \le H_1 \le \cdots \le H_n = G$; suppose that H is a normal subgroup of G, and let $K_i = H_iH/H$; $0 \le i \le n$; because of $H_{i-1} \blacktriangleright H_i$, we get $\{e\} = K_0 \le K_1 \le \cdots \le K_n = G/H$. By isomorphism theorem, we get $K_i/K_{i-1} \cong (H_iH/H)/(H_{i-1}H/H) \cong H_iH/H_{i-1}H \cong H_i \; (H_{i-1}H)/H_{i-1}H \cong H_i/H_i \cap H_{i-1}H$. Now we must prove that $H_i \cap H_{i-1}H \blacktriangleright_m H_i$. We have $H_{i-1} \blacktriangleright_m H_i$, so $\forall x, y \in H_i \exists k \in H_i; k^m x^m y^m \in H_{i-1} \le H_{i-1}H$ and $k^m x^m y^m \in H_i$ so $k^m x^m y^m \in H_i \cap H_{i-1}H$, and this means that $H_i \cap H_{i-1}H \blacktriangleright_m H_i$; thus, K_i/K_{i-1} is an (m-group).

By using isomorphism theorem, we obtain $H_i/H_i \cap H_{i-1}H \cong (H_i/H_{i-1})/(H_i \cap H_{i-1}H/H_{i-1})$, which is a homomorphic image of (m-cyclic) group, and hence K_i/K_{i-1} is (m-cyclic) and G/H must be (m-cyclic solvable). □

Theorem 9. *Let G be a group and $H \blacktriangleright G$. If H and G/H are (m-cyclic solvable), then G is (m-cyclic solvable).*

Proof. Suppose that H, G/H are (m-cyclic solvable). We have the following two (m-cyclic solvable) series: $\{e\} = H_0 \leq H_1 \leq \cdots \leq H_n = H$ and $\{H\} = \overline{K_0} \leq \overline{K_1} \leq \cdots \leq \overline{K_m} = G/H$; for each $0 \leq j \leq$ m, we can find a subgroup $K_j \leq G$ such that $K_0 = H, K_m = G, K \blacktriangleright K_{j+1}, \overline{K_j} = K_j/H$.

By isomorphism theorem, we obtain $\overline{K_j}/\overline{K_{j-1}} \cong (K_j/H)/(K_{j-1}/H) \cong K_j/K_{j-1}$, so K_j/K_{j-1} is (m-cyclic); $1 \leq j \leq m$; this implies that the series $\{e\} = H_0 \leq H_1 \leq \cdots \leq H_n = H \leq K_0 \leq K_1 \leq \cdots \leq K_m = G$ is (m-cyclic solvable). □

Theorem 10. *Let G be a finite (m-cyclic solvable) group where a prime m divides its order; then, G is solvable.*

Proof. Since every (m-cyclic solvable group) is an (m-abelian solvable), the proof holds. □

4. Sufficient Condition for Solvability

Lemma 5 (see [1]).

Let G be an (m^*-group) with m/|G|, and let $|G| = m^{k1} p_2^{k2}, \ldots, p_s^{ks}$; p_i are distinct primes for each $2 \leq i \leq s$; then,

(a) $p_2^{k2}, \ldots, p_s^{ks}/|G_m|$.

(b) G/G_m is a (p-group) with $m = p$.

(c) G is solvable if and only if G_m is solvable.

Proof

(a) For each prime p_i, the (p_i-Sylow) subgroup H_i has order p_i^{ki} with $\gcd(p_i^{ki}, \mathrm{m}) = 1$.

(i) So, $(H_i)_m = H_i \leq G_m$; then, $p_i^{ki}/|G_m|$ for each i, and thus $p_2^{k2}, \ldots, p_s^{ks}/|G_m|$.

(b) $|G/G_m| = m^k$, $k \leq k_1$, so G/G_m is a (p-group) with $m = p$.

(ii) We meant by (p-group) a group with order p^s; $s \in \in N$ and p is prime.

(c) Assume that G_m is solvable; then, G/G_m is solvable because it is a (p-group). This means that G is solvable, and the converse is clear. □

Lemma 6 (see [1]). *Let G be an (m^*-group) with m/|G|; then,*

(a) If G is simple, then it is cyclic of order m.

(b) If H►G, then $H/(H \cap G_m)$ is a (p-group) with $p = m$.

Proof

(a) We have m/|G|, so that $G \neq G_m$, but $G_m \blacktriangleright G$, so $G_m = \{e\}$, and G/G_m is a (p-group), and in this case, $G/G_m \cong$G, which means that G is a simple (p-group); thus, G is cyclic with order m.

(b) Suppose that H►G; then, $G_m \cap H \blacktriangleright H$ and $H/((H \cap G_m) \cong G_m H/G_m \leq G/G_m$, and thus $H/(H \cap G_m)$ is a (p-group). □

Remark 1. If we consider that the finite group G is (m^k-group) with $|G| = m^{k1} p_2^{k2}, \ldots, p_s^{ks}$, where $m, p_2, p_3, \ldots, p_s$ are distinct primes and $k \leq k_1$, then Lemmas (5) and 6 are still true.

Theorem 11. *Let G be a finite group. If every normal subgroup H of G is (m^*-group) where a prime m divides O(H), then G is solvable.*

Proof. G is (m^*-group); then, $G_m \blacktriangleright G$ and G/G_m is (p-group), and it is solvable. G_m is (n^* − group) where prime n divides $O(G_m)$, so $G_m/(G_m)_n$ is a (p-group) and is solvable. By the same argument, we get a series $\{e\} \leq H_1 \leq \cdots \leq H_l \leq G$ such $H_i = (H_{i+1})_m$ where prime m divides $O(H_{i+1})$ and each factor H_{i+1}/H_i is a p-group and is solvable for each i. This implies that H_1 is solvable and then H_2 is solvable and so on, and thus G is solvable.

Theorem 11 is still true if every normal subgroup is (m^k-group) with some prime m dividing its order.

The previous theorem can be described by the following form.

If G is a finite m-power closed group with respect to $m = p^k$; p is a prime, suppose that for every normal subgroup H of G, there exists a fixed positive integer $n = q^s$; q is a prime and m/$|H|$, with the following property: for each $x, y \in H$, there is $z \in H$ such that$x^n y^n = z^n$. Then, G is solvable.

The previous theorem can be considered as a new criterion to determine if a finite group G is solvable. □

Conjecture 1. *Each finite group G with odd order is m^k-group with some prime m dividing O(G).*

By Theorem 11, we can find that if this conjecture is true, then Feit–Thompson theorem holds.

Conjecture 1 is very important, since if it is true, we will get an easy proof to a famous basic theorem in algebra.

Example 2. This example is devoted to clarify the validity of our criterion in Theorem 5.

Consider $G = S_3$, the symmetric group of order 6. G is a 2 − group, since $G_2 \cong Z_3$.

The only normal subgroup of G is $H \cong Z_3$ which is a 3-group since it is abelian, and thus G is solvable according to Theorem 11.

5. Conclusion

In this article, we have introduced the concept of (m-abelian solvability) and (m-cyclic solvability) as two new generalizations of classical solvability and polycyclicity, respectively.

We have discussed some elementary properties of these concepts and proved the main result through this paper which ensures that m-abelian solvability is equivalent of solvability in finite groups if m is a prime number that divides the order of the group.

This result shows a kind of connection between primes and solvability in finite groups. An interesting question came to light according to this work. This question can be asked as follows:

If G is an infinite m-abelian solvable group for a prime m, then is G solvable?

Also, we have introduced a new sufficient condition for the solvability of finite non-simple group G based on m-power closed groups concept.

As a future research direction, m-abelian solvability can be extended to AH-subgroups defined in [8] and neutrosophic groups in [9].

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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