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*Research Article*

# On $L(h,k)$-Labeling Index of Inverse Graphs Associated with Finite Cyclic Groups

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An $L(h,k)$-labeling of a graph $G = (V, E)$ is a function $f\colon V \longrightarrow [0,\infty)$ such that the positive difference between labels of the neighbouring vertices is at least $h$ and the positive difference between the vertices separated by a distance 2 is at least $k$. The difference between the highest and lowest assigned values is the index of an $L(h,k)$-labeling. The minimum number for which the graph admits an $L(h,k)$-labeling is called the required possible index of $L(h,k)$-labeling of $G$, and it is denoted by $\lambda_k^h(G)$. In this paper, we obtain an upper bound for the index of the $L(h,k)$-labeling for an inverse graph associated with a finite cyclic group, and we also establish the fact that the upper bound is sharp. Finally, we investigate a relation between $L(h,k)$-labeling with radio labeling of an inverse graph associated with a finite cyclic group.

## 1. Introduction

The frequency assignment problem (FAP) interacts with the assignment of frequencies to locations in such a manner that there is no interruption between frequencies allotted to the neighbouring locations while attempting to reduce the index of the allocated frequencies (the difference between the highest and lowest frequency bands). There are plenty of graph theoretical models for solving FAP, but many of them are NP-hard [1]. Specifically, FAP has been represented as a graph labeling problem to allocate frequencies to transceivers in a wireless communication such as a cellular network or a radio network. With the massive growth in the required calls in the network, we must find an effective assessment of frequency in the network having a minimal index. Here, vertices represent the transceivers and lines represent couples of locations that overlap with each other.

The motivation for the study of $L(h,k)$-labelings is the allocation of radio frequency of transceivers in the interruption range [2]. This radio frequency problem can be perfectly examined by $L(h,k)$ problem. In $L(h,k)$ problem, all transceivers are allocated a frequency so that the distinction between the labels of neighbouring transceivers is at least$h$ and the distinction between transceivers separated by a distance 2 is at least $k$. The variation between the maximum and minimum allotted values is the index of an $L(h,k)$-labeling. The theoretical explanation of this graph emerges from the problem of allocating frequencies to a wireless network's transceivers to prevent any form of disruption. The geographical distance and the purpose of the atmosphere in this setting are the main factors that decide the $h$ and$k$ parameters, and $h \geq k$ is generally expected. $L(h,k)$-labelings have been studied by various authors, see [3, 4] and survey in [5]. Recently, Mitra and Bhoumik [6] provided an upper bound of the span of three specific cases of circulant graphs. Song et al. [3] proved that the labeling of $L(j,k)$ for the complete graph is the direct product. The radio labeling applies to the number of interruption permissible levels considered in the $L(2,1)$-labeling from two to the highest allowable diameter of $G$. The highest allowable distance between every couple of vertices in$G$ is denoted by $\operatorname{diam}(G)$. In [7], Chartrand presented the idea of radio labeling and discussed the radio number for cycles and paths in [8] by Liu and Zhu. For a detailed survey of graph labelings, one can refer to [9].

Since Cayley graphs, many relations between groups and graphs have been established and interest began to be on such relations. The inverse graphs associated with finite groups were recently introduced and analyzed by Alfuraidan and Zakariya [10], as a new example of such relations. They developed some important graph theoretical aspects of the inverse graphs of certain finite groups that really shine as a spotlight on algebraic aspects of the groups. They defined an interconnection in between the algebraic aspects of the finite groups and the graph theoretical aspects of the identified inverse graphs. The inverse graph was utilized to categorize certain problems of isomorphism in finite groups. In the same way, Kalaimurugan and Megeshwaran [11] explored the $Z_k$-magic index on the inverse graph.

On the contrary, Ejima et al. [12] had found the inverse graphs of dihedral and symmetric groups. Very recently, some properties of finite group invertible graphs have been investigated by Chalapathi and Kiran-Kumar [13]; invertible graphs have been established and some interesting results on them have been obtained using finite group classification. The chromatic number, girth, clique number, diameter, and size have been calculated for each finite group. Jones and Lawson [8] investigated the inverse graph of the large semigroup of a graph-related topological groupoid and the semigroup analogue called the Leavitt path algebra. Let $G$ be a finite simple connected graph whose vertex set $V(G)$ and edge set $E(G)$ with $|V(G)| = p$ and $|E(G)| = q$. The open neighbourhood $N(v)$ of a vertex $v \in V(G)$ is the set of all vertices which are adjacent to$v$. We follow [14] for graph theoretic terminology. Note that the inverse graph of a finite cyclic group $G = GS(Z_n)$ is simple and connected. To the best of our observations, for the very first time, we investigate the $L(h,k)$-labeling of the inverse graphs in this paper and we give the exact minimum span of $L(h,k)$-labeling of inverse graphs associated with finite cyclic groups.

## 2. Preliminaries

We now recall some powerful and known results used in the proofs of our new results.

*Definition 1* (see [15]). *Griggs and Yeh introduced an $L(2,1)$-labeling of a graph $G(V,E)$ as a function $f\colon V(G) \longrightarrow \{0,1,\ldots,n\}$ such that $|f(u)-f(v)| \geq 2$ if $d(u,v)=1$ and $|f(u)-f(v)| \geq 1$ if $d(u,v)=2$, where $d(u,v)$ denotes the shortest path in between the vertices $u$ and$v$. The minimum index for certain possible functions of $L(2,1)$ labeling is indicated by $\lambda_{(2,1)}(G)$ and referred to as $\lambda_{(2,1)}$ number of $G$.*

*Definition 2* (see [3]). If $h, k\,(h \geq k)$ are positive integers, then the $L(h,k)$-labeling of a graph $G = (V,E)$ is a function $f\colon V \longrightarrow [0,\infty)$ such that

$$|f(u)-f(v)| \geq \begin{cases} h, & \text{if } d(u,v)=1, \\ k, & \text{if } d(u,v)=2. \end{cases} \tag{1}$$

The maximum label attained by $f$ is referred to the index of $f$. The minimum index for a certain possible function $f$ is the $\lambda_k^h$-number of $G$ denoted by $\lambda_k^h(G)$.

*Definition 3* (see [7, 8]). A radio labeling of a graph $G(V,E)$ is a one-to-one function $f\colon V \longrightarrow [0,\infty)$ such that, for each $u,v \in V$, $|f(u)-f(v)| \geq \operatorname{diam}(G) - d(u,v) + 1$. The index of $f$ is the variation of the highest and the lowest frequencies utilized, that is, $\max_{u,v\in V}\{f(u)-f(v)\}$ for each pairs $u,v \in V$. The radio number of $G$ is the index of a radio labeling of $G$ and is denoted by $r_n(G)$.

*Definition 4* (see [10]). Let $(\Gamma, *)$ be a finite group and $S = \{u \in \Gamma | u \neq u^{-1}\}$. The inverse graph $GS(\Gamma)$ associated with $\Gamma$ as the graph whose vertex set coincides with $\Gamma$ such that two different vertices$u$ and$v$ are adjacent iff either $uv \in S$ or $vu \in S$. When $\Gamma$ is the cyclic group $Z_n$ and $S$ is the set of non-self-inverse elements of $Z_n$, $GS(\Gamma)$ is called the inverse graph of a finite cyclic group $G = GS(Z_n)$.

**Theorem 1** (see [10]). *If $\Gamma$ is a finite abelian group, which contains three or more elements, and if $S$ is a nonempty subset of the non self-invertible elements, then $GS(\Gamma)$ is a connected graph.*

**Theorem 2** (see [10]). *If $\Gamma$ is an arbitrary finite group, then the inverse graph is not complete.*

**Theorem 3** (see [10]). *A connected inverse graph has diameter two.*

## 3. Main Results

We prove the following results on the inverse graphs associated with a finite cyclic group admitting an $L(h,k)$-labeling having minimum index.

**Theorem 4.** *Let $n \geq 3$ be an odd integer. Then, the inverse graph $GS(Z_n)$ admits an $L(h,k)$-labeling with $\lambda_k^h(GS(Z_n)) \leq (n-1/2)(h+k)$ and this bound is sharp.*

*Proof.* Since$n$ is odd, the unique self-invertible element is 0. Let $G = GS(Z_n)$ be the inverse graph associated with a finite cyclic group. Let $V(G) = S_1 \cup S_2 \cup S_3$, where $S_1 = \{v_0\}$, $S_2 = \{v_1, v_2, \ldots, v_{(n-1/2)}\}$, and $S_3 = \{v_{(n+1/2)}, v_{(n+1/2)+1}, v_{(n+1/2)+2}, \ldots, v_{n-1}\}$. Define the function $f\colon V(G) \longrightarrow \{0,1,\ldots,n\}$ by

$$f(v_i) = \begin{cases} 0, & \text{if } v_0 \in S_1, \\ ih + (i-1)k, & \text{if } v_i \in S_2, \\ (n-i)(h+k), & \text{if } v_i \in S_3. \end{cases} \tag{2}$$

Case (i): if $v_0 \in S_1$ and $v_j$ are the elements of $S_2$ (or $S_3$), then $d(v_0, v_j) = 1$. We claim that $|f(v_0) - f(v_j)| \geq h$. For the function $f$, we have $h \leq |f(v_i) - f(v_j)| \leq (n - 1/2)(h + k)$.

Case (ii): if $v_i$ and $v_j$ are the elements of $S_2$ (or $S_3$), then $d(v_i, v_j) = 1$. We claim that $|f(v_i) - f(v_j)| \geq h$. We have $h + k \leq |f(v_i) - f(v_j)| \leq (n - 3/2)(h + k) \leq (n - 1/2)(h + k)$.

Case (iii): if $v_i \in S_2$ and $v_j \in S_3 - \{v_{n-i}\}$, then $d(v_i, v_j) = 1$ where, $j \neq n - i$. We claim that $|f(v_i) - f(v_j)| \geq h$. We have $h \leq |f(v_i) - f(v_j)| \leq (n - 1/2)\ (h + k) - h \leq (n - 1/2)(h + k)$.

Case (iv): if $v_i \in S_2$ and $v_j = v_{n-i} \in S_3$, then $d(v_i, v_j) = 2$, where $j = n - i$. We claim that $|f(v_i) - f(v_j)| \geq k$. We have $k \leq |f(v_i) - f(v_j)|$. Figure 1 illustrates that the bound on the graph index is sharp. □

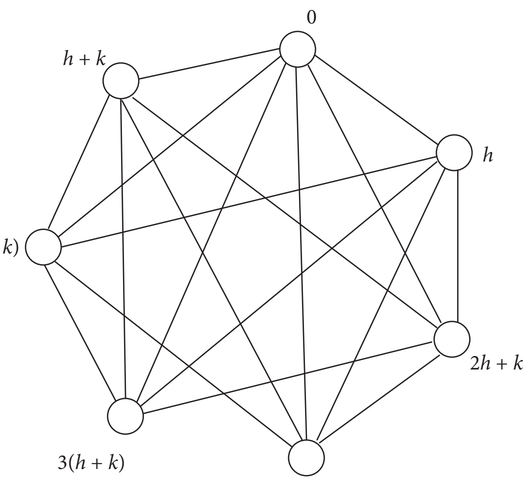


FIGURE 1: $\lambda_k^h(GS(Z_7)) = 3(h + k)$.

**Theorem 5.** *Letn be even and h and k be positive integers with $k \geq \lceil h/2 \rceil$. Then, the inverse graph $GS(Z_n)$ admits an $L(h, k)$-labeling with*

$$\lambda_k^h(GS(Z_n)) \leq \begin{cases} \dfrac{n-2}{4}(h + 3k) + k, & \text{if } n \equiv 2 \pmod 4, \\ \\ \dfrac{n}{4}(h + 3k) - k, & \text{if } n \equiv 0 \pmod 4. \end{cases} \quad (3)$$

*Also, these bounds are sharp.*

*Proof*

Case (a): let $n \equiv 2 \pmod 4$. Sincen is even, then the self-invertible elements are 0 and $(n/2)$. Let $G = GS(Z_n)$ be the inverse graph associated with a finite cyclic group. Let $V(G) = S_1 \cup S_2 \cup S_3 \cup S_4 \cup S_5 \cup S_6$, where $S_1 = \{v_0\}$ $S_2 = \{v_1, v_2, \ldots, v_{(n-2/4)}\}$, $S_3 = \{v_{(n-2/4)+1}, v_{(n-2/4)+2}, \ldots, v_{(n/2)-1}\}$, $S_4 = \{v_{(n/2)+1}, v_{(n/2)+2}, \ldots, v_{(3n-2/4)}\}$, $S_5 = \{v_{(3n-2/4)+1}, v_{(3n-2/4)+2}, \ldots, v_{n-1}\}$, and $S_6 = \{v_{(n/2)}\}$. Define the function $f\colon V(G) \longrightarrow \{0, 1, \ldots, n\}$. Then,

$$f(v_i) = \begin{cases} 0, & \text{if } v_0 \in S_1, \\ ih + (3(i-1) + 1)k, & \text{if } v_i \in S_2. \\ \left(\dfrac{n}{2} - i\right)h + \left(3\left(\dfrac{n}{2} - i\right) - 1\right)k, & \text{if } v_i \in S_3, \\ \left(i - \dfrac{n}{2}\right)h + \left(3\left(i - \dfrac{n}{2}\right)k\right), & \text{if } v_i \in S_4, \\ (n-i)h + (3(n-i) + 1)k, & \text{if } v_i \in S_5, \\ k, & \text{if } v_{(n/2)} \in S_6. \end{cases} \quad (4)$$

Subcase (i): if $v_i$ and $v_j$ are distinct elements in $S_p$ for all $2 \leq p \leq 5$, then $d(v_i, v_j) = 1$. We claim that $|f(v_i) - f(v_j)| \geq h$. We have $h + 3k \leq |f(v_i) - f(v_j)| \leq (n - 6/4)(h + 3k)$. Clearly, $h + 3k \geq h$. Hence, $|f(v_i) - f(v_j)| \geq h$.
Subcase (ii): let $v_i \in S_2$ and $v_j \in S_3$ with $i \neq j$ be the adjacent vertices iff $j \neq (n/2) - i$. Assume that $j \neq (n/2) - i$; then, $d(v_i, v_j) = 1$. For the function $f$, we have $(h + 2k) \leq |f(v_i)\ - f(v_j)| \leq (n - 6/4)(h + 3k) + k$. Clearly, $|f(v_i) - f(v_j)| \geq h$. If $j = (n/2) - i$, then $d(v_i, v_j) = 2$. We have $k = |f(v_i) - f(v_j)|$. Suppose the element $v_j \in S_4$; then, $d(v_i, v_j) = 1$. We have $h + 2k \leq |f(v_i) - f(v_j)| \leq (n - 6/4)(h + 3k) + 2k$. Clearly, $|f(v_i) - f(v_j)| \geq h$. Suppose the element $v_j \in S_5$ and $v_i$ adjacent to $v_j$ iff $j \neq n - i$. Assume that $j \neq n - i$; then, $d(v_i, v_j) = 1$. For the function $f$, we have $(h + 2k) \leq |f(v_i) - f(v_j)| \leq (n - 6/4)(h + 3k) + 3k$. If $j = n - i$, then $d(v_i, v_j) = 2$. By the function $f$, $|f(v_i) - f(v_j)| = 3k \geq k$.
Subcase (iii): if $v_i \in S_3$, then suppose $v_j \in S_4$ and $v_i$ is adjacent to $v_j$ iff $j \neq n - i$. If $j \neq n - i$, then $d(v_i, v_j) = 1$. We then have $h + 4k \leq |f(v_i)\ - f(v_j)| \leq (n - 6/4)(h + 3k) + k$. Clearly, $|f(v_i) - f(v_j)| \geq h$. If $j = n - i$, then $d(v_i, v_j) = 2$. We have $|f(v_i) - f(v_j)| = k$. Suppose that $v_j \in S_5$; then, $d(v_i, v_j) = 1$. We have $2k \leq |f(v_i) - f(v_j)| \leq (n - 2/4)(h + 3k) - k$. Clearly, $k \geq \lceil h/2 \rceil$ such that $2k \geq h$. Hence, $|f(v_i) - f(v_j)| \geq h$.
Subcase (iv): let $v_i \in S_4$ and $v_j \in S_5$ $v_i$ adjacent to $v_j$ iff $j \neq (3n/2) - i$. $j \neq (3n/2) - i$, then $d(v_i, v_j) = 1$. We then have $h + 2k \leq |f(v_i) - f(v_j)| \leq (n - 6/4)(h + 3k)$. Clearly, $|f(v_i) - f(v_j)| \geq h$. If $j = (3n/2) - i$, then $d(v_i, v_j) = 2$. We have $k = |f(v_i) - f(v_j)|$.
Subcase (v): the vertex $v_0$ is not adjacent to $v_{(n/2)}$ and $d(v_0, v_{(n/2)}) = 2$. Now, $|f(v_i) - f(v_j)| = k$. The vertex $v_0$ is adjacent to all other vertices. Then, $d(v_0, v_j) = 1$, where $j \neq (n/2)$. We have $h + k \leq |f(v_i)\ - f(v_j)| \leq (n - 2/4)(h + 3k) + k$. Suppose that the vertex $v_{(n/2)}$ is adjacent to all other vertices. Then, $d(v_{(n/2)}, v_j) = 1$,

where $j \neq 0$. Now, we have $h \leq |f(v_{(n/2)}) - f(v_j)| \leq (n-2/4)(h+3k)$.

Case (b): if $n \equiv 0 \pmod 4$. Since$n$ is even, then the self-invertible elements are 0 and $(n/2)$. Let $G = GS(Z_n)$ be the inverse graph associated with a finite cyclic group. Let $V(G) = S_1 \cup S_2 \cup S_3 \cup S_4 \cup S_5 \cup S_6 \cup S_7 \cup S_8$, where $S_1 = \{v_0\}$, $S_2 = \{v_1, v_2, \ldots, v_{(n/4)-1}\}$, $S_3 = \{v_{(n/4)+1}, v_{(n/4)+2}, \ldots, v_{(n/2)-1}\}$, $S_4 = \{v_{(n/2)+1}, v_{(n/2)+2}, \ldots, v_{(3n/4)-1}\}$, $S_5 = \{v_{(3n/4)+1}, v_{(3n/4)+2}, \ldots, v_{n-1}\}$, $S_6 = \{v_{(n/2)}\}$, $S_7 = \{v_{(n/4)}\}$, and $S_8 = \{v_{(3n/4)}\}$. Define the function $f\colon V(G) \longrightarrow \{0, 1, \ldots, n\}$. Then,

$$f(v_i) = \begin{cases} 0, & \text{if } v_0 \in S_1, \\ ih + (3(i-1)+1)k, & \text{if } v_i \in S_2, \\ \left(\dfrac{n}{2} - i\right)h + \left(3\left(\dfrac{n}{2} - i\right) - 1\right)k, & \text{if } v_i \in S_3, \\ \left(i - \dfrac{n}{2}\right)h + \left(3\left(i - \dfrac{n}{2}\right)k\right), & \text{if } v_i \in S_4, \\ (n-i)h + (3(n-i)+1)k, & \text{if } v_i \in S_5, \\ k, & \text{if } v_{(n/2)} \in S_6, \\ \dfrac{n}{4}(h+3k) - 2k, & \text{if } v_{(n/4)} \in S_7, \\ \dfrac{n}{4}(h+3k) - k, & \text{if } v_{(3n/4)} \in S_8. \end{cases} \tag{5}$$

In the case $n \equiv 0 \pmod 4$, we similarly have the following. The vertex $v_{(n/4)}$ is not adjacent to $v_{(3n/4)}$, and therefore, $d(v_{(n/4)}, v_{(3n/4)}) = 2$. Now, $|f(v_i) - f(v_j)| = k$. The vertex $v_{(n/4)}$ is adjacent to all other vertices. Then, $d(v_{(n/4)}, v_j) = 1$, where $j \neq (3n/4)$. We now have $h \leq |f(v_0) - f(v_j)| \leq (n/4)(h+3k) - 2k$. Suppose that the vertex $v_{(3n/4)}$ is adjacent to all other vertices. Then, $d(v_{(3n/4)}, v_j) = 1$, where $j \neq (n/4)$. We have $h + k \leq |f(v_{n/2}) - f(v_j)| \leq (n/4)(h+3k) - k$. Figures 2 and 3 illustrate that the bounds are sharp. □

**Theorem 6.** *Let $n$ be even and $h$ and$k$ be positive integers with $k < \lceil h/2 \rceil$. Then, the inverse graph $GS(Z_n)$ admits an $L(h,k)$-labeling with*

$$\lambda_k^h(GS(Z_n)) \leq \begin{cases} \dfrac{n-2}{2}h + \dfrac{n+2}{4}k, & \text{if } n \equiv 2 \pmod 4, \\ \left(\dfrac{n}{2} - 1\right)h + \dfrac{n+4}{4}k, & \text{if } n \equiv 0 \pmod 4. \end{cases} \tag{6}$$

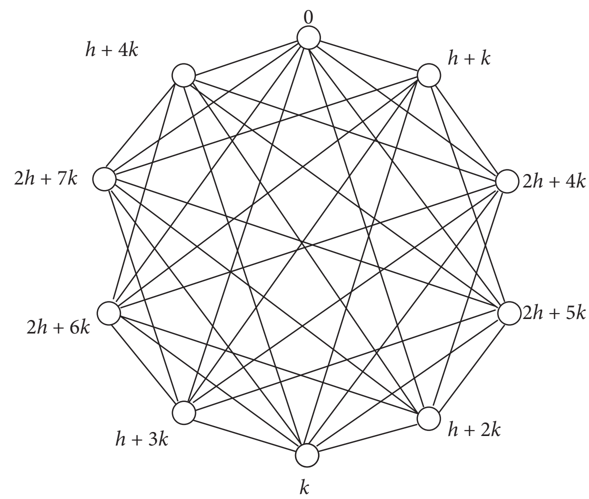


Figure 2: $\lambda_k^h(GS(Z_{10})) = 2h + 7k$.

*Finally, these bounds are sharp.*

*Proof.* We have the following cases:

Case (a): if $n \equiv 2 \pmod 4$, since $n$ is even, the self-invertible elements are 0 and $(n/2)$. Let $G = GS(Z_n)$ be the inverse graph associated with a finite cyclic group. Let $V(G) = S_1 \cup S_2 \cup S_3 \cup S_4 \cup S_5 \cup S_6$, where $S_1 = \{v_0\}$, $S_2 = \{v_1, v_2, \ldots, v_{(n-2/4)}\}$, $S_3 = \{v_{(n-2/4)+1}, v_{(n-2/4)+2}, \ldots, v_{(n/2)-1}\}$, $S_4 = \{v_{(n/2)+1}, v_{(n/2)+2}, \ldots, v_{(3n-2/4)}\}$, $S_5 = \{v_{(3n-2/4)+1}, v_{(3n-2/4)+2}, \ldots, v_{n-1}\}$, and $S_6 = \{v_{(n/2)}\}$. Define the function $f\colon V(G) \longrightarrow \{0, 1, \ldots, n\}$:

$$f(v_i) = \begin{cases} 0, & \text{if } v_0 \in S_1, \\ (2(i-1)+1)h + k, & \text{if } v_i \in S_2, \\ \left(2\left(\dfrac{n}{2} - i\right) - 1\right)h + \left(\dfrac{n}{2} - i + 1\right)k, & \text{if } v_i \in S_3, \\ \left(i - \dfrac{n}{2}\right)2h + \left(i - \dfrac{n}{2}\right)k, & \text{if } v_i \in S_4, \\ (n-i)2h + (n-i+1)k, & \text{if } v_i \in S_5, \\ k, & \text{if } v_{(n/2)} \in S_6. \end{cases} \tag{7}$$

Subcase (i): if $v_i$ and $v_j$ are distinct elements in $S_p$, for all $2 \leq p \leq 5$, then $d(v_i, v_j) = 1$. We claim that $|f(v_i) - f(v_j)| \geq h$. We have $2h + k \leq |f(v_i) - f(v_j)| \leq (n-6/4)(2h+k)$. Clearly, $2h + k \geq h$. Hence, $|f(v_i) - f(v_j)| \geq h$.
Subcase (ii): let $v_i \in S_2$ and $v_j \in S_3$ with $i \neq j$ be the adjacent vertices iff $j \neq (n/2) - i$. Assume that $j \neq (n/2) - i$; then, $d(v_i, v_j) = 1$. We have $2h + 2k \leq |f(v_i) - f(v_j)| \leq (n-6/4)(2h+k) + k$. Clearly, $|f$

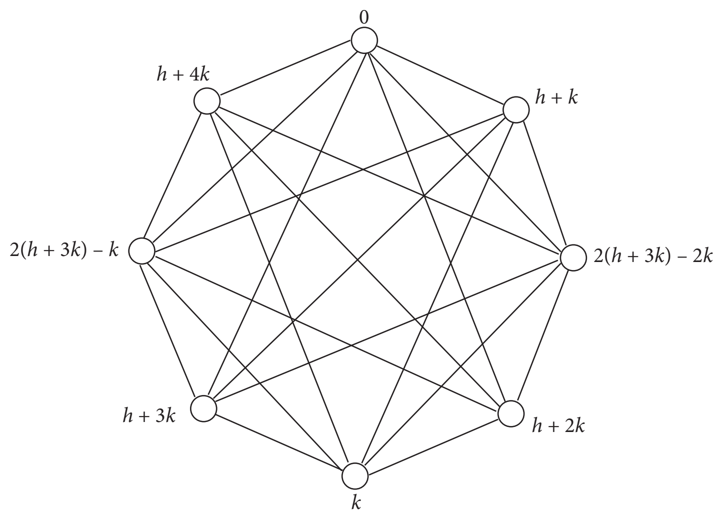


Figure 3: $\lambda_k^h(GS(Z_8)) = 2h+5k$.

$(v_i) - f(v_j)| \geq h$. If $j = (n/2) - i$, then $d(v_i, v_j) = 2$. We then have $k = |f(v_i) - f(v_j)|$. Suppose that the element $v_j \in S_4$; then, $d(v_i, v_j) = 1$. We have $h \leq |f(v_i) - f(v_j)| \leq (n-2/4)(2h+3k) - (h+k)$. Clearly, $|f(v_i) - f(v_j)| \geq h$. Suppose that the element $v_j \in S_5$, $v_i$ is adjacent to $v_j$ iff $j \neq n-i$. Assume that $j \neq n-i$, then $d(v_i, v_j) = 1$. For the function $f$, we have $h \leq |f(v_i) - f(v_j)| \leq (n-6/4)(2h+k) + (h+k)$. If $j = n-i$, then $d(v_i, v_j) = 2$. Hence, $|f(v_i) - f(v_j)| = h+k \geq k$.

Subcase(iii): If $v_i \in S_3$, suppose $v_j \in S_4$ and $v_i$ is adjacent to $v_j$ iff$j \neq n-i$. If $j \neq n-i$, then $d(v_i, v_j) = 1$. We have $h+2k \leq |f(v_i) - f(v_j)| \leq (n-6/4)\ (2h+k) + h$. Clearly, $|f(v_i) - f(v_j)| \geq h$. If $j = n-i$, then $d(v_i, v_j) = 2$. We have $|f(v_i) - f(v_j)| = h-k$. Clearly, $k < (h/2)$. Then, $2k < h$ implies that $h-k > k$. Hence, $|f(v_i) - f(v_j)| \geq k$. Suppose that $v_j \in S_5$. Then, $d(v_i, v_j) = 1$. We have $h \leq |f(v_i) - f(v_j)| \leq (n-6/4)(2h+k) + h$. Clearly, $|f(v_i) - f(v_j)| \geq h$.

Subcase (iv): let $v_i \in S_4$ and $v_j \in S_5$ with $v_i$ is adjacent to $v_j$ iff $j \neq (3n/2) - i$. $j \neq (3n/2) - i$; then, $d(v_i, v_j) = 1$. We have $2h \leq |f(v_i) - f(v_j)| \leq (n-6/4)(2h+k) + k$. Clearly, $|f(v_i) - f(v_j)| \geq h$. If $j = (3n/2) - i$, then $d(v_i, v_j) = 2$. We then have $k = |f(v_i) - f(v_j)|$.

Subcase (v): the vertex $v_0$ is not adjacent to $v_{(n/2)}$ and $d(v_0, v_{(n/2)}) = 2$. Here, $|f(v_i) - f(v_j)| = k$. The vertex $v_0$ is adjacent to all other vertices. Then, $d(v_0, v_j) = 1$, where $j \neq (n/2)$. We have $h+k \leq |f(v_0) - f(v_j)| \leq (n-2/2)h + (n+2/4)k$. Suppose that the vertex $v_{(n/2)}$ is adjacent to all other vertices. Then, $d(v_{(n/2)}, v_j) = 1$, where $j \neq 0$. We have $h \leq |f(v_{(n/2)}) - f(v_j)| \leq (n-2/2)h + (n-2/4)k$.

Case (b): if $n \equiv 0 \pmod 4$, since $n$ is even, then the self-invertible elements are 0 and $(n/2)$. Let $G = GS(Z_n)$ be the inverse graph associated with a finite cyclic group. Let $V(G) = S_1 \cup S_2 \cup S_3 \cup S_4 \cup S_5 \cup S_6 \cup S_7 \cup S_8$, where $S_1 = \{v_0\}$, $S_2 = \{v_1, v_2, \ldots, v_{(n/4)-1}\}$, $S_3 = \{v_{(n/4)+1}, v_{(n/4)+2}, \ldots, v_{(n/2)-1}\}$, $S_4 = \{v_{(n/2)+1}, v_{(n/2)+2}, \ldots, v_{(3n/4)-1}\}$, $S_5 = \{v_{(3n/4)+1}, v_{(3n/4)+2}, \ldots, v_{n-1}\}$, $S_6 = \{v_{(n/2)}\}$, $S_7 = \{v_{(n/4)}\}$, and $S_8 = \{v_{(3n/4)}\}$. Define a function $f\colon V(G) \longrightarrow \{0, 1, \ldots, n\}$. Then,

$$f(v_i) = \begin{cases} 0, & \text{if } v_0 \in S_1, \\ (2(i-1)+1)h + k, & \text{if } v_i \in S_2, \\ \left(2\left(\dfrac{n}{2} - i\right) - 1\right)h + \left(\dfrac{n}{2}, -i + 1\right)k, & \text{if } v_i \in S_3, \\ \left(i - \dfrac{n}{2}\right)2h + \left(i - \dfrac{n}{2}\right)k, & \text{if } v_i \in S_4, \\ (n-i)2h + (n-i+1)k, & \text{if } v_i \in S_5, \\ k, & \text{if } v_{(n/2)} \in S_6, \\ \left(\dfrac{n}{2} - 1\right)h + \dfrac{n}{4}k, & \text{if } v_{(n/4)} \in S_7, \\ \left(\dfrac{n}{2} - 1\right)h + \left(\dfrac{n+4}{4}\right)k, & \text{if } v_{(3n/4)} \in S_8. \end{cases} \tag{8}$$

The case $n \equiv 0 \pmod 4$ can be dealt similarly as follows. The vertex $v_{(n/4)}$ is not adjacent to $v_{(3n/4)}$ and $d(v_{(n/4)}, v_{(3n/4)}) = 2$. Then, $|f(v_i) - f(v_j)| = k$. The vertex $v_{(n/4)}$ is adjacent to all other vertices. Then, $d(v_{(n/4)}, v_j) = 1$, where $j \neq (3n/4)$. We have $h \leq |f(v_0) - f(v_j)| \leq ((n/2) - 1)h + (n/4)k$. Suppose the vertex $v_{(3n/4)}$ is adjacent to all other vertices. Then, $d(v_{(3n/4)}, v_j) = 1$, where $j \neq (n/4)$. We hence have $h+k \leq |f(v_{(n/4)}) - f(v_j)| \leq ((n/2) - 1)h + (n+4/4)k$. Figures 4 and 5 illustrate that the bounds are sharp. $\square$

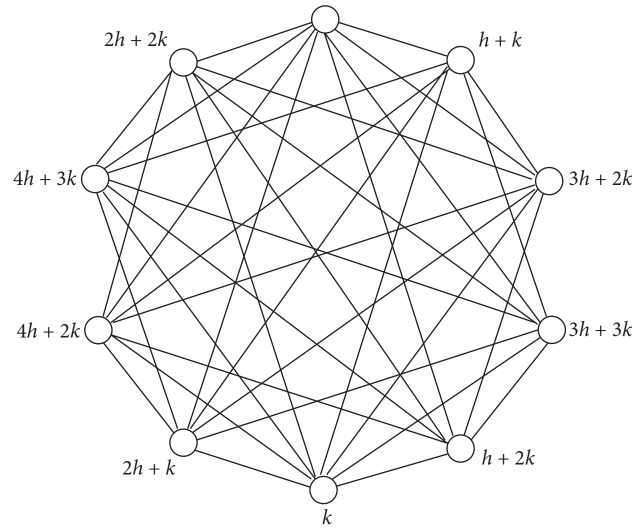


FIGURE 4: $\lambda_k^h(GS(Z_{10})) = 4h + 3k$.

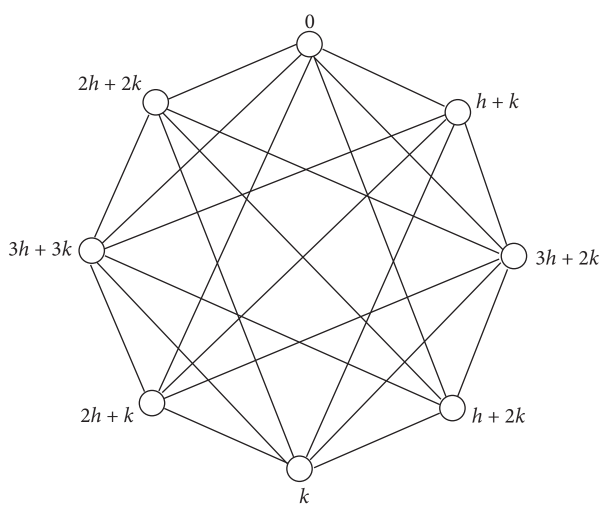


FIGURE 5: $\lambda_k^h(GS(Z_8)) = 3h + 3k$.

## 4. Relation between $L(h, k)$ and the Radio Labeling

The radio labeling idea is applied to a large number of interruption permissible levels in $L(h, k)$-labeling from $h$ to the maximum allowable diameter of $G$, where $h = 2$ and $k = 1$. By Theorem 3, $\operatorname{diam}(GS(Z_n)) = 2$. Hence, the maximum distance is 2 so that the possible distance of $GS(Z_n)$ is 1 or 2. By radio labeling given in Definition 3, a graph $G(V, E)$ is a one-to-one function $f: V \longrightarrow [0, \infty)$ such that, for every $u, v \in V$, $|f(u) - f(v)| \geq 1 + \operatorname{diam}(G) - d(u, v)$. We have the following cases.

Case (a): let $d(u, v) = 2$. If $\operatorname{diam}(GS(Z_n)) = 2$ and $d(u, v) = 2$, then, by the function of radio labeling $f$, we have $|f(u) - f(v)| \geq 1 + 2 - 2$. Clearly, $|f(v_i) - f(v_j)| \geq 1$.

Case (b): let $d(u, v) = 1$. If $\operatorname{diam}(GS(Z_n)) = 2$ and $d(u, v) = 1$, then we have $|f(u) - f(v)| \geq 1 + 2 - 1$. Clearly, $|f(v_i) - f(v_j)| \geq 2$.

By the above cases, one can easily conclude that radio labeling is the same as$L(h, k)$-labeling when $h = 2$ and $k = 1$.

Clearly, the proof of the Theorems in Section 3 is directly applicable to the radio labeling. Hence, $GS(Z_n)$ admits radio labeling with minimum index.

## 5. Conclusion and Further Research Work

In this paper, we have worked on $L(h, k)$ labeling of a new family of graphs, i.e., an inverse graph with a finite cyclic group. We provided the upper bounds of the index for five specific cases, that is, $n(\geq 3)$ is an odd integer; then, $\lambda_k^h(GS(Z_n)) \leq (n - 1/2)(h + k)$;$n$ is even integer with $k \geq \lceil h/2 \rceil$ and $n \equiv 2 \pmod 4$, then $\lambda_k^h(GS(Z_n)) \leq (n - 2/4)(h + 3k) + k$;$n$ is even integer with $k \geq \lceil h/2 \rceil$ and $n \equiv 0 \pmod 4$, then $\lambda_k^h(GS(Z_n)) \leq (n/4)(h + 3k) - k$; $n$ is even integer with $k < \lceil h/2 \rceil$ and $n \equiv 2 \pmod 4$, then $\lambda_k^h(GS(Z_n)) \leq (n - 2/2)h + (n + 2/4)k$;$n$ is even integer with $k < \lceil h/2 \rceil$ and $n \equiv 0 \pmod 4$, then $\lambda_k^h(GS(Z_n)) \leq ((n/2) - 1)h + (n + 4/4)k$. We have generalized the results and obtained the bounds for $L(h, k)$ labeling of inverse graph of a finite cyclic group of order $n$. The novelty of this research is bounds of the index which are sharp and the results are directly applicable to the radio labeling minimum index. However, the problem remains open to find $L(h, k)$-labeling with the minimum index for the inverse graphs of dihedral and symmetric groups in general. Furthermore, research is inspired to notice the $L(h, k)$ labeling of the inverse graph for an arbitrary finite group.

## Data Availability

No data were used to support this study.

## Conflicts of Interest

The authors declare that there are no conflicts of interest regarding the publication of this paper.

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