Research Article

Coordinated MT-\((s_1, s_2)\)-Convex Functions and Their Integral Inequalities of Hermite–Hadamard Type

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Received 26 January 2021; Revised 11 April 2021; Accepted 15 April 2021; Published 29 April 2021

Academic Editor: Basil Papadopoulos

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In the paper, the authors introduce a new concept of MT-\((s_1, s_2)\)-convex functions on the coordinates on the rectangle of the plane and establish some new Hermite–Hadamard-type inequalities for this kind of functions.

1. Motivations

At first, we recall several kinds of convex functions as follows.

Definition 1 (see [1]). Let \(I \subseteq \mathbb{R}\) be an interval. A nonnegative function \(f: I \to \mathbb{R}_0 = [0, \infty)\) is said to be MT-convex if the inequality
\[
f(tx + (1 - t)y) \leq \sqrt{t}f(x) + \sqrt{1 - t}f(y)
\]
holds for all \(x, y \in I\) and \(t \in (0, 1)\).

Definition 2 (see [2, 3]). Let \(s \in (0, 1]\) be a real number. A function \(f: \mathbb{R} \to \mathbb{R}_0\) is said to be \(s\)-convex in the second sense if
\[
f(tx + (1 - t)y) \leq tf(x) + (1 - t)^sf(y),
\]
for all \(x, y \in I\) and \(t \in [0, 1]\).

Definition 3 (see [4, 5]). A function \(f: \Delta = [a, b] \times [c, d] \subseteq \mathbb{R}^2 \to \mathbb{R}\) is said to be convex on the coordinates on \(\Delta\) if
\[
f(tx + (1 - t)y, \lambda x + (1 - \lambda) \omega) \leq t\lambda f(x, y) + t(1 - \lambda)f(x, \omega)
+ (1 - t)\lambda f(z, y) + (1 - t)(1 - \lambda)f(z, \omega)
\]
holds for all \(t, \lambda \in [0, 1]\) and \((x, y), (z, \omega) \in \Delta\). If the inequality (3) is reversed, then \(f\) is said to be concave on the coordinates on \(\Delta\).

Definition 4 (see [6]). We say that a function \(f: \Delta = [a, b] \times [c, d] \subseteq \mathbb{R}^2 \to \mathbb{R}_0\) is MT-convex on the coordinates on \(\Delta\) with \(a < b\) and \(c < d\), if the inequality

\[f(tx + (1 - t)y, \lambda x + (1 - \lambda) \omega) \leq \sqrt{t}f(x, y) + \sqrt{1 - t}f(x, \omega)
+ \sqrt{1 - \lambda}f(z, y) + \sqrt{1 - (1 - \lambda)}f(z, \omega)
\]
holds for all \(t, \lambda \in [0, 1]\) and \((x, y), (z, \omega) \in \Delta\).
From this, we conclude as follows:

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m therein, the HT-convexity, GT-convexity, and the \( \Delta \)-theo coordinates on \( s \) holds for all \( t, \lambda \in (0, 1) \) and \( (x, y), (z, w) \in \Delta \). If inequality (5) is reversed, then \( f \) is said to be a MT-\((s_1, s_2)\)-concave function on the coordinates on \( \Delta \).

2. Simple Properties of MT-\((s_1, s_2)\)-Convex Functions

After introduced Definition 5, now we are in a position to investigate in this section simple properties of MT-\((s_1, s_2)\)-convex functions on the coordinates on \( \Delta \).

**Proposition 1.** Let \((s_1, s_2) \in (0, 1)^2\) and \( f: \Delta = [a, b] \times [c, d] \subseteq \mathbb{R}^2 \rightarrow \mathbb{R}_0 \). If \( f \) is nonnegative and convex on the coordinates on \( \Delta \), then \( f \) is MT-convex on the coordinates on \( \Delta \), while \( f \) is also MT-\((s_1, s_2)\)-convex on the coordinates on \( \Delta \).

**Proof.** This follows from \( t^{s_1/2}/2\sqrt{1-t} \geq t^{s_1/2}/2\sqrt{1-t} \geq t \) for \( s \in (0, 1] \) and \( t \in (0, 1) \). \( \square \)

Combining the structures of Definitions 2 and 4, we introduce the notion of coordinated MT-\((s_1, s_2)\)-convex functions as follows.

**Definition 5.** For \((s_1, s_2) \in (0, 1)^2\), a function \( f: \Delta = [a, b] \times [c, d] \subseteq \mathbb{R}^2 \rightarrow \mathbb{R}_0 \) is said to be MT-\((s_1, s_2)\)-convex on the coordinates on \( \Delta \) with \( a < b \) and \( c < d \), if the inequality

\[
\frac{\sqrt{t}}{4\sqrt{(1-t)(1-\lambda)}} f(x, y) + \frac{\sqrt{t(1-\lambda)}}{4\sqrt{t}} f(x, w) + \frac{\sqrt{(1-t)}}{4\sqrt{(1-\lambda)}} f(z, y) + \frac{\sqrt{(1-t)(1-\lambda)}}{4\sqrt{t}} f(z, w) \leq \frac{\sqrt{t}t^{s_1/2}(1-\lambda)^{s_2/2}}{4\sqrt{(1-t)(1-\lambda)}} f(x, y) + \frac{\sqrt{t}t^{s_1/2}(1-\lambda)^{s_2/2}}{4\sqrt{(1-t)(1-\lambda)}} f(x, w) + \frac{(1-t)t^{s_1/2}(1-\lambda)^{s_2/2}}{4\sqrt{t}} f(z, y) + \frac{(1-t)t^{s_1/2}(1-\lambda)^{s_2/2}}{4\sqrt{t}} f(z, w)
\]

is valid for all \( t, \lambda \in (0, 1) \) and \( (x, y), (z, w) \in \Delta \).

**Example 1** (see [13], p. 104). When \( p \in (0, 1/1000) \), the functions \( f(x) = x^p \) and \( g(x) = (1 + x)^p \) for \( x \in \Delta_1 = (1, \infty) \) are MT-convex, but they are not convex on \( \Delta_1 \).

For \( m \in (0, 1/100) \), the function \( h(x) = (1 + x^2)^m \) for \( x \in [1, 3/2] \) is MT-convex, but it is not convex on the coordinates on \( \Delta_1 \).

**Remark 1.** We now discuss Examples 1 and 2 mentioned above.

For \( 0 < p < 1 \), \( c \geq 0 \), and \( I \subseteq (0, \infty) \) being a nonempty interval, the function \( \varphi(x) = (x + c)^p \) is concave on \( I \). Therefore, for \( t = 1/2 \) and all \( x, y \in \Delta \) with \( x \neq y \), we have

\[
\varphi(tx + (1-t)y) = \left(\frac{x+y+c}{2}\right)^p \geq \frac{(x+c)^p + (y+c)^p}{2} = \frac{\sqrt{t}}{2\sqrt{1-t}} \varphi(x) + \frac{\sqrt{1-t}}{2\sqrt{t}} \varphi(y).
\]

Accordingly, the function \( \varphi(x) \) is not MT-convex on \( I \).

From this, we conclude as follows:

(1) For \( p \in (0, 1/1000) \), the functions \( f(x) = x^p \) and \( g(x) = (1 + x)^p \) with respect to \( x \in \Delta_1 \) are not MT-convex on \( \Delta_1 \).

(2) For \( m \in (0, 1/100) \), the function \( h(x) = (1 + x^2)^m \) with respect to \( x \in [1, 3/2] \) is not MT-convex, but it is convex on \([1, 1/\sqrt{1-2m}]\) and is concave on \([1/\sqrt{1-2m}, 3/2]\).

(3) For \( p \in (0, 1/1000) \), the function \( f(x, y) = x^p + y^p \) with respect to \( (x, y) \in \Delta_1 \) is not MT-convex on the coordinates on \( \Delta_1 \).

**Proposition 2.** Let \( f(x, y) = x^{1/2} + (x, y) \in \mathbb{R}_+^2 = (a, b) \). Then, the function \( f(x, y) \) is MT-\((0.5, 0.02)\)-convex, but not MT-convex, on the coordinates on \( \mathbb{R}_+^2 \).
Proof. For \( s_1 = 0.5 \) and \( s_2 = 0.02 \), for \( t, \lambda \in (0, 1) \), and for \((x, y), (z, w) \in \mathbb{R}^2_+\), by Definition 5, we deduce

\[
f(tx + (1-t)z, \lambda y + (1-\lambda)w) = [tx + (1-t)z]^{1/2}
\]

\[
\begin{align*}
&= \frac{t^{0.5/2} \lambda^{0.02/2}}{4(1-t)(1-\lambda)} f(x, y) + \frac{t^{0.5/2} (1-\lambda)^{0.02/2}}{4(1-t)} f(x, w) \\
&\quad + \frac{(1-t)^{0.5/2} \lambda^{0.02/2}}{4\sqrt{t}(1-\lambda)} f(z, y) + \frac{(1-t)^{0.5/2} (1-\lambda)^{0.02/2}}{4\sqrt{t}} f(z, w) \\
&= \left[ \frac{\lambda^{0.02/2}}{2\sqrt{1-\lambda}} + (1-\lambda)^{0.02/2} \right] \left[ \frac{t^{0.5/2}}{2\sqrt{1-t}} x^{1/2} + \frac{(1-t)^{0.5/2}}{2\sqrt{t}} z^{1/2} \right].
\end{align*}
\] (7)

Making use of the inequality \( \lambda^{0.02/2}/2\sqrt{1-\lambda} + (1-\lambda)^{0.02/2}/2\sqrt{\lambda} \geq 2^{0.49} > 1.4 \) and letting \( u = x/z \) result in

\[
[tx + (1-t)z]^{1/2} = z^{1/2} [tu + (1-t)]^{1/2}
\]

\[
< 1.4z^{1/2} \left[ \frac{t^{0.5/2}}{2\sqrt{1-t}} u^{1/2} + (1-t)^{0.5/2} \right]
\]

\[
< \left[ \frac{\lambda^{0.02/2}}{2\sqrt{1-\lambda}} + (1-\lambda)^{0.02/2} \right] \left[ \frac{t^{0.5/2}}{2\sqrt{1-t}} x^{1/2} + \frac{(1-t)^{0.5/2}}{2\sqrt{t}} z^{1/2} \right].
\] (8)

This means that the function \( f(x, y) \) is MT-(0.5, 0.02)-convex on the coordinates on \( \mathbb{R}^2_+ \).

For \((x, y), (z, w) \in \mathbb{R}^2_+ \) with \( x \neq z \), taking \( t = \lambda = 1/2 \) in Definition 5 leads to

\[
f(tx + (1-t)z, \lambda y + (1-\lambda)w) = \left( \frac{x + y}{2} \right)^{1/2} > \frac{x^{1/2} + y^{1/2}}{2}
\]

\[
= \frac{\sqrt{\lambda}}{4\sqrt{(1-t)(1-\lambda)}} f(x, y) + \frac{\sqrt{t(1-\lambda)}}{4\sqrt{(1-t)}} f(x, w) \\
\quad + \frac{\sqrt{(1-t) \lambda}}{4\sqrt{t}(1-\lambda)} f(z, y) + \frac{\sqrt{(1-t)(1-\lambda)}}{4\sqrt{t\lambda}} f(z, w).
\] (9)

This means that the function \( f(x, y) \) is not MT-convex on the coordinates on \( \mathbb{R}^2_+ \). The proof of Proposition 2 is complete. \( \square \)

3. A Lemma

In order to establish integral inequalities of the Hermite–Hadamard type for MT-\((s_1, s_2)\)-convex functions on the coordinates on \( \Delta \), we need the following lemma.

Lemma 1. Let \( f: \Delta = [a, b] \times [c, d] \subseteq \mathbb{R}^2 \longrightarrow \mathbb{R} \) have partial derivatives of the second order and let \( a < b \) and \( c < d \). If \( \partial^2 f/\partial x \partial y \in L_1(\Delta) \), then

\[
I(f) = f(b, d) - \frac{1}{b-a} \int_a^b f(x, d)dx - \frac{1}{d-c} \int_c^d f(b, y)dy
\]

\[
+ \frac{1}{(b-a)(d-c)} \int_a^b \int_c^d f(x, y)dxdy
\]

\[
= (b-a)(d-c) \int_0^1 \int_0^1 (1-t)(1-\lambda) \frac{\partial^2 f(ta + (1-t)b, \lambda c + (1-\lambda)d)}{\partial x \partial y}dtd\lambda.
\] (10)

Proof. Integrating by parts gives
\[
\int_0^1 \int_0^1 (1-t)(1-\lambda) \frac{\partial^2 f (ta + (1-t)b, \lambda c + (1-\lambda)d)}{\partial x \partial y} \, dt \, d\lambda \\
= \frac{1}{a-b} \int_0^1 (1-\lambda) \left[ (1-t) \frac{\partial f (ta + (1-t)b, \lambda c + (1-\lambda)d)}{\partial y} \bigg|_{t=0}^{t=1} + \int_0^1 \frac{\partial f (ta + (1-t)b, \lambda c + (1-\lambda)d)}{\partial y} \, dt \right] \, d\lambda \\
= \frac{1}{a-b} \left[ \int_0^1 (\lambda-1) \frac{\partial f (b, \lambda c + (1-\lambda)d)}{\partial y} \, d\lambda + \int_0^1 \int_0^1 (1-\lambda) \frac{\partial f (ta + (1-t)b, \lambda c + (1-\lambda)d)}{\partial y} \, dt \, d\lambda \right] \\
= \frac{1}{(a-b)(c-d)} \left[ f (b, d) - \int_0^1 f (b, \lambda c + (1-\lambda)d) \, d\lambda \right] \\
= \frac{1}{a-b} \int_a^b f (x, d) \, dx - \frac{1}{d-c} \int_c^d f (b, y) \, dy \\
+ \frac{1}{(b-a)(d-c)} \int_c^d \int_a^b f (x, y) \, dx \, dy.
\]

The proof of Lemma 1 is complete.

\[\square\]

**Theorem 1.** Let \( f : \Delta = [a, b] \times [c, d] \subseteq \mathbb{R}^2 \longrightarrow \mathbb{R} \) for \( a < b \) and \( c < d \). If \( f \) is co-ordinated MT-(\( s_1, s_2 \))-convex on \( \Delta \) for \((s_1, s_2) \in (0, 1)^2 \) and \( f \in L_1 (\Delta) \), then

\[
\frac{1}{2^{s_1+s_2}} f \left( \frac{a + b + c + d}{2} \right) \leq \frac{1}{2^{s_1+s_2}} \left[ \frac{1}{b-a} \int_a^b f \left( x, \frac{c + d}{2} \right) \, dx + \frac{1}{d-c} \int_c^d f \left( \frac{a + b}{2}, y \right) \, dy \right] \\
\leq \frac{1}{(b-a)(d-c)} \int_c^d \int_a^b f (x, y) \, dx \, dy \\
\leq \frac{1}{2} \left[ B \left( \frac{s_1 + 2}{2}, \frac{1}{2} \right) \int_a^b \left[ f (x, c) + f (x, d) \right] \, dx + B \left( \frac{s_1 + 1}{2}, \frac{1}{2} \right) \int_c^d \left[ f (a, y) + f (b, y) \right] \, dy \right] \\
\leq B \left( \frac{(s_1 + 2)/2}{2}, \frac{1}{2} \right) B \left( \frac{(s_1 + 2)/2}{2}, \frac{1}{2} \right) \left[ f (a, c) + f (b, c) + f (a, d) + f (b, d) \right],
\]

(12)
where \( B(a, \beta) \) denotes the well-known beta function which may be defined by
\[
B(a, \beta) = \int_0^1 t^{a-1} (1-t)^{\beta-1} dt, \quad \Re(a), \Re(\beta) > 0. \tag{13}
\]

Proof. For all \( 0 < t < 1 \), we have
\[
a + b = \frac{ta + (1-t)b + (1-t)a + tb}{2},
\]
\[
c + d = \frac{1}{2} \left( \frac{c + d + c + d}{2} \right). \tag{14}
\]

Letting \( x = ta + (1-t)b, z = (1-t)a + tb, \) and \( z = w = c + \frac{d}{2} \) in (4) and using the MT- \((s_1, s_2)\)-convexity of \( f \), we obtain
\[
f\left(\frac{a + b + c + d}{2}, \frac{c + d}{2}\right) \leq \frac{1}{2(s_1 + s_2)^{2-1}} \int_0^1 f\left(\frac{ta + (1-t)b}{2}, \frac{c + d}{2}\right) + f\left((1-t)a + tb, \frac{c + d}{2}\right) \bigg] dt
\]
\[
= \frac{1}{2(s_1 + s_2)^{2-1}} (b-a) \int_a^b f\left(x, \frac{c + d}{2}\right) dx. \tag{16}
\]

By the MT- \((s_1, s_2)\)-convexity of \( f \), we obtain
\[
\frac{1}{b-a} \int_a^b f\left(x, \frac{c + d}{2}\right) dx \leq \frac{1}{2(s_1 + s_2)^{2-1}} (b-a) \int_0^1 [f(x, \lambda c + (1-\lambda)d) + f(x, (1-\lambda)c + \lambda d)] dx d\lambda
\]
\[
= \frac{1}{2(s_1 + s_2)^{2-1}} (b-a) (d-c) \int_c^d \int_a^b f(x, y) dx dy. \tag{17}
\]

From (16) and (17), it follows that
\[
\frac{1}{2^{s_2} (s_1 + s_2)^{2}} \int_0^1 f\left(\frac{a + b + c + d}{2}, \frac{c + d}{2}\right) dx \leq \frac{1}{2^{s_2} (s_1 + s_2)^{2}} (b-a) \int_0^1 \int_c^d f\left(\frac{a + b + c + d}{2}, \frac{a + b + c + d}{2}\right) dy
\]
\[
= \frac{1}{(b-a)(d-c)} \int_c^d \int_a^b f(x, y) dx dy. \tag{18}
\]

Similarly, we have
\[
\frac{1}{2^{s_2} (s_1 + s_2)^{2}} \int_0^1 f\left(x, \frac{c + d}{2}\right) dx \leq \frac{1}{2^{s_2} (s_1 + s_2)^{2}} (b-a) \int_0^1 \int_c^d f\left(x, \frac{a + b + c + d}{2}\right) dx dy
\]
\[
= \frac{1}{(b-a)(d-c)} \int_c^d \int_a^b f(x, y) dx dy. \tag{19}
\]

A combination of (18) and (19) gives the desired inequality (12).

Putting \( y = \lambda c + (1-\lambda)d \) for all \( 0 < \lambda < 1 \) and using the MT- \((s_1, s_2)\)-convexity of \( f \) reveals
\[
\frac{1}{(b-a)(d-c)} \int_c^d \int_a^b f(x, y) \, dx \, dy = \frac{1}{b-a} \int_a^b f \left( \frac{x + \lambda c + (1 - \lambda) d}{2} \right) \, dx \, d\lambda \\
\leq \frac{1}{2^{(s_1+1)/2} (b-a)} \int_0^{1/2} \left[ \lambda^{s_1/2} f(x, c) + \frac{(1 - \lambda)^{s_1/2}}{\sqrt{1 - \lambda}} f(x, d) \right] \, dx \, d\lambda \\
= B((s_2 + 2)/2, (1/2)) \int_a^b \left[ f(x, c) + f(x, d) \right] \, dx.
\]

(20)

Taking \( x = ta + (1 - t)b \) for \( 0 < t < 1 \) and employing the MT-\((s_1, s_2)\)-convexity of \( f \) leads to

\[
\frac{1}{b-a} \int_a^b f(x, d) \, dx \leq \frac{B((s_1 + 2)/2, (1/2))}{2^{(s_1+1)/2}} [f(a, d) + f(b, d)].
\]

(21)

(22)

Applying inequalities between (19) and (22) arrives at

\[
\frac{1}{(b-a)(d-c)} \int_c^d \int_a^b f(x, y) \, dx \, dy \leq \frac{B((s_2 + 2)/2, (1/2))}{2^{(s_1+1)/2} (b-a)} \int_a^b \left[ f(x, c) + f(x, d) \right] \, dx \\
\leq \frac{B((s_1 + 2)/2, (1/2))B((s_2 + 2)/2, (1/2))}{2^{(s_1+s_2)/2+1}} \left[ f(a, c) + f(b, c) + f(a, d) + f(b, d) \right].
\]

(23)

By similar argument, we can find

\[
\frac{1}{(b-a)(d-c)} \int_c^d \int_a^b f(x, y) \, dx \, dy \leq \frac{B((s_1 + 2)/2, (1/2))}{2^{(s_1+1)/2} (s_1 + 1/2)(d-c)} \int_c^d \left[ f(a, y) + f(b, y) \right] \, dy \\
\leq \frac{B((s_1 + 2)/2, (1/2))B((s_2 + 2)/2, (1/2))}{2^{(s_1+s_2)/2+1}} \left[ f(a, c) + f(b, c) + f(a, d) + f(b, d) \right].
\]

(24)

The proof of Theorem 1 is complete. \( \square \)

**Corollary 1.** Under the conditions of Theorem 1, if \( s_1 = s_2 = s \), then
\[
\frac{1}{2^{2(1-s)}} f\left(\frac{a+b+c+d}{2}\right) \leq \frac{1}{2^{2+\frac{2}{q}}} \left[ \frac{1}{b-a} \int_a^b f\left(x, \frac{c+d}{2}\right) dx + \frac{1}{d-c} \int_c^d f\left(\frac{a+b}{2}, y\right) dy \right] \\
\leq \frac{1}{(b-a)(d-c)} \int_c^d \int_a^b f(x, y) dx dy \\
\leq B((s+2)/2, (1/2)) \left[ \frac{1}{b-a} \int_a^b [f(x, c) + f(x, d)] dx + \frac{1}{d-c} \int_c^d [f(a, y) + f(b, y)] dy \right] \\
\leq \frac{[B(((s+2)/2), (1/2))]^2}{2^{2s+1}} [f(a, c) + f(b, c) + f(a, d) + f(b, d)].
\]

Theorem 2. Let \( f: \Delta = [a, b] \times [c, d] \subseteq \mathbb{R}^2 \rightarrow \mathbb{R} \) for \( a < b \) and \( c < d \) have the second partial derivatives and \((\partial^2 f / \partial x \partial y) \in L_1(\Delta)\) and let \((s_1, s_2) \in (0, 1]^2\). If 

\[
|I(f)| \leq \frac{(b-a)(d-c)}{4^{1+sq}} \left[ B(((s_1+2)/2), (3/2)) B(((s_2+2)/2), (3/2)) \right]^{1/q} \\
\cdot \left[ \frac{\partial^2 f(a, c)}{\partial x \partial y} + (s_2 + 2) \frac{\partial^2 f(a, d)}{\partial x \partial y} + (s_1 + 2) \frac{\partial^2 f(b, c)}{\partial x \partial y} + (s_1 + 2)(s_2 + 2) \frac{\partial^2 f(b, d)}{\partial x \partial y} \right]^{1/q},
\]

where \( B(\alpha, \beta) \) is the Beta function.

Proof. From Lemma 1 and Hölder's integral inequality, it follows that

\[
|I(f)| \leq (b-a)(d-c) \left[ \int_0^1 \int_0^1 (1-t)(1-\lambda) dt d\lambda \right]^{1-1/q} \\
\times \left[ \int_0^1 \int_0^1 (1-t)(1-\lambda) \left| \frac{\partial^2 f(ta + (1-t)b, \lambda c + (1-\lambda)d)}{\partial x \partial y} \right|^q dt d\lambda \right]^{1/q} \\
= \frac{(b-a)(d-c)}{4^{1-sq}} \left[ \int_0^1 \int_0^1 (1-t)(1-\lambda) \left| \frac{\partial^2 f(ta + (1-t)b, \lambda c + (1-\lambda)d)}{\partial x \partial y} \right|^q dt d\lambda \right]^{1/q}.
\]

By the coordinated \( MT\cdot(s_1, s_2)\)-convexity of \(|(\partial^2 f / \partial x \partial y)|^q\), we have
\[
\begin{aligned}
&\int_0^1 \int_0^1 (1-t)(1-\lambda) \left| \frac{\partial^2 f(ta + (1-t)b, \lambda c + (1-\lambda)d)}{\partial x \partial y} \right|^q \, dt \, d\lambda \\
\leq \frac{1}{4} \int_0^1 \int_0^1 (1-t)(1-\lambda) \frac{t^{n/2} \lambda^{n/2}}{\sqrt{(1-t)(1-\lambda)}} \left| \frac{\partial^2 f(a, c)}{\partial x \partial y} \right|^q \\
&+ \frac{t^{n/2} (1-\lambda)^{n/2}}{\sqrt{(1-t)\lambda}} \left| \frac{\partial^2 f(a, d)}{\partial x \partial y} \right|^q + \frac{(1-t)^{n/2} \lambda^{n/2}}{\sqrt{t(1-\lambda)}} \left| \frac{\partial^2 f(b, c)}{\partial x \partial y} \right|^q \\
&+ \frac{(1-t)^{n/2} (1-\lambda)^{n/2}}{\sqrt{t\lambda}} \left| \frac{\partial^2 f(b, d)}{\partial x \partial y} \right|^q \, dt \, d\lambda \\
&= \frac{B(((s_1 + 2)/2), (3/2))B(((s_2 + 2)/2), (3/2))}{4} \left| \frac{\partial^2 f(a, c)}{\partial x \partial y} \right|^q + (s_2 + 2) \left| \frac{\partial^2 f(a, d)}{\partial x \partial y} \right|^q \\
&+ (s_1 + 2) \left| \frac{\partial^2 f(b, c)}{\partial x \partial y} \right|^q + (s_1 + 2)(s_2 + 2) \left| \frac{\partial^2 f(b, d)}{\partial x \partial y} \right|^q.
\end{aligned}
\]  

(28)

Combining (27) and (28) results in (26). Theorem 2 is thus proved.

\[|I(f)| \leq \frac{(b-a)(d-c)}{4^{1+1/q}} \left[ \frac{B((s_1 + 2)/2), (3/2)}{2} \right]^{2q} \left[ \left| \frac{\partial^2 f(a, c)}{\partial x \partial y} \right|^q + (s_1 + 2) \left| \frac{\partial^2 f(a, d)}{\partial x \partial y} \right|^q \right]^{1/q} + \left| \frac{\partial^2 f(b, c)}{\partial x \partial y} \right|^q + (s_1 + 2) \left| \frac{\partial^2 f(b, d)}{\partial x \partial y} \right|^q \]  

(29)

**Theorem 3.** Let \( f: \Delta = [a, b] \times [c, d] \subseteq \mathbb{R}^2 \to \mathbb{R} \) for \( a < b \) and \( c < d \) have the second partial derivatives and \( \partial^2 f/\partial x \partial y \in L_1(\Delta) \) and let \( s_1, s_2 \in (0, 1)^2 \). If \( |(\partial^2 f/\partial x \partial y)|^q \) for \( q > 1 \) is co-ordinated MT-\((s_1, s_2)\)-convex functions on \( \Delta \), then

\[|I(f)| \leq \frac{(b-a)(d-c)}{4^{1+1/q}} \left[ \frac{q-1}{2q-1} \right]^{2-2/q} \left[ B\left( \frac{s_1 + 2}{2}, \frac{1}{2} \right) B\left( \frac{s_2 + 2}{2}, \frac{1}{2} \right) \right]^{1/q} \times \left[ \left| \frac{\partial^2 f(a, c)}{\partial x \partial y} \right|^q + \left| \frac{\partial^2 f(a, d)}{\partial x \partial y} \right|^q + \left| \frac{\partial^2 f(b, c)}{\partial x \partial y} \right|^q + \left| \frac{\partial^2 f(b, d)}{\partial x \partial y} \right|^q \right]^{1/q},\]  

(30)

where \( B(\alpha, \beta) \) is the Beta function.

**Proof.** From Lemma 1, Hölder’s integral inequality, and the coordinated MT-\((s_1, s_2)\)-convexity of \(|(\partial^2 f/\partial x \partial y)|^q\), it follows that
\[ |I(f)| \leq (b-a)(d-c) \left( \int_0^1 \int_0^1 [(1-t)(1-\lambda)]^{q(q-1)} \, dt \, d\lambda \right)^{1-1/q} \]

\[ \times \left[ \int_0^1 \int_0^1 \frac{\partial^2 f(ta + (1-t)b, \lambda c + (1-\lambda)d)^q}{\partial x \partial y} \, dt \, d\lambda \right]^{1/q} \]

\[ = (b-a)(d-c) \left( \frac{q-1}{2q-1} \right)^{2-2/q} \left[ \int_0^1 \int_0^1 \frac{\partial^2 f(ta + (1-t)b, \lambda c + (1-\lambda)d)^q}{\partial x \partial y} \, dt \, d\lambda \right]^{1/q} \]

\[ \leq \frac{(b-a)(d-c)}{4^{1/q}} \left( \frac{q-1}{2q-1} \right)^{2-2/q} \]

\[ \times \left[ \left( \int_0^1 \int_0^1 \frac{\partial^2 f(ta, \lambda c)^q}{\partial x \partial y} + \frac{\partial^2 f(ta, \lambda d)^q}{\partial x \partial y} + \frac{\partial^2 f(ta, b)^q}{\partial x \partial y} + \frac{\partial^2 f(ta, b)^q}{\partial x \partial y} \right) \right]^{1/q}. \]

Theorem 3 is thus proved. \( \square \)

**Corollary 3.** Under the conditions of Theorem 3, if \( s_1 = s_2 = s \), then

\[ |I(f)| \leq \frac{(b-a)(d-c)}{4^{1/q}} \left( \frac{q-1}{2q-1} \right)^{2-2/q} \left[ B \left( \frac{s+2}{2}, \frac{1}{2} \right) \right]^{1/q} \]

\[ \times \left[ \left( \frac{\partial^2 f(b, c)^q}{\partial x \partial y} + \frac{\partial^2 f(b, d)^q}{\partial x \partial y} + \frac{\partial^2 f(b, c)^q}{\partial x \partial y} + \frac{\partial^2 f(b, d)^q}{\partial x \partial y} \right) \right]^{1/q}. \]

**5. Conclusion**

In this paper, we conclude the following:

1. From Definition 5, we introduced a new concept of MT-\((s_1, s_2)\)-convex functions on the coordinates on the rectangle \( \Delta \) of the plane \( \mathbb{R}^2 \).

2. From Propositions 1 and 2, we investigated simple properties of MT-\((s_1, s_2)\)-convex functions on the coordinates on \( \Delta \).

3. With the help of the integral identity in Lemma 1, via Theorems 1, 2, and 3, and via Corollaries 1, 2, and 3, we established some new Hermite–Hadamard type inequalities for MT-\((s_1, s_2)\)-convex functions on the coordinates on \( \Delta \).

**Data Availability**

No data were used to support this study.

**Conflicts of Interest**

The authors declare no conflicts of interest.

**Authors’ Contributions**

The authors contributed equally to this work. All authors read and approved the final manuscript.

**Acknowledgments**

This work was partially supported by the National Natural Science Foundation of China (Grant No. 12061033), by the National Natural Science Foundation of Inner Mongolia (Grants Nos. 2018MS01023 and 2019MS01007), and by the Research Program of Science and Technology at Universities of Inner Mongolia Autonomous Region (Grant No. NJZY2019).

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