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Research Article

Coordinated MT-(s_1, s_2)-Convex Functions and Their Integral Inequalities of Hermite–Hadamard Type

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In the paper, the authors introduce a new concept of MT-(s_1, s_2)-convex functions on the coordinates on the rectangle of the plane and establish some new Hermite–Hadamard-type inequalities for this kind of functions.

1. Motivations

At first, we recall several kinds of convex functions as follows.

Definition 1 (see [1]). Let $I \subseteq \mathbb{R}$ be an interval. A nonnegative function $f: I \longrightarrow \mathbb{R}_0 = [0, \infty)$ is said to be MT-convex if the inequality

$$f(tx + (1-t)y) \le \frac{\sqrt{t}}{2\sqrt{1-t}} f(x) + \frac{\sqrt{1-t}}{2\sqrt{t}} f(y) \tag{1}$$

holds for all $x, y \in I$ and $t \in (0, 1)$.

Definition 2 (see [2, 3]). Let $s \in (0, 1]$ be a real number. A function $f: \mathbb{R} \longrightarrow \mathbb{R}_0$ is said to be s-convex in the second sense if

$$f(tx + (1-t)y) \le t^s f(x) + (1-t)^s f(y), \tag{2}$$

for all $x, y \in I$ and $t \in [0, 1]$.

Definition 3 (see [4, 5]). A function $f: \Delta = [a, b] \times [c, d] \subseteq \mathbb{R}^2 \longrightarrow \mathbb{R}$ is said to be convex on the coordinates on Δ if

$$\begin{aligned} f(tx + (1-t)z, \lambda y + (1-\lambda)w) \le{}& t\lambda f(x, y) + t(1-\lambda) f(x, w) \\ &+ (1-t)\lambda f(z, y) + (1-t)(1-\lambda) f(z, w) \end{aligned} \tag{3}$$

holds for all $t, \lambda \in [0, 1]$ and $(x, y), (z, w) \in \Delta$. If the inequality (3) is reversed, then f is said to be concave on the coordinates on Δ.

Definition 4 (see [6]). We say that a function $f: \Delta = [a, b] \times [c, d] \subseteq \mathbb{R}^2 \longrightarrow \mathbb{R}_0$ is MT-convex on the coordinates on Δ with $a < b$ and $c < d$, if the inequality

$$f(tx+(1-t)z,\lambda y+(1-\lambda)w)\leq\frac{\sqrt{t\lambda}}{4\sqrt{(1-t)(1-\lambda)}}f(x,y)+\frac{\sqrt{t(1-\lambda)}}{4\sqrt{(1-t)\lambda}}f(x,w)+\frac{\sqrt{(1-t)\lambda}}{4\sqrt{t(1-\lambda)}}f(z,y)+\frac{\sqrt{(1-t)(1-\lambda)}}{4\sqrt{t\lambda}}f(z,w) \tag{4}$$

is valid for all $t,\lambda\in(0,1)$ and $(x,y),(z,w)\in\Delta$.

In the papers [7–12] and closely related references therein, the HT-convexity, GT-convexity, and the $(s;m)$-P-convexity on the coordinates were introduced and investigated.

Combining the structures of Definitions 2 and 4, we introduce the notion of coordinated MT-(s_1,s_2)-convex functions as follows.

Definition 5. For $(s_1,s_2)\in(0,1]^2$, a function $f\colon\Delta=[a,b]\times[c,d]\subseteq\mathbb{R}^2\longrightarrow\mathbb{R}_0$ is said to be MT-(s_1,s_2)-convex on the coordinates on Δ with $a<b$ and $c<d$, if the inequality

$$f(tx+(1-t)z,\lambda y+(1-\lambda)w)\leq\frac{t^{s_1/2}\lambda^{s_2/2}}{4\sqrt{(1-t)(1-\lambda)}}f(x,y)+\frac{t^{s_1/2}(1-\lambda)^{s_2/2}}{4\sqrt{(1-t)\lambda}}f(x,w)+\frac{(1-t)^{s_1/2}\lambda^{s_2/2}}{4\sqrt{t(1-\lambda)}}f(z,y)+\frac{(1-t)^{s_1/2}(1-\lambda)^{s_2/2}}{4\sqrt{t\lambda}}f(z,w) \tag{5}$$

holds for all $t,\lambda\in(0,1)$ and $(x,y),(z,w)\in\Delta$. If inequality (5) is reversed, then f is said to be a MT-(s_1,s_2)-concave function on the coordinates on Δ.

2. Simple Properties of MT-(s_1,s_2)-Convex Functions

After introduced Definition 5, now we are in a position to investigate in this section simple properties of MT-(s_1,s_2)-convex functions on the coordinates on Δ.

Proposition 1. *Let $(s_1,s_2)\in(0,1]^2$ and $f\colon\Delta=[a,b]\times[c,d]\subseteq\mathbb{R}^2\longrightarrow\mathbb{R}_0$. If f is nonnegative and convex on the coordinates on Δ, then f is MT-convex on the coordinates on Δ, while f is also MT-(s_1,s_2)-convex on the coordinates on Δ.*

Proof. This follows from $t^{s/2}/2\sqrt{1-t}\geq t^{1/2}/2\sqrt{1-t}\geq t$ for $s\in(0,1]$ and $t\in(0,1)$. □

Example 1 (see [13], p. 104). When $p\in(0,1/1000)$, the functions $f(x)=x^p$ and $g(x)=(1+x)^p$ for $x\in\Delta_1=(1,\infty)$ are MT-convex, but they are not convex on Δ_1.

For $m\in(0,1/100)$, the function $h(x)=(1+x^2)^m$ for $x\in[1,3/2]$ is MT-convex, but it is not convex on $[1,3/2]$.

Example 2 (see [6], p. 259). When $p\in(0,1/1000)$, the function $f(x,y)=x^p+y^p$ for $(x,y)\in\Delta_1$ is MT-convex on the coordinates on Δ_1, but it is not convex on the coordinates on Δ_1.

Remark 1. We now discuss Examples 1 and 2 mentioned above.

For $0<p<1$, $c\geq0$, and $I\subseteq(0,\infty)$ being a nonempty interval, the function $\varphi(x)=(x+c)^p$ is concave on I. Therefore, for $t=1/2$ and all $x,y\in I$ with $x\neq y$, we have

$$\varphi(tx+(1-t)y)=\left(\frac{x+y}{2}+c\right)^p>\frac{(x+c)^p+(y+c)^p}{2}=\frac{\sqrt{t}}{2\sqrt{1-t}}\varphi(x)+\frac{\sqrt{1-t}}{2\sqrt{t}}\varphi(y). \tag{6}$$

Accordingly, the function $\varphi(x)$ is not MT-convex on I. From this, we conclude as follows:

(1) For $p\in(0,1/1000)$, the functions $f(x)=x^p$ and $g(x)=(1+x)^p$ with respect to $x\in\Delta_1$ are not MT-convex on Δ_1

(2) For $m\in(0,1/100)$, the function $h(x)=(1+x^2)^m$ with respect to $x\in[1,3/2]$ is not MT-convex, but it is convex on $[1,1/\sqrt{1-2m}]$ and is concave on $[1/\sqrt{1-2m},3/2]$

(3) For $p\in(0,1/1000)$, the function $f(x,y)=x^p+y^p$ with respect to $(x,y)\in\Delta_1$ is not MT-convex on the coordinates on Δ_1

Proposition 2. *Let $f(x,y)=x^{1/2}$ for $(x,y)\in\mathbb{R}_+^2=(0,\infty)^2$. Then, the function $f(x,y)$ is MT-$(0.5,0.02)$-convex, but not MT-convex, on the coordinates on $\mathbb{R}_+^2$.*

Proof. For $s_1 = 0.5$ and $s_2 = 0.02$, for $t, \lambda \in (0,1)$, and for $(x, y), (z, w) \in \mathbb{R}_+^2$, by Definition 5, we deduce

$$\begin{aligned} f(tx+(1-t)z, \lambda y+(1-\lambda)w) &= [tx+(1-t)z]^{1/2} \\ &\cdot \frac{t^{0.5/2}\lambda^{0.02/2}}{4\sqrt{(1-t)(1-\lambda)}} f(x,y) + \frac{t^{0.5/2}(1-\lambda)^{0.02/2}}{4\sqrt{(1-t)\lambda}} f(x,w) \\ &+ \frac{(1-t)^{0.5/2}\lambda^{0.02/2}}{4\sqrt{t(1-\lambda)}} f(z,y) + \frac{(1-t)^{0.5/2}(1-\lambda)^{0.02/2}}{4\sqrt{t\lambda}} f(z,w) \\ &= \left[\frac{\lambda^{0.02/2}}{2\sqrt{1-\lambda}} + \frac{(1-\lambda)^{0.02/2}}{2\sqrt{\lambda}}\right]\left[\frac{t^{0.5/2}}{2\sqrt{1-t}}x^{1/2} + \frac{(1-t)^{0.5/2}}{2\sqrt{t}}z^{1/2}\right]. \end{aligned} \tag{7}$$

Making use of the inequality $\lambda^{0.02/2}/2\sqrt{1-\lambda} + (1-\lambda)^{0.02/2}/2\sqrt{\lambda} \geq 2^{0.49} > 1.4$ and letting $u = x/z$ result in

$$\begin{aligned} [tx+(1-t)z]^{1/2} &= z^{1/2}[tu+(1-t)]^{1/2} \\ &< 1.4z^{1/2}\left[\frac{t^{0.5/2}}{2\sqrt{1-t}}u^{1/2} + \frac{(1-t)^{0.5/2}}{2\sqrt{t}}\right] \\ &< \left[\frac{\lambda^{0.02/2}}{2\sqrt{1-\lambda}} + \frac{(1-\lambda)^{0.02/2}}{2\sqrt{\lambda}}\right]\left[\frac{t^{0.5/2}}{2\sqrt{1-t}}x^{1/2} + \frac{(1-t)^{0.5/2}}{2\sqrt{t}}z^{1/2}\right]. \end{aligned} \tag{8}$$

This means that the function $f(x, y)$ is MT-$(0.5, 0.02)$-convex on the coordinates on $\mathbb{R}_+^2$.

For $(x, y), (z, w) \in \mathbb{R}_+^2$ with $x \neq z$, taking $t = \lambda = 1/2$ in Definition 5 leads to

$$\begin{aligned} f(tx+(1-t)z, \lambda y+(1-\lambda)w) &= \left(\frac{x+y}{2}\right)^{1/2} > \frac{x^{1/2}+z^{1/2}}{2} \\ &= \frac{\sqrt{t\lambda}}{4\sqrt{(1-t)(1-\lambda)}} f(x,y) + \frac{\sqrt{t(1-\lambda)}}{4\sqrt{(1-t)\lambda}} f(x,w) \\ &+ \frac{\sqrt{(1-t)\lambda}}{4\sqrt{t(1-\lambda)}} f(z,y) + \frac{\sqrt{(1-t)(1-\lambda)}}{4\sqrt{t\lambda}} f(z,w). \end{aligned} \tag{9}$$

This means that the function $f(x, y)$ is not MT-convex on the coordinates on $\mathbb{R}_+^2$. The proof of Proposition 2 is complete. □

3. A Lemma

In order to establish integral inequalities of the Hermite–Hadamard type for MT-(s_1, s_2)-convex functions on the coordinates on Δ, we need the following lemma.

Lemma 1. *Let $f\colon \Delta = [a,b] \times [c,d] \subseteq \mathbb{R}^2 \longrightarrow \mathbb{R}$ have partial derivatives of the second order and let $a < b$ and $c < d$. If $\partial^2 f/\partial x \partial y \in L_1(\Delta)$, then*

$$\begin{aligned} I(f) &= f(b,d) - \frac{1}{b-a}\int_a^b f(x,d)\mathrm{d}x - \frac{1}{d-c}\int_c^d f(b,y)\mathrm{d}y \\ &+ \frac{1}{(b-a)(d-c)}\int_c^d\int_a^b f(x,y)\mathrm{d}x\mathrm{d}y \\ &= (b-a)(d-c)\int_0^1\int_0^1 (1-t)(1-\lambda)\frac{\partial^2 f(ta+(1-t)b, \lambda c+(1-\lambda)d)}{\partial x \partial y}\mathrm{d}t\mathrm{d}\lambda. \end{aligned} \tag{10}$$

Proof. Integrating by parts gives

$$\begin{aligned}
&\int_0^1\int_0^1 (1-t)(1-\lambda)\frac{\partial^2 f(ta+(1-t)b,\lambda c+(1-\lambda)d)}{\partial x\partial y}\mathrm{d}t\mathrm{d}\lambda \\
&= \frac{1}{a-b}\int_0^1(1-\lambda)\left[(1-t)\frac{\partial f(ta+(1-t)b,\lambda c+(1-\lambda)d)}{\partial y}\Bigg|_{t=0}^{t=1}+\int_0^1\frac{\partial f(ta+(1-t)b,\lambda c+(1-\lambda)d)}{\partial y}\mathrm{d}t\right]\mathrm{d}\lambda \\
&= \frac{1}{a-b}\left[\int_0^1(\lambda-1)\frac{\partial f(b,\lambda c+(1-\lambda)d)}{\partial y}\mathrm{d}\lambda+\int_0^1\int_0^1(1-\lambda)\frac{\partial f(ta+(1-t)b,\lambda c+(1-\lambda)d)}{\partial y}\mathrm{d}t\mathrm{d}\lambda\right] \\
&= \frac{1}{(a-b)(c-d)}\left[\begin{array}{c}(\lambda-1)f(b,\lambda c+(1-\lambda)d)\Big|_{\lambda=1}^{\lambda=0}-\int_0^1 f(b,\lambda c+(1-\lambda)d)\mathrm{d}\lambda \\ +\int_0^1(1-\lambda)f(ta+(1-t)b,\lambda c+(1-\lambda)d)\Big|_{\lambda=1}^{\lambda=0}\mathrm{d}t \\ +\int_0^1\int_0^1 f(ta+(1-t)b,\lambda c+(1-\lambda)d)\mathrm{d}t\mathrm{d}\lambda\end{array}\right] \\
&= \frac{1}{(a-b)(c-d)}\left[\begin{array}{c}f(b,d)-\int_0^1 f(b,\lambda c+(1-\lambda)d)\mathrm{d}\lambda \\ -\int_0^1 f(ta+(1-t)b,d)\mathrm{d}t+\int_0^1\int_0^1 f(ta+(1-t)b,\lambda c+(1-\lambda)d)\mathrm{d}t\mathrm{d}\lambda\end{array}\right] \\
&= f(b,d)-\frac{1}{b-a}\int_a^b f(x,d)\mathrm{d}x-\frac{1}{d-c}\int_c^d f(b,y)\mathrm{d}y \\
&\quad+\frac{1}{(b-a)(d-c)}\int_c^d\int_a^b f(x,y)\mathrm{d}x\mathrm{d}y.
\end{aligned} \tag{11}$$

The proof of Lemma 1 is complete. □

4. Integral Inequalities of the Hermite–Hadamard Type

In this section, we prove some new inequalities of the Hermite–Hadamard type for co-ordinated MT-(s_1, s_2)-convex functions.

Theorem 1. *Let $f\colon \Delta=[a,b]\times[c,d]\subseteq\mathbb{R}^2\longrightarrow\mathbb{R}_+$ for $a<b$ and $c<d$. If f is coordinated MT-(s_1,s_2)-convex on Δ for $(s_1,s_2)\in(0,1]^2$ and $f\in L_1(\Delta)$, then*

$$\begin{aligned}
\frac{1}{2^{2-(s_1+s_2)}}f\left(\frac{a+b}{2},\frac{c+d}{2}\right) &\le \frac{1}{2^{2-(s_1+s_2)/2}}\left[\frac{1}{b-a}\int_a^b f\left(x,\frac{c+d}{2}\right)\mathrm{d}x+\frac{1}{d-c}\int_c^d f\left(\frac{a+b}{2},y\right)\mathrm{d}y\right] \\
&\le \frac{1}{(b-a)(d-c)}\int_c^d\int_a^b f(x,y)\mathrm{d}x\mathrm{d}y \\
&\le \frac{1}{2}\left[\frac{B((s_2+2)/2,1/2)}{2^{(s_1+1)/2}(b-a)}\int_a^b[f(x,c)+f(x,d)]\mathrm{d}x+\frac{B((s_1+2)/2),(1/2)}{2^{(s_2+1)/2}(d-c)}\int_c^d[f(a,y)+f(b,y)]\mathrm{d}y\right] \\
&\le \frac{B(((s_1+2)/2),(1/2))B(((s_2+2)/2),(1/2))}{2^{(s_1+s_2)/2+1}}[f(a,c)+f(b,c)+f(a,d)+f(b,d)],
\end{aligned} \tag{12}$$

where $B(\alpha,\beta)$ denotes the well-known beta function which may be defined by

$$B(\alpha,\beta)=\int_0^1 t^{\alpha-1}(1-t)^{\beta-1}\mathrm{d}t,\quad \mathscr{R}e(\alpha),\mathscr{R}e(\beta)>0. \tag{13}$$

Proof. For all $0<t<1$, we have

$$\begin{aligned}\frac{a+b}{2}&=\frac{ta+(1-t)b+(1-t)a+tb}{2},\\ \frac{c+d}{2}&=\frac{1}{2}\left(\frac{c+d}{2}+\frac{c+d}{2}\right).\end{aligned} \tag{14}$$

Letting $x=ta+(1-t)b$, $z=(1-t)a+tb$, and $z=w=c+d/2$ in (4) and using the MT-(s_1,s_2)-convexity of f, we obtain

$$f\left(\frac{a+b}{2},\frac{c+d}{2}\right)\le\frac{f(ta+(1-t)b,(c+d)/2)+f(1-t)a+tb,(c+d)/2}{2^{(s_1+s_2)/2}}. \tag{15}$$

Integrating the above inequality over t on $[0,1]$ and utilizing the change of the variable $x=ta+(1-t)b$ for $0<t<1$ result in

$$\begin{aligned}f\left(\frac{a+b}{2},\frac{c+d}{2}\right)&\le\frac{1}{2^{(s_1+s_2)/2}}\int_0^1\left[f\left(ta+(1-t)b,\frac{c+d}{2}\right)+f\left((1-t)a+tb,\frac{c+d}{2}\right)\right]\mathrm{d}t\\ &=\frac{1}{2^{(s_1+s_2)/2-1}(b-a)}\int_a^b f\left(x,\frac{c+d}{2}\right)\mathrm{d}x.\end{aligned} \tag{16}$$

By the MT-(s_1,s_2)-convexity of f, we obtain

$$\begin{aligned}\frac{1}{b-a}\int_a^b f\left(x,\frac{c+d}{2}\right)\mathrm{d}x&\le\frac{1}{2^{(s_1+s_2)/2}(b-a)}\int_0^1\int_a^b[f(x,\lambda c+(1-\lambda)d)+f(x,(1-\lambda)c+\lambda d)]\mathrm{d}x\mathrm{d}\lambda\\ &=\frac{1}{2^{(s_1+s_2)/2-1}(b-a)(d-c)}\int_c^d\int_a^b f(x,y)\mathrm{d}x\mathrm{d}y.\end{aligned} \tag{17}$$

From (16) and (17), it follows that

$$\begin{aligned}\frac{1}{2^{2-(s_1+s_2)}}f\left(\frac{a+b}{2},\frac{c+d}{2}\right)&\le\frac{1}{2^{1-(s_1+s_2)/2}(b-a)}\int_a^b f\left(x,\frac{c+d}{2}\right)\mathrm{d}x\\ &\le\frac{1}{(b-a)(d-c)}\int_c^d\int_a^b f(x,y)\mathrm{d}x\mathrm{d}y.\end{aligned} \tag{18}$$

Similarly, we have

$$\begin{aligned}\frac{1}{2^{2-(s_1+s_2)}}f\left(\frac{a+b}{2},\frac{c+d}{2}\right)&\le\frac{1}{2^{1-(s_1+s_2)/2}(d-c)}\int_c^d f\left(\frac{a+b}{2},y\right)\mathrm{d}y\\ &\le\frac{1}{(b-a)(d-c)}\int_c^d\int_a^b f(x,y)\mathrm{d}x\mathrm{d}y.\end{aligned} \tag{19}$$

A combination of (18) and (19) gives the desired inequality (12).

Putting $y=\lambda c+(1-\lambda)d$ for all $0<\lambda<1$ and using the MT-(s_1,s_2)-convexity of f reveals

$$\begin{aligned}\frac{1}{(b-a)(d-c)}\int_c^d\int_a^b f(x,y)\mathrm{d}x\mathrm{d}y &= \frac{1}{b-a}\int_0^1\int_a^b f\left(\frac{x+x}{2},\lambda c+(1-\lambda)d\right)\mathrm{d}x\mathrm{d}\lambda\\ &\le \frac{1}{2^{(s_1+1)/2}(b-a)}\int_0^1\int_a^b\left[\frac{\lambda^{s_2/2}}{\sqrt{1-\lambda}}f(x,c)+\frac{(1-\lambda)^{s_2/2}}{\sqrt{\lambda}}f(x,d)\right]\mathrm{d}x\mathrm{d}\lambda\\ &= \frac{B((s_2+2)/2),(1/2)}{2^{(s_1+1)/2}(b-a)}\int_a^b[f(x,c)+f(x,d)]\mathrm{d}x.\end{aligned} \tag{20}$$

Taking $x = ta + (1-t)b$ for $0 < t < 1$ and employing the MT-(s_1, s_2)-convexity of f leads to

$$\begin{aligned}\frac{1}{b-a}\int_a^b f(x,c)\mathrm{d}x &= \int_0^1 f\left(ta+(1-t)b,\frac{c+c}{2}\right)\mathrm{d}t\\ &\le \frac{1}{2^{(s_2+1)/2}}\int_0^1\left[\frac{t^{s_1/2}}{\sqrt{1-t}}f(a,c)\right.\\ &\quad\left.+\frac{(1-t)^{s_1/2}}{\sqrt{t}}f(b,c)\right]\mathrm{d}t\\ &= \frac{B((s_1+2)/2),(1/2)}{2^{(s_2+1)/2}}[f(a,c)+f(b,c)],\end{aligned} \tag{21}$$

$$\frac{1}{b-a}\int_a^b f(x,d)\mathrm{d}x \le \frac{B((s_1+2)/2),(1/2)}{2^{(s_2+1)/2}}[f(a,d)+f(b,d)]. \tag{22}$$

Applying inequalities between (19) and (22) arrives at

$$\begin{aligned}\frac{1}{(b-a)(d-c)}\int_c^d\int_a^b f(x,y)\mathrm{d}x\mathrm{d}y &\le \frac{B((s_2+2)/2),(1/2)}{2^{(s_1+1)/2}(b-a)}\int_a^b[f(x,c)+f(x,d)]\mathrm{d}x\\ &\le \frac{B(((s_1+2)/2),(1/2))B((s_2+2)/2,1/2)}{2^{(s_1+s_2)/2+1}}[f(a,c)+f(b,c)+f(a,d)+f(b,d)].\end{aligned} \tag{23}$$

By similar argument, we can find

$$\begin{aligned}\frac{1}{(b-a)(d-c)}\int_c^d\int_a^b f(x,y)\mathrm{d}x\mathrm{d}y &\le \frac{B((s_1+2)/2),(1/2)}{2^{(s_2+1)/2}(s_1+1)\Gamma(s_1+1/2)(d-c)}\int_c^d[f(a,y)+f(b,y)]\mathrm{d}y\\ &\le \frac{B(((s_1+2)/2),(1/2))B(((s_2+2)/2),(1/2))}{2^{(s_1+s_2)/2+1}}[f(a,c)+f(b,c)+f(a,d)+f(b,d)].\end{aligned} \tag{24}$$

The proof of Theorem 1 is complete. □

Corollary 1. *Under the conditions of Theorem 1, if $s_1 = s_2 = s$, then*

$$\begin{aligned}\frac{1}{2^{2(1-s)}}f\left(\frac{a+b}{2},\frac{c+d}{2}\right)&\leq\frac{1}{2^{2-s}}\left[\frac{1}{b-a}\int_a^b f\left(x,\frac{c+d}{2}\right)\mathrm{d}x+\frac{1}{d-c}\int_c^d f\left(\frac{a+b}{2},y\right)\mathrm{d}y\right]\\&\leq\frac{1}{(b-a)(d-c)}\int_c^d\int_a^b f(x,y)\mathrm{d}x\mathrm{d}y\\&\leq\frac{B(((s+2)/2),(1/2))}{2^{(s+1)/2+1}}\left[\frac{1}{b-a}\int_a^b[f(x,c)+f(x,d)]\mathrm{d}x+\frac{1}{d-c}\int_c^d[f(a,y)+f(b,y)]\mathrm{d}y\right]\\&\leq\frac{[B(((s+2)/2),(1/2))]^2}{2^{s+1}}[f(a,c)+f(b,c)+f(a,d)+f(b,d)].\end{aligned}\tag{25}$$

Theorem 2. *Let $f\colon\Delta=[a,b]\times[c,d]\subseteq\mathbb{R}^2\longrightarrow\mathbb{R}$ for $a<b$ and $c<d$ have the second partial derivatives and $(\partial^2 f/\partial x\partial y)\in L_1(\Delta)$ and let $(s_1,s_2)\in(0,1]^2$. If $|(\partial^2 f/\partial x\partial y)|^q$ for $q\geq 1$ is coordinated MT-(s_1,s_2)-convex functions on Δ, then*

$$\begin{aligned}|I(f)|\leq\frac{(b-a)(d-c)}{4^{1+1/q}}&\left[\frac{B(((s_1+2)/2),(3/2))B(((s_2+2)/2),(3/2))}{4}\right]^{1/q}\\&\cdot\left[\left|\frac{\partial^2 f(a,c)}{\partial x\partial y}\right|^q+(s_2+2)\left|\frac{\partial^2 f(a,d)}{\partial x\partial y}\right|^q+(s_1+2)\left|\frac{\partial^2 f(b,c)}{\partial x\partial y}\right|^q+(s_1+2)(s_2+2)\left|\frac{\partial^2 f(b,d)}{\partial x\partial y}\right|^q\right]^{1/q},\end{aligned}\tag{26}$$

where $B(\alpha,\beta)$ is the Beta function.

Proof. From Lemma 1 and Hölder's integral inequality, it follows that

$$\begin{aligned}|I(f)|&\leq(b-a)(d-c)\left[\int_0^1\int_0^1(1-t)(1-\lambda)\mathrm{d}t\mathrm{d}\lambda\right]^{1-1/q}\\&\quad\times\left[\int_0^1\int_0^1(1-t)(1-\lambda)\left|\frac{\partial^2 f(ta+(1-t)b,\lambda c+(1-\lambda)d)}{\partial x\partial y}\right|^q\mathrm{d}t\mathrm{d}\lambda\right]^{1/q}\\&=\frac{(b-a)(d-c)}{4^{1-1/q}}\left[\int_0^1\int_0^1(1-t)(1-\lambda)\left|\frac{\partial^2 f(ta+(1-t)b,\lambda c+(1-\lambda)d)}{\partial x\partial y}\right|^q\mathrm{d}t\mathrm{d}\lambda\right]^{1/q}.\end{aligned}\tag{27}$$

By the coordinated MT-(s_1,s_2)-convexity of $|(\partial^2 f/\partial x\partial y)|^q$, we have

$$\begin{aligned}
&\int_0^1\int_0^1 (1-t)(1-\lambda)\left|\frac{\partial^2 f(ta+(1-t)b,\lambda c+(1-\lambda)d)}{\partial x\partial y}\right|^q \mathrm{d}t\mathrm{d}\lambda \\
&\le \frac{1}{4}\int_0^1\int_0^1 (1-t)(1-\lambda)\frac{t^{s_1/2}\lambda^{s_2/2}}{\sqrt{(1-t)(1-\lambda)}}\left|\frac{\partial^2 f(a,c)}{\partial x\partial y}\right|^q \\
&\quad+\frac{t^{s_1/2}(1-\lambda)^{s_2/2}}{\sqrt{(1-t)\lambda}}\left|\frac{\partial^2 f(a,d)}{\partial x\partial y}\right|^q+\frac{(1-t)^{s_1/2}\lambda^{s_2/2}}{\sqrt{t(1-\lambda)}}\left|\frac{\partial^2 f(b,c)}{\partial x\partial y}\right|^q \\
&\quad+\frac{(1-t)^{s_1/2}(1-\lambda)^{s_2/2}}{\sqrt{t\lambda}}\left|\frac{\partial^2 f(b,d)}{\partial x\partial y}\right|^q \mathrm{d}t\mathrm{d}\lambda \\
&=\frac{B(((s_1+2)/2),(3/2))B(((s_2+2)/2),(3/2))}{4}\left|\frac{\partial^2 f(a,c)}{\partial x\partial y}\right|^q+(s_2+2)\left|\frac{\partial^2 f(a,d)}{\partial x\partial y}\right|^q \\
&\quad+(s_1+2)\left|\frac{\partial^2 f(b,c)}{\partial x\partial y}\right|^q+(s_1+2)(s_2+2)\left|\frac{\partial^2 f(b,d)}{\partial x\partial y}\right|^q.
\end{aligned} \tag{28}$$

Combining (27) and (28) results in (26). Theorem 2 is thus proved. □

Corollary 2. *Under the assumptions of Theorem 2, if $s_1 = s_2 = s$, then*

$$\begin{aligned}
|I(f)| &\le \frac{(b-a)(d-c)}{4^{1+1/q}}\left[\frac{B((s+2)/2),(3/2)}{2}\right]^{2/q} \\
&\left[\left|\frac{\partial^2 f(a,c)}{\partial x\partial y}\right|^q+(s+2)\left|\frac{\partial^2 f(a,d)}{\partial x\partial y}\right|^q+(s+2)\left|\frac{\partial^2 f(b,c)}{\partial x\partial y}\right|^q+(s+2)^2\left|\frac{\partial^2 f(b,d)}{\partial x\partial y}\right|^q\right]^{1/q}.
\end{aligned} \tag{29}$$

Theorem 3. *Let $f\colon \Delta=[a,b]\times[c,d]\subseteq\mathbb{R}^2 \longrightarrow \mathbb{R}$ for $a<b$ and $c<d$ have the second partial derivatives and $\partial^2 f/\partial x\,\partial y \in L_1(\Delta)$ and let $s_1, s_2 \in (0,1]^2$. If $|(\partial^2 f/\partial x\,\partial y)|^q$ for $q>1$ is co-ordinated MT-(s_1,s_2)-convex functions on Δ, then*

$$\begin{aligned}
|I(f)| &\le \frac{(b-a)(d-c)}{4^{1/q}}\left(\frac{q-1}{2q-1}\right)^{2-2/q}\left[B\left(\frac{s_1+2}{2},\frac{1}{2}\right)B\left(\frac{s_2+2}{2},\frac{1}{2}\right)\right]^{1/q} \\
&\times\left[\left|\frac{\partial^2 f(a,c)}{\partial x\partial y}\right|^q+\left|\frac{\partial^2 f(a,d)}{\partial x\partial y}\right|^q+\left|\frac{\partial^2 f(b,c)}{\partial x\partial y}\right|^q+\left|\frac{\partial^2 f(b,d)}{\partial x\partial y}\right|^q\right]^{1/q},
\end{aligned} \tag{30}$$

where $B(\alpha,\beta)$ is the Beta function.

Proof. From Lemma 1, Hölder's integral inequality, and the coordinated MT-(s_1,s_2)-convexity of $|(\partial^2 f/\partial x\partial y)|^q$, it follows that

$$
\begin{aligned}
|I(f)| &\leq (b-a)(d-c)\left(\int_0^1\int_0^1 [(1-t)(1-\lambda)]^{q/(q-1)}\mathrm{d}t\mathrm{d}\lambda\right)^{1-1/q} \\
&\quad\times\left[\int_0^1\int_0^1\left|\frac{\partial^2 f(ta+(1-t)b,\lambda c+(1-\lambda)d)}{\partial x\partial y}\right|^q \mathrm{d}t\mathrm{d}\lambda\right]^{1/q} \\
&= (b-a)(d-c)\left(\frac{q-1}{2q-1}\right)^{2-2/q}\left[\int_0^1\int_0^1\left|\frac{\partial^2 f(ta+(1-t)b,\lambda c+(1-\lambda)d)}{\partial x\partial y}\right|^q \mathrm{d}t\mathrm{d}\lambda\right]^{1/q} \\
&\leq \frac{(b-a)(d-c)}{4^{1/q}}\left(\frac{q-1}{2q-1}\right)^{2-2/q} \\
&\quad\times\left(\int_0^1\left[\int_0^1 \frac{t^{s_1/2}\lambda^{s_2/2}}{\sqrt{(1-t)(1-\lambda)}}\left|\frac{\partial^2 f(a,c)}{\partial x\partial y}\right|^q + \frac{t^{s_1/2}(1-\lambda)^{s_2/2}}{\sqrt{(1-t)\lambda}}\left|\frac{\partial^2 f(a,d)}{\partial x\partial y}\right|^q\right.\right. \\
&\quad\left.\left.+\frac{(1-t)^{s_1/2}\lambda^{s_2/2}}{\sqrt{t(1-\lambda)}}\left|\frac{\partial^2 f(b,c)}{\partial x\partial y}\right|^q + \frac{(1-t)^{s_1/2}(1-\lambda)^{s_2/2}}{\sqrt{t\lambda}}\left|\frac{\partial^2 f(b,d)}{\partial x\partial y}\right|^q\right]\mathrm{d}t\mathrm{d}\lambda\right)^{1/q} \\
&= \frac{(b-a)(d-c)}{4^{1/q}}\left(\frac{q-1}{2q-1}\right)^{2-2/q}\left[B\left(\frac{s_1+2}{2},\frac{1}{2}\right)B\left(\frac{s_2+2}{2},\frac{1}{2}\right)\right]^{1/q} \\
&\quad\times\left[\left|\frac{\partial^2 f(a,c)}{\partial x\partial y}\right|^q + \left|\frac{\partial^2 f(a,d)}{\partial x\partial y}\right|^q + \left|\frac{\partial^2 f(b,c)}{\partial x\partial y}\right|^q + \left|\frac{\partial^2 f(b,d)}{\partial x\partial y}\right|^q\right]^{1/q}.
\end{aligned}
\tag{31}
$$

Theorem 3 is thus proved. □

Corollary 3. *Under the conditions of Theorem 3, if $s_1 = s_2 = s$, then*

$$
\begin{aligned}
|I(f)| &\leq \frac{(b-a)(d-c)}{4^{1/q}}\left(\frac{q-1}{2q-1}\right)^{2-2/q}\left[B\left(\frac{s+2}{2},\frac{1}{2}\right)\right]^{2/q} \\
&\quad\times\left[\left|\frac{\partial^2 f(a,c)}{\partial x\partial y}\right|^q + \left|\frac{\partial^2 f(a,d)}{\partial x\partial y}\right|^q + \left|\frac{\partial^2 f(b,c)}{\partial x\partial y}\right|^q + \left|\frac{\partial^2 f(b,d)}{\partial x\partial y}\right|^q\right]^{1/q}.
\end{aligned}
\tag{32}
$$

5. Conclusion

In this paper, we conclude the following:

(1) From Definition 5, we introduced a new concept of MT-(s_1, s_2)-convex functions on the coordinates on the rectangle Δ of the plane $\mathbb{R}^2$

(2) From Propositions 1 and 2, we investigated simple properties of MT-(s_1, s_2)-convex functions on the coordinates on Δ

(3) With the help of the integral identity in Lemma 1, via Theorems 1, 2, and 3, and via Corollaries 1, 2, and 3, we established some new Hermite–Hadamard type inequalities for MT-(s_1, s_2)-convex functions on the coordinates on Δ

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare no conflicts of interest.

Authors' Contributions

The authors contributed equally to this work. All authors read and approved the final manuscript.

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