Research Article

On Some Classes with Norms of Meromorphic Function Spaces Defined by General Spherical Derivatives

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The main concerned target of this article is to define and study some concerned classes of meromorphic function spaces using the general spherical derivatives. The general Besov-type classes of meromorphic functions as well as the general normal functions are considered intensively and both are compared deeply with each other. Specifically, multiple results concerning general meromorphic-type classes as well as non-normal classes are obtained by the help of general spherical derivatives. The concerned results are proved by constructing some specific mild conditions on the sequences of points belonging to the concerned meromorphic-type classes. The obtained results generalize and improve the corresponding previous results in some concerned respects. The concerned proofs and methods are simply presented.

1. Introduction

The area of complex function spaces is fundamental and essential in many branches of pure and applied mathematics. Some decades ago, there have been obvious interests on meromorphic function classes, from concerned point of view of their singularities. For various studies on meromorphic function spaces, we may refer to all citations therein. As a concerned result, some new general classes of meromorphic functions shall be introduced by using the general spherical derivatives, which will be associated to obtain the new classes of meromorphic function spaces. Fundamental concerned properties of these concerned aforementioned meromorphic-type classes which include generalizations of meromorphic Besov spaces as well as normal function classes shall

be studied and intensively discussed. As a concerned consequence of our investigation, some relevant special cases can be pointed out. Furthermore, to capture some new generalized results under the current concerned proofs, some new concepts and definitions are introduced. Let $U = \{w: |w| < 1\}$ be the open unit disk in the complex plane $\mathbb{C}$ and let $dm(w)$ be the usual Euclidean area element on U . The symbol $M(U)$ stands for the concerned class of all meromorphic functions in U . The pseudohyperbolic metric between the points w and z is defined by $d(z, w) = |\varphi_z(w)|$. For $0 < R < 1$, assume that $U(a, R) = \{w \in U: d(w, a) < R\}$ defines the concerned pseudohyperbolic disc which is centered $a \in U$ with the specific radius R . For $0 < q < \infty$ and $0 < s < \infty$, the classes $M^\#(p, q, s)$ are defined by (see [1] pp.10)

$$M^\#(p, q, s) = \left\{ h \in M(U): \sup_{a \in U} \iint_U \left(h^\#(w) \right)^p (1 - |w|^2)^q (1 - |\varphi_a(w)|^2)^s dm(w) < \infty \right\}, \quad (1)$$

where $h^\#(w) = (|h'(w)|/(1 + |h(w)|^2))$ is the usual spherical derivative of h . The meromorphic $M^\#(q, q-2, 0)$ classes

are called the meromorphic Besov classes and denoted by $B_q^\#$, for which

$$B_q^\# = \left\{ h \in M(U): \sup_{a \in U} \iint_U \left(h^\#(w) \right)^q (1 - |w|^2)^{q-2} dm(w) < \infty \right\}. \quad (2)$$

For the analytic corresponding classes of Besov spaces, we cite [2–7]. In this article, the general meromorphic Besov-type classes always refer to the concerned classes $B^\#(q, q-2, s; n)$. Using the general spherical derivative

$(|h^{(n)}(w)|/(1 + |h(w)|^{n+1}))$ (see [8]), we give the following general meromorphic spaces.

Let $n \in \mathbb{N}$, $0 < q < \infty$, $0 < s < \infty$. Then, the general meromorphic Besov-type spaces are defined by

$$B^\#(q, q-2, s; n) = \left\{ h \in M(U): \sup_{a \in U} \int_U \left[\frac{|h^{(n)}(w)|}{1 + |h(w)|^{n+1}} \right]^q (1 - |w|^2)^{q-2} (1 - |\varphi_a(w)|^2)^s dm(w) < \infty \right\}, \quad (3)$$

where the concerned weight function is $(1 - |w|^2)^{q-2} (1 - |\varphi_a(w)|^2)^s$ and $w \in U$. Here, $\varphi_a(w)$ denotes the usual Möbius transformation $\varphi_a(w) = ((a - w)/(1 - \bar{a}w))$. Also,

$$h^{(n)}(w) = \frac{d^n h(w)}{dw^n}, \text{ that is, we have “}n\text{” times derivatives, } n \in \mathbb{N}. \quad (4)$$

The concerned meromorphic counterpart of the Bloch-type space is the class of all concerned normal functions $\mathcal{N}$ (see [1, 9]); this class of meromorphic functions can be extended to the following concerned class.

Definition 1. Assume that h is a meromorphic function in U . When

$$\|h\|_{\mathcal{N}_n} = \sup_{w \in U} (1 - |w|^2) \frac{|h^{(n)}(w)|}{1 + |h(w)|^{n+1}} < \infty, \quad (5)$$

$h \in \mathcal{N}_n$ of concerned normal functions.

Definition 2. Suppose that the function h stands for a concerned meromorphic function in U . The concerned sequence of points $\{a_m\} (|a_m| \rightarrow 1)$ in U is called a $q_{(N,n)}$ -sequence if

$$\lim_{m \rightarrow \infty} \frac{|h^{(n)}(w)|}{1 + |h(w)|^{n+1}} (1 - |a_m|^2) = +\infty. \quad (6)$$

In Definitions 2, by letting $n = 1$, we obtain the class of all usual normal functions $\mathcal{N}$ (see [1, 9]). For more interesting various studies on different meromorphic function classes, we refer to [10–15] and others. The following definitions can be introduced.

Definition 3. Assume that h is a meromorphic function in U . For $2 < q < \infty$ and $0 < s < \infty$, the concerned sequence of points $\{a_m\} (|a_m| \rightarrow 1)$ in U is called a $b_{(q,n)}$ -sequence if

$$\lim_{m \rightarrow \infty} \iint_U \left[\frac{|h^{(n)}(w)|}{1 + |h(w)|^{n+1}} \right]^q (1 - |w|^2)^{q-2} (1 - |\varphi_{a_m}(w)|^2)^s dm(w) = +\infty. \quad (7)$$

2. Families of $b_{(q,n)}$ - and $q_{(N,n)}$ -Type Sequences

Theorem 1. Let $h \in M(U)$. Suppose that $\{a_m\}$ defines the $q_{(N,n)}$ -type sequence, thus any sequence of points $\{c_m\}$ in U , such that $d(a_m, c_m) \rightarrow 0$ is a $b_{(q,n)}$ -type sequence for all values of q with $2 < q < \infty$.

Proof. In view of [16], we can find two concerned sequences $\{c_m\} \subset U$ and $\{d_m\} \subset \mathbb{R}^+$, with $d(a_m, c_m) \rightarrow 0$ with

$$\frac{d_m}{(1 - |c_m|^2)} \rightarrow 0, \quad (8)$$

where the concerned sequence of functions $\{h_m(t)\} = \{h(b_m + d_m t)\}$ has uniformly converging type on

each concerned compact subset of $\mathbb{C}$ to a concerned non-constant meromorphic function $G(t)$. Thus,

$$\begin{aligned}
 & \sup_{c_m \in U} \iint_U \left[\frac{|h^{(n)}(w)|}{1 + |h(w)|^{n+1}} \right]^q (1 - |w|^2)^{q-2} (1 - |\varphi_{c_m}(w)|^2)^s dm(w) \\
 & \geq \iint_{U(c_m, (1/e))} \left[\frac{|h^{(n)}(w)|}{1 + |h(w)|^{n+1}} \right]^q (1 - |w|^2)^{q-2} (1 - |\varphi_{c_m}(w)|^2)^s dm(w) \\
 & \geq \iint_{U(0,r)} \left[\frac{|h_m^{(n)}(t)|}{1 + |h_m(t)|^{n+1}} \right]^q (1 - |c_m + d_m t|^2)^{q-2} (1 - |\varphi_{c_m}(c_m + d_m t)|^2)^s d_m^{2-q} dm(t) \\
 & = \iint_{U(0,r)} \left[\frac{|h_m^{(n)}(t)|}{1 + |h_m(t)|^{n+1}} \right]^q \left(\frac{1 - |c_m + d_m t|^2}{d_m} \right)^{q-2} \times \left(\left| \frac{c_m - (c_m + d_m t)}{1 - \bar{c}_m(c_m + d_m t)} \right|^2 \right)^s dm(t) \\
 & = \iint_{U(0,r)} \left[\frac{|h_m^{(n)}(t)|}{1 + |h_m(t)|^{n+1}} \right]^q \left(\frac{1 - |c_m + d_m t|^2}{d_m} \right)^{q-2} \times \left(1 - \left| \frac{1}{((1 - |c_m|^2)/d_m t) - \bar{c}_m} \right|^2 \right)^s dm(t).
 \end{aligned} \tag{9}$$

Using the uniform convergence techniques, we deduce that

$$\iint_{U(0,r)} \left[\frac{|h_m^{(n)}(t)|}{1 + |h_m(t)|^{n+1}} \right]^q dm(t) \longrightarrow \iint_{U(0,r)} \left(G_n^\#(t) \right)^q dm(t), \tag{10}$$

where the last defined integral is positive, since $G(t)$ is a concerned nonconstant meromorphic function. Further, by (6), when $m \rightarrow \infty$, we conclude that

$$1 - \left| \frac{1}{((1 - |c_m|^2)/d_m t) - \bar{c}_m} \right|^2 \longrightarrow 1. \tag{11}$$

Therefore, we can obtain that

$$\iint_{U(0,r)} \left[\frac{|h_m^{(n)}(t)|}{1 + |h_m(t)|^{n+1}} \right]^q \left(\frac{1 - |a_m + d_m t|^2}{d_m} \right)^{q-2} \left(1 - \left| \frac{1}{((1 - |a_m|^2)/d_m t) - \bar{a}_m} \right|^2 \right)^s dm(t) \longrightarrow \infty. \tag{12}$$

Hence, when $2 < q < \infty$, we have that

$$\iint_U \left[\frac{|h^{(n)}(w)|}{1 + |h(w)|^{n+1}} \right]^q (1 - |w|^2)^{q-2} (1 - |\varphi_{c_m}(w)|^2)^s dm(w) \longrightarrow \infty, \tag{13}$$

Thus, $\{c_m\} \in U$ is a $b_{(q,n)}$ -type sequence for all q , with $2 < q < \infty$. The proof of Theorem 1 is completely finished. $\square$

Theorem 2. We can find a concerned non-normal function h and a concerned sequence $\{a_m\}$ in U which is a $b_{(q,n)}$ -sequence for all q , with $2 < q < \infty$, whereas $\{a_m\}$ is not a $q_{(N,n)}$ -sequence.

Proof. Assume that the function $h(w) = \exp(i/(1-w))$ is a non-normal function where $i = \sqrt{-1}$. Considering the concerned sequence $\{c_m\} = \{m^2/(1+m^2)\}$, after simple computation, we deduce that

$$\lim_{m \rightarrow \infty} (1 - |c_m|^2) \frac{|h^{(n)}(c_m)|}{1 + |h(c_m)|^{n+1}} = +\infty. \tag{14}$$

Applying Theorem 1 for any concerned sequence of specific points $\{a_m\}$ in U , with $d(a_m, c_m) \rightarrow 0$, we get

$$\begin{aligned}
 & \lim_{m \rightarrow \infty} \iint_U \left[\frac{|h^{(n)}(w)|}{1 + |h(w)|^{n+1}} \right]^q (1 - |w|^2)^{q-2} \\
 & \cdot (1 - |\varphi_{a_m}(w)|^2)^s dm(w) = +\infty,
 \end{aligned} \tag{15}$$

for all q , with $2 < q < \infty$. Let $\{a_m\} = \{(m^2/(1+m^2)) - (i/(m+m^3))\}$, and note that $d(a_m, c_m) \rightarrow 0$. But

$$\lim_{m \rightarrow \infty} (1 - |a_m|^2) \frac{|h^{(n)}(a_m)|}{1 + |h(a_m)|^{n+1}} = 0. \tag{16}$$

Hence, the concerned sequence $\{a_m\}$ is our needed sequence of points. $\square$

Theorem 3. Let $h \in M(U)$ and suppose that $2 < q_1 < q < \infty$ and $0 < s_1 < s < \infty$. For a concerned sequence of points $\{a_m\}$ in the disc U , when

$$\lim_{m \rightarrow \infty} \iint_U \left[\frac{|h^{(n)}(w)|}{1 + |h(w)|^{n+1}} \right]^q (1 - |w|^2)^{q-2} (1 - |\varphi_{a_m}(w)|^2)^s dm(w) = +\infty, \quad (17)$$

$$\lim_{m \rightarrow \infty} \iint_U \left[\frac{|h^{(n)}(w)|}{1 + |h(w)|^{n+1}} \right]^{q_1} (1 - |w|^2)^{q_1-2} (1 - |\varphi_{a_m}(w)|^2)^{s_1} dm(w) = +\infty. \quad (18)$$

Proof. When condition (17) holds, then for $2 < q_1 < q < \infty$ and $0 < s_1 < s < \infty$, using the known inequality of Hölder, we conclude that

$$\begin{aligned} & \iint_U \left[\frac{|h^{(n)}(w)|}{1 + |h(w)|^{n+1}} \right]^{q_1} (1 - |w|^2)^{q_1-2} (1 - |\varphi_{a_m}(w)|^2)^{s_1} dm(w) \\ & \leq \left(\iint_U \left[\frac{|h^{(n)}(w)|}{1 + |h(w)|^{n+1}} \right]^q (1 - |w|^2)^{q-2} (1 - |\varphi_{a_m}(w)|^2)^s dm(w) \right)^{(q_1/q)} \\ & \quad \times \left(\iint_U (1 - |\varphi_{a_m}(w)|^2)^{(s_1 - (sq_1/q))} (1 - |w|^2)^{(q - q_1) - 2} dm(w) \right)^{(1 - (q_1/q))} \\ & = \left(\iint_U \left[\frac{|h^{(n)}(w)|}{1 + |h(w)|^{n+1}} \right]^q (1 - |w|^2)^{q-2} (1 - |\varphi_{a_m}(w)|^2)^s dm(w) \right)^{(q_1/q)} \\ & \quad \times \left(\iint_U (1 - |w|^2)^{((s_1 q - sq_1)/(q - q_1) - 2)} dm(w) \right)^{(1 - (q_1/q))}. \end{aligned} \quad (19)$$

It is obvious to see that $((s_1 q - sq_1)/(q - q_1) - 2) = (\eta - 2) > -1$, for $\eta > 1$, and we obtain

$$\iint_U (1 - |w|^2)^{((s_1 q - sq_1)/(q - q_1) - 2)} dm(w) = \iint_U (1 - |w|^2)^{(\eta - 2)} dm(w) < C_1 < \infty, \quad (20)$$

where $C_1 > 0$.
 $B^\#(q_1, q_1 - 2, s_1; n)$.

Therefore, $B^\#(q, q - 2, s; n) \subset$

Thus, the following inequality can be followed:

$$\begin{aligned} & \iint_U \left[\frac{|h^{(n)}(w)|}{1 + |h(w)|^{n+1}} \right]^{q_1} (1 - |w|^2)^{q_1-2} (1 - |\varphi_{a_m}(w)|^2)^{s_1} dm(w) \\ & \geq \iint_U \left[\frac{|h^{(n)}(w)|}{1 + |h(w)|^{n+1}} \right]^q (1 - |w|^2)^{q-2} (1 - |\varphi_{a_m}(w)|^2)^s dm(w) = +\infty. \end{aligned} \quad (21)$$

Then, condition (18) must be verified. Thus, the proof is established completely. $\square$

Remark 1. Using the specific condition (17), we deduce that the function h not in the classes $B^\#(q, q-2, s; n)$, this because the concerned meromorphic classes $B^\#(q, q-2, s; n)$ have a specific nesting property and the meromorphic function h is not belong to the meromorphic classes

$B^\#(q_1, q_1-2, s_1)$ when $2 < q_1 < q < \infty$ and $0 < s_1 < s < \infty$. Nevertheless, Theorem 3 shows further details on this case which clearing that the similar concerned sequence of points $\{a_m\}$, for which $B^\#(q, q-2, s; n)$ -condition can be excluded, also it excludes the $B^\#(q_1, q_1-2, s_1; n)$ -condition.

Remark 2. From the concerned proof of Theorem 3, we can clearly show that for a fixed ρ_0 , $0 < \rho_0 < 1$ and $\rho > 0$, when

$$\lim_{m \rightarrow \infty} \iint_{U(a_m, \rho_0)} \left[\frac{|h^{(n)}(w)|}{1 + |h(w)|^{n+1}} \right]^q (1 - |w|^2)^{q-2} (1 - |\varphi_{a_m}(w)|^2)^s dm(w) = +\infty. \quad (22)$$

Thus, we can find a concerned sequence of points $\{c_m\}$ in $\Delta_\rho^n = \{w: (1 - |\varphi_a(w)|^2) > \rho\}$, for which

$$\lim_{m \rightarrow \infty} (1 - |c_m|^2) \frac{|h^{(n)}(c_m)|}{1 + |h(c_m)|^{n+1}} = +\infty. \quad (23)$$

Theorem 4. Let $h \in M(U)$. For a concerned sequence of points $\{a_m\} \subset U$, when

$$\lim_{m \rightarrow \infty} (1 - |a_m|^2) \frac{|h^{(n)}(a_m)|}{1 + |h(a_m)|^{n+1}} = +\infty, \quad (24)$$

for the same concerned sequence $\{a_m\}$, we have

$$\lim_{m \rightarrow \infty} \iint_{U(a_m, \rho)} \left[\frac{|h^{(n)}(w)|}{1 + |h(w)|^{n+1}} \right]^q (1 - |w|^2)^{q-2} (1 - |\varphi_{a_m}(w)|^2)^s dm(w) = +\infty, \quad (25)$$

for all values of q, s where $2 < q < \infty$ and $0 < s < \infty$ as well as ρ , with $0 < \rho < 1$.

Proof. Assume that condition (25) holds. Then, we have ρ_0 , $0 < \rho_0 < 1$, such that

$$\lim_{m \rightarrow \infty} \sup \iint_{U(a_m, \rho_0)} \left[\frac{|h^{(n)}(w)|}{1 + |h(w)|^{n+1}} \right]^q (1 - |w|^2)^{q-2} (1 - |\varphi_{a_m}(w)|^2)^s dm(w) = K < +\infty. \quad (26)$$

Thus, we can find a concerned subsequence $\{a_{m_k}\}$ of $\{a_m\}$, for which

$$\iint_{U(a_{m_k}, \rho_0)} \left[\frac{|h^{(n)}(w)|}{1 + |h(w)|^{n+1}} \right]^q (1 - |w|^2)^{q-2} (1 - |\varphi_{a_{m_k}}(w)|^2)^s dm(w) \leq K + 1, \quad (27)$$

this can be verified for sufficiently large k . Let ρ_1 , $0 < \rho_1 < \rho_0$, $U(a_{m_k}, \rho_1) = \{w \in U: |\varphi_{a_{m_k}}(w)| < \rho_1\}$, which verifies that

$$\frac{K + 1}{(1 - \rho^2)^{s+q-2}} < \frac{\pi}{2}. \quad (28)$$

This implies that

$$\iint_{U(a_{m_k}, \rho_1)} \left[\frac{|h^{(n)}(w)|}{1 + |h(w)|^{n+1}} \right]^q dm(w) \leq \frac{K + 1}{(1 - \rho_1^2)^{s+q-2}} < \frac{\pi}{2}, \quad (29)$$

where $(1 - |\varphi_{a_{m_k}}(w)|^2) \geq (1 - \rho_1^2)$. Applying the theorem of Dufresny (see [15]), we deduce that

$$(1 - |a_{m_k}|^2) \frac{|h^{(n)}(a_{m_k})|}{1 + |h(a_{m_k})|^{n+1}} \leq \frac{1}{\rho_1}, \quad (30)$$

and this is a contradiction of the concerned assumption. Therefore, the concerned proof of Theorem 4 is finished. $\square$

Theorem 5. Suppose that $h \in M(U)$. For $q, s \in (0, \infty)$, we can find a concerned sequence of points $\{a_m\} \subset U$, for which

$$\lim_{m \rightarrow \infty} \iint_U \left[\frac{|h^{(n)}(w)|}{1 + |h(w)|^{n+1}} \right]^q (1 - |w|^2)^{q-2} \left(1 - |\varphi_{a_m}(w)|^2 \right)^s dm(w) = +\infty. \quad (31)$$

Hence, for any concerned sequence of points $\{c_m\}$ in U such that $d(a_m, c_m) \rightarrow 0$, we have

$$\lim_{m \rightarrow \infty} \iint_U \left[\frac{|h^{(n)}(w)|}{1 + |h(w)|^{n+1}} \right]^q (1 - |w|^2)^{q-2} \left(1 - |\varphi_{c_m}(w)|^2 \right)^s dm(w) = +\infty. \quad (32)$$

Proof. Let K_1 and K_2 be two specific positive constants satisfying $K_2 < K_1$. Suppose that

$$\begin{aligned} U_{K_1}^m &= \left\{ w : \left(1 - |\varphi_{a_m}(w)|^2 \right) > K_1 \right\} \text{ and} \\ U_{K_2}^m &= \left\{ w : \left(1 - |\varphi_{a_m}(w)|^2 \right) > K_2 \right\}. \end{aligned} \quad (33)$$

Thus, when $w \in U_{K_1}^m$ and $w \in U \setminus U_{K_2}^m$, we have that $C(1 - |\varphi_{a_m}(w)|^2) \leq (1 - |\varphi_{c_m}(w)|^2)^2$ for some specific constant $C > 0$. Thus, for all m , we conclude that

$$\begin{aligned} & \iint_{U \setminus U_{K_2}^m} \left[\frac{|h^{(n)}(w)|}{1 + |h(w)|^{n+1}} \right]^q (1 - |w|^2)^{q-2} \left(1 - |\varphi_{c_m}(z)|^2 \right)^s dm(w) \\ & \geq C \iint_{U \setminus U_{K_2}^m} \left[\frac{|h^{(n)}(w)|}{1 + |h(w)|^{n+1}} \right]^q (1 - |w|^2)^{q-2} \left(1 - |\varphi_{a_m}(w)|^2 \right)^s dm(w). \end{aligned} \quad (34)$$

This inequality holds for any concerned sequence of points $\{c_m\}$ in U with $d(a_m, c_m) \rightarrow 0$. When

$$\lim_{m \rightarrow \infty} \sup \iint_{U \setminus U_{K_2}^m} \left[\frac{|h^{(n)}(w)|}{1 + |h(w)|^{n+1}} \right]^q (1 - |w|^2)^{q-2} \left(1 - |\varphi_{a_m}(w)|^2 \right)^s dm(w) = +\infty. \quad (35)$$

Therefore, using (10), we obtain

$$\lim_{m \rightarrow \infty} \sup \iint_{U \setminus U_{K_2}^m} \left[\frac{|h^{(n)}(w)|}{1 + |h(w)|^{n+1}} \right]^q (1 - |w|^2)^{q-2} \left(1 - |\varphi_{c_m}(w)|^2 \right)^s dm(w) = +\infty, \quad (36)$$

If

$$\lim_{m \rightarrow \infty} \sup \iint_{U_{K_2}^m} \left[\frac{|h^{(n)}(w)|}{1 + |h(w)|^{n+1}} \right]^q (1 - |w|^2)^{q-2} \left(1 - |\varphi_{a_m}(w)|^2 \right)^s dm(w) = +\infty. \quad (37)$$

Thus, we can consider two cases. $\square$

Case 1. In this case, we can find a concerned sequence of points $\{c_m\}$ in $U_{K_2}^m$, such that $d(a_m, c_m) \rightarrow 0$, for which

$$\lim_{m \rightarrow \infty} (1 - |c_m|^2) \frac{|h^{(n)}(c_m)|}{1 + |h(c_m)|^{n+1}} = +\infty, \quad (38)$$

or we can consider the following case.

Case 2. We can find $R_0, 0 < R_0 < e^{-K_2}$; also, there exists $\lambda > 0$, for which

$$(1 - |w|^2) \frac{|h^{(n)}(w)|}{1 + |h(w)|^{n+1}} \leq \lambda, \quad (39)$$

where we consider all $w \in U(a_m, R_0)$. If Case 1 is verified, by Theorem 1, for the aforementioned sequence $\{c_m\}$ such that $d(a_m, c_m) \rightarrow 0$, we deduce that

$$\lim_{m \rightarrow \infty} \sup \iint_U \left[\frac{|h^{(n)}(w)|}{1 + |h(w)|^{n+1}} \right]^q (1 - |w|^2)^{q-2} \left(1 - |\varphi_{c_m}(w)|^2 \right)^s dm(w) = +\infty. \quad (40)$$

This is because $d(a_m, c_m) \rightarrow 0$. Also, when Case 2 holds, using the same concerned conclusions for the concerned weight functions, we obtain that necessarily

$$\lim_{m \rightarrow \infty} \sup \iint_U \left[\frac{|h^{(n)}(w)|}{1 + |h(w)|^{n+1}} \right]^q (1 - |w|^2)^{q-2} \left(1 - |\varphi_{c_m}(w)|^2 \right)^s dm(w) = +\infty. \quad (41)$$

This is the end of the concerned proof.

Now we are dealing with the following interesting question:

Assume that $2 < q < \infty$ for any concerned sequence $\{a_m\}$ and assume also that

$$\lim_{m \rightarrow \infty} \iint_U \left[\frac{|h^{(n)}(w)|}{1 + |h(w)|^{n+1}} \right]^q (1 - |w|^2)^{q-2} \left(1 - |\varphi_{a_m}(w)|^2 \right)^s dm(w) = +\infty. \quad (42)$$

Is the following equation correct?

$$\lim_{m \rightarrow \infty} \sup \iint_U \left[\frac{|h^{(n)}(w)|}{1 + |h(w)|^{n+1}} \right]^{q_1} (1 - |w|^2)^{q_1-2} \left(1 - |\varphi_{a_m}(w)|^2 \right)^s dm(w) = +\infty, \quad (43)$$

for q_1 with $q < q_1$.

We give the answer of this important question by introducing Theorem 6 with its concerned proof.

Definition 4. Let $\{a_m\}$ be any concerned sequence of points in U ; then, $\{a_m\}$ is said to be a $m_{(q,n)}$ -sequence, when

$$\lim_{m \rightarrow \infty} \sup \iint_U \left[\frac{|h^{(n)}(w)|}{1 + |h(w)|^{n+1}} \right]^q (1 - |\varphi_{a_m}(w)|^2)^q dm(w) = +\infty. \quad (44)$$

Theorem 6. Let $q \in (2, \infty)$ and assume that

$$\lim_{m \rightarrow \infty} \iint_U \left[\frac{|h^{(n)}(w)|}{1 + |h(w)|^{n+1}} \right]^q (1 - |w|^2)^{q-2} \left(1 - |\varphi_{a_m}(w)|^2 \right)^s dm(w) = +\infty. \quad (45)$$

When the concerned sequence of points $\{a_m\}$ in U is not a concerned $m_{(q,n)}$ -sequence, for any q_1 with $q < q_1$, we conclude that

$$\lim_{m \rightarrow \infty} \iint_U \left[\frac{|h^{(n)}(w)|}{1 + |h(w)|^{n+1}} \right]^{q_1} (1 - |w|^2)^{q_1-2} \left(1 - |\varphi_{a_m}(w)|^2 \right)^s dm(w) = +\infty. \quad (46)$$

Proof. As in [1], we can deduce that $\mathcal{N}_n \cap B_{q,q-2;n}^\#$. Also, we have

- (i) $B^\#(q, q-2, s; n) \subset \mathcal{N}_n$ for all q , with $2 < q < \infty$ and $0 < s < 1$.
- (ii) For all values of $q, q_1, 2 < q < \infty, 0 < s < 1$ with $q_1 + s > 1$, we have that

$$\bigcup_{2 < q < q_1} B^\#(q, q-2, s; n) \subsetneq B^\#(q_1, q_1-2, s; n). \quad (47)$$

Therefore, it is obvious to get that

$$\lim_{m \rightarrow \infty} \iint_U \left[\frac{|h^{(n)}(w)|}{1 + |h(w)|^{n+1}} \right]^{q_1} (1 - |w|^2)^{q_1-2} \left(1 - |\varphi_{a_m}(w)|^2 \right)^s dm(w) = +\infty. \quad (48)$$

Remark 3. The recent developments of fractional calculus as well as its applications are more essential to complex function spaces with the specific arbitrary fractional order derivatives. For some recent interesting studies on the subject of fractional calculus, we can here refer to [17–19] and others. To the best of our knowledge, a few number of manuscripts researched some certain classes of analytic function spaces by the help of general fractional derivatives (see [20]). For further research work, the following specific interesting question can be considered:

How one can define and study the Besov spaces of general meromorphic functions by using the general fractional derivatives?

3. Conclusions

Certain concerned weighted classes of meromorphic function spaces using the general spherical derivatives are studied and discussed in this article. The general Besov-type classes of meromorphic functions as well as the general normal functions are considered intensively and both are compared deeply with each other. For a concerned non-normal function h , the concerned families of points $\{a_m\}$ and $\{c_m\}$, for which

$$\lim_{m \rightarrow \infty} (1 - |a_m|^2) \frac{|h^{(n)}(a_m)|}{1 + |h(a_m)|^{n+1}} = +\infty, \quad n \in \mathbb{N}, \quad (49)$$

$$\lim_{m \rightarrow \infty} \iint_U \left[\frac{|h^{(n)}(w)|}{1 + |h(w)|^{n+1}} \right]^q (1 - |w|^2)^{q-2} \left(1 - |\varphi_{a_m}(w)|^2 \right)^s dm(w) = +\infty$$

are introduced and discussed. Several connections between families (sequences) of $b_{(q,n)}$ and $q_{(N,n)}$ type are established. The obtained results improve, extend, and generalize numerous results in [21–23].

Remark 4. Quite recently there are some important enjoyable research studies on hyperbolic function classes (see [24, 25]). For more interesting research, how we can construct some workable conditions on some hyperbolic-type

sequences of points that make guarantee to belong to some specific hyperbolic-type classes?

Data Availability

No data were applied or considered to support this study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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