

Research Article

Hadamard and Fejér–Hadamard Inequalities for Further Generalized Fractional Integrals Involving Mittag-Leffler Functions

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Received 7 January 2021; Revised 19 February 2021; Accepted 24 February 2021; Published 16 March 2021

Academic Editor: Ahmet Ocak Akdemir

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In this paper, generalized versions of Hadamard and Fejér–Hadamard type fractional integral inequalities are obtained. By using generalized fractional integrals containing Mittag-Leffler functions, some well-known results for convex and harmonically convex functions are generalized. The results of this paper are connected with various published fractional integral inequalities.

1. Introduction

First we give definitions of fractional integral operators which are useful in establishing the results of this paper. In the following, we give fractional integral operators defined by Andrić et al. in [1] via an extended generalized Mittag-Leffler function in their kernels.

Definition 1 (see [1]). Let $\omega, \tau, \delta, \rho, c \in \mathbb{C}$, $\Re(\tau), \Re(\delta) > 0$, $\Re(c) > \Re(\rho) > 0$ with $\bar{p} \geq 0$, $\sigma, r > 0$ and $0 < k \leq r + \sigma$. Let $\varphi \in L_1[\varepsilon_1, \varepsilon_2]$ and $x \in [\varepsilon_1, \varepsilon_2]$. Then, the generalized fractional integral operators $e_{\sigma, \tau, \delta, \omega, \varepsilon_1^+}^{\rho, r, k, c} \varphi$ and $e_{\sigma, \tau, \delta, \omega, \varepsilon_2^-}^{\rho, r, k, c} \varphi$ are defined by

$$\begin{aligned} \left(e_{\sigma, \tau, \delta, \omega, \varepsilon_1^+}^{\rho, r, k, c} \varphi \right) (x; \bar{p}) &= \int_{\varepsilon_1}^x (x-t)^{\tau-1} E_{\sigma, \tau, \delta}^{\rho, r, k, c}(\omega(x-t)^\sigma; \bar{p}) \varphi(t) dt, \\ \left(e_{\sigma, \tau, \delta, \omega, \varepsilon_2^-}^{\rho, r, k, c} \varphi \right) (x; \bar{p}) &= \int_x^{\varepsilon_2} (t-x)^{\tau-1} E_{\sigma, \tau, \delta}^{\rho, r, k, c}(\omega(t-x)^\sigma; \bar{p}) \varphi(t) dt, \end{aligned} \quad (1)$$

where

$$E_{\sigma, \tau, \delta}^{\rho, r, k, c}(t; \bar{p}) = \sum_{n=0}^{\infty} \frac{\beta_{\bar{p}}(\rho + nk, c - \rho)(c)_{nk} t^n}{\beta(\rho, c - \rho) \Gamma(\sigma n + \tau) (\delta)_{nr}}, \quad (2)$$

is the extended generalized Mittag-Leffler function and $\beta_{\bar{p}}$ is the extension of beta function which is defined as follows:

$$\beta_{\bar{p}}(x, y) = \int_0^1 t^{x-1} (1-t)^{y-1} e^{-\bar{p}/t(1-t)} dt, \quad (3)$$

where x, y, \bar{p} are positive real numbers.

Recently, Farid defined elegantly a unified integral operator in [2] (see, also [3]) as follows.

Definition 2. Let $\varphi, \theta: [\varepsilon_1, \varepsilon_2] \rightarrow \mathbb{R}$, $0 < \varepsilon_1 < \varepsilon_2$ be the functions such that φ be positive and $\varphi \in L_1[\varepsilon_1, \varepsilon_2]$ and θ be a differentiable and strictly increasing function. Also, let χ/x be an increasing function on $[\varepsilon_1, \infty)$ and $\omega, \tau, \delta, \rho, c \in \mathbb{C}$, $\Re(\tau), \Re(\delta) > 0$, $\Re(c) > \Re(\rho) > 0$ with $\bar{p} \geq 0$, $\sigma, r > 0$, and $0 < k \leq r + \sigma$.

Then, for $x \in [\varepsilon_1, \varepsilon_2]$, the integral operators $(\theta Y_{\sigma, \tau, \delta, \varepsilon_1^+}^{\chi, \rho, r, k, c} \varphi)$ and $(\theta Y_{\sigma, \tau, \delta, \varepsilon_2^-}^{\chi, \rho, r, k, c} \varphi)$ are defined by

$$\begin{aligned} \left({}_{\theta} \Upsilon_{\sigma, \tau, \delta, \varepsilon_1^+}^{\rho, r, k, c} \varphi \right) (x; \bar{p}) &= \int_{\varepsilon_1}^x \frac{\chi(\theta(x) - \theta(t))}{\theta(x) - \theta(t)} E_{\sigma, \tau, \delta}^{\rho, r, k, c} (\omega(\theta(x) - \theta(t))^\sigma; \bar{p}) \varphi(t) d(\theta(t)), \\ \left({}_{\theta} \Upsilon_{\sigma, \tau, \delta, \varepsilon_2^-}^{\rho, r, k, c} \varphi \right) (x; \bar{p}) &= \int_x^{\varepsilon_2} \frac{\chi(\theta(t) - \theta(x))}{\theta(t) - \theta(x)} E_{\sigma, \tau, \delta}^{\rho, r, k, c} (\omega(\theta(t) - \theta(x))^\sigma; \bar{p}) \varphi(t) d(\theta(t)). \end{aligned} \tag{4}$$

The following definition of generalized fractional integral operators containing extended Mittag-Leffler function in the kernel can be extracted from Definition 2. It is generalization of Definition 1 by a monotonically increasing function.

Definition 3. Let $\varphi, \theta: [\varepsilon_1, \varepsilon_2] \rightarrow \mathbb{R}$, $0 < \varepsilon_1 < \varepsilon_2$ be the functions such that φ be positive and $\varphi \in L_1[\varepsilon_1, \varepsilon_2]$ and θ be a differentiable and strictly increasing function. Also, let $\omega, \tau, \delta, \rho, c \in \mathbb{C}$, $\Re(\tau), \Re(\delta) > 0$, $\Re(c) > \Re(\rho) > 0$ with $\bar{p} \geq 0$, $\sigma, r > 0$ and $0 < k \leq r + \sigma$. Then, for $x \in [\varepsilon_1, \varepsilon_2]$, fractional integral operators are defined by

$$\left({}_{\theta} \Upsilon_{\sigma, \tau, \delta, \omega \varepsilon_1^+}^{\rho, r, k, c} \varphi \right) (x; \bar{p}) = \int_{\varepsilon_1}^x (\theta(x) - \theta(t))^{\tau-1} E_{\sigma, \tau, \delta}^{\rho, r, k, c} (\omega(\theta(x) - \theta(t))^\sigma; \bar{p}) \varphi(t) d(\theta(t)), \tag{5}$$

$$\left({}_{\theta} \Upsilon_{\sigma, \tau, \delta, \omega \varepsilon_2^-}^{\rho, r, k, c} \varphi \right) (x; \bar{p}) = \int_x^{\varepsilon_2} (\theta(t) - \theta(x))^{\tau-1} E_{\sigma, \tau, \delta}^{\rho, r, k, c} (\omega(\theta(t) - \theta(x))^\sigma; \bar{p}) \varphi(t) d(\theta(t)). \tag{6}$$

The following remark provides connection of Definition 3 with existing fractional integral operators.

generalized fractional integral operators are utilized to get main results of this paper.

Remark 1

- (i) If we set $\bar{p} = 0$ and $\theta(x) = x$ in equations (5) and (6), then these reduce to fractional integral operators defined by Salim and Faraj in [4].
- (ii) If we set $\delta = r = 1$ and $\theta(x) = x$ in equations (5) and (6), then these reduce to the fractional integral operators ${}_{\theta} \Upsilon_{\sigma, \tau, 1, \omega, \varepsilon_1^+}^{\rho, 1, k, c}$ and ${}_{\theta} \Upsilon_{\sigma, \tau, 1, \omega, \varepsilon_2^-}^{\rho, 1, k, c}$ containing generalized Mittag-Leffler function $E_{\sigma, \tau, 1}^{\rho, 1, k, c}(t; \bar{p})$ defined by Rahman et al. in [5].
- (iii) If we take $\bar{p} = 0, \delta = r = 1$ and $\theta(x) = x$ in equations (5) and (6), then these reduce to fractional integral operators containing extended generalized Mittag-Leffler function introduced by Srivastava and Tomovski in [6].
- (iv) If we set $\bar{p} = 0, \delta = r = k = 1$ and $\theta(x) = x$ in equations (5) and (6), then these reduce to fractional integral operators defined by Prabhaker in [7].
- (v) For $\bar{p} = \omega = 0$ and $\theta(x) = x$ in equations (5) and (6), these reduce to renowned Riemann–Liouville fractional integral operators [8].

The Riemann–Liouville fractional integrals for a function $\varphi \in L_1[\varepsilon_1, \varepsilon_2]$ of order $\tau \in \mathbb{R}$ ($\tau > 0$) are defined by

$$\begin{aligned} I_{\varepsilon_1^+}^\tau \varphi(x) &= \frac{1}{\Gamma(\tau)} \int_{\varepsilon_1}^x (x - t)^{\tau-1} \varphi(t) dt, \quad x > \varepsilon_1, \\ I_{\varepsilon_2^-}^\tau \varphi(x) &= \frac{1}{\Gamma(\tau)} \int_x^{\varepsilon_2} (t - x)^{\tau-1} \varphi(t) dt, \quad x < \varepsilon_2. \end{aligned} \tag{7}$$

After introducing generalized fractional integral operators, now we define notions of functions for which

Definition 4 (see [9]). A function $\varphi: [\varepsilon_1, \varepsilon_2] \rightarrow \mathbb{R}$ is said to be convex if

$$\varphi(tx_1 + (1 - t)x_2) \leq t\varphi(x_1) + (1 - t)\varphi(x_2), \tag{8}$$

holds for all $x_1, x_2 \in [\varepsilon_1, \varepsilon_2]$ and $t \in [0, 1]$.

Definition 5 (see [10]). Let I be an interval such that $I \subseteq \mathbb{R}_+$. Then, a function $\varphi: I \rightarrow \mathbb{R}$ is said to be harmonically convex, if

$$\varphi\left(\frac{ab}{ta + (1 - t)b}\right) \leq t\varphi(b) + (1 - t)\varphi(a), \tag{9}$$

holds for all $a, b \in I$ and $t \in [0, 1]$.

Definition 6 (see [11]). Let $J \subset (0, \infty)$ be a real interval and $p \in \mathbb{R} \setminus \{0\}$. Then, a function $\varphi: J \rightarrow \mathbb{R}$ is said to be p -convex, if

$$\varphi\left([t\varepsilon_1^p + (1 - t)\varepsilon_2^p]^{1/p}\right) \leq t\varphi(\varepsilon_1) + (1 - t)\varphi(\varepsilon_2), \tag{10}$$

holds for $\varepsilon_1, \varepsilon_2 \in J$ and $t \in [0, 1]$.

It is easy to see that for $p = 1$ and $p = -1$, the p -convexity reduces to convexity and harmonical convexity, respectively.

Definition 7 (see [11]). Let $p \in \mathbb{R} \setminus \{0\}$. Then, a function $\varphi: [\varepsilon_1, \varepsilon_2] \subset (0, \infty) \rightarrow \mathbb{R}$ is said to be p -symmetric with respect to $[(\varepsilon_1^p + \varepsilon_2^p)/2]^{1/p}$ if

$$\varphi(t^{1/p}) = \varphi\left([\varepsilon_1^p + \varepsilon_2^p - t]^{1/p}\right), \tag{11}$$

holds, for $t \in [\varepsilon_1, \varepsilon_2]$.

Convex functions are equivalently studied by the Hadamard inequality.

Theorem 1. *Let $\varphi: [\varepsilon_1, \varepsilon_2] \rightarrow \mathbb{R}$ be a convex function such that $\varepsilon_1 < \varepsilon_2$. Then, the following inequality holds:*

$$\varphi\left(\frac{\varepsilon_1 + \varepsilon_2}{2}\right) \leq \frac{1}{\varepsilon_2 - \varepsilon_1} \int_{\varepsilon_1}^{\varepsilon_2} \varphi(x) dx \leq \frac{\varphi(\varepsilon_1) + \varphi(\varepsilon_2)}{2}. \quad (12)$$

The Fejér–Hadamard inequality is a weighted version of the Hadamard inequality given by Fejér in [12].

Theorem 2. *Let $\varphi: [\varepsilon_1, \varepsilon_2] \rightarrow \mathbb{R}$ be a convex function and $g: [\varepsilon_1, \varepsilon_2] \rightarrow \mathbb{R}$ be non-negative, integrable, and symmetric about $(\varepsilon_1 + \varepsilon_2)/2$. Then, the following inequality holds:*

$$\begin{aligned} \varphi\left(\frac{\varepsilon_1 + \varepsilon_2}{2}\right) \int_{\varepsilon_1}^{\varepsilon_2} g(x) dx &\leq \int_{\varepsilon_1}^{\varepsilon_2} \varphi(x) g(x) dx \\ &\leq \frac{\varphi(\varepsilon_1) + \varphi(\varepsilon_2)}{2} \int_{\varepsilon_1}^{\varepsilon_2} g(x) dx. \end{aligned} \quad (13)$$

In recent decades, the Hadamard and the Fejér–Hadamard fractional integral inequalities have been studied extensively for different kinds of convex functions (see [1, 3, 13–21]). In this paper, we find Hadamard and Fejér–Hadamard inequalities for a generalized fractional integral operator involving an extended generalized Mittag-Leffler function.

In the upcoming section, we give two versions of the Hadamard inequality as well as two versions of the Fejér–Hadamard inequality. Their special cases are also discussed along with noticing connections with published results.

2. Main Results

First we give the following version of the Hadamard inequality.

Theorem 3. *Let $\varphi, \theta: [\varepsilon_1, \varepsilon_2] \subset (0, \infty) \rightarrow \mathbb{R}$, Range $(\theta) \subset [\varepsilon_1, \varepsilon_2]$ be the functions such that φ be positive and $\varphi \in L_1[\varepsilon_1, \varepsilon_2]$, and θ be a differentiable and strictly increasing function. If φ is p -convex, $p \in \mathbb{R} \setminus \{0\}$, then the following inequalities for fractional integral operators (5) and (6) hold:*

(i) If $p > 0$, then

$$\begin{aligned} &\varphi\left(\left(\frac{\theta^p(\varepsilon_1) + \theta^p(\varepsilon_2)}{2}\right)^{1/p}\right) \left({}_{\theta Y_{\sigma, \tau, \delta, \omega', \theta^{-1}(\theta^p(\varepsilon_1))_+}^{\rho, r, k, c}} 1\right) (\theta^{-1}(\theta^p(\varepsilon_2)); \bar{p}) \\ &\leq \frac{1}{2} \left(\left({}_{\theta Y_{\sigma, \tau, \delta, \omega', \theta^{-1}(\theta^p(\varepsilon_1))_+}^{\rho, r, k, c}} \varphi \circ \psi\right) (\theta^{-1}(\theta^p(\varepsilon_2)); \bar{p}) \right. \\ &\quad \left. + \left({}_{\theta Y_{\sigma, \tau, \delta, \omega', \theta^{-1}(\theta^p(\varepsilon_2))_-}^{\rho, r, k, c}} \varphi \circ \psi\right) (\theta^{-1}(\theta^p(\varepsilon_1)); \bar{p}) \right) \\ &\leq \frac{\varphi(\theta(\varepsilon_1)) + \varphi(\theta(\varepsilon_2))}{2} \left({}_{\theta Y_{\sigma, \tau, \delta, \omega', \theta^{-1}(\theta^p(\varepsilon_1))_+}^{\rho, r, k, c}} 1\right) (\theta^{-1}(\theta^p(\varepsilon_2)); \bar{p}), \end{aligned} \quad (14)$$

where $\omega' = \omega/(\theta^p(\varepsilon_2) - \theta^p(\varepsilon_1))^\sigma$ and $\psi(t) = \theta^{1/p}(t)$ for all $t \in [\varepsilon_1^p, \varepsilon_2^p]$.

(ii) If $p < 0$, then

$$\begin{aligned} &\varphi\left(\left(\frac{\theta^p(\varepsilon_1) + \theta^p(\varepsilon_2)}{2}\right)^{1/p}\right) \left({}_{\theta Y_{\sigma, \tau, \delta, \omega', \theta^{-1}(\theta^p(\varepsilon_1))_-}^{\rho, r, k, c}} 1\right) (\theta^{-1}(\theta^p(\varepsilon_2)); \bar{p}) \\ &\leq \frac{1}{2} \left(\left({}_{\theta Y_{\sigma, \tau, \delta, \omega', \theta^{-1}(\theta^p(\varepsilon_2))_+}^{\rho, r, k, c}} \varphi \circ \psi\right) (\theta^{-1}(\theta^p(\varepsilon_1)); \bar{p}) \right. \\ &\quad \left. + \left({}_{\theta Y_{\sigma, \tau, \delta, \omega', \theta^{-1}(\theta^p(\varepsilon_1))_-}^{\rho, r, k, c}} \varphi \circ \psi\right) (\theta^{-1}(\theta^p(\varepsilon_2)); \bar{p}) \right) \\ &\leq \frac{\varphi(\theta(\varepsilon_1)) + \varphi(\theta(\varepsilon_2))}{2} \left({}_{\theta Y_{\sigma, \tau, \delta, \omega', \theta^{-1}(\theta^p(\varepsilon_1))_-}^{\rho, r, k, c}} 1\right) (\theta^{-1}(\theta^p(\varepsilon_2)); \bar{p}), \end{aligned} \quad (15)$$

where $\omega' = \omega/(\theta^p(\varepsilon_1) - \theta^p(\varepsilon_2))^\sigma$ and $\psi(t) = \theta^{1/p}(t)$ for all $t \in [\varepsilon_2^p, \varepsilon_1^p]$.

Proof. (i) Since φ is p -convex over $[\varepsilon_1, \varepsilon_2]$, for all $x, y \in I$, we have

$$\varphi\left(\left(\frac{\theta^p(x) + \theta^p(y)}{2}\right)^{1/p}\right) \leq \frac{\varphi(\theta(x)) + \varphi(\theta(y))}{2}. \quad (16)$$

Setting $\theta(x) = (t\theta^p(\varepsilon_1) + (1-t)\theta^p(\varepsilon_2))^{1/p}$ and $\theta(y) = (t\theta^p(\varepsilon_2) + (1-t)\theta^p(\varepsilon_1))^{1/p}$ in above inequality, we have

$$\begin{aligned} &\varphi\left(\left(\frac{\theta^p(\varepsilon_1) + \theta^p(\varepsilon_2)}{2}\right)^{1/p}\right) \\ &\leq \frac{\varphi\left((t\theta^p(\varepsilon_1) + (1-t)\theta^p(\varepsilon_2))^{1/p}\right) + \varphi\left((t\theta^p(\varepsilon_2) + (1-t)\theta^p(\varepsilon_1))^{1/p}\right)}{2}. \end{aligned} \quad (17)$$

Multiplying both sides of (17) by $2t^{\tau-1}E_{\sigma,\tau,\delta}^{\rho,r,k,c}(\omega t^\sigma; \bar{p})$ and then integrating over $[0, 1]$, we have

$$\begin{aligned} &2\varphi\left(\left(\frac{\theta^p(\varepsilon_1) + \theta^p(\varepsilon_2)}{2}\right)^{1/p}\right) \int_0^1 t^{\tau-1}E_{\sigma,\tau,\delta}^{\rho,r,k,c}(\omega t^\sigma; \bar{p})dt \\ &\leq \int_0^1 t^{\tau-1}E_{\sigma,\tau,\delta}^{\rho,r,k,c}(\omega t^\sigma; \bar{p})\varphi\left((t\theta^p(\varepsilon_1) + (1-t)\theta^p(\varepsilon_2))^{1/p}\right)dt \\ &\quad + \int_0^1 t^{\tau-1}E_{\sigma,\tau,\delta}^{\rho,r,k,c}(\omega t^\sigma; \bar{p})\varphi\left((t\theta^p(\varepsilon_2) + (1-t)\theta^p(\varepsilon_1))^{1/p}\right)dt. \end{aligned} \quad (18)$$

By choosing $\theta(x) = t\theta^p(\varepsilon_1) + (1-t)\theta^p(\varepsilon_2)$ and $\theta(y) = t\theta^p(\varepsilon_2) + (1-t)\theta^p(\varepsilon_1)$ in (18), we have

$$\begin{aligned} &2\varphi\left(\left(\frac{\theta^p(\varepsilon_1) + \theta^p(\varepsilon_2)}{2}\right)^{1/p}\right) \int_{\theta^{-1}(\theta^p(\varepsilon_1))}^{\theta^{-1}(\theta^p(\varepsilon_2))} (\theta^p(\varepsilon_2) - \theta(x))^{\tau-1}E_{\sigma,\tau,\delta}^{\rho,r,k,c}(\omega'(\theta^p(\varepsilon_2) - \theta(x))^\sigma; \bar{p})1d(\theta(x)) \\ &\leq \int_{\theta^{-1}(\theta^p(\varepsilon_1))}^{\theta^{-1}(\theta^p(\varepsilon_2))} (\theta^p(\varepsilon_2) - \theta(x))^{\tau-1}E_{\sigma,\tau,\delta}^{\rho,r,k,c}(\omega'(\theta^p(\varepsilon_2) - \theta(x))^\sigma; \bar{p})\varphi(\theta(x)^{1/p})d(\theta(x)) \\ &\quad + \int_{\theta^{-1}(\theta^p(\varepsilon_1))}^{\theta^{-1}(\theta^p(\varepsilon_2))} (\theta(y) - \theta^p(\varepsilon_1))^{\tau-1}E_{\sigma,\tau,\delta}^{\rho,r,k,c}(\omega'(\theta(y) - \theta^p(\varepsilon_1))^\sigma; \bar{p})\varphi(\theta(y)^{1/p})d(\theta(y)), \end{aligned} \quad (19)$$

where $\omega' = \omega/(\theta^p(\varepsilon_2) - \theta^p(\varepsilon_1))^\sigma$.

This implies

$$\begin{aligned} &2\varphi\left(\left(\frac{\theta^p(\varepsilon_1) + \theta^p(\varepsilon_2)}{2}\right)^{1/p}\right) \left(\mathcal{E}_{\sigma,\tau,\delta,\omega',\theta^{-1}(\theta^p(\varepsilon_1))_+}^{\rho,r,k,c}1\right) (\theta^{-1}(\theta^p(\varepsilon_2)); \bar{p}) \\ &\leq \left(\mathcal{I}_{\theta^p(\varepsilon_2),\sigma,\tau,\delta,\omega',\theta^{-1}(\theta^p(\varepsilon_1))_+}^{\rho,r,k,c}\varphi \circ \psi\right) (\theta^{-1}(\theta^p(\varepsilon_2)); \bar{p}) \\ &\quad + \left(\mathcal{I}_{\theta^p(\varepsilon_1),\sigma,\tau,\delta,\omega',\theta^{-1}(\theta^p(\varepsilon_2))_-}^{\rho,r,k,c}\varphi \circ \psi\right) (\theta^{-1}(\theta^p(\varepsilon_1)); \bar{p}). \end{aligned} \quad (20)$$

To prove the second inequality of (14), again from p -convexity of φ over $[\varepsilon_1, \varepsilon_2]$ and for $t \in [0, 1]$, we have

$$\begin{aligned} &\varphi\left((t\theta^p(\varepsilon_1) + (1-t)\theta^p(\varepsilon_2))^{1/p}\right) \\ &\quad + \varphi\left((t\theta^p(\varepsilon_2) + (1-t)\theta^p(\varepsilon_1))^{1/p}\right) \leq \varphi(\theta(\varepsilon_1)) + \varphi(\theta(\varepsilon_2)). \end{aligned} \quad (21)$$

Multiplying both sides of (21) by $t^{\tau-1}E_{\sigma,\tau,\delta}^{\rho,r,k,c}(\omega t^\sigma; \bar{p})$ and then integrating over $[0, 1]$, we have

$$\begin{aligned} &\int_0^1 t^{\tau-1}E_{\sigma,\tau,\delta}^{\rho,r,k,c}(\omega t^\sigma; \bar{p})\varphi\left((t\theta^p(\varepsilon_1) + (1-t)\theta^p(\varepsilon_2))^{1/p}\right)dt \\ &\quad + \int_0^1 t^{\tau-1}E_{\sigma,\tau,\delta}^{\rho,r,k,c}(\omega t^\sigma; \bar{p})\varphi\left((t\theta^p(\varepsilon_2) + (1-t)\theta^p(\varepsilon_1))^{1/p}\right)dt \\ &\leq (\varphi(\theta(\varepsilon_1)) + \varphi(\theta(\varepsilon_2))) \int_0^1 t^{\tau-1}E_{\sigma,\tau,\delta}^{\rho,r,k,c}(\omega t^\sigma; \bar{p})dt. \end{aligned} \quad (22)$$

Setting $\theta(x) = t\theta^p(\varepsilon_1) + (1-t)\theta^p(\varepsilon_2)$ and $\theta(y) = t\theta^p(\varepsilon_2) + (1-t)\theta^p(\varepsilon_1)$ in (22), we have

$$\begin{aligned} & \left({}_{\theta} \Upsilon_{\sigma, \tau, \delta, \omega', \theta^{-1}(\theta^p(\varepsilon_1))_+}^{\rho, r, k, c} \varphi \circ \psi \right) (\theta^{-1}(\theta^p(\varepsilon_2)); \bar{p}) \\ & + \left({}_{\theta} \Upsilon_{\sigma, \tau, \delta, \omega', \theta^{-1}(\theta^p(\varepsilon_2))_-}^{\rho, r, k, c} \varphi \circ \psi \right) (\theta^{-1}(\theta^p(\varepsilon_1)); \bar{p}) \\ & \leq (\varphi(\theta(\varepsilon_1)) + \varphi(\theta(\varepsilon_2))) \\ & \cdot \left({}_{\theta} \Upsilon_{\sigma, \tau, \delta, \omega', \theta^{-1}(\theta^p(\varepsilon_1))_+}^{\rho, r, k, c} 1 \right) (\theta^{-1}(\theta^p(\varepsilon_2)); \bar{p}). \end{aligned} \tag{23}$$

By combining (20) and (23), we get (14).

(ii) Proof is similar to the proof of (i). □

Remark 2

- (i) By setting $\bar{p} = \omega = 0$ and $\theta = I$, Theorem 9 of [11] is obtained.
- (ii) By setting $\bar{p} = 0$, $p = -1$, and $\theta = I$, Theorem 2.1 of [22] is obtained.
- (iii) By setting $\theta = I$ and $p = -1$, Theorem 2.1 of [23] is obtained.
- (iv) By setting $\omega = \bar{p} = 0$, $p = -1$, and $\theta = I$, Theorem 4 of [18] is obtained.
- (v) By setting $p = -1$, Theorem 2.1 of [24] is obtained.
- (vi) By setting $p = -1$ and $\psi(x) = x$, Corollary 2.3 of [24] is obtained.

Corollary 1. *In (15), if we take $\omega = \bar{p} = 0$, $p = -1$, and $\theta = I$, then we get the following Hadamard inequality for the RL fractional integrals:*

$$\begin{aligned} \varphi\left(\frac{2\varepsilon_1\varepsilon_2}{\varepsilon_1 + \varepsilon_2}\right) & \leq \frac{\Gamma(\tau + 1)}{2} \left(\frac{\varepsilon_1\varepsilon_2}{\varepsilon_2 - \varepsilon_1}\right)^\tau \\ & \cdot \left(\left(I_{1|\varepsilon_1}^\tau - \varphi \circ \psi \right) \left(\frac{1}{\varepsilon_2} \right) + \left(I_{1|\varepsilon_1}^\tau + \varphi \circ \psi \right) \left(\frac{1}{\varepsilon_1} \right) \right) \\ & \leq \frac{\varphi(\varepsilon_1) + \varphi(\varepsilon_2)}{2}. \end{aligned} \tag{24}$$

Now we obtain Fejér–Hadamard type fractional integral inequalities for p -convex function via generalized fractional

integral operators; for this, first we prove the following lemma.

Lemma 1. *Let $\varphi, \theta: [\varepsilon_1, \varepsilon_2] \subset (0, \infty) \rightarrow \mathbb{R}$, Range $(\theta) \subset [\varepsilon_1, \varepsilon_2]$ be the functions such that φ be positive and $\varphi \in L_1[\varepsilon_1, \varepsilon_2]$, $\varepsilon_1 < \varepsilon_2$, and θ be a differentiable and strictly increasing function. If φ is p -convex, $p \in \mathbb{R} \setminus \{0\}$, and $\varphi(\theta^{1/p}(x)) = \varphi((\theta^p(\varepsilon_1) + \theta^p(\varepsilon_2) - \theta(x))^{1/p})$, then for generalized fractional integral operators (5) and (6), we have*

(i) If $p > 0$, then

$$\begin{aligned} & \left({}_{\theta} \Upsilon_{\sigma, \tau, \delta, \omega, \theta^{-1}(\theta^p(\varepsilon_1))_+}^{\rho, r, k, c} \varphi \circ \psi \right) (\theta^{-1}(\theta^p(\varepsilon_2)); \bar{p}) \\ & = \left({}_{\theta} \Upsilon_{\sigma, \tau, \delta, \omega, \theta^{-1}(\theta^p(\varepsilon_2))_-}^{\rho, r, k, c} \varphi \circ \psi \right) (\theta^{-1}(\theta^p(\varepsilon_1)); \bar{p}) \\ & = \frac{1}{2} \left(\left({}_{\theta} \Upsilon_{\sigma, \tau, \delta, \omega, \theta^{-1}(\theta^p(\varepsilon_1))_+}^{\rho, r, k, c} \varphi \circ \psi \right) (\theta^{-1}(\theta^p(\varepsilon_2)); \bar{p}) \right. \\ & \quad \left. + \left({}_{\theta} \Upsilon_{\sigma, \tau, \delta, \omega, \theta^{-1}(\theta^p(\varepsilon_2))_-}^{\rho, r, k, c} \varphi \circ \psi \right) (\theta^{-1}(\theta^p(\varepsilon_1)); \bar{p}) \right), \end{aligned} \tag{25}$$

with $\psi(t) = \theta^{1/p}(t)$, $t \in [\varepsilon_1^p, \varepsilon_2^p]$.

(ii) If $p < 0$, then

$$\begin{aligned} & \left({}_{\theta} \Upsilon_{\sigma, \tau, \delta, \omega, \theta^{-1}(\theta^p(\varepsilon_2))_+}^{\rho, r, k, c} \varphi \circ \psi \right) (\theta^{-1}(\theta^p(\varepsilon_1)); \bar{p}) \\ & = \left({}_{\theta} \Upsilon_{\sigma, \tau, \delta, \omega, \theta^{-1}(\theta^p(\varepsilon_1))_-}^{\rho, r, k, c} \varphi \circ \psi \right) (\theta^{-1}(\theta^p(\varepsilon_2)); \bar{p}) \\ & = \frac{1}{2} \left(\left({}_{\theta} \Upsilon_{\sigma, \tau, \delta, \omega, \theta^{-1}(\theta^p(\varepsilon_2))_+}^{\rho, r, k, c} \varphi \circ \psi \right) (\theta^{-1}(\theta^p(\varepsilon_1)); \bar{p}) \right. \\ & \quad \left. + \left({}_{\theta} \Upsilon_{\sigma, \tau, \delta, \omega, \theta^{-1}(\theta^p(\varepsilon_1))_-}^{\rho, r, k, c} \varphi \circ \psi \right) (\theta^{-1}(\theta^p(\varepsilon_2)); \bar{p}) \right), \end{aligned} \tag{26}$$

with $\psi(t) = \theta^{1/p}(t)$, $t \in [\varepsilon_2^p, \varepsilon_1^p]$.

Proof. (i) By definition of generalized fractional integral operators (5) and (6), we have

$$\begin{aligned} & \left({}_{\theta} \Upsilon_{\sigma, \tau, \delta, \omega, \theta^{-1}(\theta^p(\varepsilon_1))_+}^{\rho, r, k, c} \varphi \circ \psi \right) (\theta^{-1}(\theta^p(\varepsilon_2)); \bar{p}) \\ & = \int_{\theta^{-1}(\theta^p(\varepsilon_1))}^{\theta^{-1}(\theta^p(\varepsilon_2))} (\theta^p(\varepsilon_2) - \theta(x))^{\tau-1} E_{\sigma, \tau, \delta}^{\rho, r, k, c}(\omega(\theta^p(\varepsilon_2) - \theta(x))^\sigma; \bar{p}) \varphi \circ \psi(x) d(\theta(x)) \\ & = \int_{\theta^{-1}(\theta^p(\varepsilon_1))}^{\theta^{-1}(\theta^p(\varepsilon_2))} (\theta^p(\varepsilon_2) - \theta(x))^{\tau-1} E_{\sigma, \tau, \delta}^{\rho, r, k, c}(\omega(\theta^p(\varepsilon_2) - \theta(x))^\sigma; \bar{p}) \varphi(\theta(x)^{1/p}) d(\theta(x)) \\ & = \int_{\theta^{-1}(\theta^p(\varepsilon_1))}^{\theta^{-1}(\theta^p(\varepsilon_2))} (\theta^p(\varepsilon_2) - \theta(x))^{\tau-1} E_{\sigma, \tau, \delta}^{\rho, r, k, c}(\omega(\theta^p(\varepsilon_2) - \theta(x))^\sigma; \bar{p}) \varphi\left((\theta^p(\varepsilon_1) + \theta^p(\varepsilon_2) - \theta(x))^{1/p}\right) d(\theta(x)). \end{aligned} \tag{27}$$

Setting $\theta(t) = \theta^p(\varepsilon_1) + \theta^p(\varepsilon_2) - \theta(x)$ in the above equation and using $\varphi(\theta^{1/p}(x)) = \varphi((\theta^p(\varepsilon_1) + \theta^p(\varepsilon_2) - \theta(x))^{1/p})$, we have

$$\begin{aligned} & \left({}_{\theta} \Upsilon_{\sigma, \tau, \delta, \omega, \theta^{-1}(\theta^p(\varepsilon_1))_+}^{\rho, r, k, c} \varphi \circ \psi \right) (\theta^{-1}(\theta^p(\varepsilon_2)); \bar{p}) \\ &= \int_{\theta^{-1}(\theta^p(\varepsilon_1))}^{\theta^{-1}(\theta^p(\varepsilon_2))} (\theta(t) - \theta^p(\varepsilon_1))^{\tau-1} E_{\sigma, \tau, \delta}^{\rho, r, k, c} (\omega(\theta(t) - \theta^p(\varepsilon_1))^\sigma; \bar{p}) \varphi(\theta(t)^{1/p}) d(\theta(t)) \\ &= \int_{\theta^{-1}(\theta^p(\varepsilon_1))}^{\theta^{-1}(\theta^p(\varepsilon_2))} (\theta(t) - \theta^p(\varepsilon_1))^{\tau-1} E_{\sigma, \tau, \delta}^{\rho, r, k, c} (\omega(\theta(t) - \theta^p(\varepsilon_1))^\sigma; \bar{p}) \varphi \circ \psi(t) d(\theta(t)). \end{aligned} \quad (28)$$

This implies

$$\begin{aligned} & \left({}_{\theta} \Upsilon_{\sigma, \tau, \delta, \omega, \theta^{-1}(\theta^p(\varepsilon_1))_+}^{\rho, r, k, c} \varphi \circ \psi \right) (\theta^{-1}(\theta^p(\varepsilon_2)); \bar{p}) \\ &= \left({}_{\theta} \Upsilon_{\sigma, \tau, \delta, \omega, \theta^{-1}(\theta^p(\varepsilon_2))_-}^{\rho, r, k, c} \varphi \circ \psi \right) (\theta^{-1}(\theta^p(\varepsilon_1)); \bar{p}). \end{aligned} \quad (29)$$

By adding $({}_{\theta} \Upsilon_{\sigma, \tau, \delta, \omega, \theta^{-1}(\theta^p(\varepsilon_1))_+}^{\rho, r, k, c} \varphi \circ \psi)(\theta^{-1}(\theta^p(\varepsilon_2)); \bar{p})$ on both sides of (29), we have

$$\begin{aligned} & 2 \left({}_{\theta} \Upsilon_{\sigma, \tau, \delta, \omega, \theta^{-1}(\theta^p(\varepsilon_1))_+}^{\rho, r, k, c} \varphi \circ \psi \right) (\theta^{-1}(\theta^p(\varepsilon_2)); \bar{p}) \\ &= \left({}_{\theta} \Upsilon_{\sigma, \tau, \delta, \omega, \theta^{-1}(\theta^p(\varepsilon_1))_+}^{\rho, r, k, c} \varphi \circ \psi \right) (\theta^{-1}(\theta^p(\varepsilon_2)); \bar{p}) \\ &+ \left({}_{\theta} \Upsilon_{\sigma, \tau, \delta, \omega, \theta^{-1}(\theta^p(\varepsilon_2))_-}^{\rho, r, k, c} \varphi \circ \psi \right) (\theta^{-1}(\theta^p(\varepsilon_1)); \bar{p}). \end{aligned} \quad (30)$$

From equations (29) and (30), the required result can be obtained.

(ii) Proof is on the same lines as the proof of (i). \square

Theorem 4. Let $\varphi, \theta, h: [\varepsilon_1, \varepsilon_2] \subset (0, \infty) \rightarrow \mathbb{R}$, Range (θ) , Range $(h) \subset [\varepsilon_1, \varepsilon_2]$ be the functions such that φ be positive and $\varphi \in L_1(\varepsilon_1, \varepsilon_2)$, $\varepsilon_1 < \varepsilon_2$, and θ be a differentiable and strictly increasing function where h is a non-negative and integrable function. If φ is p -convex, $p \in \mathbb{R} \setminus \{0\}$, and $\varphi(\theta^{1/p}(x)) = \varphi((\theta^p(\varepsilon_1) + \theta^p(\varepsilon_2) - \theta(x))^{1/p})$, then the following inequalities for generalized fractional integral operators (5) and (6) hold:

(i) If $p > 0$, then

$$\begin{aligned} & \varphi \left(\left(\frac{\theta^p(\varepsilon_1) + \theta^p(\varepsilon_2)}{2} \right)^{1/p} \right) \left(\left({}_{\theta} \Upsilon_{\sigma, \tau, \delta, \omega', \theta^{-1}(\theta^p(\varepsilon_2))_+}^{\rho, r, k, c} h \circ \psi \right) (\theta^{-1}(\theta^p(\varepsilon_1)); \bar{p}) \right. \\ &+ \left. \left({}_{\theta} \Upsilon_{\sigma, \tau, \delta, \omega', \theta^{-1}(\theta^p(\varepsilon_1))_-}^{\rho, r, k, c} h \circ \psi \right) (\theta^{-1}(\theta^p(\varepsilon_2)); \bar{p}) \right) \\ &\leq \left({}_{\theta} \Upsilon_{\sigma, \tau, \delta, \omega', \theta^{-1}(\theta^p(\varepsilon_2))_+}^{\rho, r, k, c} \varphi h \circ \psi \right) (\theta^{-1}(\theta^p(\varepsilon_1)); \bar{p}) \\ &+ \left({}_{\theta} \Upsilon_{\sigma, \tau, \delta, \omega', \theta^{-1}(\theta^p(\varepsilon_1))_-}^{\rho, r, k, c} \varphi h \circ \psi \right) (\theta^{-1}(\theta^p(\varepsilon_2)); \bar{p}) \\ &\leq \frac{\varphi(\theta(\varepsilon_1)) + \varphi(\theta(\varepsilon_2))}{2} \left(\left({}_{\theta} \Upsilon_{\sigma, \tau, \delta, \omega', \theta^{-1}(\theta^p(\varepsilon_2))_+}^{\rho, r, k, c} h \circ \psi \right) (\theta^{-1}(\theta^p(\varepsilon_1)); \bar{p}) \right. \\ &\left. + \left({}_{\theta} \Upsilon_{\sigma, \tau, \delta, \omega', \theta^{-1}(\theta^p(\varepsilon_1))_-}^{\rho, r, k, c} h \circ \psi \right) (\theta^{-1}(\theta^p(\varepsilon_2)); \bar{p}) \right), \end{aligned} \quad (32)$$

$$\begin{aligned} & \varphi \left(\left(\frac{\theta^p(\varepsilon_1) + \theta^p(\varepsilon_2)}{2} \right)^{1/p} \right) \\ &\cdot \left(\left({}_{\theta} \Upsilon_{\sigma, \tau, \delta, \omega', \theta^{-1}(\theta^p(\varepsilon_1))_+}^{\rho, r, k, c} h \circ \psi \right) (\theta^{-1}(\theta^p(\varepsilon_2)); \bar{p}) \right) \\ &+ \left(\left({}_{\theta} \Upsilon_{\sigma, \tau, \delta, \omega', \theta^{-1}(\theta^p(\varepsilon_2))_-}^{\rho, r, k, c} h \circ \psi \right) (\theta^{-1}(\theta^p(\varepsilon_1)); \bar{p}) \right) \\ &\leq \left({}_{\theta} \Upsilon_{\sigma, \tau, \delta, \omega', \theta^{-1}(\theta^p(\varepsilon_1))_+}^{\rho, r, k, c} \varphi h \circ \psi \right) (\theta^{-1}(\theta^p(\varepsilon_2)); \bar{p}) \\ &+ \left({}_{\theta} \Upsilon_{\sigma, \tau, \delta, \omega', \theta^{-1}(\theta^p(\varepsilon_2))_-}^{\rho, r, k, c} \varphi h \circ \psi \right) (\theta^{-1}(\theta^p(\varepsilon_1)); \bar{p}) \\ &\leq \frac{\varphi(\theta(\varepsilon_1)) + \varphi(\theta(\varepsilon_2))}{2} \\ &\cdot \left(\left({}_{\theta} \Upsilon_{\sigma, \tau, \delta, \omega', \theta^{-1}(\theta^p(\varepsilon_1))_+}^{\rho, r, k, c} h \circ \psi \right) (\theta^{-1}(\theta^p(\varepsilon_2)); \bar{p}) \right) \\ &+ \left(\left({}_{\theta} \Upsilon_{\sigma, \tau, \delta, \omega', \theta^{-1}(\theta^p(\varepsilon_2))_-}^{\rho, r, k, c} h \circ \psi \right) (\theta^{-1}(\theta^p(\varepsilon_1)); \bar{p}) \right), \end{aligned} \quad (31)$$

where $\omega' = \omega / (\theta^p(\varepsilon_2) - \theta^p(\varepsilon_1))^\sigma$ and $\psi(t) = \theta^{1/p}(t)$ for all $t \in [\varepsilon_1^p, \varepsilon_2^p]$.

(ii) If $p < 0$, then

where $\omega' = \omega / (\theta^p(\varepsilon_1) - \theta^p(\varepsilon_2))^\sigma$ and $\psi(t) = \theta^{1/p}(t)$ for all $t \in [\varepsilon_2^p, \varepsilon_1^p]$.

Proof. (i) Multiplying both sides of (17) by $2t^{\tau-1}E_{\sigma,\tau,\delta}^{\rho,r,k,c}(\omega t^\sigma; \bar{p})h((t\theta^p(\varepsilon_1) + (1-t)\theta^p(\varepsilon_2))^{1/p})$ and then integrating over $[0, 1]$, we have

$$\begin{aligned} & 2\varphi\left(\left(\frac{\theta^p(\varepsilon_1) + \theta^p(\varepsilon_2)}{2}\right)^{1/p}\right) \int_0^1 t^{\tau-1}E_{\sigma,\tau,\delta}^{\rho,r,k,c}(\omega t^\sigma; \bar{p})h\left((t\theta^p(\varepsilon_1) + (1-t)\theta^p(\varepsilon_2))^{1/p}\right)dt \\ & \leq \int_0^1 t^{\tau-1}E_{\sigma,\tau,\delta}^{\rho,r,k,c}(\omega t^\sigma; \bar{p})\varphi\left((t\theta^p(\varepsilon_1) + (1-t)\theta^p(\varepsilon_2))^{1/p}\right)h\left((t\theta^p(\varepsilon_1) + (1-t)\theta^p(\varepsilon_2))^{1/p}\right)dt \\ & \quad + \int_0^1 t^{\tau-1}E_{\sigma,\tau,\delta}^{\rho,r,k,c}(\omega t^\sigma; \bar{p})\varphi\left((t\theta^p(\varepsilon_2) + (1-t)\theta^p(\varepsilon_1))^{1/p}\right)h\left((t\theta^p(\varepsilon_1) + (1-t)\theta^p(\varepsilon_2))^{1/p}\right)dt. \end{aligned} \tag{33}$$

By choosing $\theta(x) = t\theta^p(\varepsilon_1) + (1-t)\theta^p(\varepsilon_2)$, that is, $\theta^p(\varepsilon_1) + \theta^p(\varepsilon_2) - \theta(x) = t\theta^p(\varepsilon_2) + (1-t)\theta^p(\varepsilon_1)$, in (33)

and using $\varphi(\theta^{1/p}(x)) = \varphi((\theta^p(\varepsilon_1) + \theta^p(\varepsilon_2) - \theta(x))^{1/p})$, we have

$$\begin{aligned} & 2\varphi\left(\left(\frac{\theta^p(\varepsilon_1) + \theta^p(\varepsilon_2)}{2}\right)^{1/p}\right) \\ & \quad \cdot \int_{\theta^{-1}(\theta^p(\varepsilon_1))}^{\theta^{-1}(\theta^p(\varepsilon_2))} (\theta^p(\varepsilon_2) - \theta(x))^{\tau-1}E_{\sigma,\tau,\delta}^{\rho,r,k,c}(\omega'(\theta^p(\varepsilon_2) - \theta(x))^\sigma; \bar{p})h \circ \psi(x)d(\theta(x)) \\ & \leq \int_{\theta^{-1}(\theta^p(\varepsilon_1))}^{\theta^{-1}(\theta^p(\varepsilon_2))} (\theta^p(\varepsilon_2) - \theta(x))^{\tau-1}E_{\sigma,\tau,\delta}^{\rho,r,k,c}(\omega'(\theta^p(\varepsilon_2) - \theta(x))^\sigma; \bar{p})\varphi h \circ \psi(x)d(\theta(x)) \\ & \quad + \int_{\theta^{-1}(\theta^p(\varepsilon_1))}^{\theta^{-1}(\theta^p(\varepsilon_2))} (\theta(x) - \theta^p(\varepsilon_1))^{\tau-1}E_{\sigma,\tau,\delta}^{\rho,r,k,c}(\omega'(\theta(x) - \theta^p(\varepsilon_1))^\sigma; \bar{p})\varphi h \circ \psi(x)d(\theta(x)). \end{aligned} \tag{34}$$

This implies

$$\begin{aligned} & 2\varphi\left(\left(\frac{\theta^p(\varepsilon_1) + \theta^p(\varepsilon_2)}{2}\right)^{1/p}\right)\left({}_{\theta^{-1}(\theta^p(\varepsilon_1))_+}Y_{\sigma,\tau,\delta,\omega',\theta^{-1}(\theta^p(\varepsilon_1))_+}^{\rho,r,k,c}h \circ \psi\right)(\theta^{-1}(\theta^p(\varepsilon_2)); \bar{p}) \\ & \leq \left({}_{\theta^{-1}(\theta^p(\varepsilon_1))_+}Y_{\sigma,\tau,\delta,\omega',\theta^{-1}(\theta^p(\varepsilon_1))_+}^{\rho,r,k,c}\varphi h \circ \psi\right)(\theta^{-1}(\theta^p(\varepsilon_2)); \bar{p}) \\ & \quad + \left({}_{\theta^{-1}(\theta^p(\varepsilon_2))_-}Y_{\sigma,\tau,\delta,\omega',\theta^{-1}(\theta^p(\varepsilon_2))_-}^{\rho,r,k,c}\varphi h \circ \psi\right)(\theta^{-1}(\theta^p(\varepsilon_1)); \bar{p}). \end{aligned} \tag{35}$$

Using Lemma 1 (i) in above inequality, we have

$$\begin{aligned} & \varphi\left(\left(\frac{\theta^p(\varepsilon_1) + \theta^p(\varepsilon_2)}{2}\right)^{1/p}\right)\left(\left({}_{\theta}Y_{\sigma,\tau,\delta,\omega',\theta^{-1}(\theta^p(\varepsilon_1))_+}^{\rho,r,k,c} h \circ \psi\right)(\theta^{-1}(\theta^p(\varepsilon_2)); \bar{p})\right) \\ & + \left({}_{\theta}Y_{\sigma,\tau,\delta,\omega',\theta^{-1}(\theta^p(\varepsilon_2))_-}^{\rho,r,k,c} h \circ \psi\right)(\theta^{-1}(\theta^p(\varepsilon_1)); \bar{p}) \\ & \leq \left({}_{\theta}Y_{\sigma,\tau,\delta,\omega',\theta^{-1}(\theta^p(\varepsilon_1))_+}^{\rho,r,k,c} \varphi h \circ \psi\right)(\theta^{-1}(\theta^p(\varepsilon_2)); \bar{p}) \\ & + \left({}_{\theta}Y_{\sigma,\tau,\delta,\omega',\theta^{-1}(\theta^p(\varepsilon_2))_-}^{\rho,r,k,c} \varphi h \circ \psi\right)(\theta^{-1}(\theta^p(\varepsilon_1)); \bar{p}). \end{aligned} \tag{36}$$

To prove the second inequality of (31), multiplying both sides of (21) by $t^{\tau-1}E_{\sigma,\tau,\delta}^{\rho,r,k,c}(\omega t^\sigma; \bar{p})h((t\theta^p(\varepsilon_1) + (1-t)\theta^p(\varepsilon_2))^{1/p})$ and then integrating over $[0, 1]$, we have

$$\begin{aligned} & \int_0^1 t^{\tau-1} E_{\sigma,\tau,\delta}^{\rho,r,k,c}(\omega t^\sigma; \bar{p}) \varphi\left((t\theta^p(\varepsilon_1) + (1-t)\theta^p(\varepsilon_2))^{1/p}\right) h\left((t\theta^p(\varepsilon_1) + (1-t)\theta^p(\varepsilon_2))^{1/p}\right) dt \\ & + \int_0^1 t^{\tau-1} E_{\sigma,\tau,\delta}^{\rho,r,k,c}(\omega t^\sigma; \bar{p}) \varphi\left((t\theta^p(\varepsilon_2) + (1-t)\theta^p(\varepsilon_1))^{1/p}\right) h\left((t\theta^p(\varepsilon_1) + (1-t)\theta^p(\varepsilon_2))^{1/p}\right) dt \\ & \leq (\varphi(\theta(\varepsilon_1)) + \varphi(\theta(\varepsilon_2))) \int_0^1 t^{\tau-1} E_{\sigma,\tau,\delta}^{\rho,r,k,c}(\omega t^\sigma; \bar{p}) h\left((t\theta^p(\varepsilon_1) + (1-t)\theta^p(\varepsilon_2))^{1/p}\right) dt. \end{aligned} \tag{37}$$

Setting $\theta(x) = t\theta^p(\varepsilon_1) + (1-t)\theta^p(\varepsilon_2)$ and using $\varphi(\theta^{1/p}(x)) = \varphi((\theta^p(\varepsilon_1) + \theta^p(\varepsilon_2) - \theta(x))^{1/p})$ in (37) and after simplification, we have

$$\begin{aligned} & \left({}_{\theta}Y_{\sigma,\tau,\delta,\omega',\theta^{-1}(\theta^p(\varepsilon_1))_+}^{\rho,r,k,c} \varphi h \circ \psi\right)(\theta^{-1}(\theta^p(\varepsilon_2)); \bar{p}) \\ & + \left({}_{\theta}Y_{\sigma,\tau,\delta,\omega',\theta^{-1}(\theta^p(\varepsilon_2))_-}^{\rho,r,k,c} \varphi h \circ \psi\right)(\theta^{-1}(\theta^p(\varepsilon_1)); \bar{p}) \\ & \leq (\varphi(\theta(\varepsilon_1)) + \varphi(\theta(\varepsilon_2))) \left({}_{\theta}Y_{\sigma,\tau,\delta,\omega',\theta^{-1}(\theta^p(\varepsilon_1))_+}^{\rho,r,k,c} h \circ \psi\right)(\theta^{-1}(\theta^p(\varepsilon_2)); \bar{p}). \end{aligned} \tag{38}$$

Using Lemma 1 (i), inequality (38) becomes

$$\begin{aligned} & \left({}_{\theta}Y_{\sigma,\tau,\delta,\omega',\theta^{-1}(\theta^p(\varepsilon_1))_+}^{\rho,r,k,c} \varphi h \circ \psi\right)(\theta^{-1}(\theta^p(\varepsilon_2)); \bar{p}) \\ & + \left({}_{\theta}Y_{\sigma,\tau,\delta,\omega',\theta^{-1}(\theta^p(\varepsilon_2))_-}^{\rho,r,k,c} \varphi h \circ \psi\right)(\theta^{-1}(\theta^p(\varepsilon_1)); \bar{p}) \\ & \leq \frac{(\varphi(\theta(\varepsilon_1)) + \varphi(\theta(\varepsilon_2)))}{2} \\ & \cdot \left(\left({}_{\theta}Y_{\sigma,\tau,\delta,\omega',\theta^{-1}(\theta^p(\varepsilon_1))_+}^{\rho,r,k,c} h \circ \psi\right)(\theta^{-1}(\theta^p(\varepsilon_2)); \bar{p})\right) \\ & + \left({}_{\theta}Y_{\sigma,\tau,\delta,\omega',\theta^{-1}(\theta^p(\varepsilon_2))_-}^{\rho,r,k,c} h \circ \psi\right)(\theta^{-1}(\theta^p(\varepsilon_1)); \bar{p}). \end{aligned} \tag{39}$$

By combining (36) and (39), we get (31).

(ii) Proof is similar to the proof of (i) by using (ii) of Lemma 1. \square

Remark 3

- (i) By setting $\bar{p} = 0$ and $\theta = I$, Theorem 2.2 of [25] is obtained.
- (ii) By setting $\bar{p} = \omega = 0$ and $\theta = I$, Theorem 9 of [11] is obtained.
- (iii) By setting $\bar{p} = 0$, $h(x) = 1$, $p = -1$, and $\theta = I$, Theorem 2.1 of [22] is obtained.
- (iv) By setting $\theta = I$, $h(x) = 1$, and $p = -1$, Theorem 2.1 of [23] is obtained.
- (v) By setting $\omega = \bar{p} = 0$, $h(x) = 1$, $p = -1$, and $\theta = I$, Theorem 4 of [18] is obtained.
- (vi) By setting $p = -1$, Theorem 2.5 of [24] is obtained.

Corollary 2. *If we put $p = -1$, $\bar{p} = 0$, and $\theta = I$ in Theorem 4 (ii), we get the following Fejér–Hadamard inequalities for harmonically convex function via generalized fractional integral operators:*

$$\begin{aligned} & \varphi\left(\frac{2\varepsilon_1\varepsilon_2}{\varepsilon_1+\varepsilon_2}\right)\left(\left(\theta Y_{\sigma,\tau,\delta,\omega',(1/\varepsilon_2)}^{\rho,r,k,c}h\circ\psi\right)\left(\frac{1}{\varepsilon_1}\right)+\left(\theta Y_{\sigma,\tau,\delta,\omega',(1/\varepsilon_1)}^{\rho,r,k,c}h\circ\psi\right)\left(\frac{1}{\varepsilon_2}\right)\right) \\ & \leq\left(\theta Y_{\sigma,\tau,\delta,\omega',(1/\varepsilon_2)}^{\rho,r,k,c}\varphi h\circ\psi\right)\left(\frac{1}{\varepsilon_1}\right)+\left(\theta Y_{\sigma,\tau,\delta,\omega',(1/\varepsilon_1)}^{\rho,r,k,c}\varphi h\circ\psi\right)\left(\frac{1}{\varepsilon_2}\right) \\ & \leq\frac{\varphi(\varepsilon_1)+\varphi(\varepsilon_2)}{2}\left(\left(\theta Y_{\sigma,\tau,\delta,\omega',(1/\varepsilon_2)}^{\rho,r,k,c}h\circ\psi\right)\left(\frac{1}{\varepsilon_1}\right)+\left(\theta Y_{\sigma,\tau,\delta,\omega',(1/\varepsilon_1)}^{\rho,r,k,c}h\circ\psi\right)\left(\frac{1}{\varepsilon_2}\right)\right). \end{aligned} \tag{40}$$

Now we give another version of the Hadamard inequality.

Theorem 5. Let $\varphi, \theta: [\varepsilon_1, \varepsilon_2] \subset (0, \infty) \rightarrow \mathbb{R}$, Range $(\theta) \subset [\varepsilon_1, \varepsilon_2]$ be the functions such that φ be positive and

$\varphi \in L_1[\varepsilon_1, \varepsilon_2]$, $\varepsilon_1 < \varepsilon_2$, and θ be a differentiable and strictly increasing function. If φ is p -convex, $p \in \mathbb{R} \setminus \{0\}$, then for generalized fractional integral operators (5) and (6), we have

(i) If $p > 0$, then

$$\begin{aligned} & \varphi\left(\left(\frac{\theta^p(\varepsilon_1)+\theta^p(\varepsilon_2)}{2}\right)^{1/p}\right)\left(\theta Y_{\sigma,\tau,\delta,2^\sigma\omega',\theta^{-1}((\theta^p(\varepsilon_1)+\theta^p(\varepsilon_2))/2)_+}^{\rho,r,k,c}1\right)\left(\theta^{-1}(\theta^p(\varepsilon_2));\bar{p}\right) \\ & \leq\left(\theta Y_{\sigma,\tau,\delta,2^\sigma\omega',\theta^{-1}((\theta^p(\varepsilon_1)+\theta^p(\varepsilon_2))/2)_+}^{\rho,r,k,c}\varphi\circ\psi\right)\left(\theta^{-1}(\theta^p(\varepsilon_2));\bar{p}\right) \\ & \quad +\left(\theta Y_{\sigma,\tau,\delta,2^\sigma\omega',\theta^{-1}((\theta^p(\varepsilon_1)+\theta^p(\varepsilon_2))/2)_-}^{\rho,r,k,c}\varphi\circ\psi\right)\left(\theta^{-1}(\theta^p(\varepsilon_1));\bar{p}\right) \\ & \leq\frac{\varphi(\theta(\varepsilon_1))+\varphi(\theta(\varepsilon_2))}{2}\left(\theta Y_{\sigma,\tau,\delta,2^\sigma\omega',\theta^{-1}((\theta^p(\varepsilon_1)+\theta^p(\varepsilon_2))/2)_+}^{\rho,r,k,c}1\right)\left(\theta^{-1}(\theta^p(\varepsilon_2));\bar{p}\right), \end{aligned} \tag{41}$$

where $\omega' = \omega/(\theta^p(\varepsilon_2) - \theta^p(\varepsilon_1))^\sigma$ and $\psi(t) = \theta^{1/p}(t)$ for all $t \in [\varepsilon_1^p, \varepsilon_2^p]$.

(ii) If $p < 0$, then

$$\begin{aligned} & \varphi\left(\left(\frac{\theta^p(\varepsilon_1)+\theta^p(\varepsilon_2)}{2}\right)^{1/p}\right)\left(\theta Y_{\sigma,\tau,\delta,2^\sigma\omega',\theta^{-1}((\theta^p(\varepsilon_1)+\theta^p(\varepsilon_2))/2)_-}^{\rho,r,k,c}1\right)\left(\theta^{-1}(\theta^p(\varepsilon_2));\bar{p}\right) \\ & \leq\left(\theta Y_{\sigma,\tau,\delta,2^\sigma\omega',\theta^{-1}((\theta^p(\varepsilon_1)+\theta^p(\varepsilon_2))/2)_+}^{\rho,r,k,c}\varphi\circ\psi\right)\left(\theta^{-1}(\theta^p(\varepsilon_1));\bar{p}\right) \\ & \quad +\left(\theta Y_{\sigma,\tau,\delta,2^\sigma\omega',\theta^{-1}((\theta^p(\varepsilon_1)+\theta^p(\varepsilon_2))/2)_-}^{\rho,r,k,c}\varphi\circ\psi\right)\left(\theta^{-1}(\theta^p(\varepsilon_2));\bar{p}\right) \\ & \leq\frac{\varphi(\theta(\varepsilon_1))+\varphi(\theta(\varepsilon_2))}{2}\left(\theta Y_{\sigma,\tau,\delta,2^\sigma\omega',\theta^{-1}((\theta^p(\varepsilon_1)+\theta^p(\varepsilon_2))/2)_-}^{\rho,r,k,c}1\right)\left(\theta^{-1}(\theta^p(\varepsilon_2));\bar{p}\right), \end{aligned} \tag{42}$$

where $\omega' = \omega/(\theta^p(\varepsilon_1) - \theta^p(\varepsilon_2))^\sigma$ and $\psi(t) = \theta^{1/p}(t)$ for all $t \in [\varepsilon_2^p, \varepsilon_1^p]$.

Proof. (i) Setting $\theta(x) = ((t/2)\theta^p(\varepsilon_1) + ((2-t)/2)\theta^p(\varepsilon_2))^{1/p}$ and $\theta(y) = ((t/2)\theta^p(\varepsilon_2) + ((2-t)/2)\theta^p(\varepsilon_1))^{1/p}$ in (12), we have

$$\begin{aligned} & \varphi\left(\left(\frac{\theta^p(\varepsilon_1) + \theta^p(\varepsilon_2)}{2}\right)^{1/p}\right) \\ & \leq \frac{\varphi\left(\left((t/2)\theta^p(\varepsilon_1) + ((2-t)/2)\theta^p(\varepsilon_2)\right)^{1/p}\right) + \varphi\left(\left((t/2)\theta^p(\varepsilon_2) + ((2-t)/2)\theta^p(\varepsilon_1)\right)^{1/p}\right)}{2}. \end{aligned} \tag{43}$$

Multiplying both sides of (43) by $2t^{\tau-1}E_{\sigma,\tau,\delta}^{\rho,r,k,c}(\omega t^\sigma; \bar{p})$ and then integrating over $[0, 1]$, we have

$$\begin{aligned} & 2\varphi\left(\left(\frac{\theta^p(\varepsilon_1) + \theta^p(\varepsilon_2)}{2}\right)^{1/p}\right) \int_0^1 t^{\tau-1}E_{\sigma,\tau,\delta}^{\rho,r,k,c}(\omega t^\sigma; \bar{p})dt \\ & \leq \int_0^1 t^{\tau-1}E_{\sigma,\tau,\delta}^{\rho,r,k,c}(\omega t^\sigma; \bar{p})\varphi\left(\left(\frac{t}{2}\theta^p(\varepsilon_1) + ((2-t)/2)\theta^p(\varepsilon_2)\right)^{1/p}\right)dt \\ & \quad + \int_0^1 t^{\tau-1}E_{\sigma,\tau,\delta}^{\rho,r,k,c}(\omega t^\sigma; \bar{p})\varphi\left(\left(\frac{t}{2}\theta^p(\varepsilon_2) + ((2-t)/2)\theta^p(\varepsilon_1)\right)^{1/p}\right)dt. \end{aligned} \tag{44}$$

By choosing $\theta(x) = (t/2)\theta^p(\varepsilon_1) + ((2-t)/2)\theta^p(\varepsilon_2)$ and $\theta(y) = (t/2)\theta^p(\varepsilon_2) + ((2-t)/2)\theta^p(\varepsilon_1)$ in (44) and by (5) and (6), we get first inequality of (41).

To prove the second inequality of (41), again from p -convexity of φ over $[\varepsilon_1, \varepsilon_2]$ and for $t \in [0, 1]$, we have

$$\begin{aligned} & \varphi\left(\left(\frac{t}{2}\theta^p(\varepsilon_1) + \left(\frac{2-t}{2}\right)\theta^p(\varepsilon_2)\right)^{1/p}\right) \\ & \quad + \varphi\left(\left(\frac{t}{2}\theta^p(\varepsilon_2) + \left(\frac{2-t}{2}\right)\theta^p(\varepsilon_1)\right)^{1/p}\right) \\ & \leq \varphi(\theta(\varepsilon_1)) + \varphi(\theta(\varepsilon_2)). \end{aligned} \tag{45}$$

Multiplying both sides of (45) by $t^{\tau-1}E_{\sigma,\tau,\delta}^{\rho,r,k,c}(\omega t^\sigma; \bar{p})$ and then integrating over $[0, 1]$, we have

$$\begin{aligned} & \int_0^1 t^{\tau-1}E_{\sigma,\tau,\delta}^{\rho,r,k,c}(\omega t^\sigma; \bar{p})\varphi\left(\left(\frac{t}{2}\theta^p(\varepsilon_1) + \left(\frac{2-t}{2}\right)\theta^p(\varepsilon_2)\right)^{1/p}\right)dt \\ & \quad + \int_0^1 t^{\tau-1}E_{\sigma,\tau,\delta}^{\rho,r,k,c}(\omega t^\sigma; \bar{p})\varphi\left(\left(\frac{t}{2}\theta^p(\varepsilon_2) + \left(\frac{2-t}{2}\right)\theta^p(\varepsilon_1)\right)^{1/p}\right)dt \\ & \leq (\varphi(\theta(\varepsilon_1)) + \varphi(\theta(\varepsilon_2))) \int_0^1 t^{\tau-1}E_{\sigma,\tau,\delta}^{\rho,r,k,c}(\omega t^\sigma; \bar{p})dt. \end{aligned} \tag{46}$$

Setting $\theta(x) = (t/2)\theta^p(\varepsilon_1) + ((2-t)/2)\theta^p(\varepsilon_2)$ and $\theta(y) = (t/2)\theta^p(\varepsilon_2) + ((2-t)/2)\theta^p(\varepsilon_1)$ in (46) and using (5) and (6), we get second inequality of (41).

(ii) Proof is similar to the proof of (i). □

Remark 4

- (i) By setting $\bar{p} = 0$, $p = -1$, and $\theta = I$, Theorem 2.3 of [22] is obtained.
- (ii) By setting $\theta = I$ and $p = -1$, Theorem 2.3 of [23] is obtained.
- (iii) By setting $p = -1$, Theorem 2.7 of [24] is obtained.

Now we obtain another Fejér–Hadamard type fractional integral inequality for p -convex function via generalized fractional integral operators (5) and (6).

Theorem 6. Let $\varphi, \theta, h: [\varepsilon_1, \varepsilon_2] \subset (0, \infty) \rightarrow \mathbb{R}$, Range (θ) , Range $(h) \subset [\varepsilon_1, \varepsilon_2]$ be the functions such that φ be positive and $\varphi \in L_1(\varepsilon_1, \varepsilon_2)$, $\varepsilon_1 < \varepsilon_2$, and θ be a differentiable and strictly increasing function where h is a non-negative and integrable function. If φ is p -convex, $p \in \mathbb{R} \setminus \{0\}$, and $\varphi(\theta^{1/p}(x)) = \varphi((\theta^p(\varepsilon_1) + \theta^p(\varepsilon_2) - \theta(x))^{1/p})$, then the following inequalities for generalized fractional integral operators (5) and (6) hold:

- (i) If $p > 0$, then

$$\begin{aligned}
 & \varphi\left(\left(\frac{\theta^p(\varepsilon_1) + \theta^p(\varepsilon_2)}{2}\right)^{1/p}\right) \left(\theta \Upsilon_{\sigma,\tau,\delta,2^\sigma\omega',\theta^{-1}((\theta^p(\varepsilon_1)+\theta^p(\varepsilon_2))/2)_+}^{\rho,r,k,c} h \circ \psi\right) (\theta^{-1}(\theta^p(\varepsilon_2)); \bar{p}) \\
 & \leq \left(\theta \Upsilon_{\sigma,\tau,\delta,2^\sigma\omega',\theta^{-1}((\theta^p(\varepsilon_1)+\theta^p(\varepsilon_2))/2)_+}^{\rho,r,k,c} \varphi h \circ \psi\right) (\theta^{-1}(\theta^p(\varepsilon_2)); \bar{p}) \\
 & \quad + \left(\theta \Upsilon_{\sigma,\tau,\delta,2^\sigma\omega',\theta^{-1}((\theta^p(\varepsilon_1)+\theta^p(\varepsilon_2))/2)_-}^{\rho,r,k,c} \varphi h \circ \psi\right) (\theta^{-1}(\theta^p(\varepsilon_1)); \bar{p}) \\
 & \leq \frac{\varphi(\theta(\varepsilon_1)) + \varphi(\theta(\varepsilon_2))}{2} \left(\theta \Upsilon_{\sigma,\tau,\delta,2^\sigma\omega',\theta^{-1}((\theta^p(\varepsilon_1)+\theta^p(\varepsilon_2))/2)_+}^{\rho,r,k,c} h \circ \psi\right) (\theta^{-1}(\theta^p(\varepsilon_2)); \bar{p}),
 \end{aligned} \tag{47}$$

where $\omega' = \omega/(\theta^p(\varepsilon_2) - \theta^p(\varepsilon_1))^\sigma$ and $\psi(t) = \theta^{1/p}(t)$ for all $t \in [\varepsilon_1^p, \varepsilon_2^p]$. (ii) If $p < 0$, then

$$\begin{aligned}
 & \varphi\left(\left(\frac{\theta^p(\varepsilon_1) + \theta^p(\varepsilon_2)}{2}\right)^{1/p}\right) \left(\theta \Upsilon_{\sigma,\tau,\delta,2^\sigma\omega',\theta^{-1}((\theta^p(\varepsilon_1)+\theta^p(\varepsilon_2))/2)_-}^{\rho,r,k,c} h \circ \psi\right) (\theta^{-1}(\theta^p(\varepsilon_2)); \bar{p}) \\
 & \leq \left(\theta \Upsilon_{\sigma,\tau,\delta,2^\sigma\omega',\theta^{-1}((\theta^p(\varepsilon_1)+\theta^p(\varepsilon_2))/2)_+}^{\rho,r,k,c} \varphi h \circ \psi\right) (\theta^{-1}(\theta^p(\varepsilon_1)); \bar{p}) \\
 & \quad + \left(\theta \Upsilon_{\sigma,\tau,\delta,2^\sigma\omega',\theta^{-1}((\theta^p(\varepsilon_1)+\theta^p(\varepsilon_2))/2)_-}^{\rho,r,k,c} \varphi h \circ \psi\right) (\theta^{-1}(\theta^p(\varepsilon_2)); \bar{p}) \\
 & \leq \frac{\varphi(\theta(\varepsilon_1)) + \varphi(\theta(\varepsilon_2))}{2} \left(\theta \Upsilon_{\sigma,\tau,\delta,2^\sigma\omega',\theta^{-1}((\theta^p(\varepsilon_1)+\theta^p(\varepsilon_2))/2)_-}^{\rho,r,k,c} h \circ \psi\right) (\theta^{-1}(\theta^p(\varepsilon_2)); \bar{p}),
 \end{aligned} \tag{48}$$

where $\omega' = \omega/(\theta^p(\varepsilon_1) - \theta^p(\varepsilon_2))^\sigma$ and $\psi(t) = \theta^{1/p}(t)$ for all $t \in [\varepsilon_2^p, \varepsilon_1^p]$.

Multiplying (43) by $2t^{\tau-1}E_{\sigma,\tau,\delta}^{\rho,r,k,c}(\omega t^\sigma; \bar{p})h(((t/2)\theta^p(\varepsilon_1) + ((2-t)/2)\theta^p(\varepsilon_2))^{1/p})$ and then integrating over $[0, 1]$, we have

Proof. (i)

$$\begin{aligned}
 & 2\varphi\left(\left(\frac{\theta^p(\varepsilon_1) + \theta^p(\varepsilon_2)}{2}\right)^{1/p}\right) \\
 & \cdot \int_0^1 t^{\tau-1} E_{\sigma,\tau,\delta}^{\rho,r,k,c}(\omega t^\sigma; \bar{p}) h\left(\left(\frac{t}{2}\theta^p(\varepsilon_1) + \left(\frac{2-t}{2}\right)\theta^p(\varepsilon_2)\right)^{1/p}\right) dt \\
 & \leq \int_0^1 t^{\tau-1} E_{\sigma,\tau,\delta}^{\rho,r,k,c}(\omega t^\sigma; \bar{p}) \varphi\left(\left(\frac{t}{2}\theta^p(\varepsilon_1) + \left(\frac{2-t}{2}\right)\theta^p(\varepsilon_2)\right)^{1/p}\right) h\left(\left(\frac{t}{2}\theta^p(\varepsilon_1) + \left(\frac{2-t}{2}\right)\theta^p(\varepsilon_2)\right)^{1/p}\right) dt \\
 & \quad + \int_0^1 t^{\tau-1} E_{\sigma,\tau,\delta}^{\rho,r,k,c}(\omega t^\sigma; \bar{p}) \varphi\left(\left(\frac{t}{2}\theta^p(\varepsilon_2) + \left(\frac{2-t}{2}\right)\theta^p(\varepsilon_1)\right)^{1/p}\right) h\left(\left(\frac{t}{2}\theta^p(\varepsilon_1) + \left(\frac{2-t}{2}\right)\theta^p(\varepsilon_2)\right)^{1/p}\right) dt.
 \end{aligned} \tag{49}$$

By choosing $\theta(x) = (t/2)\theta^p(\varepsilon_1) + ((2-t)/2)\theta^p(\varepsilon_2)$, that is, $\theta^p(\varepsilon_1) + \theta^p(\varepsilon_2) - \theta(x) = (t/2)\theta^p(\varepsilon_2) + ((2-t)/2)\theta^p(\varepsilon_1)$, in (49) and using the condition $\varphi(\theta^{1/p}(x)) = \varphi((\theta^p(\varepsilon_1) + \theta^p(\varepsilon_2) - \theta(x))^{1/p})$, one can get first inequality of (47).

To prove the second inequality of (47), multiplying both sides of (45) by $t^{\tau-1}E_{\sigma,\tau,\delta}^{\rho,r,k,c}(\omega t^\sigma; \bar{p})h(((t/2)\theta^p(\varepsilon_1) + ((2-t)/2)\theta^p(\varepsilon_2))^{1/p})$ and then integrating over $[0, 1]$, we have

$$\begin{aligned}
& \int_0^1 t^{\tau-1} E_{\sigma, \tau, \delta}^{\rho, r, k, c}(\omega t^\sigma; \bar{p}) \varphi \left(\left(\frac{t}{2} \theta^p(\varepsilon_1) + \left(\frac{2-t}{2} \right) \theta^p(\varepsilon_2) \right)^{1/p} \right) h \left(\left(\frac{t}{2} \theta^p(\varepsilon_1) + \left(\frac{2-t}{2} \right) \theta^p(\varepsilon_2) \right)^{1/p} \right) dt \\
& + \int_0^1 t^{\tau-1} E_{\sigma, \tau, \delta}^{\rho, r, k, c}(\omega t^\sigma; \bar{p}) \varphi \left(\left(\frac{t}{2} \theta^p(\varepsilon_2) + \left(\frac{2-t}{2} \right) \theta^p(\varepsilon_1) \right)^{1/p} \right) h \left(\left(\frac{t}{2} \theta^p(\varepsilon_1) + \left(\frac{2-t}{2} \right) \theta^p(\varepsilon_2) \right)^{1/p} \right) dt \\
& \leq (\varphi(\theta(\varepsilon_1)) + \varphi(\theta(\varepsilon_2))) \\
& \cdot \int_0^1 t^{\tau-1} E_{\sigma, \tau, \delta}^{\rho, r, k, c}(\omega t^\sigma; \bar{p}) h \left(\left(\frac{t}{2} \theta^p(\varepsilon_1) + \left(\frac{2-t}{2} \right) \theta^p(\varepsilon_2) \right)^{1/p} \right) dt.
\end{aligned} \tag{50}$$

Setting $\theta(x) = (t/2)\theta^p(\varepsilon_1) + ((2-t)/2)\theta^p(\varepsilon_2)$ in (38) and using the condition $\varphi(\theta^{1/p}(x)) = \varphi((\theta^p(\varepsilon_1) + \theta^p(\varepsilon_2) - \theta(x))^{1/p})$, one can get second inequality of (47).

(ii) Proof is similar to the proof of (i). \square

Remark 5

(i) By setting $p = -1$, $\bar{p} = 0$, and $\theta = I$, Theorem 2.6 of [22] is obtained.

(ii) By setting $\theta = I$ and $p = -1$, Theorem 2.6 of [23] is obtained.

(iii) By setting $p = -1$, Theorem 2.10 of [24] is obtained.

Corollary 3. *When we set $p = -1$, $\omega = \bar{p} = 0$, and $\theta = I$ in Theorem 6, then we get the following inequalities via RL fractional integrals.*

$$\begin{aligned}
& \varphi \left(\frac{2\varepsilon_1 \varepsilon_2}{\varepsilon_1 + \varepsilon_2} \right) \left(\left(I_{((\varepsilon_1 + \varepsilon_2)/2\varepsilon_1 \varepsilon_2)^+}^\tau h \circ \psi \right) \left(\frac{1}{\varepsilon_1} \right) + \left(I_{((\varepsilon_1 + \varepsilon_2)/2\varepsilon_1 \varepsilon_2)^-}^\tau h \circ \psi \right) \left(\frac{1}{\varepsilon_2} \right) \right) \\
& \leq \left(I_{((\varepsilon_1 + \varepsilon_2)/2\varepsilon_1 \varepsilon_2)^+}^\tau \varphi h \circ \psi \right) \left(\frac{1}{\varepsilon_1} \right) + \left(I_{((\varepsilon_1 + \varepsilon_2)/2\varepsilon_1 \varepsilon_2)^-}^\tau \varphi h \circ \psi \right) \left(\frac{1}{\varepsilon_2} \right) \\
& \leq \frac{\varphi(\varepsilon_1) + \varphi(\varepsilon_2)}{2} \left(\left(I_{((\varepsilon_1 + \varepsilon_2)/2\varepsilon_1 \varepsilon_2)^+}^\tau h \circ \psi \right) \left(\frac{1}{\varepsilon_1} \right) + \left(I_{((\varepsilon_1 + \varepsilon_2)/2\varepsilon_1 \varepsilon_2)^-}^\tau h \circ \psi \right) \left(\frac{1}{\varepsilon_2} \right) \right).
\end{aligned} \tag{51}$$

3. Conclusion

We have established Hadamard and Fejér–Hadamard fractional integral inequalities for generalized fractional integrals of p -convex functions. The results of this paper hold simultaneously for convex and harmonically convex functions for different fractional integral operators containing Mittag-Leffler functions.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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