Research Article

Approximation Properties of Generalized λ -Bernstein–Stancu-Type Operators

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The present study introduces generalized λ -Bernstein–Stancu-type operators with shifted knots. A Korovkin-type approximation theorem is given, and the rate of convergence of these types of operators is obtained for Lipschitz-type functions. Then, a Voronovskaja-type theorem was given for the asymptotic behavior for these operators. Finally, numerical examples and their graphs were given to demonstrate the convergence of $G_{m,\lambda}^{\alpha,\beta}(f, x)$ to $f(x)$ with respect to m values.

1. Introduction

The Bernstein operators, which are positive linear operators, are of great importance for the theory of approximation. In [1], Bernstein operators were introduced by Bernstein to prove the Weierstrass approximation theorem. For $g \in C[0, 1]$, the classical Bernstein operators are given as

$$B_m(g, x) = \sum_{j=0}^m g\left(\frac{j}{m}\right) q_{m,j}(x), \quad x \in [0, 1], \quad (1)$$

where the Bernstein basis functions $q_{m,j}(x)$ are defined as

$$q_{m,j}(x) := \binom{m}{j} x^j (1-x)^{m-j}; \quad j = 0, 1, \dots, m. \quad (2)$$

There are many generalizations of the $B_m(g, x)$ operators. In [2], Gadjiev and Ghorbanalizadeh gave Bernstein–Stancu operators with shifted knots:

$$S_{m,\alpha,\beta}(g, x) = \left(\frac{m+\beta_2}{m}\right) \sum_{j=0}^m g\left(\frac{j+\alpha_1}{m+\beta_1}\right) p_{m,j}(x), \quad (3)$$

where $x \in A_m := [\alpha_2/(m+\beta_2), (m+\alpha_2)/(m+\beta_2)]$ and

$p_{m,j}(x) := \binom{m}{j} (x - \alpha_2/(m+\beta_2))^j ((m+\alpha_2)/(m+\beta_2) - x)^{m-j}$ for $j = 0, 1, \dots, m$, and α_i and β_i are positive real numbers, and $0 \leq \alpha_i \leq \beta_i$ for $i = 1, 2$. In the case of $\alpha_i = \beta_i = 0$ for $i = 1, 2$, operator (3) reduces to the classical Bernstein operator (1). In addition, the case of $\alpha_2 = \beta_2 = 0$ was handled and examined by Stancu in [3] and these operators were called the classical Bernstein–Stancu operators. A Durrmeyer variant of the Bernstein–Stancu operators in (3) was studied by Dinlemez Kantar and Ergelen in [4], and a Voronovskaja-type approximation theorem for these operators was given. Several studies were conducted on some approximation properties, and asymptotic-type results were given for these operators in [5–14].

In [5], Cai et al. introduced the Bernstein operators with shape parameter $\lambda \in [-1, 1]$ as follows:

$$B_m^\lambda(g, x) = \sum_{j=0}^m g\left(\frac{j}{m}\right) \tilde{q}_{m,j}(\lambda, x), \quad m \in \mathbb{N}, \quad (4)$$

where $x \in [0, 1]$ and $\tilde{q}_{m,j}(\lambda, x)$ are Bézier basis functions with shape parameter $\lambda \in [-1, 1]$ defined by

$$\begin{aligned}\tilde{q}_{m,0}(\lambda, x) &:= q_{m,0}(x) - (\lambda/(m+1))q_{m+1,1}(x), \\ \tilde{q}_{m,j}(\lambda, x) &:= q_{m,j}(x) + \lambda \left[\frac{m-2j+1}{m^2-1}q_{m+1,j}(x) - \frac{m-2j-1}{m^2-1}q_{m+1,j+1}(x) \right]; \quad j = 1, 2, \dots, m-1, \\ \tilde{q}_{m,m}(\lambda, x) &:= q_{m,m}(x) - \frac{\lambda}{m+1}q_{m+1,m}(x).\end{aligned}\tag{5}$$

Later in [6], Cai proposed Kantorovich-type λ -Bernstein operators, as well as their Bézier variant, and examined the approximation results. Kantorovich-type λ -Bernstein operators were also studied by Acu et al. in [7], and they considered the approximation properties and asymptotic-type results.

Recently, in [8], Srivastava et al. constructed λ -Bernstein-Stancu operators defined by

$$B_{m,\alpha,\beta}^\lambda(g, x) = \sum_{j=0}^m g\left(\frac{j+\alpha}{m+\beta}\right) \tilde{q}_{m,j}(\lambda, x), \quad m \in \mathbb{N}, \tag{6}$$

where $x \in [0, 1]$ and Bézier basis functions $\tilde{q}_{m,j}(\lambda, x)$ with shape parameter λ are defined in (5).

In the present study, we introduce the following generalized λ -Bernstein-Stancu operators with shifted knots for $g \in C[0, 1]$:

$$G_{m,\lambda}^{\alpha,\beta}(g, x) = \left(\frac{m+\beta_2}{m}\right)^m \sum_{j=0}^m g\left(\frac{j+\alpha_1}{m+\beta_1}\right) \tilde{p}_{m,j}(\lambda, x), \tag{7}$$

where α_i and β_i are positive real numbers satisfying $0 \leq \alpha_i \leq \beta_i$ for $i = 1, 2$ and Bézier basis functions $\tilde{p}_{m,j}(\lambda, x)$ with shape parameter $\lambda \in [-1, 1]$ are defined by

$$\begin{aligned}\tilde{p}_{m,0}(\lambda, x) &:= p_{m,0}(x) - \frac{\lambda}{m+1}p_{m+1,1}(x), \\ \tilde{p}_{m,j}(\lambda, x) &:= p_{m,j}(x) + \lambda \left[\frac{m-2j+1}{m^2-1}p_{m+1,j}(x) - \frac{m-2j-1}{m^2-1}p_{m+1,j+1}(x) \right]; \quad j = 1, 2, \dots, m-1, \\ \tilde{p}_{m,m}(\lambda, x) &:= p_{m,m}(x) - \frac{\lambda}{m+1}p_{m+1,m}(x),\end{aligned}\tag{8}$$

where $p_{m,j}(x) := \binom{m}{j} (x - \alpha_2/(m+\beta_2))^j ((m+\alpha_2)/(m+\beta_2) - x)^{m-j}$ for $j = 0, 1, \dots, m$, α_i and β_i are positive real numbers, $0 \leq \alpha_i \leq \beta_i$ for $i = 1, 2$, and $x \in A_m := [\alpha_2/(m+\beta_2), (m+\alpha_2)/(m+\beta_2)]$. We give approximation properties and Voronovskaja-type approximation theorem for asymptotic behavior of operator (7). When $\alpha_i = \beta_i = 0$ for $i = 1, 2$ and $\lambda = 0$, operator (7) reduces to the classical Bernstein operator (1). When $\lambda = 0$, it reduces

to the Bernstein-Stancu operator (3). When $\alpha_i = \beta_i = 0$ for $i = 1, 2$, it reduces to the Bernstein operator with shape parameter $\lambda \in [-1, 1]$ (4). In the following section, several lemmas are given to prove the main results.

2. Some Preliminary Results

Lemma 1. *For generalized λ -Bernstein-Stancu operators with shifted knots, we have the following equalities:*

$$\begin{aligned}
(i) \quad & G_{m,\lambda}^{\alpha,\beta}(1, x) = 1, \\
(ii) \quad & G_{m,\lambda}^{\alpha,\beta}(s, x) = \frac{m + \beta_2}{m + \beta_1} \left(x - \frac{\alpha_2}{m + \beta_2} \right) + \frac{\alpha_1}{m + \beta_1}, \\
& + \lambda \left\{ \frac{1}{(m + \beta_1)(m - 1)} \left[\frac{m}{m + \beta_2} - 2 \left(x - \frac{\alpha_2}{m + \beta_2} \right) + \left(\frac{m + \beta_2}{m} \right)^m \left(\left(x - \frac{\alpha_2}{m + \beta_2} \right)^{m+1} \right. \right. \right. \\
& \left. \left. \left. + (\alpha_1 - 1) \left(\frac{m + \alpha_2}{m + \beta_2} - x \right)^{m+1} \right) \right] + \frac{\alpha_1}{m + \beta_1} \left(\frac{m + \beta_2}{m} \right)^m \left(x - \frac{\alpha_2}{m + \beta_2} \right) \left(\frac{m + \alpha_2}{m + \beta_2} - x \right)^m \right\}. \\
(iii) \quad & G_{m,\lambda}^{\alpha,\beta}(s^2, x) = \frac{m + \beta_2}{(m + \beta_1)^2} \left[\frac{(m - 1)(m + \beta_2)}{m} \left(x - \frac{\alpha_2}{m + \beta_2} \right)^2 + (1 + 2\alpha_1) \left(x - \frac{\alpha_2}{m + \beta_2} \right) \right] \\
& + \frac{\alpha_1^2}{(m + \beta_1)^2} + \lambda \left\{ \frac{1}{(m + \beta_1)^2(m - 1)} \left[-\frac{m}{m + \beta_2} + 2m \left(x - \frac{\alpha_2}{m + \beta_2} \right) - 4m \left(x - \frac{\alpha_2}{m + \beta_2} \right)^2 \right. \right. \\
& \left. \left. + \left(\frac{m + \beta_2}{m} \right)^m \left((2m + 1) \left(x - \frac{\alpha_2}{m + \beta_2} \right)^{m+1} + \left(\frac{m + \alpha_2}{m + \beta_2} - x \right)^{m+1} \right) \right] \right. \\
& \left. + \frac{\alpha_1}{(m + \beta_1)^2(m - 1)} \left[\frac{2m}{m + \beta_2} - 4 \left(x - \frac{\alpha_2}{m + \beta_2} \right) + \left(\frac{m + \beta_2}{m} \right)^m \left(2 \left(x - \frac{\alpha_2}{m + \beta_2} \right)^{m+1} \right. \right. \right. \\
& \left. \left. \left. + (\alpha_1 - 2) \left(\frac{m + \alpha_2}{m + \beta_2} - x \right)^{m+1} \right) \right] + \frac{\alpha_1^2}{(m + \beta_1)^2} \left(\frac{m + \beta_2}{m} \right)^m \left(x - \frac{\alpha_2}{m + \beta_2} \right) \left(\frac{m + \alpha_2}{m + \beta_2} - x \right)^m \right\}, \\
(iv) \quad & G_{m,\lambda}^{\alpha,\beta}(s^3, x) = \frac{m + \beta_2}{(m + \beta_1)^3} \left[\frac{(m - 1)(m - 2)(m + \beta_2)^2}{m^2} \left(x - \frac{\alpha_2}{m + \beta_2} \right)^3 \right. \\
& \left. + \frac{3(m - 1)(1 + \alpha_1)(m + \beta_2)}{m} \left(x - \frac{\alpha_2}{m + \beta_2} \right)^2 + (1 + 3\alpha_1(1 + \alpha_1)) \left(x - \frac{\alpha_2}{m + \beta_2} \right) \right] \\
& + \frac{\alpha_1^3}{(m + \beta_1)^3} + \lambda \left\{ \frac{3m}{(m + \beta_1)^3} \left[\left(x - \frac{\alpha_2}{m + \beta_2} \right)^2 - 2 \left(x - \frac{\alpha_2}{m + \beta_2} \right)^3 \right] + \frac{1}{(m + \beta_1)^3(m - 1)} \left[\frac{m}{m + \beta_2} \right. \right. \\
& \left. \left. - 2 \left(x - \frac{\alpha_2}{m + \beta_2} \right) - 6m \left(x - \frac{\alpha_2}{m + \beta_2} \right)^2 + \left(\frac{m + \beta_2}{m} \right)^m \left((3m^2 + 3m + 1) \left(x - \frac{\alpha_2}{m + \beta_2} \right)^{m+1} \right. \right. \right. \\
& \left. \left. \left. - \left(\frac{m + \alpha_2}{m + \beta_2} - x \right)^{m+1} \right) \right] + \frac{\alpha_1^2}{(m + \beta_1)^3} \left(\frac{m + \beta_2}{m} \right)^m \left[-3 \left(x - \frac{\alpha_2}{m + \beta_2} \right)^{m+1} \right. \right. \\
& \left. \left. + \alpha_1 \left(x - \frac{\alpha_2}{m + \beta_2} \right) \left(\frac{m + \alpha_2}{m + \beta_2} - x \right)^m \right] + \frac{\alpha_1}{(m + \beta_1)^3(m - 1)} \left[\frac{3(\alpha_1 - 1)m}{m + \beta_2} + 6m \left(x - \frac{\alpha_2}{m + \beta_2} \right) \right. \right. \\
& \left. \left. - (12m - 6\alpha_1) \left(x - \frac{\alpha_2}{m + \beta_2} \right)^2 + \left(\frac{m + \beta_2}{m} \right)^m \left((\alpha_1^2 - 3(\alpha_1 - 1)) \left(\frac{m + \alpha_2}{m + \beta_2} - x \right)^{m+1} \right. \right. \right. \\
& \left. \left. \left. + 6(m + 1) \left(x - \frac{\alpha_2}{m + \beta_2} \right)^{m+1} \right) \right] + \frac{3\alpha_1}{(m + \beta_1)^3(m + 1)} \left(\frac{m + \beta_2}{m} \right)^m \left(x - \frac{\alpha_2}{m + \beta_2} \right)^{m+1} \right\}
\end{aligned}$$

$$\begin{aligned}
& + \frac{6\alpha_1^2 m}{(m+\beta_1)^3(m^2-1)} \left(\frac{m+\beta_2}{m} \right)^m \left(x - \frac{\alpha_2}{m+\beta_2} \right)^{m+1} \Big\}, \\
(v) G_{m,\lambda}^{\alpha,\beta}(s^4, x) &= \frac{m+\beta_2}{(m+\beta_1)^4} \left[\frac{(m-1)(m-2)(m-3)(m+\beta_2)^3}{m^3} \left(x - \frac{\alpha_2}{m+\beta_2} \right)^4 \right. \\
& + \frac{2(m-1)(m-2)(2\alpha_1+3)(m+\beta_2)^2}{m^2} \left(x - \frac{\alpha_2}{m+\beta_2} \right)^3 + \frac{(m-1)(6\alpha_1(\alpha_1+2)+7)(m+\beta_2)}{m} \left(x - \frac{\alpha_2}{m+\beta_2} \right)^2 \\
& + \left. \left(2(2\alpha_1^3+3\alpha_1^2+2\alpha_1)+1 \right) \left(x - \frac{\alpha_2}{m+\beta_2} \right) \right] + \frac{\alpha_1^4}{(m+\beta_1)^4} \\
& + \lambda \left\{ \frac{2}{(m+\beta_1)^4} \left[\left(x - \frac{\alpha_2}{m+\beta_2} \right) + (3m+22) \left(x - \frac{\alpha_2}{m+\beta_2} \right)^2 + 2m(m-8) \left(x - \frac{\alpha_2}{m+\beta_2} \right)^3 \right. \right. \\
& - 4m(m-2) \left(x - \frac{\alpha_2}{m+\beta_2} \right)^4 + (2m^2+5m-23) \left(\frac{m+\beta_2}{m} \right)^m \left(x - \frac{\alpha_2}{m+\beta_2} \right)^{m+1} \Big] \\
& + \frac{1}{(m+\beta_1)^4(m-1)} \left[\frac{-m}{m+\beta_2} + 2 \left(x - \frac{\alpha_2}{m+\beta_2} \right) + 44 \left(x - \frac{\alpha_2}{m+\beta_2} \right)^2 \right. \\
& - \left. \left(\frac{m+\beta_2}{m} \right)^m \left(45 \left(x - \frac{\alpha_2}{m+\beta_2} \right)^{m+1} - \left(\frac{m+\alpha_2}{m+\beta_2} - x \right)^{m+1} \right) \right] \\
& + \frac{\alpha_1}{(m+\beta_1)^4} \left[12\alpha_1 \left(x - \frac{\alpha_2}{m+\beta_2} \right) + 12m \left(x - \frac{\alpha_2}{m+\beta_2} \right)^2 - 24m \left(x - \frac{\alpha_2}{m+\beta_2} \right)^3 \right. \\
& + \left. \left(\frac{m+\beta_2}{m} \right)^m \left(12(m+2+\alpha_1) \left(x - \frac{\alpha_2}{m+\beta_2} \right)^{m+1} + \alpha_1^3 \left(x - \frac{\alpha_2}{m+\beta_2} \right) \left(\frac{m+\alpha_2}{m+\beta_2} - x \right)^m \right) \right] \\
& + \frac{\alpha_1}{(m+\beta_1)^4(m-1)} \left[2(2\alpha_1^2-3\alpha_1+2) \left(\frac{m}{m+\beta_2} - 2 \left(x - \frac{\alpha_2}{m+\beta_2} \right) \right) \right. \\
& - 24m(\alpha_1+1) \left(x - \frac{\alpha_2}{m+\beta_2} \right)^2 + \left(\frac{m+\beta_2}{m} \right)^m \left(2(2\alpha_1^2+9\alpha_1+14) \left(x - \frac{\alpha_2}{m+\beta_2} \right)^{m+1} \right. \\
& + \left. \left. \left. \left(\alpha_1^3 - 4\alpha_1^2 + 6\alpha_1 - 4 \right) \left(\frac{m+\alpha_2}{m+\beta_2} - x \right)^{m+1} \right) \right] \right] \Big\}.
\end{aligned}$$

(9)

Proof. If we use Bézier basis functions (8) in λ -Bernstein–Stancu operators (7), we obtain

$$\begin{aligned}
 G_{m,\lambda}^{\alpha,\beta}(1,x) &= \left(\frac{m+\beta_2}{m}\right)^m \sum_{j=0}^m \tilde{p}_{m,j}(\lambda,x) = \left(\frac{m+\beta_2}{m}\right)^m \sum_{j=0}^m \tilde{p}_{m,j}(x) \\
 &= \left(\frac{m+\beta_2}{m}\right)^m \left(\frac{m}{m+\beta_2}\right)^m = 1, \\
 G_{m,\lambda}^{\alpha,\beta}(s,x) &= \left(\frac{m+\beta_2}{m}\right)^m \sum_{j=0}^m \left(\frac{j+\alpha_1}{m+\beta_1}\right) \tilde{p}_{m,j}(\lambda,x) \\
 &= \left(\frac{m+\beta_2}{m}\right)^m \left[\frac{\alpha_1}{m+\beta_1} \left(P_{m,0}(x) - \frac{\lambda}{m+1} P_{m+1,1}(x) \right) \right. \\
 &\quad + \sum_{j=1}^{m-1} \left(\frac{j+\alpha_1}{m+\beta_1} \right) \left(P_{m,j}(x) + \lambda \left(\frac{m-2j+1}{m^2-1} P_{m+1,j}(x) - \frac{m-2j-1}{m^2-1} P_{m+1,j+1}(x) \right) \right) \\
 &\quad \left. + \frac{m+\alpha_1}{m+\beta_1} \left(P_{m,m}(x) - \frac{\lambda}{m+1} P_{m+1,m}(x) \right) \right] \\
 &= \left(\frac{m+\beta_2}{m}\right)^m \sum_{j=0}^m \left(\frac{j+\alpha_1}{m+\beta_1}\right) P_{m,j}(x) + \lambda \{ \Upsilon_{1,m}^{\alpha,\beta}(x) - \Upsilon_{2,m}^{\alpha,\beta}(x) \},
 \end{aligned} \tag{10}$$

where

$$\begin{aligned}
 \Upsilon_{1,m}^{\alpha,\beta}(x) &= \left(\frac{m+\beta_2}{m}\right)^m \sum_{j=0}^m \left(\frac{j+\alpha_1}{m+\beta_1}\right) \frac{m-2j+1}{m^2-1} P_{m+1,j}(x), \\
 \Upsilon_{2,m}^{\alpha,\beta}(x) &= \left(\frac{m+\beta_2}{m}\right)^m \sum_{j=1}^{m-1} \left(\frac{j+\alpha_1}{m+\beta_1}\right) \frac{m-2j-1}{m^2-1} P_{m+1,j+1}(x).
 \end{aligned} \tag{11}$$

Thanks to the linearity of Bernstein–Stancu operators (3), we obtain

$$\left(\frac{m+\beta_2}{m}\right)^m \sum_{j=0}^m \frac{j+\alpha_1}{m+\beta_1} P_{m,j}(x) = \frac{m+\beta_2}{m+\beta_1} \left(x - \frac{\alpha_2}{m+\beta_2} \right) + \frac{\alpha_1}{m+\beta_1}. \tag{12}$$

Now, we will compute $\Upsilon_{1,m}^{\alpha,\beta}(x)$ and $\Upsilon_{2,m}^{\alpha,\beta}(x)$

$$\begin{aligned}
 \Upsilon_{1,m}^{\alpha,\beta}(x) &= \left(\frac{m+\beta_2}{m}\right)^m \sum_{j=0}^m \left(\frac{j+\alpha_1}{m+\beta_1}\right) \frac{m-2j+1}{m^2-1} P_{m+1,j}(x) \\
 &= \left(\frac{m+\beta_2}{m}\right)^m \frac{1}{(m+\beta_1)(m-1)} \left[(m+1) \left(x - \frac{\alpha_2}{m+\beta_2} \right) \sum_{j=0}^{m-1} P_{m,j} + \alpha_1 \sum_{j=0}^m P_{m+1,j}(x) \right] \\
 &\quad - \left(\frac{m+\beta_2}{m}\right)^m \frac{2(m+1)}{(m+\beta_1)(m^2-1)} \left[m \left(x - \frac{\alpha_2}{m+\beta_2} \right)^2 \sum_{j=0}^{m-2} P_{m-1,j}(x) \right. \\
 &\quad \left. + (1+\alpha_1) \left(x - \frac{\alpha_2}{m+\beta_2} \right) \sum_{j=0}^{m-1} P_{m,j}(x) \right]
 \end{aligned}$$

$$\begin{aligned}
&= \frac{1}{m + \beta_1} \left[\left(x - \frac{\alpha_2}{m + \beta_2} \right) - \left(\frac{m + \beta_2}{m} \right)^m \left(x - \frac{\alpha_2}{m + \beta_2} \right)^{m+1} \right] \\
&\quad + \frac{2m}{(m + \beta_1)(m - 1)} \left[- \left(x - \frac{\alpha_2}{m + \beta_2} \right)^2 + \left(\frac{m + \beta_2}{m} \right)^m \left(x - \frac{\alpha_2}{m + \beta_2} \right)^{m+1} \right] \\
&\quad + \frac{\alpha_1}{(m + \beta_1)(m - 1)} \left[-2 \left(x - \frac{\alpha_2}{m + \beta_2} \right) - \left(\frac{m + \beta_2}{m} \right)^m \left(x - \frac{\alpha_2}{m + \beta_2} \right)^{m+1} + \frac{m}{m + \beta_2} \right], \\
Y_{2,m}^{\alpha,\beta}(x) &= \left(\frac{m + \beta_2}{m} \right)^m \sum_{j=1}^{m-1} \left(\frac{j + \alpha_1}{m + \beta_1} \right) \frac{m - 2j - 1}{m^2 - 1} p_{m+1,j+1}(x) \\
&= \left(\frac{m + \beta_2}{m} \right)^m \frac{1}{(m + \beta_1)(m + 1)} \left[(m + 1) \left(x - \frac{\alpha_2}{m + \beta_2} \right) \sum_{j=1}^{m-1} p_{m,j}(x) - \sum_{j=1}^{m-1} p_{m+1,j+1}(x) \right. \\
&\quad \left. + \alpha_1 \sum_{j=1}^{m-1} p_{m+1,j+1}(x) \right] - \left(\frac{m + \beta_2}{m} \right)^m \frac{2}{(m + \beta_1)(m^2 - 1)} \left[m(m + 1) \left(x - \frac{\alpha_2}{m + \beta_2} \right)^2 \sum_{j=0}^{m-2} p_{m-1,j}(x) \right. \\
&\quad \left. - (m + 1) \left(x - \frac{\alpha_2}{m + \beta_2} \right) \sum_{j=1}^{m-1} p_{m,j}(x) + \sum_{j=1}^{m-1} p_{m+1,j+1}(x) \right. \\
&\quad \left. + \alpha_1 (m + 1) \left(x - \frac{\alpha_2}{m + \beta_2} \right) \sum_{j=1}^{m-1} p_{m,j}(x) - \alpha_1 \sum_{j=1}^{m-1} p_{m+1,j+1}(x) \right] \\
&= \frac{1}{m + \beta_1} \left[\left(x - \frac{\alpha_2}{m + \beta_2} \right) - \left(\frac{m + \beta_2}{m} \right)^m \left(x - \frac{\alpha_2}{m + \beta_2} \right)^{m+1} \right] \\
&\quad - \frac{1}{(m + \beta_1)(m + 1)} \left[\frac{m}{m + \beta_2} - \left(\frac{m + \beta_2}{m} \right)^m \left(\left(\frac{m + \alpha_2}{m + \beta_2} - x \right)^{m+1} + \left(x - \frac{\alpha_2}{m + \beta_2} \right)^{m+1} \right) \right] \\
&\quad - \frac{2}{(m + \beta_1)(m^2 - 1)} \left[\frac{m}{m + \beta_2} - \left(\frac{m + \beta_2}{m} \right)^m \left(\left(\frac{m + \alpha_2}{m + \beta_2} - x \right)^{m+1} + (m + 1) \left(x - \frac{\alpha_2}{m + \beta_2} \right) \left(\frac{m + \alpha_2}{m + \beta_2} - x \right)^m \right. \right. \\
&\quad \left. \left. + \left(x - \frac{\alpha_2}{m + \beta_2} \right)^{m+1} \right) \right] - \frac{\alpha_1}{m + \beta_1} \left(\frac{m + \beta_2}{m} \right)^m \left(x - \frac{\alpha_2}{m + \beta_2} \right) \left(\frac{m + \alpha_2}{m + \beta_2} - x \right)^m \\
&\quad + \frac{\alpha_1}{(m + \beta_1)(m + 1)} \left[\frac{m}{m + \beta_2} - \left(\frac{m + \beta_2}{m} \right)^m \left(\left(\frac{m + \alpha_2}{m + \beta_2} - x \right)^{m+1} + \left(x - \frac{\alpha_2}{m + \beta_2} \right)^{m+1} \right) \right] \\
&\quad + \frac{2}{(m + \beta_1)(m - 1)} \left[\left(x - \frac{\alpha_2}{m + \beta_2} \right) - \left(\frac{m + \beta_2}{m} \right)^m \left(\left(x - \frac{\alpha_2}{m + \beta_2} \right) \left(\frac{m + \alpha_2}{m + \beta_2} - x \right)^m \right. \right. \\
&\quad \left. \left. + \left(x - \frac{\alpha_2}{m + \beta_2} \right)^{m+1} \right) \right] - \frac{2m}{(m + \beta_1)(m - 1)} \left[\left(x - \frac{\alpha_2}{m + \beta_2} \right)^2 - \left(\frac{m + \beta_2}{m} \right)^m \left(x - \frac{\alpha_2}{m + \beta_2} \right)^{m+1} \right] \\
&\quad - \frac{2\alpha_1}{(m + \beta_1)(m - 1)} \left[\left(x - \frac{\alpha_2}{m + \beta_2} \right) - \left(\frac{m + \beta_2}{m} \right)^m \left(x - \frac{\alpha_2}{m + \beta_2} \right)^{m+1} \right] \\
&\quad + \frac{2\alpha_1}{(m + \beta_1)(m^2 - 1)} \left[\frac{m}{m + \beta_2} - \left(\frac{m + \beta_2}{m} \right)^m \left(\left(\frac{m + \alpha_2}{m + \beta_2} - x \right)^{m+1} + \left(x - \frac{\alpha_2}{m + \beta_2} \right)^{m+1} \right) \right]. \tag{13}
\end{aligned}$$

Using (12), $\Upsilon_{1,m}^{\alpha,\beta}(x)$, and $\Upsilon_{2,m}^{\alpha,\beta}(x)$ in $G_{m,\lambda}^{\alpha,\beta}(s, x)$, we obtain the following equation:

$$\begin{aligned} G_{m,\lambda}^{\alpha,\beta}(s, x) &= \frac{m + \beta_2}{m + \beta_1} \left(x - \frac{\alpha_2}{m + \beta_2} \right) + \frac{\alpha_1}{m + \beta_1} \\ &+ \lambda \left\{ \frac{1}{(m + \beta_1)(m - 1)} \left[\frac{m}{m + \beta_2} - 2 \left(x - \frac{\alpha_2}{m + \beta_2} \right) + \left(\frac{m + \beta_2}{m} \right)^m \left(x - \frac{\alpha_2}{m + \beta_2} \right)^{m+1} \right. \right. \\ &\left. \left. + (\alpha_1 - 1) \left(\frac{m + \alpha_2}{m + \beta_2} - x \right)^{m+1} \right] \right\} + \frac{\alpha_1}{m + \beta_1} \left(\frac{m + \beta_2}{m} \right)^m \left(x - \frac{\alpha_2}{m + \beta_2} \right) \left(\frac{m + \alpha_2}{m + \beta_2} - x \right)^m \}. \end{aligned} \quad (14)$$

Then, we have the following equality for the third moment by using the linearity of $G_{m,\lambda}^{\alpha,\beta}(g(s), x)$:

$$\begin{aligned} G_{m,\lambda}^{\alpha,\beta}(s^2, x) &= \left(\frac{m + \beta_2}{m} \right)^m \sum_{j=0}^m \left(\frac{j + \alpha_1}{m + \beta_1} \right)^2 \tilde{P}_{m,j}(\lambda, x) \\ &= \left(\frac{m + \beta_2}{m} \right)^m \left[\left(\frac{\alpha_1}{m + \beta_1} \right)^2 \left(p_{m,0}(x) - \frac{\lambda}{m + 1} p_{m+1,1}(x) \right) \right. \\ &\quad + \sum_{j=1}^{m-1} \left(\frac{j + \alpha_1}{m + \beta_1} \right)^2 \left(p_{m,j}(x) + \lambda \left(\frac{m - 2j + 1}{m^2 - 1} p_{m+1,j}(x) - \frac{m - 2j - 1}{m^2 - 1} p_{m+1,j+1}(x) \right) \right) \\ &\quad \left. + \left(\frac{m + \alpha_1}{m + \beta_1} \right)^2 \left(p_{m,m}(x) - \frac{\lambda}{m + 1} p_{m+1,m}(x) \right) \right] \\ &= \left(\frac{m + \beta_2}{m} \right)^m \sum_{j=0}^m \left(\frac{j + \alpha_1}{m + \beta_1} \right)^2 p_{m,j}(x) + \lambda \{ \Upsilon_{3,m}^{\alpha,\beta}(x) - \Upsilon_{4,m}^{\alpha,\beta}(x) \}, \end{aligned} \quad (15)$$

where

$$\begin{aligned} \Upsilon_{3,m}^{\alpha,\beta}(x) &= \left(\frac{m + \beta_2}{m} \right)^m \sum_{j=0}^m \left(\frac{j + \alpha_1}{m + \beta_1} \right)^2 \frac{m - 2j + 1}{m^2 - 1} p_{m+1,j}(x), \\ \Upsilon_{4,m}^{\alpha,\beta}(x) &= \left(\frac{m + \beta_2}{m} \right)^m \sum_{j=1}^{m-1} \left(\frac{j + \alpha_1}{m + \beta_1} \right)^2 \frac{m - 2j - 1}{m^2 - 1} p_{m+1,j+1}(x). \end{aligned} \quad (16)$$

From the linearity of the Bernstein–Stancu operators (3) and Bernstein basis functions, we obtain

$$\begin{aligned} &\left(\frac{m + \beta_2}{m} \right)^m \sum_{j=0}^m \left(\frac{j + \alpha_1}{m + \beta_1} \right)^2 p_{m,j}(x) \\ &= \frac{m + \beta_2}{(m + \beta_1)^2} \left[\frac{(m - 1)(m + \beta_2)(x - \alpha_2/(m + \beta_2))^2}{m} + (1 + 2\alpha_1) \left(x - \frac{\alpha_2}{m + \beta_2} \right) \right] + \frac{\alpha_1^2}{(m + \beta_1)^2}. \end{aligned} \quad (17)$$

Next, we compute $\Upsilon_{3,m}^{\alpha,\beta}(x)$ and $\Upsilon_{4,m}^{\alpha,\beta}(x)$

$$\begin{aligned}
\Upsilon_{3,m}^{\alpha,\beta}(x) &= \left(\frac{m+\beta_2}{m}\right)^m \sum_{j=0}^m \left(\frac{j+\alpha_1}{m+\beta_1}\right)^2 \frac{m-2j+1}{m^2-1} p_{m+1,j}(x) \\
&= \left(\frac{m+\beta_2}{m}\right)^m \frac{1}{(m+\beta_1)^2(m-1)} \left[m(m+1) \left(x - \frac{\alpha_2}{m+\beta_2}\right)^2 \sum_{j=0}^{m-2} p_{m-1,j}(x) \right. \\
&\quad \left. + (m+1)(1+2\alpha_1) \left(x - \frac{\alpha_2}{m+\beta_2}\right) \sum_{j=0}^{m-1} p_{m,j}(x) + \alpha_1^2 \sum_{j=0}^m p_{m+1,j}(x) \right] \\
&\quad - \left(\frac{m+\beta_2}{m}\right)^m \frac{1}{(m+\beta_1)^2(m^2-1)} \left[2(m+1)m(m-1) \left(x - \frac{\alpha_2}{m+\beta_2}\right)^3 \sum_{j=0}^{m-3} p_{m-2,j}(x) \right. \\
&\quad \left. + 2(m+1)m(3+2\alpha_1) \left(x - \frac{\alpha_2}{m+\beta_2}\right)^2 \sum_{j=0}^{m-2} p_{m-1,j}(x) \right. \\
&\quad \left. + 2(m+1)(1+2\alpha_1+\alpha_1^2) \left(x - \frac{\alpha_2}{m+\beta_2}\right) \sum_{j=0}^{m-1} p_{m,j}(x) \right] \\
&= \frac{1}{(m+\beta_1)^2} \left[\left(x - \frac{\alpha_2}{m+\beta_2}\right) - 2m \left(x - \frac{\alpha_2}{m+\beta_2}\right)^3 + (2m-1) \left(\frac{m+\beta_2}{m}\right)^m \left(x - \frac{\alpha_2}{m+\beta_2}\right)^{m+1} \right] \\
&\quad + \frac{m(m-5)}{(m+\beta_1)^2(m-1)} \left[\left(x - \frac{\alpha_2}{m+\beta_2}\right)^2 - \left(\frac{m+\beta_2}{m}\right)^m \left(x - \frac{\alpha_2}{m+\beta_2}\right)^{m+1} \right] + \frac{2\alpha_1}{(m+\beta_1)^2} \left(x - \frac{\alpha_2}{m+\beta_2}\right) \\
&\quad + \frac{\alpha_1}{(m+\beta_1)^2(m-1)} \left[2(m+1) \left(\frac{m+\beta_2}{m}\right)^m \left(x - \frac{\alpha_2}{m+\beta_2}\right)^{m+1} + \alpha_1 \left(\left(\frac{m+\beta_2}{m}\right)^m \left(x - \frac{\alpha_2}{m+\beta_2}\right)^{m+1} \right. \right. \\
&\quad \left. \left. - 2 \left(x - \frac{\alpha_2}{m+\beta_2}\right) + \frac{m}{m+\beta_2} \right) - 4m \left(x - \frac{\alpha_2}{m+\beta_2}\right)^2 \right] \\
\Upsilon_{4,m}^{\alpha,\beta}(x) &= \left(\frac{m+\beta_2}{m}\right)^m \sum_{j=1}^{m-1} \left(\frac{j+\alpha_1}{m+\beta_1}\right)^2 \frac{m-2j-1}{m^2-1} p_{m+1,j+1}(x) \\
&= \left(\frac{m+\beta_2}{m}\right)^m \frac{1}{(m+\beta_1)^2(m+1)} \left[m(m+1) \left(x - \frac{\alpha_2}{m+\beta_2}\right)^2 \sum_{j=0}^{m-2} p_{m-1,j}(x) \right. \\
&\quad \left. - (m+1)(1-2\alpha_1) \left(x - \frac{\alpha_2}{m+\beta_2}\right) \sum_{j=1}^{m-1} p_{m,j}(x) + (1-2\alpha_1+\alpha_1^2) \sum_{j=1}^{m-1} p_{m+1,j+1}(x) \right] \\
&\quad - \left(\frac{m+\beta_2}{m}\right)^m \frac{2(m+1)}{(m+\beta_1)^2(m^2-1)} \left[m(m-1) \left(x - \frac{\alpha_2}{m+\beta_2}\right)^3 \sum_{j=0}^{m-3} p_{m-2,j}(x) \right. \\
&\quad \left. + 2\alpha_1 m \left(x - \frac{\alpha_2}{m+\beta_2}\right)^2 \sum_{j=0}^{m-2} p_{m-1,j}(x) + (1-2\alpha_1+\alpha_1^2) \left(x - \frac{\alpha_2}{m+\beta_2}\right) \sum_{j=1}^{m-1} p_{m,j}(x) \right]
\end{aligned}$$

$$\begin{aligned}
& + \left(\frac{m + \beta_2}{m} \right)^m \frac{2(1 - 2\alpha_1 + \alpha_1^2)}{(m + \beta_1)^2(m^2 - 1)} \sum_{j=1}^{m-1} p_{m+1,j+1}(x) \\
& = \frac{1}{(m + \beta_1)^2} \left[- \left(x - \frac{\alpha_2}{m + \beta_2} \right) + m \left(x - \frac{\alpha_2}{m + \beta_2} \right)^2 - 2m \left(x - \frac{\alpha_2}{m + \beta_2} \right)^3 \right. \\
& \quad + (m + 1) \left(\frac{m + \beta_2}{m} \right)^m \left(x - \frac{\alpha_2}{m + \beta_2} \right)^{m+1} \left. \right] + \frac{1}{(m + \beta_1)^2(m + 1)} \left[\frac{m}{m + \beta_2} \right. \\
& \quad - \left. \left(\frac{m + \beta_2}{m} \right)^m \left(\left(x - \frac{\alpha_2}{m + \beta_2} \right)^{m+1} + \left(\frac{m + \alpha_2}{m + \beta_2} - x \right)^{m+1} \right) \right] \\
& \quad + \frac{2}{(m + \beta_1)^2(m - 1)} \left[- \left(x - \frac{\alpha_2}{m + \beta_2} \right) + \left(\frac{m + \beta_2}{m} \right)^m \left(x - \frac{\alpha_2}{m + \beta_2} \right)^{m+1} \right] \\
& \quad + \frac{2}{(m + \beta_1)^2(m^2 - 1)} \left[\frac{m}{m + \beta_2} - \left(\frac{m + \beta_2}{m} \right)^m \left(\left(x - \frac{\alpha_2}{m + \beta_2} \right)^{m+1} + \left(\frac{m + \alpha_2}{m + \beta_2} - x \right)^{m+1} \right) \right] \\
& \quad + \frac{\alpha_1}{(m + \beta_1)^2} \left[2 \left(x - \frac{\alpha_2}{m + \beta_2} \right) + \left(\frac{m + \beta_2}{m} \right)^m \left(2 \left(x - \frac{\alpha_2}{m + \beta_2} \right)^{m+1} - \alpha_1 \left(x - \frac{\alpha_2}{m + \beta_2} \right) \left(\frac{m + \alpha_2}{m + \beta_2} - x \right)^m \right) \right] \\
& \quad + \frac{\alpha_1(\alpha_1 - 2)}{(m + \beta_1)^2(m + 1)} \left[\frac{m}{m + \beta_2} - \left(\frac{m + \beta_2}{m} \right)^m \left(\left(x - \frac{\alpha_2}{m + \beta_2} \right)^{m+1} + \left(\frac{m + \alpha_2}{m + \beta_2} - x \right)^{m+1} \right) \right] \\
& \quad - \frac{2\alpha_1}{(m + \beta_1)^2(m - 1)} \left[(\alpha_1 - 2) \left(x - \frac{\alpha_2}{m + \beta_2} \right) + 2m \left(x - \frac{\alpha_2}{m + \beta_2} \right)^2 - \alpha_1 \left(\frac{m + \beta_2}{m} \right)^m \left(x - \frac{\alpha_2}{m + \beta_2} \right)^{m+1} \right] \\
& \quad + \frac{2\alpha_1(\alpha_1 - 2)}{(m + \beta_1)^2(m^2 - 1)} \left[\frac{m}{m + \beta_2} - \left(\frac{m + \beta_2}{m} \right)^m \left(\left(x - \frac{\alpha_2}{m + \beta_2} \right)^{m+1} + \left(\frac{m + \alpha_2}{m + \beta_2} - x \right)^{m+1} \right) \right]. \tag{18}
\end{aligned}$$

Combining (17), $\Upsilon_{3,m}^{\alpha,\beta}(x)$, and $\Upsilon_{4,m}^{\alpha,\beta}(x)$, we obtain the result for $G_{m,\lambda}^{\alpha,\beta}(s^2, x)$.

Using the same technique in the above moments, we obtain $G_{m,\lambda}^{\alpha,\beta}(s^3, x)$ and $G_{m,\lambda}^{\alpha,\beta}(s^4, x)$. $\square$

Corollary 1. Using Lemma 1, we obtain the following inequalities of central moments $G_{m,\lambda}^{\alpha,\beta}((s - x)^n, x)$ for $n = 1, 2$ and for fixed $x \in [0, 1]$ and $\lambda \in [-1, 1]$:

$$\begin{aligned}
G_{m,\lambda}^{\alpha,\beta}(s - x, x) &= \frac{\beta_2 - \beta_1}{m + \beta_1} x + \frac{\alpha_1 - \alpha_2}{m + \beta_1} \\
&+ \lambda \left\{ \frac{-2 + ((m + \beta_2)/m)^m (x - (\alpha_2/(m + \beta_2)))^m + (\alpha_1 m - 2\alpha_1 + 1)((m + \beta_2)/m)^m (((m + \alpha_2)/(m + \beta_2)) - x)^m}{(m + \beta_1)(m - 1)} x \right. \\
&+ \frac{m + 2\alpha_2}{(m + \beta_1)(m - 1)(m + \beta_2)} - \frac{\alpha_2((m + \beta_2)/m)^m (x - (\alpha_2/(m + \beta_2)))^m}{(m + \beta_1)(m - 1)(m + \beta_2)} \\
&\left. + \frac{((\alpha_1 - \alpha_1\alpha_2 - 1)m + 2\alpha_1\alpha_2 - \alpha_2)((m + \beta_2)/m)^m ((m + \alpha_2)/(m + \beta_2) - x)^m}{(m + \beta_1)(m - 1)(m + \beta_2)} \right\}
\end{aligned}$$

$$\begin{aligned}
&\leq \frac{\beta_2 + \beta_1}{m + \beta_1} x + \frac{\alpha_1 + \alpha_2}{m + \beta_1} \\
&+ \lambda \left\{ \frac{2 + ((m + \beta_2)/m)^m (x - (\alpha_2/(m + \beta_2)))^m + (\alpha_1 m + 2\alpha_1 + 1)((m + \beta_2)/m)^m (((m + \alpha_2)/(m + \beta_2)) - x)^m}{(m + \beta_1)(m - 1)} x \right. \\
&+ \frac{m + 2\alpha_2}{(m + \beta_1)(m - 1)(m + \beta_2)} + \frac{\alpha_2 ((m + \beta_2)/m)^m (x - (\alpha_2/(m + \beta_2)))^m}{(m + \beta_1)(m - 1)(m + \beta_2)} \\
&\left. + \frac{((\alpha_1 + \alpha_1\alpha_2 + 1)m + 2\alpha_1\alpha_2 + \alpha_2)((m + \beta_2)/m)^m ((m + \alpha_2)/(m + \beta_2) - x)^m}{(m + \beta_1)(m - 1)(m + \beta_2)} \right\} := \Psi_{m,\lambda}^{\alpha,\beta}(x)
\end{aligned}$$

$$\begin{aligned}
G_{m,\lambda}^{\alpha,\beta}((s-x)^2, x) &= \frac{-m^2 + (\beta_2^2 - 2\beta_2\beta_1 - 2\beta_2 + \beta_1^2)m - \beta_2^2}{m(m + \beta_1)^2} x^2 \\
&+ \frac{m^2 + (\beta_2 - 2\beta_2\alpha_2 + 2\alpha_2 + 2\beta_2\alpha_1 + 2\alpha_2\beta_1 - 2\beta_1\alpha_1)m + 2\beta_2\alpha_2}{m(m + \beta_1)^2} x + \frac{(\alpha_2^2 + \alpha_1^2 - \alpha_2 - 2\alpha_1\alpha_2)m - \alpha_2^2}{m(m + \beta_1)^2} \\
&+ \lambda \left\{ \left[\frac{4\beta_1}{(m + \beta_1)^2(m - 1)} \right. \right. \\
&- \left. \frac{2((m + \beta_2)/m)^m (x - \alpha_2/(m + \beta_2))^m - (2\alpha_1 m - 4\alpha_1 + 2)((m + \beta_2)/m)^m ((m + \alpha_2)/(m + \beta_2) - x)^m}{(m + \beta_1)(m - 1)} \right] x^2 \\
&+ \left[\frac{(2\beta_2 - 2\beta_1 + 4\alpha_2 - 4\alpha_1)m - 4\alpha_1\beta_2 - 4\alpha_2\beta_1}{(m + \beta_1)^2(m - 1)(m + \beta_2)} + \left(\frac{2m^2 + (2\alpha_1 + 2\alpha_2 + 2\beta_2 + 1)m}{(m + \beta_1)^2(m - 1)(m + \beta_2)} \right. \right. \\
&+ \left. \left. \frac{\beta_2 + 2\alpha_1\beta_2 + 2\alpha_2\beta_1}{(m + \beta_1)^2(m - 1)(m + \beta_2)} \right) \left(\frac{m + \beta_2}{m} \right)^m \left(x - \frac{\alpha_2}{m + \beta_2} \right)^m + \left(\frac{(\alpha_1^2 + 2\alpha_1\alpha_2 - 2\alpha_1 + 2)m^2}{(m + \beta_1)^2(m - 1)(m + \beta_2)} \right. \right. \\
&+ \left. \left. \frac{(\alpha_1^2\beta_2 - 2\alpha_1^2 + 2\alpha_1\alpha_2\beta_1 - 4\alpha_1\alpha_2 - 2\alpha_1\beta_1 + 2\beta_1 + 2\alpha_1 + 2\alpha_2 - 1)m - 2\alpha_1^2\beta_2 + 2\alpha_1\beta_2 - \beta_2 + 2\alpha_2\beta_1}{(m + \beta_1)^2(m - 1)(m + \beta_2)} \right. \right. \\
&+ \left. \left. \frac{-4\alpha_1\alpha_2}{(m + \beta_1)^2(m - 1)(m + \beta_2)} \right) \left(\frac{m + \beta_2}{m} \right)^m \left(\frac{m + \alpha_2}{m + \beta_2} - x \right)^m \right] x + \frac{(2\alpha_1 - 2\alpha_2 - 1)m^2}{(m + \beta_1)^2(m + \beta_2)(m - 1)} \\
&+ \frac{(2\alpha_1\beta_2 - 2\alpha_2\beta_2 + 4\alpha_1\alpha_2 - 4\alpha_2^2 - \beta_2)m + 4\alpha_1\alpha_2\beta_2}{(m + \beta_1)^2(m + \beta_2)(m - 1)} \\
&- \left. \frac{(2\alpha_2 m + 2\alpha_1\alpha_2 + \alpha_2)((m + \beta_2)/m)^m (x - (\alpha_2/(m + \beta_2)))^m}{(m + \beta_1)^2(m + \beta_2)(m - 1)} \right\}
\end{aligned}$$

$$\begin{aligned}
& + \frac{\left((\alpha_1^2 - \alpha_1^2 \alpha_2 - 2\alpha_1 + 1)m + 2\alpha_1^2 \alpha_2 - 2\alpha_1 \alpha_2 + \alpha_2 \right) \left((m + \beta_2)/m \right)^m \left((m + \alpha_2)/(m + \beta_2) - x \right)^m}{(m + \beta_1)^2 (m + \beta_2) (m - 1)} \Big\} \\
& \leq \frac{m^2 + (\beta_2^2 + 2\beta_2 \beta_1 + 2\beta_2 + \beta_2^2)m + \beta_2^2}{m(m + \beta_1)^2} x^2 + \frac{m^2 + (\beta_2 + 2\beta_2 \alpha_2 + 2\alpha_2 + 2\beta_2 \alpha_1 + 2\alpha_2 \beta_1 + 2\beta_1 \alpha_1)m + 2\beta_2 \alpha_2}{m(m + \beta_1)^2} x \\
& + \frac{(\alpha_2^2 + \alpha_1^2 + \alpha_2 + 2\alpha_1 \alpha_2)m + \alpha_2^2}{m(m + \beta_1)^2} + \lambda \left\{ \left[\frac{4\beta_1}{(m + \beta_1)^2 (m - 1)} + \frac{2 \left((m + \beta_2)/m \right)^m (x - \alpha_2/(m + \beta_2))^m}{(m + \beta_1)(m - 1)} \right. \right. \\
& + \left. \left. \frac{(2\alpha_1 m + 4\alpha_1 + 2) \left((m + \beta_2)/m \right)^m \left((m + \alpha_2)/(m + \beta_2) - x \right)^m}{(m + \beta_1)(m - 1)} \right] x^2 + \left[\frac{(2\beta_2 + 2\beta_1 + 4\alpha_2 + 4\alpha_1)m + 4\alpha_1 \beta_2 + 4\alpha_2 \beta_1}{(m + \beta_1)^2 (m - 1)(m + \beta_2)} \right. \right. \\
& + \left. \left. \frac{(2m^2 + (2\alpha_1 + 2\alpha_2 + 2\beta_2 + 1)m + \beta_2 + 2\alpha_1 \beta_2 + 2\alpha_2 \beta_1) \left((m + \beta_2)/m \right)^m (x - \alpha_2/(m + \beta_2))^m}{(m + \beta_1)^2 (m - 1)(m + \beta_2)} \right. \right. \\
& + \left. \left. \frac{\left((\alpha_1^2 + 2\alpha_1 \alpha_2 + 2\alpha_1 + 2)m^2 + (\alpha_1^2 \beta_2 + 2\alpha_1^2 + 2\alpha_1 \alpha_2 \beta_1 + 4\alpha_1 \alpha_2 + 2\alpha_1 \beta_1 + 2\beta_1 + 2\alpha_1 + 2\alpha_2 + 1)m \right)}{(m + \beta_1)^2 (m - 1)(m + \beta_2)} \right. \right. \\
& + \left. \left. \frac{2\alpha_1^2 \beta_2 + 2\alpha_1 \beta_2 + \beta_2 + 2\alpha_2 \beta_1 + 4\alpha_1 \alpha_2}{(m + \beta_1)^2 (m - 1)(m + \beta_2)} \right) \left(\frac{m + \beta_2}{m} \right)^m \left(\frac{m + \alpha_2}{m + \beta_2} - x \right)^m \right] x \\
& + \frac{(2\alpha_1 + 2\alpha_2 + 1)m^2}{(m + \beta_1)^2 (m + \beta_2) (m - 1)} + \frac{(2\alpha_1 \beta_2 + 2\alpha_2 \beta_2 + 4\alpha_1 \alpha_2 + 4\alpha_2^2 + \beta_2)m + 4\alpha_1 \alpha_2 \beta_2}{(m + \beta_1)^2 (m + \beta_2) (m - 1)} \\
& + \frac{(2\alpha_2 m + 2\alpha_1 \alpha_2 + \alpha_2) \left((m + \beta_2)/m \right)^m (x - \alpha_2/(m + \beta_2))^m}{(m + \beta_1)^2 (m + \beta_2) (m - 1)} \\
& + \frac{\left((\alpha_1^2 + \alpha_1^2 \alpha_2 + 2\alpha_1 + 1)m + 2\alpha_1^2 \alpha_2 + 2\alpha_1 \alpha_2 + \alpha_2 \right) \left((m + \beta_2)/m \right)^m \left((m + \alpha_2)/(m + \beta_2) - x \right)^m}{(m + \beta_1)^2 (m + \beta_2) (m - 1)} \Big\} \\
& := \Phi_{m,\lambda}^{\alpha,\beta}(x). \tag{19}
\end{aligned}$$

Using Lemma 1 and the linearity of $G_{m,\lambda}^{\alpha,\beta}(g(s), x)$, we have the following Corollary 2.

Corollary 2. We obtain the following equalities:

- (i) $\lim_{m \rightarrow \infty} G_{m,\lambda}^{\alpha,\beta}(s - x, x) = (\beta_2 - \beta_1)x + \alpha_1 - \alpha_2$
- (ii) $\lim_{m \rightarrow \infty} G_{m,\lambda}^{\alpha,\beta}((s - x)^2, x) = x - x^2$
- (iii) $\lim_{m \rightarrow \infty} G_{m,\lambda}^{\alpha,\beta}((s - x)^4, x) = 3x^4 - 6x^3 + (24\alpha_2^2 - 24\alpha_2 \alpha_1 - 6\alpha_1^2 + 3)x^2$

3. Convergence Properties of $G_{m,\lambda}^{\alpha,\beta}$

For the asymptotic behavior of $G_{m,\lambda}^{\alpha,\beta}(g, x)$ operators, we give the following Korovkin-type approximation theorem.

Theorem 1. If $g \in C[0, 1]$ and $\lambda \in [-1, 1]$, then $G_{m,\lambda}^{\alpha,\beta}(g, x)$ operators converge uniformly to g on $[0, 1]$, where $C[0, 1]$ is a Banach space of all continuous functions on $[0, 1]$ with norm $\|g\| := \sup_{x \in [0, 1]} |g(x)|$.

Proof. Using the equalities (i), (ii), and (iii) of Lemma 1, we obtain

$$\lim_{m \rightarrow \infty} \|G_{m,\lambda}^{\alpha,\beta}(s^i, x) - x^i\| = 0, \quad i = 0, 1, 2. \tag{20}$$

Therefore, the proof is completed using Korovkin theorem.

For $\varepsilon > 0$, the Peetre K-functional is defined as

$$K_2(g, \varepsilon) := \inf_{h \in C^2[0, 1]} [\|g - h\| + \varepsilon \|h''\|], \tag{21}$$

where $C^2[0, 1] := \{h \in C[0, 1]: h', h'' \in C[0, 1]\}$. For $g \in C[0, 1]$, the second-order modulus of continuity is defined as

$$\varphi_2(g, \sqrt{\varepsilon}) := \sup_{0 < l \leq \sqrt{\varepsilon}} \sup_{x, x+l, x+2l \in [0, 1]} |g(x+2l) - 2g(x+l) + g(x)|. \quad (22)$$

By [9, Theorem 2.4], there exists an absolute constant $C > 0$ such that

$$K_2(g, \varepsilon) \leq C\varphi_2(g, \sqrt{\varepsilon}). \quad (23)$$

Also, the usual modulus of continuity of $g \in C[0, 1]$ is defined as $\varphi(g, \varepsilon) := \sup_{0 < l \leq \varepsilon} \sup_{x, x+l \in [0, 1]} |g(x+l) - g(x)|$. $\square$

Theorem 2. We obtain the following inequality for $g \in C[0, 1]$ and $\lambda \in [-1, 1]$:

$$\begin{aligned} |G_{m,\lambda}^{\alpha,\beta}(g, x) - g(x)| &\leq 4C\varphi_2\left(g, \sqrt{\frac{(\Psi_{m,\lambda}^{\alpha,\beta})^2(x) + \Phi_{m,\lambda}^{\alpha,\beta}(x)}{8}}\right) \\ &\quad + \varphi(g, \Psi_{m,\lambda}^{\alpha,\beta}(x)), \end{aligned} \quad (24)$$

where $\Psi_{m,\lambda}^{\alpha,\beta}(x)$ and $\Phi_{m,\lambda}^{\alpha,\beta}(x)$ are defined in Corollary 1.

Proof. Let us define the following operators:

$$\begin{aligned} \tilde{G}_{m,\lambda}^{\alpha,\beta}(g, x) &= G_{m,\lambda}^{\alpha,\beta}(g, x) - g\left(\frac{m+\beta_2}{m+\beta_1}x + \frac{\alpha_1-\alpha_2}{m+\beta_1}\right) + \lambda\left\{\frac{1}{(m+\beta_1)(m-1)}\left[\frac{m}{m+\beta_2} - 2\left(x - \frac{\alpha_2}{m+\beta_2}\right)\right.\right. \\ &\quad \left.\left. + \left(\frac{m+\beta_2}{m}\right)^m\left(\left(x - \frac{\alpha_2}{m+\beta_2}\right)^{m+1} + (\alpha_1-1)\left(\frac{m+\alpha_2}{m+\beta_2} - x\right)^{m+1}\right)\right]\right. \\ &\quad \left. + \frac{\alpha_1}{m+\beta_1}\left(\frac{m+\beta_2}{m}\right)^m\left(x - \frac{\alpha_2}{m+\beta_2}\right)\left(\frac{m+\alpha_2}{m+\beta_2} - x\right)^m\right\} + g(x). \end{aligned} \quad (25)$$

From the linearity of $\tilde{G}_{m,\lambda}^{\alpha,\beta}(g, x)$ and the equalities (i) and (ii) of Lemma 1, we obtain

$$\tilde{G}_{m,\lambda}^{\alpha,\beta}(s-x, x) = 0. \quad (26)$$

Using Taylor's expansion for $h \in C^2[0, 1]$, we write

$$h(s) = h(x) + h'(x)(s-x) + \int_x^s (s-u)h''(u)du. \quad (27)$$

Applying generalized λ -Bernstein-Stancu operators to both sides of (27) and using (26), we yield

$$\begin{aligned} \tilde{G}_{m,\lambda}^{\alpha,\beta}(h, x) &= h(x) + \tilde{G}_{m,\lambda}^{\alpha,\beta}\left(\int_x^s (s-u)h''(u)du, x\right) \\ \left|\tilde{G}_{m,\lambda}^{\alpha,\beta}(h, x) - h(x)\right| &\leq \left|\int_x^{G_{m,\lambda}^{\alpha,\beta}(s,x)} (G_{m,\lambda}^{\alpha,\beta}(s,x) - u)h''(u)du\right| + \left|G_{m,\lambda}^{\alpha,\beta}\left(\int_x^s (s-u)h''(u)du, x\right)\right| \\ &\leq \frac{\|h''\|}{2}(G_{m,\lambda}^{\alpha,\beta}(s-x, x))^2 + \frac{\|h''\|}{2}G_{m,\lambda}^{\alpha,\beta}((s-x)^2, x) \\ &\leq \frac{\|h''\|}{2}\left[(\Psi_{m,\lambda}^{\alpha,\beta})^2(x) + \Phi_{m,\lambda}^{\alpha,\beta}(x)\right]. \end{aligned} \quad (28)$$

And, we find

$$\begin{aligned}
 |G_{m,\lambda}^{\alpha,\beta}(g, x) - g(x)| &\leq |\tilde{G}_{m,\lambda}^{\alpha,\beta}(g - h, x) - (g - h)(x)| + |\tilde{G}_{m,\lambda}^{\alpha,\beta}(g, x) - g(x)| \\
 &\quad + |g(G_{m,\lambda}^{\alpha,\beta}(s, x)) - g(x)| \\
 &\leq 4\|g - h\| + \left[(\Psi_{m,\lambda}^{\alpha,\beta})^2(x) + \Phi_{m,\lambda}^{\alpha,\beta}(x) \right] \frac{\|h''\|}{2} + \varphi(g, \Psi_{m,\lambda}^{\alpha,\beta}(x)).
 \end{aligned} \tag{29}$$

Therefore, if we take infimum on the right side of (29), overall $h \in C^2[0, 1]$, we obtain

$$\begin{aligned}
 |G_{m,\lambda}^{\alpha,\beta}(g, x) - g(x)| &\leq 4K_2 \left(g, \frac{(\Psi_{m,\lambda}^{\alpha,\beta})^2(x) + \Phi_{m,\lambda}^{\alpha,\beta}(x)}{8} \right) \\
 &\quad + \varphi(g, \Psi_{m,\lambda}^{\alpha,\beta}(x)).
 \end{aligned} \tag{30}$$

Using inequality (23), we have

$$\begin{aligned}
 |G_{m,\lambda}^{\alpha,\beta}(g, x) - g(x)| &\leq 4C\varphi_2 \left(g, \sqrt{\frac{(\Psi_{m,\lambda}^{\alpha,\beta})^2(x) + \Phi_{m,\lambda}^{\alpha,\beta}(x)}{8}} \right) \\
 &\quad + \varphi(g, \Psi_{m,\lambda}^{\alpha,\beta}(x)).
 \end{aligned} \tag{31}$$

Thus, Theorem 2 is proved. $\square$

Remark 1. Since $\lim_{m \rightarrow \infty} \Psi_{m,\lambda}^{\alpha,\beta}(x) = 0$ and $\lim_{m \rightarrow \infty} \Phi_{m,\lambda}^{\alpha,\beta}(x) = 0$ for $\forall x \in [0, 1]$, these limits give us a rate of pointwise convergence of the operators $G_{m,\lambda}^{\alpha,\beta}(g, x)$ to $g(x)$.

The space of the Lipschitz-type functions is defined as

$$\text{Lip}_\kappa(\xi) = \{g \in C[0, 1]: |g(y) - g(x)| \leq \kappa|y - x|^\xi\}; \tag{32}$$

$x, y \in \mathbb{R},$

where $\kappa > 0$ and $0 < \xi \leq 1$ [10].

In the following theorem, we obtain the rate of convergence of generalized λ -Bernstein–Stancu operators $G_{m,\lambda}^{\alpha,\beta}(g, x)$ for functions in $\text{Lip}_\kappa(\xi)$.

Theorem 3. If $g \in \text{Lip}_\kappa(\xi)$, $x \in [0, 1]$, and $\lambda \in [-1, 1]$, then we have

$$|G_{m,\lambda}^{\alpha,\beta}(g, x) - g(x)| \leq \kappa \left[\Phi_{m,\lambda}^{\alpha,\beta}(x) \right]^{(\xi/2)}, \tag{33}$$

where $\Phi_{m,\lambda}^{\alpha,\beta}(x)$ is defined in Corollary 1.

Proof. Because $g \in \text{Lip}_\kappa(\xi)$ and $G_{m,\lambda}^{\alpha,\beta}(g, x)$ are linear positive operators, we obtain the inequality by using Hölder's inequality:

$$\begin{aligned}
 |G_{m,\lambda}^{\alpha,\beta}(g, x) - g(x)| &\leq G_{m,\lambda}^{\alpha,\beta}(|g(s) - g(x)|, x) = \left(\frac{m + \beta_2}{m} \right)^m \sum_{j=0}^m \tilde{p}_{m,j}(\lambda, x) \left| g\left(\frac{j + \alpha_1}{m + \beta_1} \right) - g(x) \right| \\
 &\leq \left(\frac{m + \beta_2}{m} \right)^m \kappa \sum_{j=0}^m \tilde{p}_{m,j}(\lambda, x) \left| \frac{j + \alpha_1}{m + \beta_1} - x \right|^\xi \\
 &\leq \kappa \sum_{j=0}^m \left[\left(\frac{m + \beta_2}{m} \right)^m \tilde{p}_{m,j}(\lambda, x) \left(\frac{j + \alpha_1}{m + \beta_1} - x \right)^2 \right]^{(\xi/2)} \left[\left(\frac{m + \beta_2}{m} \right)^m \tilde{p}_{m,j}(\lambda, x) \right]^{(2-\xi)/2} \\
 &\leq \kappa \left[\sum_{j=0}^m \left(\frac{m + \beta_2}{m} \right)^m \tilde{p}_{m,j}(\lambda, x) \left(\frac{j + \alpha_1}{m + \beta_1} - x \right)^2 \right]^{(\xi/2)} \left[\sum_{j=0}^m \left(\frac{m + \beta_2}{m} \right)^m \tilde{p}_{m,j}(\lambda, x) \right]^{(2-\xi)/2} \\
 &\leq \kappa \left[\Phi_{m,\lambda}^{\alpha,\beta}(x) \right]^{(\xi/2)}.
 \end{aligned} \tag{34}$$

Hence, we proved Theorem 3.

FIGURE 1: Convergence of $G_{m-1}^{1,2}(f, x)$ to $f(x)$ with respect to m values. FIGURE 2: Convergence of $G_{m,1}^{1,2}(f, x)$ to $f(x)$ with respect to m values.

Finally, we give the main result of the article in Theorem 4. □

Theorem 4. If $g \in C[0, 1]$, then for every $x \in (0, 1)$ and $\lambda \in [-1, 1]$, we obtain

$$\lim_{m \rightarrow \infty} m \left[G_{m,\lambda}^{\alpha,\beta}(g, x) - g(x) \right] = g'(x) \{ (\beta_2 - \beta_1)x + \alpha_1 - \alpha_2 \} + \frac{g''(x)}{2} (x - x^2), \quad (35)$$

where $g''(x)$ exists.

Proof. Using Taylor's formula, we can write the following equation for a fixed $x \in [0, 1]$:

$$g(s) = g(x) + g'(x)(s - x) + \frac{g''(x)}{2}(s - x)^2 + r(s, x)(s - x)^2, \quad (36)$$

where $r(s, x)$ is Peano form of the remainder, $r(s, x) \in C[0, 1]$, and $\lim_{s \rightarrow x} r(s, x) = 0$. If we apply $G_{m,\lambda}^{\alpha,\beta}(g, x)$ to (36), then we have

FIGURE 3: Convergence of $G_{m,-1}^{10,11}(f, x)$ to $f(x)$ with respect to m values. FIGURE 4: Convergence of $G_{m,1}^{10,11}(f, x)$ to $f(x)$ with respect to m values.

$$\begin{aligned}
 G_{m,\lambda}^{\alpha,\beta}(g, x) - g(x) &= g'(x)G_{m,\lambda}^{\alpha,\beta}((s-x), x) + \frac{g''(x)}{2}G_{m,\lambda}^{\alpha,\beta}((s-x)^2, x), \\
 &+ G_{m,\lambda}^{\alpha,\beta}(r(s, x)(s-x)^2, x),
 \end{aligned}
 \tag{37}$$

FIGURE 5: Convergences of $G_{m,1}^{10,11}(f, x)$, $B_{20,10,11}^1(f, x)$, and $B_{20}^1(f, x)$ to $f(x)$.

and we obtain

$$\begin{aligned} \lim_{m \rightarrow \infty} m \left[G_{m,\lambda}^{\alpha,\beta}(g, x) - g(x) \right] &= g'(x) \lim_{m \rightarrow \infty} m G_{m,\lambda}^{\alpha,\beta}((s-x), x) \\ &+ \frac{g''(x)}{2} \lim_{m \rightarrow \infty} m G_{m,\lambda}^{\alpha,\beta}((s-x)^2, x) + \lim_{m \rightarrow \infty} m G_{m,\lambda}^{\alpha,\beta}(r(s, x)(s-x)^2, x). \end{aligned} \quad (38)$$

Using Cauchy-Schwarz inequality, we obtain

$$\begin{aligned} G_{m,\lambda}^{\alpha,\beta}(r(s, x)(s-x)^2, x) &\leq \left(G_{m,\lambda}^{\alpha,\beta}(r^2(s, x), x) \right)^{(1/2)} \\ &\cdot \left(G_{m,\lambda}^{\alpha,\beta}((s-x)^4, x) \right)^{(1/2)}. \end{aligned} \quad (39)$$

Because $\lim_{s \rightarrow x} r(s, x) = 0$ and $G_{m,\lambda}^{\alpha,\beta}((s-x)^4, x)$ are finite operators, we have

$$\lim_{m \rightarrow \infty} m G_{m,\lambda}^{\alpha,\beta}(r(s, x)(s-x)^2, x) = 0. \quad (40)$$

In the end, by using the equalities (i) and (ii) of Corollary 2 and (40) in (38), we yield

$$\begin{aligned} \lim_{m \rightarrow \infty} m \left[G_{m,\lambda}^{\alpha,\beta}(g, x) - g(x) \right] &= g'(x) [(\beta_2 - \beta_1)x + \alpha_1 - \alpha_2] \\ &+ \frac{g''(x)}{2} (x - x^2). \end{aligned} \quad (41)$$

Thus, we proved Theorem 4. $\square$

4. Numerical Examples

In this section, we show the theoretical results demonstrated in the previous sections by the following example.

Example 1. Let the trigonometric function $f(x) = 1 - \cos(4e^x)$ for $x \in A_m := [(\alpha_2/m + \beta_2), (m + \alpha_2/m + \beta_2)]$, $0 \leq \alpha_i \leq \beta_i$ ($i = 1, 2$), and $m = 10, 20, 50, 100$. First, let us choose $\alpha_1 = \alpha_2 = 1$ and $\beta_1 = \beta_2 = 2$.

For $\lambda = -1$, the graphs of $G_{m,\lambda}^{\alpha,\beta}(f, x)$ and $f(x)$ are shown in Figure 1.

For $\lambda = 1$, the graphs of $G_{m,\lambda}^{\alpha,\beta}(f, x)$ and $f(x)$ are shown in Figure 2.

Now, let us choose $\alpha_1 = \alpha_2 = 10$ and $\beta_1 = \beta_2 = 11$. The graph of $f(x)$ and the graph of $G_{m,\lambda}^{\alpha,\beta}(f, x)$ with $\lambda = -1$ and $\lambda = 1$ are shown in Figures 3 and 4, respectively.

As a result, Figure 5 reveals that the curve of $G_{20,1}^{\alpha,\beta}(f, x)$ with $\alpha_1 = \alpha_2 = 10$ and $\beta_1 = \beta_2 = 11$ studied in the article approaches the curve of the function $f(x)$ much better than the curves of the operators $B_{20}^1(f, x)$ and $B_{20,10,11}^1(f, x)$ defined in [5, 8], respectively.

Data Availability

The data used to support the findings of this study are included within the article.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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