Research Article

Oriented by Characteristic Roots Reduced Matrices in the Class of Semiscalarly Equivalent

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A set of polynomial 3×3-matrices of simple structure has been singled out, for which a so-called oriented by certain characteristic roots reduced matrix is established in the class of semiscalarly equivalent. The invariants of such reduced matrices and the conditions of their semiscalar equivalence are indicated. The obtained results will also be applied to the problem of similarity of sets of numerical matrices.

1. Introduction

By Theorem 1 (see Section 4 in [1] and [2]), matrix $F(x) \in M(n, \mathbf{C}[x])$ of full rank by transformation

$$F(x) \longrightarrow PF(x)Q(x) = G(x), \qquad (1)$$

where $P \in GL(n, C)$ and $Q(x) \in GL(n, C[x])$ are reduced to the lower triangular form with invariant factors on the main diagonal. We consider the case when $F(x)$ is a 3×3-matrix of simple structure and its first invariant factor is equal to one.

The latter condition does not limit the generality in the sense that everything stated in this article can be extended to 3×3-matrices, which after reduction (i.e., division of all their elements) by the first invariant factor become matrices of simple structure. By the definition of a matrix of simple structure, all its elementary divisors are linear. This concept is taken from [1], where it is introduced for polynomial matrices of arbitrary degree by analogy with the concept of numerical matrices (above the field) of simple structure with [3]. Note that the numerical matrices of a simple structure in [4] are called simple and in [5] are nondefective. Next, we will use the terminology from [1, 2], namely, the transformation (1) and the matrices $F(x)$ and $G(x)$ will be called semiscalarly equivalent (abbreviation: ssk.e., notation:

$F(x) \approx G(x)$). Our task is to construct in the class $\{PF(x)Q(x)\}$ a lower triangular matrix with invariant factors on the main diagonal and some predefined properties, to find a complete system of its invariants and to establish on this basis the conditions of ssk.e. for two such matrices. This problem for 2×2-matrices in the general case is solved by the author in [6] and for 3×3-matrices with one characteristic root in [7].

2. Preliminary Information

In class $\{PF(x)Q(x)\}$, there is a matrix of the form

$$A(x) = \left\| \begin{matrix} 1 & 0 & 0 \\ a_1(x) & \phi_1(x) & 0 \\ a_3(x) & a_2(x) & \phi_2(x) \end{matrix} \right\|, \qquad (2)$$

where $\deg a_1(x) < \deg \phi_1(x)$, $\deg a_2(x), \deg a_3(x) < \deg \phi_2(x)$, and $\phi_1(x)$ divides $a_2(x)$ and $\phi_2(x)$. Let us denote $\phi_{12}(x): = \phi_2(x)/\phi_1(x)$ and $a_2'(x): = a_2(x)/\phi_1(x)$. We assume that $\deg \phi_1(x), \deg \phi_{12}(x) > 1$. Otherwise, $A(x)$ in class $\{PF(x)Q(x)\}$ can be chosen so that $a_1(x) \equiv 0$ or $a_2(x) \equiv 0$. In these cases, the task of finding the conditions ssk.e. matrices are greatly simplified. Hereinafter, the notation M_1 and M_2 will mean the sets of roots of polynomials $\phi_1(x)$ and $\phi_{12}(x)$, respectively. Recall that element

$\alpha \in M_1 \cup M_2$ is called the characteristic root of the matrix $A(x)$ and $M_1 \cup M_2$ for class $\{PF(x)Q(x)\}$ is defined unambiguously (i.e., does not depend on the choice of matrix $A(x)$). Obviously, $A(x)$ in class $\{PF(x)Q(x)\}$ can be chosen so that, for some $\alpha_0 \in M_1$, the following condition

$$a_1(\alpha_0) = a_3(\alpha_0) = 0 \qquad (3)$$

is satisfied. In the future, this condition will be mandatory for $A(x)$ and for all triangular matrices of class $\{PF(x)Q(x)\}$. Hereinafter, record $(a(x), b(x))$ will mean the greatest common divisor of polynomials $a(x)$ and $b(x)$. We will use the corresponding notation for the greatest common divisor of three polynomials.

Proposition 1. *The greatest common divisor* $(a_1(x), a_3(x), \phi_1(x))$ *for class* $\{PF(x)Q(x)\}$ *does not depend on the choice of the matrix* $A(x)$ *with a fixed characteristic root* $\alpha_0 \in M_1$.

Proof. Let the matrix

$$B(x) = \begin{Vmatrix} 1 & 0 & 0 \\ b_1(x) & \phi_1(x) & 0 \\ b_3(x) & b_2(x) & \phi_2(x) \end{Vmatrix}, \qquad (4)$$

where $\deg b_1(x) < \deg \phi_1(x)$, $\deg b_2(x), \deg b_3(x) < \deg \phi_2(x)$, and $b_1(\alpha_0) = b_3(\alpha_0) = 0$, belongs to class $\{PF(x)Q(x)\}$.

Then, for some matrices $\|s_{ij}\|_1^3 \in GL(3, C)$ and $\|r_{ij}(x)\|_1^3 \in GL(3, C[x])$, the equality

$$\|s_{ij}\|_1^3 \begin{Vmatrix} 1 & 0 & 0 \\ a_1(x) & \phi_1(x) & 0 \\ a_3(x) & a_2(x) & \phi_2(x) \end{Vmatrix} = \begin{Vmatrix} 1 & 0 & 0 \\ b_1(x) & \phi_1(x) & 0 \\ b_3(x) & b_2(x) & \phi_2(x) \end{Vmatrix} \|r_{ij}(x)\|_1^3 \qquad (5)$$

holds. From the latter, we can write

$$r_{11}(x) = s_{11} + s_{12}a_1(x) + s_{13}a_3(x), \qquad (6)$$

$$s_{21} + s_{22}a_1(x) + s_{23}a_3(x) = b_1(x)r_{11}(x) + \phi_1(x)r_{21}(x), \qquad (7)$$

$$s_{31} + s_{32}a_1(x) + s_{33}a_3(x) = b_3(x)r_{11}(x) + b_2(x)r_{21}(x) + \phi_2(x)r_{31}(x). \qquad (8)$$

If we put in (7) and (8) $x = \alpha_0$, we get $s_{21} = 0$ and $s_{31} = 0$, respectively. Since $\phi_1(x)$ divides $r_{12}(x)$ and $r_{13}(x)$, then $(r_{11}(x), \phi_1(x)) = 1$. Therefore, from (7) and (8), we have that $(a_1(x), a_3(x), \phi_1(x))$ divides $b_1(x)$ and $b_3(x)$. Given that ssk.e. has the property of symmetry (moreover, ssk.e. is the ratio of equivalence), we can write that $(b_1(x), b_3(x), \phi_1(x))$ divides $a_1(x)$ and $a_3(x)$. Therefore, $(a_1(x), a_3(x), \phi_1(x)) = (b_1(x), b_3(x), \phi_1(x))$. The proposition is proved.

In this study, we will limit ourselves to the case when $(a_1(x), a_3(x), \phi_1(x)) = \phi_1(x)$. Then, $a_1(x) \equiv 0$ and $\phi_1(x)$ divides $a_3(x)$. Considering the opposite case, when

$(a_1(x), a_3(x), \phi_1(x)) \neq \phi_1(x)$, is the subject of another study. $\square$

3. Invariants of Triangular Matrices

Proposition 2. *If* $(a_1(x), a_3(x), \phi_1(x)) = \phi_1(x)$, *then* $(a_3(x), \phi_2(x))$ *for class* $\{PF(x)Q(x)\}$ *does not depend on the choice of matrix* $A(x)$.

Proof. Let $B(x)$ (4) be some other matrix of class $\{PF(x)Q(x)\}$ for which a condition similar to (3) holds. Since $a_1(x) = b_1(x) \equiv 0$, then, from (6) and (7), we have $r_{11}(x) = s_{11} + s_{13}a_3(x)$ and $r_{21}(x) = s_{23}a_3(x)/\phi_1(x)$. Then, (8), where $s_{31} = 0_1$, can be written as

$$s_{33}a_3(x) = b_3(x)(s_{11} + s_{13}a_3(x)) + s_{23}b_2(x)\frac{a_3(x)}{\phi_1(x)} + \phi_2(x)r_{31}(x). \qquad (9)$$

Since $s_{11} \neq 0$, the latter shows that $(a_3(x), \phi_2(x))$ divides $b_3(x)$. Therefore, given the symmetry of the ratio nsk.e., we have that $(b_3(x), \phi_2(x))$ divides $a_3(x)$ and finally $(a_3(x), \phi_2(x)) = (b_3(x), \phi_2(x))$. The proposition is proved.

We divide the set M_2 of roots of polynomial $\phi_{12}(x)$ into subsets according to the following principle: we assign elements $\alpha_1, \alpha_2 \in M_2$ to the same subset if condition $\|a_3(\alpha_1) \ a_2'(\alpha_1)\| = \|a_3(\alpha_2) \ a_2'(\alpha_2)\|$ holds for $A(x)$. Thus, we get some partition of the set M_2:

$$M_2 = K_1 \cup \ldots \cup K_m. \qquad (10)$$
$\square$

Proposition 3. *Partition (10) of the set* M_2 *for class* $\{PF(x)Q(x)\}$ *does not depend on the choice of matrix* $A(x)$ *if* $(a_1(x), a_3(x), \phi_1(x)) = \phi_1(x)$.

Proof. Suppose that, in addition to $A(x)$, the matrix $B(x)$ (4) with a condition similar to (3) also belongs to class $\{PF(x)Q(x)\}$. Then, for some $\|s_{ij}\|_1^3 \in GL(3, C)$ and $\|r_{ij}(x)\|_1^3 \in GL(3, C[x])$, equation (5) holds, from which we have (9) and

$$s_{32} + s_{33}a_2'(x) = b_3(x)(s_{12} + s_{13}a_2'(x)) + b_2'(x) \cdot (s_{22} + s_{23}a_2'(x)) + \phi_{12}(x)r_{32}(x), \qquad (11)$$

where $b_2'(x) := b_2(x)/\phi_1(x)$. Let α_1 and α_2 belong to one of the subsets of (10), for example, to K_1. If $x = \alpha_1$ and $x = \alpha_2$ is put alternately and the obtained equations are subtracted in (9), then we get $(b_3(\alpha_1) - b_3(\alpha_2))(s_{11} + s_{13}a_3(\alpha_1)) + (b_2'(\alpha_1) - b_2'(\alpha_2))s_{23}a_3(\alpha_1) = 0$.

Similarly, from (11), we obtain $(b_3(\alpha_1) - b_3(\alpha_2))(s_{12} + s_{13}a_2'(\alpha_1)) + (b_2'(\alpha_1) - b_{22}'(\alpha_2))(s_{22} + s_{23}a_2'(\alpha_1)) = 0$.

The last two equations are written in matrix form $\|b_3(\alpha_1) - b_3(\alpha_2) \ b_2'(\alpha_1) - b_2'(\alpha_2)\| \|s_{11} + s_{13}a_3(\alpha_1) \ s_{12} + s_{13}a_2'(\alpha_1)s_{23}a_3(\alpha_1)s_{22} + s_{23}a_2'(\alpha_1)\| = 0$.

It is easy to see that $\det \begin{Vmatrix} s_{11} + s_{13}a_3(\alpha_1) & s_{12} + s_{13}a_2'(\alpha_1) \\ s_{23}a_3(\alpha_1) & s_{22} + s_{23}a_2'(\alpha_1) \end{Vmatrix} = \det \|r_{ij}(\alpha_1)\|_1^2$.

Since in $\|r_{ij}(x)\|_1^3$, for α_1, we have $r_{13}(\alpha_1) = r_{23}(\alpha_1) = 0$, then $\|r_{ij}(\alpha_1)\|_1^2 \neq 0$. Therefore, $\|b_3(\alpha_1)\ \ b_2'(\alpha_1)\| = \|b_3(\alpha_2)\ \ b_2'(\alpha_2)\|$. The proposition is proved.

If, in (10), $m = 1$, then the elements $a_2(x)$ and $a_3(x)$ in $A(x)$ are determined to the nearest multiplier, and then, $A(x)$ can be chosen so that $a_2(x) \equiv 0$. In this case, $A(x)$ can be considered canonical in class $\{PF(x)Q(x)\}$. Therefore, we consider the case when $m > 1$. □

4. Reduction to a Special Triangular Form

Proposition 4. *Suppose that, for $A(x)$ (2), the conditions hold:* $(a_1(x), a_3(x), \phi_1(x)) = \phi_1(x)$, *in* (10), $m > 1$ *and $a_3(\beta_1)$ or $a_3(\beta_2)$ are nonzero for $\beta_1 \in K_1$ and $\beta_2 \in K_2$. Then, in the class $\{PF(x)Q(x)\}$, there is a matrix $B(x)$ (4), in which $b_3(\beta_1) \neq b_3(\beta_2)$ and $b_2(\beta_1) = b_2(\beta_2) = 0$.*

Proof. We can assume that $a_2(\beta_1) \equiv 0$. Otherwise, this is achieved by adding the second row of matrix $A(x)$ multiplied by some number to its third row. If $a_3(\beta_1) = a_3(\beta_2) \neq 0$ and $a_2(\beta_2) \neq 0$, then, in the first step from $A(x)$, we move on to such a matrix $B(x)$ (4) in class $\{PF(x)Q(x)\}$ that $b_3(\beta_1) \neq b_3(\beta_2)$. To do this, choose $c_1 \neq 0$ so that $1 + c_1 a_2'(\beta_i) \neq 0$ for every $\beta_i \in M_2$. Then, from the congruence $a_2'(x) - b_2'(x)(1 + c_1 a_2'(x)) \equiv 0 \pmod{\phi_{12}(x)}$, we find $b_2'(x)$, $\deg b_2'(x) < \deg \phi_{12}(x)$. Obviously, $b_2'(\beta_1) = 0$ and $b_2'(\beta_2) \neq 0$. Next, from the congruence $a_3(x) - b_3(x) - c_1 b_2'(x) a_3(x) \equiv 0 \pmod{\phi_2(x)}$, we find $b_3(x)$, $\deg b_3(x) < \deg \phi_2(x)$. It is easy to see that $b_3(\beta_1) = a_3(\beta_1)$ and $b_3(\beta_2) \neq a_3(\beta_2)$, since $c_1 b_2'(\beta_2) a_3(\beta_2) \neq 0$. Based on the above selected c_1 and obtained $b_2'(x)$ and $b_3(x)$, we construct matrices $\|s_{ij}\|_1^3 = \begin{Vmatrix} 1 & 0 & 0 \\ 0 & 1 & c_1 \\ 0 & 0 & 1 \end{Vmatrix}$ and $B(x)$ (4), where $b_2(x) = b_2'(x)\phi_1(x)$, $b_1(x) \equiv 0$, and $\|r_{ij}(x)\|_1^3 = \begin{Vmatrix} 1 & 0 & 0 \\ c_1 a_3(x)/\phi_1(x) & 1 + c_1 a_2'(x) & c_1 \phi_{12}(x) \\ a_3(x)(1 - c_1 b_2'(x)) - b_3(x)/\phi_2(x) & a_2'(x)(1 - c_1 b_2'(x)) - b_2'(x)/\phi_{12}(x) & 1 - c_1 b_2'(x) \end{Vmatrix}$

$\|r_{ij}(x)\|_1^3 \in GL(3, \mathbf{C}[x])$. We make sure that the constructed matrices $\|s_{ij}\|_1^3$, $B(x)$, $\|r_{ij}(x)\|_1^3$ together with the original matrix $A(x)$ satisfy equality (5). In order not to enter new notations, we assume that conditions $a_3(\beta_1) \neq a_3(\beta_2)$ and $a_2(\beta_1) = 0$ are satisfied for $A(x)$. In the second step in class $\{PF(x)Q(x)\}$ from $A(x)$, we pass to matrix $B(x)$ with the properties specified in proposition. To do this, first, from equation $(a_3(\beta_2) - a_3(\beta_1))c_2 = a_2'(\beta_2)$, we find $c_2 \neq 0$ and then, from congruence, $b_2'(x) + c_2(a_3(x) - a_3(\beta_1)) - a_2'(x) \equiv 0 \pmod{\phi_{21}(x)}$, we find $b_2'(x)$, $\deg b_2'(x) < \deg \phi_{12}(x)$. According to the found c_2, $b_2'(x)$, we construct matrices $\|s_{ij}\|_1^3 = \begin{Vmatrix} 1 & c_2 & 0 \\ 0 & 1 & 0 \\ 0 & c_2 & 1 \end{Vmatrix}$ and $B(x)$ (4), where $b_1(x) \equiv 0$, $b_2(x) = b_2'(x)\phi_1(x)$, $b_3(x) = a_3(x)$, and $\|r_{ij}(x)\|_1^3 = \begin{Vmatrix} 1 & c_2\phi_1(x) & 0 \\ 0 & 1 & 0 \\ 0 & a_2'(x) - c_2(a_3(x) - a_3(\beta_1)) - b_2'(x)/\phi_{12}(x) & 1 \end{Vmatrix} \in GL(3, C[x])$.

For the original matrix $A(x)$ and constructed $\|s_{ij}\|_1^3$, $B(x)$, and $\|r_{ij}(x)\|_1^3$, we check the truth of equality (5). Since $b_2'(\beta_1) = b_2'(\beta_2) = 0$, then $B(x)$ is the required matrix in class $\{PF(x)Q(x)\}$. The proposition is proved. □

Proposition 5. *If in matrices $A(x)$ (2) and $B(x)$ (4) of class $\{PF(x)Q(x)\}$ the conditions of Proposition 4 are satisfied and, for $\beta_1, \beta_2 \in M_2$, we have $a_3(\beta_1) \neq a_3(\beta_2)$, $b_3(\beta_1) \neq b_3(\beta_2)$, and $a_2(\beta_1) = a_2(\beta_2) = b_2(\beta_1) = b_2(\beta_2) = 0$, then the matrix $\|s_{ij}\|_1^3$ in (5) is the upper triangular, where $s_{12} = 0$. In addition, $(a_2'(x), \phi_{12}(x)) = (b_2'(x), \phi_{12}(x))$.*

Proof. If we put in (11) $x = \beta_1$ and $x = \beta_2$, we have $\begin{cases} s_{32} + s_{12}b_3(\beta_1) = 0, \\ s_{32} + s_{12}b_3(\beta_2) = 0, \end{cases}$ where we immediately get $s_{32} = s_{12} = 0$ because $b_3(\beta_1) \neq b_3(\beta_2)$. Then, (11) will take the form

$$s_{33}a_2'(x) = s_{13}a_2'(x)b_3(x) + b_2'(x)(s_{22} + s_{23}a_2'(x)) + \phi_{12}(x)r_{32}(x). \tag{12}$$

From (12), it is seen that $(a_2'(x), \phi_{12}(x)) = (b_2'(x), \phi_{12}(x))$, since $s_{22}, s_{33} \neq 0$. The proposition is proved.

Denote by N_1 and N_2 the subsets of the set M_2, for elements $\beta_k \in N_1$ and $\gamma_l \in N_2$ of which we have $a_3(\beta_k) \neq 0$ and $a_2(\gamma_l) \neq 0$, respectively. Propositions 2 and 5 imply the following. □

Corollary 1. *Subsets $N_1 = \{\beta_k, \beta_k \in M_2, a_3(\beta_k) \neq 0\}$ and $N_2 = \{\gamma_l, \gamma_l \in M_2, a_2(\gamma_l) \neq 0\}$ for class $\{PF(x)Q(x)\}$ for fixed $\beta_1, \beta_2 \in M_2$ do not depend on the choice of the matrix $A(x)$, which satisfies the conditions of Proposition 5.*

Since we have imposed the condition that, in (10), $m > 1$, then N_1 is not an empty set. If N_2 is empty, then $a_2(x) \equiv 0$. In this case, the problem of ssk.e. for 3×3-matrices is reduced to a similar problem for 2×2-matrices. The latter, as mentioned in Section 1, is resolved in [6]. Therefore, in the future, we assume that N_2 is not an empty set.

Let $A(x)$ satisfy the conditions of Proposition 5. It is clear that one of the values $a_3(\beta_1)$ and $a_3(\beta_2)$ is nonzero. Let $a_3(\beta_1) \neq 0$. We can also assume that, in $A(x)$, one of the following conditions is fulfilled: $a_3(\beta_3) = a_2'(\beta_3) = 1$ for some $\beta_3 \in N_1 \cap N_2$ if the intersection $N_1 \cap N_2$ is not empty or $a_3(\beta_3) = a_2'(\beta_3) = 1$ for some $\beta_3 \in N_2$ otherwise. Each of these conditions for $A(x)$ is achieved by multiplying the first two rows and the first two columns by the corresponding numerical factors. Such a matrix $A(x)$ is called *oriented by characteristic roots $\beta_1, \beta_2, \beta_3$ reduced matrix*.

Proposition 6. *If the intersection $N_1 \cap N_2$ for oriented by the same characteristic roots $\beta_1, \beta_2, \beta_3$ reduced matrices $A(x)$ (2) and $B(x)$ (4) of class $\{PF(x)Q(x)\}$ is not empty, then $s_{11} = s_{22}$ in the matrix $\|s_{ij}\|_1^3$ of equation (5).*

Proof. According to Proposition 5, the matrix $\|s_{ij}\|_1^3$ in question is an upper triangular, and $s_{12} = 0$. Equations (9) and (12) hold for elements of matrices $A(x)$ and $B(x)$. From them, we can get a congruence:

$$s_{11}a_2'(x)b_3(x) - s_{22}b_2'(x)a_3(x) \equiv 0\,(\mathrm{mod}\,\phi_{12}(x)). \quad (13)$$

If we put, in (13), $x = \beta_3$, we immediately get $s_{11} - s_{22} = 0$. The proposition is proved. $\qquad\square$

5. Conditions of Semiscalar Equivalence of Reduced Matrices

Theorem 1. *Let $A(x)$ (2) and $B(x)$ (4) be oriented by the same characteristic roots $\beta_1, \beta_2, \beta_3$ reduced matrices and*
$$\Omega_A(x) := \det\left\|\begin{matrix} a_3(x) & a_2'(x) & 1 \\ a_3(\beta_1) & 0 & 1 \\ 1 & 1 & 1 \end{matrix}\right\|, \; \Omega_B(x) := \det\left\|\begin{matrix} b_3(x) & b_2'(x) & 1 \\ b_3(\beta_1) & 0 & 1 \\ 1 & 1 & 1 \end{matrix}\right\|.$$

If, for the sets N_1 and N_2 defined in the corollary, the intersection $N_1 \cap N_2$ is not empty, then the matrices $A(x)$ and $B(x)$ are ssk.e. if and only if the following conditions are met:

(i) *$a_2'(\alpha)/a_3(\alpha) = b_2'(\alpha)/b_3(\alpha)$ for each $\alpha \in N_1 \cap N_2$.*

(ii) *Values $\Omega_A(\beta)$, $\Omega_B(\beta)$ are simultaneously equal to zero or are nonzero for each $\beta \in N_1 \cup N_2$.*

(iii) *If $\Omega_A(\beta_i) \neq 0$ for some $\beta_i \in N_1$ or $\Omega_A(\gamma_j) \neq 0$ for some $\gamma_j \in N_2$, then*

$$\frac{\Omega_A(\beta_k)a_3(\beta_i)}{\Omega_A(\beta_i)a_3(\beta_k)} = \frac{\Omega_B(\beta_k)b_3(\beta_i)}{\Omega_B(\beta_i)b_3(\beta_k)},$$
$$\frac{\Omega_A(\gamma_l)a_3(\beta_i)}{\Omega_A(\beta_i)a_2'(\gamma_l)} = \frac{\Omega_B(\gamma_l)b_3(\beta_i)}{\Omega_B(\beta_i)b_2'(\gamma_l)}, \quad (14)$$

or

$$\frac{\Omega_A(\beta_k)a_2'(\gamma_j)}{\Omega_A(\gamma_j)a_3(\beta_k)} = \frac{\Omega_B(\beta_k)b_2'(\gamma_j)}{\Omega_B(\gamma_j)b_3(\beta_k)},$$
$$\frac{\Omega_A(\gamma_l)a_2'(\gamma_j)}{\Omega_A(\gamma_j)a_2'(\gamma_l)} = \frac{\Omega_B(\gamma_l)b_2'(\gamma_j)}{\Omega_B(\gamma_j)b_2'(\gamma_l)}, \quad (15)$$

for each $\beta_k \in N_1$ and $\gamma_l \in N_2$.

If the intersection $N_1 \cap N_2$ is empty, then the matrices $A(x)$ and $B(x)$ are ssk.e. if and only if the following conditions are met:

(1) *Values $a_3(\beta_k) - 1$ and $b_3(\beta_k) - 1$, as well as values $a_2'(\gamma_l) - 1$ and $b_2'(\gamma_l) - 1$, are simultaneously equal to zero or are nonzero for each $\beta_k \in N_1$ and $\gamma_l \in N_2$.*

(2) *If $a_3(\beta_i) \neq 1$ for some $\beta_i \in N_1$, then*

$$\frac{(a_3(\beta_k) - 1)a_3(\beta_i)}{(a_3(\beta_i) - 1)a_3(\beta_k)} = \frac{(b_3(\beta_k) - 1)b_3(\beta_i)}{(b_3(\beta_i) - 1)b_3(\beta_k)}, \quad (16)$$

for every $\beta_k \in N_1$, and if $a_2'(\gamma_j) \neq 1$ for some $\gamma_j \in N_2$, then

$$\frac{(a_2'(\gamma_l) - 1)a_2'(\gamma_j)}{(a_2'(\gamma_j) - 1)a_2'(\gamma_l)} = \frac{(b_2'(\gamma_l) - 1)b_2'(\gamma_j)}{(b_2'(\gamma_j) - 1)b_2'(\gamma_l)}, \quad (17)$$

for each $\gamma_l \in N_2$.

Proof (necessity). Let $A(x) \approx B(x)$. Then, for the elements of matrices $A(x)$ and $B(x)$ are equations (9) and (12), in which according to Proposition 6 in the absence of intersection of sets N_1 and N_2 we have $s_{11} = s_{22}$. Therefore, from (13), we immediately have (i). If in equations (9) and (12) put $x = \beta_1$, $x = \beta_3$, $x = \beta_k$, and $x = \gamma_l$, where β_k and γ_l mean an arbitrary element of N_1 and N_2, respectively, the result can be written in the matrix form as follows:

$$\left\|\begin{matrix} \dfrac{1}{a_3(\beta_1)} & -\dfrac{1}{b_3(\beta_1)} & 1 & 0 \\ 1 & -1 & 1 & 1 \\ \dfrac{1}{a_3(\beta_k)} & -\dfrac{1}{b_3(\beta_k)} & 1 & \dfrac{b_2'(\beta_k)}{b_3(\beta_k)} \\ \dfrac{1}{a_2'(\gamma_l)} & -\dfrac{1}{b_2'(\gamma_l)} & \dfrac{a_3(\gamma_l)}{a_2'(\gamma_l)} & 1 \end{matrix}\right\| \left\|\begin{matrix} s_{11} \\ s_{33} \\ s_{13} \\ s_{23} \end{matrix}\right\| = 0. \quad (18)$$

Hence, we have

$$\left\|\begin{matrix} \dfrac{1}{a_3(\beta_1)} & -\dfrac{1}{b_3(\beta_1)} & 1 & 0 \\ 1 - \dfrac{1}{a_3(\beta_1)} & -\left(1 - \dfrac{1}{b_3(\beta_1)}\right) & 0 & 1 \\ \dfrac{\Omega_A(\beta_k)}{a_3(\beta_k)a_3(\beta_1)} & -\dfrac{\Omega_B(\beta_k)}{b_3(\beta_k)b_3(\beta_1)} & 0 & 0 \\ \dfrac{\Omega_A(\gamma_l)}{a_2'(\gamma_l)a_3(\beta_1)} & -\dfrac{\Omega_B(\gamma_l)}{b_2'(\gamma_l)b_3(\beta_1)} & 0 & 0 \end{matrix}\right\| \left\|\begin{matrix} s_{11} \\ s_{33} \\ s_{13} \\ s_{23} \end{matrix}\right\| = 0. \quad (19)$$

From the last equality (ii), since s_{11}, $s_{33} \neq 0$. If $\Omega_A(\beta_i) \neq 0$ for some $\beta_i \in N_1$, then, from equality,
$$\left\|\begin{matrix} \Omega_A(\beta_i)/a_3(\beta_i)a_3(\beta_1) & \Omega_B(\beta_i)/b_3(\beta_i)b_3(\beta_1) \\ \Omega_A(\beta_k)/a_3(\beta_k)a_3(\beta_1) & \Omega_B(\beta_k)/b_3(\beta_k)b_3(\beta_1) \\ \Omega_A(\gamma_l)/a_2'(\gamma_l)a_3(\beta_1) & \Omega_B(\gamma_l)/b_2'(\gamma_l)b_3(\beta_1) \end{matrix}\right\| \left\|\begin{matrix} s_{11} \\ s_{33} \end{matrix}\right\| = 0.$$

For arbitrary $\beta_k \in N_1$ and $\gamma_l \in N_2$, we have each equality with (14), since s_{11}, $s_{33} \neq 0$. If $\Omega_A(\gamma_j) \neq 0$ for some $\gamma_j \in N_2$, then, for any $\beta_k \in N_1$ and $\gamma_l \in N_2$, similarly we have each of equation (15).

Now, let the sets N_1, N_2 not intersect and $\beta_k \in N_1$ and $\gamma_l \in N_2$. If we put $x = \beta_1$, $x = \beta_k$ and $x = \beta_3$, $x = \gamma_l$, respectively, in (9) and (12), we obtain equations that can be written as

$$\left\|\begin{array}{ccccc} 1 & 0 & -1 & 1 & 0 \\ \dfrac{1}{a_3(\beta_k)} & 0 & -\dfrac{1}{b_3(\beta_k)} & 1 & 0 \\ 0 & 1 & -1 & 0 & 1 \\ 0 & \dfrac{1}{a_2'(\gamma_l)} & -\dfrac{1}{b_2'(\gamma_l)} & 0 & 1 \end{array}\right\| \left\|\begin{array}{c} s_{11} \\ s_{22} \\ s_{33} \\ s_{13} \\ s_{23} \end{array}\right\| = 0. \qquad (20)$$

Hence, we have

$$\left\|\begin{array}{ccc} 1 - 1/a_3(\beta_k) & 0 & -(1 - 1/b_3(\beta_k)) \\ 0 & 1 - 1/a_2'(\gamma_l) & -(1 - 1/b_2'(\gamma_l)) \end{array}\right\| \left\|\begin{array}{c} s_{11} \\ s_{22} \\ s_{33} \end{array}\right\| = 0.$$

Since $s_{11}, s_{22}, s_{33} \neq 0$, then, from the last equality, we obtain (j). If $a_3(\beta_i) \neq 1$, for some $\beta_i \in N_1$, then, from equality,

$$\left\|\begin{array}{ccc} 1 - \dfrac{1}{a_3(\beta_i)} & 0 & -\left(1 - \dfrac{1}{b_3(\beta_i)}\right) \\ 1 - \dfrac{1}{a_3(\beta_k)} & 0 & -\left(1 - \dfrac{1}{b_3(\beta_k)}\right) \end{array}\right\| \left\|\begin{array}{c} s_{11} \\ s_{22} \\ s_{33} \end{array}\right\| = 0. \qquad (21)$$

We get (16) for each $\beta_k \in N_1$. Similarly, if $a_2'(\gamma_j) \neq 1$, for some $\gamma_j \in N_2$, then, from equality,

$$\left\|\begin{array}{ccc} 0 & 1 - \dfrac{1}{a_2'(\gamma_j)} & -\left(1 - \dfrac{1}{b_2'(\gamma_j)}\right) \\ 0 & 1 - \dfrac{1}{a_2'(\gamma_l)} & -\left(1 - \dfrac{1}{b_2'(\gamma_l)}\right) \end{array}\right\| \left\|\begin{array}{c} s_{11} \\ s_{22} \\ s_{33} \end{array}\right\| = 0. \qquad (22)$$

We obtain (17) for every $\gamma_l \in N_2$. The necessity is proved. $\qquad\square$

5.1. Sufficiency. Let the sets N_1, N_2 intersect. Consider (19) as an equation with the unknown $\left\| s_{11}\ \ s_{33}\ \ s_{13}\ \ s_{23} \right\|^T$. We mean that β_k and γ_l acquire all values from sets N_1 and N_2, respectively. Therefore, the matrix on the left side of this equation may have more rows than four. Under the conditions of the theorem, the rank of this matrix is equal to 2 if $\Omega_A(\beta_k) = \Omega_A(\gamma_l) = 0$ for every $\beta_k \in N_1$ and $\gamma_l \in N_2$. Otherwise, when $\Omega_A(\beta_i) \neq 0$ for some $\beta_i \in N_1$ or $\Omega_A(\gamma_j) \neq 0$ for some $\gamma_j \in N_2$, the rank of the matrix is 3. In each case, this equation has a solution $\left\| s_{110}\ \ s_{330}\ \ s_{130}\ \ s_{230} \right\|^T$ with nonzero first two components s_{110}, s_{330}. If we take into account (i), then, from (19), we can proceed to the equivalent equation (18). Since, as already mentioned, β_k and γ_l acquire all values from N_1 and N_2, respectively; based on the solution $\left\| s_{110}\ \ s_{330}\ \ s_{130}\ \ s_{230} \right\|^T$ of equation (18), we can write such a system of congruences:

$$\begin{cases} s_{330}a_3(x) - b_3(x)(s_{110} + s_{130}a_3(x)) - s_{230}a_3(x)b_2'(x) \equiv 0 \,(\mathrm{mod}\phi_{12}(x)), \\ s_{330}a_2'(x) - b_2'(x)(s_{220} + s_{230}a_2'(x)) - s_{130}a_2'(x)b_3(x) \equiv 0 \,(\mathrm{mod}\phi_{12}(x)), \end{cases} \qquad (23)$$

where $s_{220} = s_{110}$ and $s_{110}, s_{330} \neq 0$. The first of the congruences is performed modulo $\phi_2(x) = \phi_1(x)\phi_{12}(x)$ because $(\phi_1(x), \phi_{12}(x)) = 1$ and $\phi_1(x)$ divides $a_3(x)$ and $b_3(x)$. Also, we can pass on module $\phi_2(x)$ and in the second congruence after its multiplication by $\phi_1(x)$. So, we obtain $\{s_{330}a_3(x) - b_3(x)(s_{110} + s_{130}a_3 \quad (x)) - s_{230}a_3(x)b_2'(x) \equiv 0\,(\mathrm{mod}\phi_2 \ (x)), s_{330}a_2(x) - b_2(x)(s_{220} + s_{230}a_2' \ (x)) - s_{130}a_2 (x)b_3(x) \equiv 0\,(\mathrm{mod}\phi_2(x))$.

From the elements of solution $\left\| s_{110}\ \ s_{330}\ \ s_{130}\ \ s_{230} \right\|^T$, we construct a matrix $\|s_{ij}\|_1^3 = \left\|\begin{array}{ccc} s_{110} & 0 & s_{130} \\ 0 & s_{220} & s_{230} \\ 0 & 0 & s_{330} \end{array}\right\|$, where, as before, $s_{220} = s_{110}$, and the matrix $\|r_{ij}(x)\|_1^3$, where $r_{11}(x) = s_{110} + s_{130}a_3(x)$, $r_{12}(x) = s_{130}a_2(x)$, $r_{13}(x) = s_{130}\phi_2(x)$, $r_{21}(x) = s_{230}a_3(x)/\phi_1(x) \in C[x]$, $r_{22}(x) = s_{220} + s_{230}a_2'(x)$, $r_{23}(x) = s_{230}\phi_{12}(x)$, $r_{31}(x) = (s_{330}a_3(x) - b_3(x) \ (s_{110} + s_{130}a_3(x)) - s_{230}a_3(x)b_2'(x))/\phi_2(x) \in C[x], r_{31}(x) = (s_{330}a_2 (x) - b_2(x)(s_{220} + s_{230}a_2'(x)) - s_{130}a_2(x)b_3(x))/\phi_2(x) \in C [x]$, and $r_{33}(x) = s_{330} - s_{130}b_3(x) - s_{230}b_2'(x)$.

For the constructed matrices $\|s_{ij}\|_1^3$ and $\|r_{ij}(x)\|_1^3$ and for the original $A(x)$ and $B(x)$, we are convinced of the truth of equality (5). This means that $A(x) \approx B(x)$.

Now, let set that N_1 and N_2 do not intersect. Consider (21) and (22) together as a system of two equations with the unknown $\left\| s_{11}\ \ s_{22}\ \ s_{33} \right\|^T$. Note that, in this system β_k and γ_l, acquire all values from N_1 and N_2, respectively. Under the conditions (1) and (2) of the theorem, the rank of the matrix of this system does not exceed 2. It is also easy to see that this system has a solution of $\left\| s_{110}\ \ s_{220}\ \ s_{330} \right\|^T$ with nonzero of all three of its components, i.e., $s_{110}, s_{220}, s_{330} \neq 0$. If we consider (20) as an equation with the unknown $\left\| s_{11}\ \ s_{22}\ \ s_{33}\ \ s_{13}\ \ s_{23} \right\|^T$ and put in this equation $\left\| s_{110}\ \ s_{220}\ \ s_{330} \right\|^T$ instead of $\left\| s_{11}\ \ s_{22}\ \ s_{33} \right\|^T$, we can unambiguously find $s_{13} = s_{130}$ and $s_{23} = s_{230}$. This will find the solution $\left\| s_{110}\ \ s_{220}\ \ s_{330}\ \ s_{130}\ \ s_{230} \right\|^T$ of equation (20) with nonzero of first three components. Based on this solution, we can write a system of congruences (23). Then, by the same considerations as in the previous case, we come to the conclusion that $A(x) \approx B(x)$. The theorem is proved.

6. Conclusions

In this paper, a special form is established for the selected set of matrices of simple structure, which is achieved with the help of ssk.e. transformations and is called the oriented by

characteristic roots reduced matrix (Proposition 4). The matrix of this special shape is lower triangular and has invariant factors on the main diagonal and some predefined properties. Propositions 1–3 and 5 and Corollary indicate the invariants of the reduced matrix. Propositions 5 and 6 set out the form of the left transform matrix, which together with some right transform (polynomial) matrix translates one reduced matrix into another within class $\{PF(x)Q(x)\}$. This matrix has an upper triangular form with a zero element in position $(1, 2)$. In some cases, it has the same first two diagonal elements. The theorem specifies the necessary and sufficient conditions under which the reduced matrices are ssk.e. Thus, Propositions 1–3 and 5, corollary, and theorem indicate a complete system of invariants of the reduced matrix with respect to ssk.e. It is clear that theorem can be applied to establish ssk.e for arbitrary matrices $F(x)$ and $G(x)$ (not reduced) if they belong to the set selected in this paper. Namely, $F(x)$ and $G(x)$ are ssk.e. if and only if their oriented by the same characteristic roots reduced matrices $A(x)$ and $B(x)$ satisfy the conditions of theorem. Also, as already mentioned in Section 1, everything described in this article can be extended to 3×3-matrices, which after division by the first invariant factor become matrices of simple structure. The proved theorem can be applied to the problem of classification of finite sets of numerical matrices with accuracy to similarity. In particular, let sets F and G of numerical (from the field) 3×3-matrices be given. On the matrices of these sets, as on the coefficients, we construct matrix polynomials $F(x)$ and $G(x)$ with single higher coefficients. Suppose that, after division by the first invariant factor of the polynomial matrices $F(x)$ and $G(x)$ obtained in this way, we have particles $F'(x)$ and $G'(x)$ of simple structure. Then, for the similarity of sets F and G, it is necessary and sufficient to satisfy the conditions of theorem for consolidated matrices $A(x)$ and $B(x)$ such that $A(x) \approx F'(x)$ and $B(x) \approx G'(x)$.

Data Availability

The data used to support the findings of the study are given within the article as references.

Conflicts of Interest

The author declares that there are no conflicts of interest.

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