Research Article

Some New Results of Mitrinović–Cusa’s and Related Inequalities Based on the Interpolation and Approximation Method

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In this paper, new refinements and improvements of Mitrinović–Cusa’s and related inequalities are presented. First, we give new polynomial bounds for $\sin(x)$ and $\cos(x)$ functions using the interpolation and approximation method. Based on the obtained results of the above two functions, we establish new bounds for Mitrinović–Cusa’s, Wilker’s, Huygens’, Wu–Srivastava’s, and Neuman–Sándor’s inequalities. The analysis results show that our bounds are tighter than the previous methods.

1. Introduction

Trigonometric inequalities play an important role in pure and applied mathematics and have been used in various fields. This study starts from the following inequality:

$$(\cos(x))^{(1/3)} < \sin(x) < \frac{\sin(x)}{x} < \frac{2 + \cos(x)}{3}, \quad 0 < x < \frac{\pi}{2}, \quad (1)$$

which is the focus of many researchers. The left-side inequality of (1) was first proved by Mitrinović [1, 2], which is called as Mitrinović–s inequality. The right-side inequality of (1) was proposed by the German philosopher and theologian Nicolaus de Cusa and proved explicitly by Huygens [3]. Therefore, inequality (1) is called as Mitrinović–Cusa’s inequality.

Huygens’ inequality is

$$2 \frac{\sin(x)}{x} + \frac{\tan(x)}{x} > 3, \quad 0 < x < \frac{\pi}{2}, \quad (2)$$

which is a consequence of inequality (1) [3].

The following inequality

$$\left(\frac{\sin(x)}{x}\right)^2 + \frac{\tan(x)}{x} > 2, \quad 0 < x < \frac{\pi}{2}, \quad (3)$$

due to Wilker, is called as Wilker’s inequality [4].

Wu and Srivastava [5] proved the following inequality:

$$\left(\frac{x}{\sin(x)}\right)^2 + \frac{x}{\tan(x)} > 2, \quad 0 < x < \frac{\pi}{2}. \quad (4)$$

Neuman and Sándor [6] presented the following inequality:

$$3 \frac{x}{\sin(x)} + \cos(x) > 4, \quad 0 < x < \frac{\pi}{2}. \quad (5)$$

Inequalities (1)–(5) have attracted the attention of many scholars [7–32].

Bhayo and Sándor [33] obtained the following results, for $x \in (0, (\pi/2))$:

$$\frac{2 + \cos(x) - (x/\pi)^3}{3} < \frac{\sin(x)}{x} < \frac{2 + \cos(x) - (x/\pi)^4}{3}, \quad (6)$$

$$\frac{\sin(x)}{x} < \frac{\pi^2 - x^2}{\pi^2 + \alpha x^2} < \frac{2 + \cos(x)}{3} < \frac{\pi^2 - x^2}{\pi^2 + \beta x^2}, \quad (7)$$

where $\alpha = (\pi^2/6) - 1$ and $\beta = (1/2)$ are the best possible constants.

Yang [34] gave the following inequalities:

$$\begin{aligned} \frac{\sin(x)}{x} &> (\cos(px))^{1/(3p^2)}, \quad 0 < x < \frac{\pi}{2}, p \in [p_1, 1], \\ \frac{\sin(x)}{x} &< (\cos(qx))^{1/(3q^2)}, \quad 0 < x < \frac{\pi}{2}, q \in (0, (\sqrt{5}/5)], \end{aligned} \quad (8)$$

where $p_1 = 0.45346830977067 \dots$ is the unique root of equation $\ln(2/\pi) - (1/3p^2)\ln \cos(p\pi/2) = 0$ in $p \in (0, 1]$.

Yang [35] gave further results:

$$\begin{aligned} \frac{2p + (p+3)\cos(x)}{(3p+1) + 2\cos(x)} &< (\cos(x))^{(1/3)} < \frac{2q + (q+3)\cos(x)}{(3q+1) + 2\cos(x)} \\ &< \frac{\sin(x)}{x} \\ &< \frac{2r + (r+3)\cos(x)}{(3r+1) + 2\cos(x)} < \frac{2 + \cos(x)}{3} \\ &< \frac{2s + (s+3)\cos(x)}{(3s+1) + 2\cos(x)}, \end{aligned} \quad (9)$$

where $x \in (0, (\pi/2))$, $p = 0$, $q \in [1, (\pi-3)^{-1}]$, $r \in [9, \infty]$, and $s \in [-\infty, -1]$.

Sumner et al. [36] obtained a further result of Wilker's inequality as follows:

$$2 + \frac{16}{\pi^4} x^3 \tan(x) < \left(\frac{\sin(x)}{x} \right)^2 + \frac{\tan(x)}{x} < 2 + \frac{8}{45} x^3 \tan(x), \quad 0 < x < \frac{\pi}{2}. \quad (10)$$

Furthermore, $(16/\pi^4)$ and $(8/45)$ are the best constants.

Chen and Cheung [37] proved the following inequalities, for $x \in (0, (\pi/2))$:

$$3 + \frac{3}{20} x^3 \tan(x) < 2 \frac{\sin(x)}{x} + \frac{\tan(x)}{x} < 3 + \left(\frac{2}{\pi} \right)^4 x^3 \tan(x), \quad (11)$$

$$3 + \frac{3}{20} x^4 + \frac{3}{56} x^5 \tan(x) < 2 \frac{\sin(x)}{x} + \frac{\tan(x)}{x} < 3 + \frac{3}{20} x^4 + \left(\frac{2}{\pi} \right)^6 x^5 \tan(x), \quad (12)$$

$$2 + \frac{8}{45} x^4 + \frac{16}{315} x^5 \tan(x) < \left(\frac{\sin(x)}{x} \right)^2 + \frac{\tan(x)}{x} < 2 + \frac{8}{45} x^4 + \left(\frac{2}{\pi} \right)^6 x^5 \tan(x), \quad (13)$$

$$2 + \frac{8}{45} x^4 + \frac{16}{315} x^6 + \frac{104}{4725} x^7 \tan(x) < \left(\frac{\sin(x)}{x} \right)^2 + \frac{\tan(x)}{x} < 2 + \frac{8}{45} x^4 + \frac{16}{315} x^6 + \left(\frac{2}{\pi} \right)^8 x^7 \tan(x), \quad (14)$$

$$\left(\frac{x}{\sin(x)} \right)^2 + \frac{x}{\tan(x)} < 2 + \frac{2}{45} x^3 \tan(x), \quad (15)$$

$$\left(\frac{x}{\sin(x)} \right)^2 + \frac{x}{\tan(x)} < 2 + \frac{2}{45} x^4 + \frac{8}{945} x^5 \tan(x), \quad (16)$$

where $(3/20)$, $(2/\pi)^4$, $(3/56)$, $(2/\pi)^6$, $(16/315)$, $(104/4725)$, $(2/\pi)^8$, $(2/45)$, and $(8/945)$ are the best possible constants.

Nenezic et al. [38] proved the following inequalities, for $x \in (0, (\pi/2))$:

$$\begin{aligned}
2 + \left(\frac{8}{45} - \frac{8}{945}x^2\right)x^3 \tan(x) &< \left(\frac{\sin(x)}{x}\right)^2 + \frac{\tan(x)}{x} \\
&< 2 + \left(\frac{8}{45} - \frac{8}{945}x^2 + \frac{480\pi^6 - 40320\pi^4 + 3628800}{14175\pi^8}x^4\right)x^3 \tan(x),
\end{aligned} \tag{17}$$

$$\begin{aligned}
2 + \left(\frac{16}{\pi^4} + \left(\frac{160}{\pi^5} - \frac{16}{\pi^3}\right)\left(\frac{\pi}{2} - x\right)\right)x^3 \tan(x) &< \left(\frac{\sin(x)}{x}\right)^2 + \frac{\tan(x)}{x} \\
&< 2 + \left(\frac{16}{\pi^4} + \left(\frac{160}{\pi^5} - \frac{16}{\pi^3}\right)\left(\frac{\pi}{2} - x\right) + \left(\frac{960}{\pi^6} - \frac{96}{\pi^4}\right)\left(\frac{\pi}{2} - x\right)^2\right)x^3 \tan(x).
\end{aligned} \tag{18}$$

Chen and Paris [39] gave the following inequalities, for $x \in (0, (\pi/2))$:

$$3 + \left(\frac{3}{20} + \frac{1}{280}x^2 + \frac{23}{33600}x^4\right)x^3 \tan(x) < 2 \frac{\sin(x)}{x} + \frac{\tan(x)}{x} < 3 + \left(\frac{3}{20} + \frac{1}{280}x^2 + \frac{17920 - 168\pi^4 - \pi^6}{70\pi^8}x^4\right)x^3 \tan(x), \tag{19}$$

$$2 + \left(\frac{8}{45} - \frac{8}{945}x^2 + \frac{16}{14175}x^4\right)x^3 \tan(x) < \left(\frac{\sin(x)}{x}\right)^2 + \frac{\tan(x)}{x} < 2 + \left(\frac{8}{45} - \frac{8}{945}x^2 + \frac{241920 - 2688\pi^4 + 32\pi^6}{945\pi^8}x^4\right)x^3 \tan(x), \tag{20}$$

$$2 + \left(\frac{2}{45} - \frac{2}{315}x^2 - \frac{224 - 8\pi^2}{315\pi^4}x^4\right)x^3 \tan(x) < \left(\frac{x}{\sin(x)}\right)^2 + \frac{x}{\tan(x)} < 2 + \left(\frac{2}{45} - \frac{2}{315}x^2 - \frac{4}{1575}x^4\right)x^3 \tan(x). \tag{21}$$

Wang [40] proved the following inequality:

$$\begin{aligned}
2 + \frac{2}{45}x^3 \sin(x) &< \left(\frac{x}{\sin(x)}\right)^2 + \frac{x}{\tan(x)} < 2 \\
&+ \left(\frac{2}{\pi} - \frac{16}{\pi^3}\right)x^3 \sin(x), \quad 0 < x < \frac{\pi}{2},
\end{aligned} \tag{22}$$

where $(2/45)$ and $(2/\pi) - (16/\pi^3)$ are the best possible constants.

Jiang et al. [41] gave the following inequality:

$$4 + \frac{1}{10}x^3 \sin(x) < 3 \frac{x}{\sin(x)} + \cos(x) < 4 + \frac{12\pi - 32}{\pi^3}x^3 \sin(x),$$

where $(1/10)$ and $(12\pi - 32/\pi^3)$ are the best possible numbers.

In particular, Mortici [42] gave the improvements of inequalities (1)–(5), for every $0 < x < (\pi/2)$:

$$-\frac{x^4}{15} < \cos(x) - \left(\frac{\sin(x)}{x}\right)^3 < -\frac{x^4}{15} + \frac{23x^6}{1890}, \tag{24}$$

$$-\frac{1}{180}x^4 < \frac{\sin(x)}{x} - \frac{\cos(x) + 2}{3} < -\frac{1}{180}x^4 + \frac{1}{3780}x^6, \tag{25}$$

$$3 + \left(\frac{3}{20}x^4 - \frac{3}{140}x^6\right) \frac{1}{\cos(x)} < 2 \frac{\sin(x)}{x} + \frac{\tan(x)}{x} < 3 + \frac{3}{20} \frac{x^4}{\cos(x)}, \tag{26}$$

$$2 + \left(\frac{8}{45}x^4 - \frac{8}{105}x^6\right) \frac{1}{\cos(x)} < \left(\frac{\sin(x)}{x}\right)^2 + \frac{\tan(x)}{x} < 2 + \frac{8}{45} \frac{x^4}{\cos(x)}, \tag{27}$$

$$\left(\frac{x}{\sin(x)}\right)^2 + \frac{x}{\tan(x)} > 2 + \frac{2}{45}x^4, \quad (28)$$

$$3\frac{x}{\sin(x)} + \cos(x) > 4 + \frac{1}{10}x^4 + \frac{1}{210}x^6. \quad (29)$$

Malešević et al. [43] further improved the above results of inequalities (24)–(29), for $x \in (0, (\pi/2))$ and $n \in \mathbb{N}$:

$$\sum_{k=2}^{2n} (-1)^k A(x) x^{2k} < \cos(x) - \left(\frac{\sin(x)}{x}\right)^3 < \sum_{k=2}^{2n+1} (-1)^k A(k) x^{2k}, \quad (30)$$

$$\sum_{k=2}^{2n} (-1)^{k+1} B(x) x^{2k} < \frac{\sin(x)}{x} - \frac{\cos(x) + 2}{3} < \sum_{k=2}^{2n+1} (-1)^{k+1} B(k) x^{2k}, \quad (31)$$

$$3 + \frac{1}{\cos(x)} \sum_{k=2}^{2n+1} (-1)^k C(k) x^{2k} < 2 \frac{\sin(x)}{x} + \frac{\tan(x)}{x} < 3 + \frac{1}{\cos(x)} \sum_{k=2}^{2n} (-1)^k C(k) x^{2k}, \quad (32)$$

$$2 + \frac{1}{\cos(x)} \sum_{k=2}^{2n+1} (-1)^k D(x) x^{2k} < \left(\frac{\sin(x)}{x}\right)^2 + \frac{\tan(x)}{x} < 2 + \frac{1}{\cos(x)} \sum_{k=2}^{2n} (-1)^k D(k) x^{2k}, \quad (33)$$

$$2 + \sum_{k=2}^{m-1} \frac{|\mathbf{B}_{2k}| (2k-2) 4^k}{(2k)!} x^{2k} + \left(\frac{2x}{\pi}\right)^{2m} \left(\frac{\pi^2}{4} - 2 - \sum_{k=2}^{m-1} \frac{|\mathbf{B}_{2k}| (2k-2) 4^k}{(2k)!} \left(\frac{\pi}{2}\right)^{2k}\right) > \left(\frac{x}{\sin(x)}\right)^2 + \frac{x}{\tan(x)} > 2 + \sum_{k=2}^m \frac{|\mathbf{B}_{2k}| (2k-2) 4^k}{(2k)!} x^{2k}, \quad (34)$$

$$\begin{aligned} & 4 + \sum_{k=1}^{m-1} \frac{3|\mathbf{B}_{2k}| (2^{2k} - 2) + (-1)^k}{(2k)!} x^{2k} + \left(\frac{2x}{\pi}\right)^{2m} \left(\frac{3\pi}{2} - 4 - \sum_{k=1}^{m-1} \frac{3|\mathbf{B}_{2k}| (2^{2k} - 2) + (-1)^k}{(2k)!} \left(\frac{\pi}{2}\right)^{2k}\right) \\ & > 3 \frac{x}{\sin(x)} + \cos(x) > 4 + \sum_{k=1}^m \frac{3|\mathbf{B}_{2k}| (2^{2k} - 2) + (-1)^k}{(2k)!} x^{2k}, \end{aligned} \quad (35)$$

where $A(k) = (3^{2k+3} - 32k^3 - 96k^2 - 88k - 27/4(2k+3)!)$, $B(k) = (2/3)(k-1/(2k+1)!)$, $C(k) = 2(4^k - 3k - 1/(2k+1)!)$, $D(k) = (1/4)(-9 + 3^{2k+2} - 40k - 32k^2/(2k+2)!)$, and $\mathbf{B}_i$ are Bernoulli's numbers.

The goal of this paper is to obtain some new inequalities which provide generalizations of inequalities (1)–(5). In order to provide the new bounds of Mitrinović–Cusa's and related inequalities, we introduce a method called the interpolation and approximation. This method has been successfully applied to prove and approximate a wide category of trigonometric inequalities 26, 27, 44–46.

In this paper, we first give the new polynomial bounds of $\sin c(x)$ and $\cos(x)$ functions using the interpolation and approximation method. We obtain the new bounds of Mitrinović–Cusa's and related inequalities based on the new bounds of the above two functions. The related inequalities include Wilker's, Huygens', Wu–Srivastava's, and Neuman–Sandor's inequalities. At the same time, we also directly use the interpolation and approximation method to

get the upper and lower bounds of Mitrinović–Cusa's inequality. The analysis results show that our bounds are tighter than the previous conclusions.

2. Main Results

Firstly, we introduce the following theorem of interpolation and approximation which is very useful for our proof [47].

Theorem 1. Let $w_0, w_1, \dots, w_r$ be $r+1$ distinct points in $[a, b]$ and $n_0, n_1, \dots, n_r$ be $r+1$ integers ≥ 0 . Let $N = n_0 + \dots + n_r + r$. Suppose that $g(t)$ is a polynomial of degree N such that

$$g^{(i)}(w_j) = f^{(i)}(w_j), \quad i = 0, \dots, n_j, j = 0, \dots, r. \quad (36)$$

Then, there exists $\xi(t) \in [a, b]$ such that

$$f(t) - g(t) = \frac{f^{(N+1)}(\xi(t))}{(N+1)!} \prod_{i=0}^r (t - w_i)^{n_i+1}. \quad (37)$$

Next, we try to derive the novel polynomial bounds of $\sin c(x)$ and $\cos(x)$ functions using the above interpolation and approximation theorem.

Theorem 2. For $x \in (0, \pi/2)$, we have that

$$\begin{aligned} s_l(x) &= 1 + b_2x^2 + b_3x^3 + b_4x^4 + b_5x^5 + b_6x^6 + b_7x^7 + b_8x^8 \leq \sin c(x) \\ &\leq 1 + a_1x + a_2x^2 + a_3x^3 + a_4x^4 + a_5x^5 + a_6x^6 + a_7x^7 + a_8x^8 = s_u(x), \end{aligned} \quad (38)$$

where

$$\begin{aligned} a_1 &= \frac{1}{\pi^2} \left[-5768 + 2048\sqrt{2} + 8019\frac{\sqrt{3}}{2} - \left(\frac{303}{2}2 + 512\sqrt{2} + 243\sqrt{3} \right) \pi \right], \\ a_2 &= \frac{1}{\pi^3} [146932 - 45056\sqrt{2} - 107163\sqrt{3} + (7331/2 + 13312\sqrt{2} + 5832\sqrt{3})\pi], \\ a_3 &= \frac{1}{\pi^4} [-1542128 + 395264\sqrt{2} + 1192644\sqrt{3} - (39696 + 143872\sqrt{2} + 58563\sqrt{3})\pi], \\ a_4 &= \frac{1}{\pi^5} [8694420 - 1769472\sqrt{2} - 7155864\sqrt{3} + (236620 + 837632\sqrt{2} + 319302\sqrt{3})\pi], \\ a_5 &= \frac{1}{\pi^6} [-28539504 + 4276224\sqrt{2} + 24978456\sqrt{3} - (828216 + 2838528\sqrt{2} + 1021572\sqrt{3})\pi], \\ a_6 &= \frac{1}{\pi^7} [54670032 - 5308416\sqrt{2} - 50703408\sqrt{3} + (1694232 + 5603328\sqrt{2} + 1918728\sqrt{3})\pi], \\ a_7 &= \frac{1}{\pi^8} [-56681856 + 2654208\sqrt{2} + 55427328\sqrt{3} - (1871424 + 5971968\sqrt{2} + 1959552\sqrt{3})\pi], \\ a_8 &= \frac{1}{\pi^9} [24572160 - 25194240\sqrt{3} + (860544 + 2654208\sqrt{2} + 839808\sqrt{3})\pi], \\ b_2 &= \frac{1}{\pi^3} \left[-14572 + 12288\sqrt{2} + 5103\sqrt{3} - \left(\frac{1153}{2} + 1024\sqrt{2} + 972\sqrt{3} \right) \pi \right], \\ b_3 &= \frac{1}{\pi^4} \left[355544 - 278528\sqrt{2} - 252963\sqrt{3}/2 + \left(\frac{20295}{2} + 24576\sqrt{2} + 21384\sqrt{3} \right) \pi \right], \\ b_4 &= \frac{1}{\pi^5} [-3429916 + 2535424\sqrt{2} + 1272105\sqrt{3} - (81833 + 238592\sqrt{2} + 191484\sqrt{3})\pi], \\ b_5 &= \frac{1}{\pi^6} [16935408 - 11870208\sqrt{2} - 6632442\sqrt{3} + (366210 + 1198080\sqrt{2} + 894240\sqrt{3})\pi], \\ b_6 &= \frac{1}{\pi^7} [-45416304 + 30228480\sqrt{2} + 18869436\sqrt{3} - (934596 + 3280896\sqrt{2} + 2297808\sqrt{3})\pi], \\ b_7 &= \frac{1}{\pi^8} [62923392 - 39813120\sqrt{2} - 27713664\sqrt{3} + (1270080 + 4644864\sqrt{2} + 3079296\sqrt{3})\pi], \\ b_8 &= \frac{1}{\pi^9} [-35230464 + 21233664\sqrt{2} + 16376256\sqrt{3} - (710208 + 2654208\sqrt{2} + 1679616\sqrt{3})\pi]. \end{aligned} \quad (39)$$

Proof. Let $e_{sl}(x) = \sin c(x) - 1 - b_2x^2 - b_3x^3 - b_4x^4 - b_5x^5 - b_6x^6 - b_7x^7 - b_8x^8$ and $e_{su}(x) = \sin c(x) - 1 - a_1x - a_2x^2 - a_3x^3 - a_4x^4 - a_5x^5 - a_6x^6 - a_7x^7 - a_8x^8$; then, $e_{sl}^{(9)}(x) = e_{su}^{(9)}(x) = \sin c^{(9)}(x)$. It is easy to see that

$$\sin c^{(9)}(x) = \frac{(-9x^8 + 504x^6 - 15120x^4 + 181440x^2 - 362880)\sin(x) + (x^9 - 72x^7 + 3024x^5 - 60480x^3 + 362880x)\cos(x)}{x^{10}}. \quad (40)$$

We let $g(x) = (-9x^8 + 504x^6 - 15120x^4 + 181440x^2 - 362880)\sin(x) + (x^9 - 72x^7 + 3024x^5 - 60480x^3 + 362880x)\cos(x)$; then,

$$g'(x) = -x^9 \sin(x) < 0, \quad 0 < x < \frac{\pi}{2}. \quad (41)$$

Therefore, $g(x)$ is a monotone decreasing function in $(0, (\pi/2))$, and we have $g(x) \leq g(0) = 0$; then, $\sin c^{(9)}(x) \leq 0$ for $x \in (0, (\pi/2))$.

By the definition of $e_{sl}(x)$ and $e_{su}(x)$, we have

$$\begin{aligned} e_{su}(0) &= e_{su}\left(\frac{\pi}{6}\right) = e_{su}\left(\frac{\pi}{4}\right) = e_{su}\left(\frac{\pi}{3}\right) = e_{su}\left(\frac{\pi}{2}\right) = e'_{su}\left(\frac{\pi}{6}\right) = e'_{su}\left(\frac{\pi}{4}\right) = e'_{su}\left(\frac{\pi}{3}\right) = e'_{su}\left(\frac{\pi}{2}\right) = 0, \\ e_{sl}(0) &= e_{sl}\left(\frac{\pi}{6}\right) = e_{sl}\left(\frac{\pi}{4}\right) = e_{sl}\left(\frac{\pi}{3}\right) = e_{sl}\left(\frac{\pi}{2}\right) = e'_{sl}(0) = e'_{sl}\left(\frac{\pi}{6}\right) = e'_{sl}\left(\frac{\pi}{4}\right) = e'_{sl}\left(\frac{\pi}{3}\right) = 0. \end{aligned} \quad (42)$$

By Theorem 1, there exist $\zeta_j, j = 1, 2$, such that

$$\begin{aligned} e_{su}(x) &= \frac{\sin c^{(9)}(\zeta_1)}{9!} x \left(x - \frac{\pi}{6}\right)^2 \left(x - \frac{\pi}{4}\right)^2 \left(x - \frac{\pi}{3}\right)^2 \left(x - \frac{\pi}{2}\right)^2 \leq 0, \\ e_{sl}(x) &= \frac{\sin c^{(9)}(\zeta_2)}{9!} x^2 \left(x - \frac{\pi}{6}\right)^2 \left(x - \frac{\pi}{4}\right)^2 \left(x - \frac{\pi}{3}\right)^2 \left(x - \frac{\pi}{2}\right)^2 \geq 0, \end{aligned} \quad (43)$$

which mean the conclusions are valid.

The proof of Theorem 2 is completed. $\square$

Theorem 3. For $x \in (0, (\pi/2))$, we have that

$$\begin{aligned} c_l(x) &= 1 + d_2x^2 + d_3x^3 + d_4x^4 + d_5x^5 + d_6x^6 + d_7x^7 + d_8x^8 \leq \cos(x) \\ &\leq 1 + c_1x + c_2x^2 + c_3x^3 + c_4x^4 + c_5x^5 + c_6x^6 + c_7x^7 + c_8x^8 = c_u(x), \end{aligned} \quad (44)$$

where

$$\begin{aligned} c_1 &= \frac{1}{\pi} \left[1185 - 1215\sqrt{3} + \left(\frac{83}{2} + 128\sqrt{2} + 81\frac{\sqrt{3}}{2} \right) \pi \right], \\ c_2 &= \frac{1}{\pi^2} \left[-\frac{64111}{2} + 2048\sqrt{2} + 30618\sqrt{3} - \left(1000 + 3328\sqrt{2} + 2187\frac{\sqrt{3}}{2} \right) \pi \right], \\ c_3 &= \frac{1}{\pi^3} \left[357852 - 45056\sqrt{2} - 319059\sqrt{3} + \left(\frac{20179}{2} + 35968\sqrt{2} + 12312\sqrt{3} \right) \pi \right], \\ c_4 &= \frac{1}{\pi^4} \left[-2148668 + 395264\sqrt{2} + 1790424\sqrt{3} - (55319 + 209408\sqrt{2} + 74844\sqrt{3})\pi \right], \end{aligned}$$

$$\begin{aligned}
c_5 &= \frac{1}{\pi^5} [7497936 - 1769472\sqrt{2} - 5860188\sqrt{3} + (178146 + 709632\sqrt{2} + 265032\sqrt{3})\pi], \\
c_6 &= \frac{1}{\pi^6} [-15206904 + 4276224\sqrt{2} + 11209104\sqrt{3} - (337140 + 1400832\sqrt{2} + 546264\sqrt{3})\pi], \\
c_7 &= \frac{1}{\pi^7} [16604352 - 5308416\sqrt{2} - 11617344\sqrt{3} + (347328 + 1492992\sqrt{2} + 606528\sqrt{3})\pi], \\
c_8 &= \frac{1}{\pi^8} [-7537536 + 2654208\sqrt{2} + 5038848\sqrt{3} - (150336 + 663552\sqrt{2} + 279936\sqrt{3})\pi], \\
d_2 &= \frac{1}{\pi^2} \left[\frac{2249}{2} + 2048\sqrt{2} - 3402\sqrt{3} + \left(162 + 256\sqrt{2} + 81\frac{\sqrt{3}}{2} \right) \pi \right], \\
d_3 &= \frac{1}{\pi^3} \left[-32013 - 45056\sqrt{2} + 80676\sqrt{3} - \left(3564 + 6144\sqrt{2} + 2025\frac{\sqrt{3}}{2} \right) \pi \right], \\
d_4 &= \frac{1}{\pi^4} [342202 + 395264\sqrt{2} - 763506\sqrt{3} + (31914 + 59648\sqrt{2} + 10287\sqrt{3})\pi], \\
d_5 &= \frac{1}{\pi^5} [-1844604 - 1769472\sqrt{2} + 3718872\sqrt{3} - (149040 + 299520\sqrt{2} + 54270\sqrt{3})\pi], \\
d_6 &= \frac{1}{\pi^6} [5355216 + 4276224\sqrt{2} - 9873576\sqrt{3} + (382968 + 820224\sqrt{2} + 156492\sqrt{3})\pi], \\
d_7 &= \frac{1}{\pi^7} [-7967808 - 5308416\sqrt{2} + 13576896\sqrt{3} - (513216 + 1161216\sqrt{2} + 233280\sqrt{3})\pi], \\
d_8 &= \frac{1}{\pi^8} [4748544 + 2654208\sqrt{2} - 7558272\sqrt{3} + (279936 + 663552\sqrt{2} + 139968\sqrt{3})\pi].
\end{aligned} \tag{45}$$

Proof. Let $e_{cl}(x) = \cos(x) - 1 - d_2x^2 - d_3x^3 - d_4x^4 - d_5x^5 - d_6x^6 - d_7x^7 - d_8x^8$ and $e_{cu}(x) = \cos(x) - 1 - c_1x - c_2x^2 - c_3x^3 - c_4x^4 - c_5x^5 - c_6x^6 - c_7x^7 - c_8x^8$; then, $e_{cl}^{(9)}(x) = e_{cu}^{(9)}(x) = \cos^{(9)}(x)$.

It is obvious that $\cos^{(9)}(x) = -\sin(x) \leq 0$ for $x \in (0, (\pi/2))$.

By the definition of $e_{cl}(x)$ and $e_{cu}(x)$, we have

$$\begin{aligned}
e_{cu}(0) &= e_{cu}\left(\frac{\pi}{6}\right) = e_{cu}\left(\frac{\pi}{4}\right) = e_{cu}\left(\frac{\pi}{3}\right) = e_{cu}\left(\frac{\pi}{2}\right) = e_{cu}'\left(\frac{\pi}{6}\right) = e_{cu}'\left(\frac{\pi}{4}\right) = e_{cu}'\left(\frac{\pi}{3}\right) = e_{cu}'\left(\frac{\pi}{2}\right) = 0, \\
e_{cl}(0) &= e_{cl}\left(\frac{\pi}{6}\right) = e_{cl}\left(\frac{\pi}{4}\right) = e_{cl}\left(\frac{\pi}{3}\right) = e_{cl}\left(\frac{\pi}{2}\right) = e_{cl}'(0) = e_{cl}'\left(\frac{\pi}{6}\right) = e_{cl}'\left(\frac{\pi}{4}\right) = e_{cl}'\left(\frac{\pi}{3}\right) = 0.
\end{aligned} \tag{46}$$

By Theorem 1, there exist $\eta_j, j = 1, 2$, such that

$$\begin{aligned}
e_{cu}(x) &= \frac{\cos^{(9)}(\eta_1)}{9!} x \left(x - \frac{\pi}{6}\right)^2 \left(x - \frac{\pi}{4}\right)^2 \left(x - \frac{\pi}{3}\right)^2 \left(x - \frac{\pi}{2}\right)^2 \leq 0, \\
e_{cl}(x) &= \frac{\cos^{(9)}(\eta_2)}{9!} x^2 \left(x - \frac{\pi}{6}\right)^2 \left(x - \frac{\pi}{4}\right)^2 \left(x - \frac{\pi}{3}\right)^2 \left(x - \frac{\pi}{2}\right)^2 \geq 0,
\end{aligned} \tag{47}$$

which mean the conclusions are valid.

The proof of Theorem 3 is completed. $\square$

We propose the following refinements and improvements of inequalities (1)–(5). $\square$

Theorem 4. For $x \in (0, (\pi/2))$, we have that

$$s_l(x) - (c_u(x))^{(1/3)} \leq \frac{\sin(x)}{x} - (\cos(x))^{(1/3)} \leq s_u(x) - (c_l(x))^{(1/3)}, \quad (48)$$

$$(s_l(x))^3 - c_u(x) \leq \left(\frac{\sin(x)}{x}\right)^3 - \cos(x) \leq (s_u(x))^3 - c_l(x), \quad (49)$$

$$\frac{3s_l(x) - c_u(x) - 2}{3} \leq \frac{\sin(x)}{x} - \frac{2 + \cos(x)}{3} \leq \frac{3s_u(x) - c_l(x) - 2}{3}, \quad (50)$$

$$2s_l(x) + \frac{s_l(x)}{c_u(x)} \leq 2\frac{\sin(x)}{x} + \frac{\tan(x)}{x} \leq 2s_u(x) + \frac{s_u(x)}{c_l(x)}, \quad (51)$$

$$(s_l(x))^2 + \frac{s_l(x)}{c_u(x)} \leq \left(\frac{\sin(x)}{x}\right)^2 + \frac{\tan(x)}{x} \leq (s_u(x))^2 + \frac{s_u(x)}{c_l(x)}, \quad (52)$$

$$\frac{1}{(s_u(x))^2} + \frac{c_l(x)}{s_u(x)} \leq \left(\frac{x}{\sin(x)}\right)^2 + \frac{x}{\tan(x)} \leq \frac{1}{(s_l(x))^2} + \frac{c_u(x)}{s_l(x)}, \quad (53)$$

$$\frac{3}{s_u(x)} + c_l(x) \leq 3\frac{x}{\sin(x)} + \cos(x) \leq \frac{3}{s_l(x)} + c_u(x), \quad (54)$$

where $s_l(x)$, $s_u(x)$, $c_l(x)$, and $c_u(x)$ are defined in Theorems 2 and 3.

Proof. First, we prove inequality (44). By Theorems 2 and 3, we have

$$s_l(x) \leq \frac{\sin(x)}{x} \leq s_u(x), \quad (55)$$

$$c_l(x) \leq \cos(x) \leq c_u(x). \quad (56)$$

Therefore, we have $(c_l(x))^{(1/3)} \leq (\cos(x))^{(1/3)} \leq (c_u(x))^{(1/3)}$; then, $s_l(x) - (c_u(x))^{(1/3)} \leq (\sin(x)/x) - (\cos(x))^{(1/3)} \leq s_u(x) - (c_l(x))^{(1/3)}$. Inequality (44) is proved.

Second, we give the proof of inequality (48). For the same reason, we have $(s_l(x))^3 < (\sin(x)/x)^3 < (s_u(x))^3$ and $-c_u(x) \leq -\cos(x) \leq -c_l(x)$; then, $(s_l(x))^3 - c_u(x) \leq (\sin(x)/x)^3 - \cos(x) \leq (s_u(x))^3 - c_l(x)$. That is to say, inequality (48) holds.

Third, we prove inequality (49).

It is easy to see that $(2 + c_l(x)/3) \leq (2 + \cos(x)/3) \leq (2 + c_u(x)/3)$; then, inequality (49) is valid.

Fourth, we prove inequality (50). By Theorem 2, we have $c_l(x) = 1 + d_2x^2 + d_3x^3 + d_4x^4 + d_5x^5 + d_6x^6 + d_7x^7 + d_8x^8$, where $d_i, i = 2, 3, \dots, 8$, are defined in Theorem 2. Then, $c_l(x)$ can be rewritten as

$$c_l(x) = (x - x_1)(x - x_2)(x - x_3)(x - x_4)(x - x_5) \cdot (x - x_6)(x - x_7)(x - x_8), \quad (57)$$

where $x_1 \approx -7.01106143$, $x_2 \approx 5.18438798 + 2.25095656i$, $x_3 \approx 5.18438798 - 2.25095656i$, $x_4 \approx 4.42109817$, $x_5 \approx -4.444313683 + 1.43538013i$, $x_6 \approx -4.444313683 - 1.43538013i$, $x_7 \approx 1.57079633$, $x_8 \approx -1.56923028$.

Then, $x \in (0, (\pi/2))$, and we have $c_l(x) > 0$. For the same reason, we can prove $s_l(x) > 0, x \in (0, (\pi/2))$. It is obvious that $c_u(x) > 0$ and $s_u(x) > 0$.

By inequality (55) and the results of $c_l(x) > 0$, $s_l(x) > 0$, $c_u(x) > 0$, and $s_u(x) > 0$, we have

$$2s_l(x) \leq 2\frac{\sin(x)}{x} \leq 2s_u(x), \quad (58)$$

$$\frac{s_l(x)}{c_u(x)} \leq \frac{\tan(x)}{x} \leq \frac{s_u(x)}{c_l(x)}.$$

Hence, inequality (50) holds.

Fifth, we prove inequality (51). By the result of $s_l(x) > 0$ and $s_u(x) > 0$, we have $(s_l(x))^2 < (\sin(x)/x)^2 < (s_u(x))^2$. And because of the above results of $(s_l(x)/c_u(x)) \leq (\tan(x)/x) \leq (s_u(x)/c_l(x))$, we have that inequality (51) holds.

Sixth, we prove inequality (52). By the result of $s_l(x) > 0$ and $s_u(x) > 0$, we have $(s_l(x))^2 < (\sin(x)/x)^2 < (s_u(x))^2$. And because of the results of $(s_l(x)/c_u(x)) \leq (\tan(x)/x) \leq (s_u(x)/c_l(x))$ and $s_l(x) > 0, s_u(x) > 0, c_l(x) > 0, c_u(x) > 0$, we have $(c_l(x)/s_u(x)) < (x/\tan(x)) < (c_u(x)/s_l(x))$. Therefore, inequality (52) is proved.

At last, we prove inequality (53). By the results of $s_l(x) > 0, s_u(x) > 0$ and inequality (54), we have $1/s_u(x) < x/\sin(x) < 1/s_l(x)$. And because of inequality (55), inequality (53) holds.

The proof of Theorem 4 is completed. $\square$

Theorems 2 and 3 give the new bounds of $\sin c(x)$ and $\cos(x)$ based on the interpolation and approximation method. Theorem 4 gives the improvements and refinements of Mitrinović–Cusa's and related inequalities using the new bounds of $\sin c(x)$ and $\cos(x)$. Next, we try to obtain the new improvements of Mitrinović–Cusa's

inequalities directly based on the interpolation and approximation method. $\square$

Theorem 5. For $x \in (0, (\pi/2))$, we have

$$g_2x^2 + g_3x^3 + g_4x^4 + g_5x^5 + g_6x^6 \leq \frac{\sin(x)}{x} - \frac{2 + \cos(x)}{3} \leq f_1x + f_2x^2 + f_3x^3 + f_4x^4 + f_5x^5 + f_6x^6, \quad (59)$$

where

$$\begin{aligned} f_1 &= \frac{1}{\pi^2} \left[208 - 2560\sqrt{2} + 5103\frac{\sqrt{3}}{2} - \left(\frac{753}{2} - 128\sqrt{2} \right) \pi - \left(\frac{4}{3} + 32\frac{\sqrt{2}}{3} + 27\frac{\sqrt{3}}{2} \right) \pi^2 \right], \\ f_2 &= \frac{1}{\pi^3} \left[-3296 + 37888\sqrt{2} - 37179\sqrt{3} + \left(\frac{16304}{3} - 5888\frac{\sqrt{2}}{3} \right) \pi + \left(64/3 + 448\frac{\sqrt{2}}{3} + 405\frac{\sqrt{3}}{2} \right) \pi^2 \right], \\ f_3 &= \frac{1}{\pi^4} \left[20560 - 217600\sqrt{2} + 211410\sqrt{3} - \left(30962 - 34688\frac{\sqrt{2}}{3} \right) \pi - \left(\frac{404}{3} + 2464\frac{\sqrt{2}}{3} + 1188\sqrt{3} \right) \pi^2 \right], \\ f_4 &= \frac{1}{\pi^5} \left[-62976 + 608256\sqrt{2} - 586116\sqrt{3} + \left(\frac{258824}{3} - 98816\sqrt{2}/3 \right) \pi + \left(\frac{1256}{3} + 6656\frac{\sqrt{2}}{3} + 3402\sqrt{3} \right) \pi^2 \right], \\ f_5 &= \frac{1}{\pi^6} \left[94464 - 829440\sqrt{2} + 793152\sqrt{3} - (117472 - 45568\sqrt{2})\pi - (640 + 2944\sqrt{2} + 4752\sqrt{3})\pi^2 \right], \\ f_6 &= \frac{1}{\pi^7} \left[-55296 + 442368\sqrt{2} - 419904\sqrt{3} + (62592 - 24576\sqrt{2})\pi + (384 + 1536\sqrt{2} + 2592\sqrt{3})\pi^2 \right], \\ g_2 &= \frac{1}{\pi^3} \left[32 - 3072\sqrt{2} + 3645\sqrt{3} - \left(\frac{1768}{3} - 256\frac{\sqrt{2}}{3} \right) \pi - \left(64\frac{\sqrt{2}}{3} + 27\frac{\sqrt{3}}{2} \right) \pi^2 \right], \\ g_3 &= \frac{1}{\pi^4} \left[-448 + 40960\sqrt{2} - 92583\sqrt{3}/2 + \left(\frac{14129}{2} - 4096\frac{\sqrt{2}}{3} \right) \pi + \left(256\sqrt{2} + 351\frac{\sqrt{3}}{2} \right) \pi^2 \right], \\ g_4 &= \frac{1}{\pi^5} \left[2336 - 195584\sqrt{2} + 215055\sqrt{3} - \left(\frac{95839}{3} - 21760\frac{\sqrt{2}}{3} \right) \pi - \left(3392\frac{\sqrt{2}}{3} + 837\sqrt{3} \right) \pi^2 \right], \\ g_5 &= \frac{1}{\pi^6} \left[-5376 + 399360\sqrt{2} - 431568\sqrt{3} + (63248 - 15872\sqrt{2})\pi + (2176\sqrt{2} + 1728\sqrt{3})\pi^2 \right], \\ g_6 &= \frac{1}{\pi^7} \left[4608 - 294912\sqrt{2} + 314928\sqrt{3} - (45840 - 12288\sqrt{2})\pi - (1536\sqrt{2} + 1296\sqrt{3})\pi^2 \right]. \end{aligned} \quad (60)$$

Proof. Let $f_{sc}(x) = (\sin(x)/x) - (2 + \cos(x)/3)$, $e_l(x) = f_{sc}(x) - g_2x^2 - g_3x^3 - g_4x^4 - g_5x^5 - g_6x^6$, and $e_u(x) = f_{sc}(x) - f_1x - f_2x^2 - f_3x^3 - f_4x^4 - f_5x^5 - f_6x^6$; then,

$$\begin{aligned} e_l^{(7)}(x) &= e_u^{(7)}(x) = \sin c^{(7)}(x) - \frac{1}{3} \sin(x) \\ &= \frac{(7x^6 - 210x^4 + 2520x^2 - 5040)\sin(x) + (-x^7 + 42x^5 - 840x^3 + 5040x)\cos(x)}{x^8} - \frac{1}{3} \sin(x). \end{aligned} \quad (61)$$

TABLE 1: Maximum errors between $(\sin(x)/x)^3 - \cos(x)$ and its bounds (Mitrinović's inequality).

Method	Error	
	MaxError _{low}	MaxError _{upp}
Mortici [42] (inequality (24))	3.4808×10^{-2}	1.4744×10^{-1}
Malešević et al. [43] (inequality (30), $n=2$)	5.6770×10^{-4}	5.2314×10^{-3}
The results of this paper (inequality (48))	3.2409×10^{-8}	3.1707×10^{-8}

TABLE 2: Maximum errors between $(\sin(x)/x) - (2 + \cos(x)/3)$ and its bounds (Cusa's inequality).

Method	Error	
	MaxError _{low}	MaxError _{upp}
Bhayo and sandor [33] (inequality (6))	1.1614×10^{-2}	9.1985×10^{-3}
Mortici [42] (inequality (25))	3.7644×10^{-3}	1.9749×10^{-4}
Malešević et al. [43] (inequality (31), $n=2$)	5.9603×10^{-6}	1.1839×10^{-7}
The results of this paper (inequality (49))	1.0803×10^{-8}	1.0701×10^{-8}
The results of this paper (inequality (56))	5.8244×10^{-7}	3.2174×10^{-6}

TABLE 3: Maximum errors between $2(\sin(x)/x) + (\tan(x)/x)$ and its bounds (Huygens' inequality).

Method	Error	
	MaxError _{low}	MaxError _{upp}
Huygens [3] (inequality (2))	7.9812×10^2	—
Chen and Cheung [37] (inequality (11))	6.9173×10^1	1.0539×10^{-1}
Chen and Cheung [37] (inequality (12))	1.5550×10^2	2.0761×10^{-1}
Chen and Paris [39] (inequality (19))	6.1807×10^0	8.3747×10^{-3}
Mortici [42] (inequality (26))	2.9643×10^{-3}	3.4633×10^2
Malešević et al. [43] (inequality (32), $n=2$)	1.3520×10^0	4.4188×10^0
The results of this paper (inequality (50))	3.8724×10^{-7}	7.1581×10^{-4}

TABLE 4: Maximum errors between $(\sin(x)/x)^2 + (\tan(x)/x)$ and its bounds (Wilker's inequality).

Method	Error	
	MaxError _{low}	MaxError _{upp}
Wilker and Porblem [4] (inequality (3))	7.9826×10^2	—
Sumner et al. [36] (inequality (10))	2.6424×10^{-2}	6.5686×10^1
Chen and Cheung [37] (inequality (13))	1.8874×10^2	2.4457×10^{-1}
Chen and Cheung [37] (inequality (14))	1.4653×10^2	1.9695×10^{-1}
Nenezic et al. [38] (inequality (17))	3.5720×10^1	3.9383×10^{-3}
Nenezic et al. [38] (inequality (18))	5.7271×10^{-3}	3.0139×10^{-3}
Chen and Paris [39] (inequality (20))	2.3923×10^0	3.9383×10^{-3}
Mortici [42] (inequality (27))	8.7474×10^2	5.5813×10^2
Malešević et al. [43] (inequality (33), $n=2$)	3.5293×10^0	2.8103×10^1
The results of this paper (inequality (51))	3.8636×10^{-7}	7.1581×10^{-4}

We let $h(x) = (7x^6 - 210x^4 + 2520x^2 - 5040)\sin(x) + (-x^7 + 42x^5 - 840x^3 + 5040x)\cos(x)$; then, we have $h'(x) = -5x^7 - x^8 \cos(x) < 0$, $x \in (0, (\pi/2))$. Then, $h(x)$ is a monotone decreasing function in $(0, (\pi/2))$, and we have $h(x) \leq h(0) = 0$; then, $\sin c^{(7)}(x) \leq 0$ for $x \in (0, (\pi/2))$. Therefore, we have $e_l^{(7)}(x) = e_u^{(7)}(x) = f_{sc}^{(7)}(x) = \sin c^{(7)}(x) - (1/3)\sin(x) < 0$.

By the definition of $e_l(x)$ and $e_u(x)$, we have

$$e_l(0) = e_l\left(\frac{\pi}{4}\right) = e_l\left(\frac{\pi}{3}\right) = e_l\left(\frac{\pi}{2}\right) = e_l'(0) = e_l'\left(\frac{\pi}{4}\right) = e_l'\left(\frac{\pi}{3}\right) = 0,$$

$$e_u(0) = e_u\left(\frac{\pi}{4}\right) = e_u\left(\frac{\pi}{3}\right) = e_u\left(\frac{\pi}{2}\right) = e_u'\left(\frac{\pi}{4}\right) = e_u'\left(\frac{\pi}{3}\right) = e_u'\left(\frac{\pi}{2}\right) = 0. \quad (62)$$

By Theorem 1, there exist θ_j , $j = 1, 2$, such that

TABLE 5: Maximum errors between $(x/\sin(x))^2 + (x/\tan(x))$ and its bounds (Wu and Srivastava's inequality).

Method	Error	
	MaxError _{low}	MaxError _{upp}
Wu and Srivastava [5] (inequality (4))	4.6615×10^{-1}	—
Chen and Cheung [37] (inequality (15))	—	2.1552×10^2
Chen and Cheung [37] (inequality (16))	—	1.0111×10^2
Chen and Paris [39] (inequality (21))	1.0560×10^{-1}	6.4478×10^1
Wang [40] (inequality (22))	2.9416×10^{-1}	9.4934×10^{-2}
Mortici [42] (inequality (28))	1.9612×10^{-1}	—
Malešević et al. [43] (inequality (34), $m=3$)	6.9338×10^{-2}	9.6995×10^{-3}
The results of this paper (inequality (52))	4.8271×10^{-8}	3.2457×10^{-8}

TABLE 6: Maximum errors between $3(x/\sin(x)) + (x/\tan(x))$ and its bounds (Neuman and Sandor's inequality).

Method	Error	
	MaxError _{low}	MaxError _{upp}
Neuman and Sandor [6] (inequality (5))	7.1080×10^{-1}	—
Jiang et al. [41] (inequality (23))	3.2381×10^{-1}	7.4286×10^{-2}
Mortici [42] (inequality (29))	3.1910×10^{-2}	—
Malešević et al. [43] (inequality (35), $m=3$)	63.1910×10^{-2}	4.1471×10^{-3}
The results of this paper (inequality (53))	3.2982×10^{-8}	3.2411×10^{-8}

$$e_l(x) = \frac{f_{sc}^{(7)}(\theta_1)}{7!} x^2 \left(x - \frac{\pi}{4}\right)^2 \left(x - \frac{\pi}{3}\right)^2 \left(x - \frac{\pi}{2}\right) \geq 0, \quad (63)$$

$$e_u(x) = \frac{f_{sc}^{(7)}(\theta_2)}{7!} x \left(x - \frac{\pi}{4}\right)^2 \left(x - \frac{\pi}{3}\right)^2 \left(x - \frac{\pi}{2}\right)^2 \leq 0,$$

which mean the conclusions are valid. $\square$

3. Conclusions and Analysis

In this paper, we give the new results of Mitrinović–Cusa's and related inequalities, including Wilker's, Huygens', Wu–Srivastava's, and Neuman–Sandor's inequalities. Theorems 2 and 3 present the novel polynomial bounds of $\sin c(x)$ and $\cos(x)$ functions using the interpolation and approximation method. Theorem 4 gives the new refinements and improvements of the above five inequalities based on the results of $\sin c(x)$ and $\cos(x)$. In order to compare our results with the previous methods, we introduce the concept of the maximum error. The maximum error is the most important index to measure the upper and lower bounds of an inequality (a function). MaxError_{low} denotes the maximum error between a function and its lower bound. Similarly, MaxError_{upp} denotes the maximum error between a function and its upper bound. Tables 1–6 give the maximum errors of the upper and lower bounds of the above five inequalities. We consider both sides of Mitrinović–Cusa's inequality separately, which is why there are six tables for five inequalities. It is obvious that the results of this paper are superior to the previous conclusions.

For Mitrinović's inequality, many researchers focused on the equivalent form of this inequality, that is, the inequality $(\sin(x)/x)^3 - \cos(x) > 0$. Table 1 gives the comparison of the maximum errors between $(\sin(x)/x)^3 - \cos(x) > 0$ and its

bounds for different methods. In inequality (40), the maximum error between the function $(\sin(x)/x)^3 - \cos(x) > 0$ and the lower bound is only 3.2409×10^{-8} , and the maximum error of the upper bound is 3.1707×10^{-8} . It is easy to see that the maximum errors of this paper are the smallest of all methods. For Cusa's inequality, Wilker's inequality, Huygens' inequality, Wu–Srivastava's inequality, and Neuman–Sandor's inequality, the same conclusions can be obtained from Tables 2 to 6.

In this paper, we not only consider the equivalent form of Mitrinović's inequality but also consider the lower and upper bounds of the function $(\sin(x)/x) - \cos(x)^{(1/3)}$ directly. Inequality (44) gives the new improvement of Mitrinović's inequality. The maximum error between the function $(\sin(x)/x) - \cos(x)^{(1/3)}$ and the lower bound is only 1.0834×10^{-8} , and the maximum error of the upper bound is 6.8129×10^{-8} .

Tables 1–6 show that the results of this paper are far superior to the previous conclusions. Among the previous conclusions, only the conclusions of Malešević et al. [43] are close to the results of this paper for Cusa's inequality in Table 2. In this paper, the maximum error between $(\sin(x)/x) - (2 + \cos(x)/3)$ and its lower bound is 1.0803×10^{-8} , and the maximum error between $(\sin(x)/x) - (2 + \cos(x)/3)$ and its upper bound is 1.0701×10^{-8} . The maximum error of the lower bound is 5.9603×10^{-6} , and the maximum error of the upper bound is 1.1839×10^{-7} in Malešević et al. [43]. Although the maximum errors of Malešević et al. [43] are close to our results, we can find that the degree of the bounds is 8 in inequality (49), and the degree of the upper bound reaches 10 in inequality (33) (when $n=2$).

Another advantage of our results is that the form of the bound is relatively simple. The bounds are all polynomial

functions, and there are no other functions. In previous conclusions, many bounds contain not only polynomial functions but also trigonometric functions, for example, $\sin(x)$, $\cos(x)$, and $\tan(x)$. In general, the polynomial function as the upper and lower bounds of the trigonometric function is more effective than the trigonometric function as the upper and lower bounds of the trigonometric function.

At the end of this paper, we try to obtain the bounds of $(\sin(x)/x) - (2 + \cos(x)/3)$ directly using the interpolation and approximation theorem. Theorem 5 gives the new upper and lower bounds of the function $(\sin(x)/x) - (2 + \cos(x)/3)$. The last row of Table 2 gives the maximum errors of inequality (56). The maximum error between the function $(\sin(x)/x) - (2 + \cos(x)/3)$ and the lower bound is 5.8244×10^{-7} , and the maximum error of the upper bound is 3.2174×10^{-6} . Table 2 shows that the maximum errors are close to the results of inequality (49). Although the maximum errors of are bigger than the maximum errors of inequality (49), the degree of the bounds is 6 in inequality (56), and the degree of the bounds is 8 in inequality (49). Similarly, we can obtain the upper and lower bounds of other functions directly using the interpolation and approximation theorem.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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