

Research Article

Random Cyclic Triangle-Free Graphs of Prime Order

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Received 25 February 2021; Accepted 28 July 2021; Published 9 August 2021

Academic Editor: Efthymios G. Tsionas

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Cyclic triangle-free process (CTFP) is the cyclic analog of the triangle-free process. It begins with an empty graph of order n and generates a cyclic graph of order n by iteratively adding parameters, chosen uniformly at random, subject to the constraint that no triangle is formed in the cyclic graph obtained, until no more parameters can be added. The structure of a cyclic triangle-free graph of the prime order is different from that of composite integer order. Cyclic graphs of prime order have better properties than those of composite number order, which enables generating cyclic triangle-free graphs more efficiently. In this paper, a novel approach to generating cyclic triangle-free graphs of prime order is proposed. Based on the cyclic graphs of prime order, obtained by the CTFP and its variant, many new lower bounds on $R(3, t)$ are computed, including $R(3, 34) \geq 230$, $R(3, 35) \geq 242$, $R(3, 36) \geq 252$, $R(3, 37) \geq 264$, $R(3, 38) \geq 272$. Our experimental results demonstrate that all those related best known lower bounds, except the bound on $R(3, 34)$, are improved by 5 or more.

1. Introduction

Ramsey theory [1] has played an important branch in combinatorics, which spans numerous diverse areas of mathematics. Many research efforts have been devoted into computing Ramsey numbers and their generalizations [2–5].

Let S be a set of integers $S \subseteq \{1, \dots, \lfloor n/2 \rfloor\}$ ($n \in \mathbb{Z}^+$ and $n \geq 5$), a graph with the vertex set $V = \{1, \dots, n\}$ and the edge set $E = \{(x, y) \mid \min\{|x - y|, n - |x - y|\} \in S\}$ is a cyclic graph of order n , namely $G_n(S)$. S is the parameter set of $G_n(S)$. Let s and t be two positive integers, the Ramsey number for s and t , denoted by $R(s, t)$, is the minimum positive integer N such that every graph of order N contains either an s -clique or a t -independent set. There are many results and open problems in Ramsey theory in terms of computing lower bounds for Ramsey numbers (see [6]).

The Ramsey number $R(3, t)$ is an important topic in Ramsey theory. In [7], Calkin et al. gave theoretical motivation for searching for lower bound for Ramsey numbers based on cyclic graphs of prime order, and provided additional computational evidence that primes tend to perform

better than composites. The analysis in [7] does not focus on Ramsey numbers of form $R(3, t)$. For $R(s, t)$, in [7] it was shown that standard expected value arguments cannot be used to give bounds on $R(s, t)$ that are exponential in $\min\{s, t\}$, but in [8] Alon and Orlicsky proved by more sophisticated arguments that random cyclic graphs nonetheless give bounds on $R(k, k)$ of order $e^{\sqrt{k}}$.

The triangle-free process is an important tool in studying the asymptotic lower bound on $R(3, t)$. The triangle-free process was used in studying the asymptotic lower bound for $R(3, t)$ in [9, 10]. Cyclic triangle-free process (CTFP) is the cyclic analog of the triangle-free process, which is used to generate cyclic graphs of a certain order. To generate a cyclic graph of order n , the process starts with an empty graph of order n and iteratively adding random parameters, chosen uniformly at random, conformed to the constraint that no triangle is formed in the obtained cyclic graph, until no more parameters can be added [11].

In our previous work [11], CTFP was applied to study lower bounds on $R(3, t)$. Because of the symmetry of cyclic graphs, it is easier to compute the independence numbers of

cyclic graphs than those of non-cyclic graphs with the same orders and edge density. In this paper, the previous work is extended by including an approach to generating cyclic triangle-free graphs of prime order. The experimental results demonstrate that generation of cyclic triangle-free graphs is much more efficient compared to the previous work [11].

By employing our approach, it is feasible to generate large amount of cyclic triangle-free graphs and improve some previous best known lower bounds on $R(3, t)$, including $R(3, 42) \geq 312$, $R(3, 44) \geq 320$, $R(3, 45) \geq 337$, $R(3, 46) \geq 348$, $R(3, 47) \geq 360$, $R(3, 49) \geq 376$ and $R(3, 52) \geq 401$.

The remaining parts of this paper are organized as follows. In Section 2, definitions of Ramsey numbers, as well as some known results on cyclic triangle-free graphs and $R(3, t)$, are introduced. The sizes of the parameter sets of cyclic triangle-free graphs of prime order obtained by the CTFP are studied in Section 3. Furthermore, new lower bounds on $R(3, t)$ for small t are given in Section 4. Section 5 concludes the paper, and discusses a problem on cyclic triangle-free graphs and $R(3, t)$.

2. Preliminaries

In this section, we firstly introduce some basic concepts and notations used in this paper. Then, some basic known results on $R(3, t)$ are presented. Finally, cyclic triangle-free process (CTFP) is discussed in details.

2.1. Definitions and Notations. All graphs considered in this paper are finite and undirected graphs. The complete graph of order n ($n \in \mathbb{Z}^+$) is denoted by K_n . K_3 represents a triangle. For a positive integer d , if every vertex in G is adjacent to d vertices, then G is called d -regular. The clique number of graph G , denoted by $cl(G)$, is the cardinality of the largest clique in G . The independence number of graph G , denoted by $\alpha(G)$, is the cardinality of the largest independent set in G . A clique of order k is called a k -clique, and an independent set of order k is called a k -independent set.

Let s and t be two positive integers, the Ramsey number $R(s, t)$ is the smallest positive integer n such that every graph of order n contains either an s -clique or a t -independent set. In accordance with the well-known Ramsey theorem [12], it is known that $R(s, t)$ is finite. An (s, t) -graph is a graph that contains neither an s -clique nor a t -independent set. Therefore there is an (s, t) -graph of order $R(s, t) - 1$, but there is not an (s, t) -graph of order $R(s, t)$.

The triangle-free process begins with E_n , an empty graph of order n , and iteratively adds edges chosen uniformly at random subject to the constraint that no triangle is formed, until no more edge can be added. The triangle-free process ends with a maximal triangle-free graph. The cyclic triangle-free process, i.e., CTFP, begins with E_n , and generate a cyclic graph of order n by iteratively adding parameters, chosen uniformly at random, subject to the constraint that no triangle is formed in the cyclic graph obtained, until no more parameters can be added.

For any real number a , we use $\lfloor a \rfloor$ to designate the largest integer that is smaller than or equal to a . Similarly, $\lceil a \rceil$ is used to designate the smallest integer that is larger than or equal to a . Given an integer $n \geq 5$, suppose $S \subseteq \{1, 2, \dots, \lfloor n/2 \rfloor\}$, and let G be a graph with the vertex set $V(G) = \{1, \dots, n\}$ and the edge set $E(G) = \{(x, y) \mid \min\{|x - y|, n - |x - y|\} \in S\}$, a graph G is called a cyclic graph of order n , namely $G_n(S)$. S is called the parameter set of $G_n(S)$.

2.2. Some Basic Known Results on $R(3, t)$ and the Triangle-Free Process. Although $R(3, t)$ is simple among Ramsey numbers $R(s, t)$, it can be difficult when t becomes large. The best known asymptotic lower bound on $R(3, t)$ is $R(3, t) \geq ((1/4) + o(1))(t^2/\log t)$, obtained by Bohman and Keevash [9] in 2013 and by Pontiveros et al. [10] independently and simultaneously. In 2020, the work of [10] was updated and published as [13]. The triangle-free process was used in both [9, 10]. This asymptotic lower bound on $R(3, t)$ was obtained by proving the following theorem.

Theorem 1 ([9, 10]). *Let G be the maximal triangle-free graph of order n at which the triangle-free process terminates. With high probability, G has independence number at most $(1 + o(1))\sqrt{2n \log n}$.*

The best known asymptotic lower bound on $R(3, t)$, given in [9, 10], is far from the best known upper bound $(1 + o(1))(t^2/\log t)$, which was proved by Shearer in [14]. The exact value of $R(3, t)$ is known only for positive integer $t \leq 9$. For larger small positive integer $t \leq 38$, the best known lower bound on $R(3, t)$ was obtained no later than 2017, as described in the survey [15]. Most of these best known lower bounds on small $R(3, t)$ were obtained by finding cyclic $(3, t)$ -graphs. The best known lower bound on $R(3, t)$ for any integer $t \in \{27, 28, \dots, 34\}$ cited in [15] was obtained in [16] based on cyclic triangle-free graphs.

In [17], $R(3, s + t - 1) \geq R(3, s) + R(3, t) + s - 2$ was proved for integers s and t , where $2 \leq s \leq t$. The subcase $s = 2$, $R(3, t + 1) \geq R(3, t) + 3$, was proved in [18] in 1989. This bound is weak in general, but it is difficult to be improved.

2.3. The Cyclic Triangle-Free Process. Most best known lower bounds on small $R(3, t)$ were obtained by cyclic graphs of which the orders are composite integers. Some relevant results can be found in [16, 19, 20]. For prime orders between 120 and 260, there is only one best known lower bound obtained by cyclic graphs of prime order, which is $R(3, 33) > 223$ (i.e., the lower bound on $R(3, 33)$ is 224) given in [16].

There is still no known interesting general lower bound on $R(3, t)$ given by random cyclic graphs that is better than all linear ones. Thus, it is interesting to know if we can obtain better lower bounds on Ramsey numbers of the form $R(3, t)$ by computing more triangle-free cyclic graphs generated by the CTFP. In particular, it would be interesting to study lower bounds on $R(3, t)$ based on cyclic graphs of prime

order. The structure of cyclic triangle-free graphs can be quite different between the prime order and the composite integer order cases. For instance, if n is an odd prime, then for any cyclic graph $G_n(S_1)$ and any $i \in \{1, 2, \dots, \lfloor n/2 \rfloor\}$, there will be a cyclic graph $G_n(S_2)$ that is isomorphic to $G_n(S_1)$ such that $1 \in S_2$.

In our previous work, new lower bounds on some small Ramsey numbers of form $R(3, t)$, including $R(3, 35) \geq 237$, $R(3, 36) \geq 245$, $R(3, 37) \geq 255$ and $R(3, 38) \geq 267$, were obtained in [11] by the CTFP, which improved the best known lower bounds in [15]. More lower bounds were obtained by the CTFP in [11], including $R(3, 41) \geq 291$, $R(3, 42) \geq 300$, $R(3, 43) \geq 309$, $R(3, 44) \geq 316$ and $R(3, 48) \geq 362$.

It is difficult to give interesting lower bounds on small $R(3, t)$ by the triangle-free process. For example, we have generated 10^5 graphs of order 300 by the triangle-free process, and without difficulty, we found that all of them contain 45-independent sets. On the other hand, we have found a cyclic triangle-free graph of order 307 and independence number 40 by the CTFP. More discussions on this would be given in Section 4.

In [9, 10], it was proved that with high probability, every vertex of G has degree $(1 + o(1))\sqrt{(1/2)n \log n}$, for the maximal triangle-free graph G of order n at which the triangle-free process terminates. As shown in the computing results in [11], when n is not large, the graphs generated by the CTFP, compared to those generated by the triangle-free process, has more edges and smaller independence numbers with high probability. When n is large, the difference between the numbers of edges in graphs generated by the CTFP, as well as those of edges in graphs generated by the triangle-free process, may become smaller.

We focus on improving the best known lower bounds on small $R(3, t)$ based on cyclic triangle-free graphs of prime order. In particular, we are interesting in finding triangle-free graphs with small independence numbers. Since the degree of a cyclic triangle-free graph is closely related to its independence number, we study the sizes of parameter sets of cyclic graphs obtained by the CTFP in the next section.

3. The Size of Parameter Set of Cyclic Triangle-Free Graphs of Prime Order

Similar to the proof performed on the triangle-free process in [9, 10] (including Theorem 1 in Section 2.1), proving theorems on the CTFP can be difficult. Since cyclic triangle-free graphs are not well understood by now, more details on the structure of those graphs are needed.

3.1. Small Parameter Sets of Some Cyclic Triangle-Free Graphs of Prime Orders. As the order of all triangle-free graphs considered are of odd prime orders, the degree equals to the twice of the number of parameters, which is a lower bound on the independence numbers. When generating cyclic triangle-free graphs of prime orders range from 223 to 401, we have found maximal cyclic triangle-free graphs with different parameter sets. Data on sizes of parameter sets are useful in computation experiment design. The sizes of some

maximal cyclic triangle-free graphs with small parameter sets are listed in Table 1. With only one exception (i.e., 151), the results on prime orders ranging from 127 to 211 are the same to those in [11].

3.2. Computing Time Through an Example. The Maximum Independence Number Problem is NP-hard even if restricted to cyclic graphs [21]. It can be difficult to compute the independence number of a large cyclic graph of which the density of edges is low. For instance, it is difficult to compute the independence number of a random cyclic triangle-free graph when the order is larger than 500.

Suppose that among all maximal cyclic triangle-free graphs of order n , the proportion of graphs with independence number smaller than t is x . Suppose that t is not too small such that $x > 0$ holds. If we generate y maximal cyclic triangle-free graphs randomly, then the probability that at least one graph among them has independence number smaller than t is $1 - (1 - x)^y$. When $x > 0$ and y approaches infinite, the probability tends towards 1. When $y = \lceil 1/x \rceil$ and positive x is very small, the probability is $1 - (1 - x)^y \approx 1 - (1/e)$, where e is the base of natural logarithm. To find a cyclic triangle-free graph of order n and independence number smaller than t with large probability, sufficient amount of cyclic triangle-free graphs should be generated. Therefore, to enable handling with large amount of graphs, the computation of graphs should be fast enough.

For any integer n between 121 and 401, generating a cyclic graph of order n by the CTFP costs about one second in average. However, generating a cyclic triangle-free graph G with small parameter set, can spend much more time in average. When the order becomes larger, generating a cyclic graph by the CTFP can take longer time in average. On the other hand, finding a t -independent set in a given cyclic triangle-free graph G can be much easier when t is smaller.

We discuss the computing time through an example below. By using CTFP, it is easy to find a cyclic triangle-free graph G of order 313 such that $\alpha(G) = 43$. However, among 15000 cyclic graphs generated, only one cyclic graph G of order 313 with $\alpha(G) = 42$ is found. All of the cyclic graphs contain 20 or fewer parameters. For the triangle-free graphs of order 313, whose number of parameters is at most 20, the computation time is about one minute in average. On the other hand, almost for any graph among them, a 42-independent set can be found within one second, and we know that they can not be used to prove $R(3, 42) > 313$.

Generating random cyclic triangle-free graphs by the CTFP or similar methods quickly is important in computing new lower bounds on small $R(3, t)$. Our experiment result shows that this is not difficult when the order n is an odd prime.

3.3. A Method on Generating Random Cyclic Triangle-Free Graphs of Prime Order. Suppose that p is an odd prime and $p > 120$. Let $G_1 = G_p(S_1)$ ($|S_1| > 10$) be a cyclic triangle-free graph, $S_0 = \{a, a + 1, \dots, b\}$ and $a = \lceil p/3 \rceil$ and $b = p - 1/2$, $G_p(S_0)$ is a triangle-free graph.

TABLE 1: Sizes of small parameter sets of some cyclic triangle-free graphs with prime orders.

p	127	131	137	139	149	151	157	163	167	173
α	9	9	9	10	10	10	11	11	11	12
p	179	181	191	193	197	199	211	223	227	229
α	12	12	13	13	13	13	13	14	14	14
p	233	239	241	251	257	263	269	271	277	281
α	14	14	14	15	15	15	15	16	16	16
p	283	293	307	311	313	317	331	337	347	349
α	16	17	17	17	17	18	18	18	19	19
p	353	359	367	373	379	383	389	397	401	
α	19	19	19	20	20	20	20	21	20	

Based on the data on graphs generated by the CTFP, we found that among cyclic graphs $G = G_p(S)$ of order p generated that have x parameters, the number of parameters in $S \cap S_0$ equals to $\lfloor x/3 \rfloor$ or $\lfloor x/3 \rfloor + 1$ with high probability. This can be proved via Theorem 2.

Theorem 2. *Suppose that p is a prime and $p > 120$, and let $G_1 = G_p(S_1)$ be a cyclic triangle-free graph, where $|S_1| = k > 10$. Let $S_0 = \{a, a + 1, \dots, b\}$, where $a = \lfloor p/3 \rfloor$ and $b = p - 1/2$, there is a parameter set S such that $G = G_p(S)$ is isomorphic to G_1 , and $|S \cap S_0| \geq \lfloor |S_1|/3 \rfloor$.*

Proof. Let $T = S \cup \{i \mid p - i \in S\}$ and $x \pmod{p}$ denote $y \in \{0, 1, \dots, p - 1\}$ such that $y \equiv x \pmod{p}$, i.e., $y - x = cp$ (c is an integer). Let $T_i = \{ij \pmod{p} \mid j \in T\}$, and $S_i = T_i \cap \{1, 2, \dots, (p - 1)/2\}$ for any $i \in \{1, 2, \dots, p - 1\}$, $T_1 = T$ and $|T_i| = 2k$ for any $i \in \{1, 2, \dots, p - 1\}$. For any $i \in \{1, 2, \dots, p - 1\}$, $G_p(S_i)$ is isomorphic to $G_p(S_1)$. For any $i \in T$, if j runs over $\{1, 2, \dots, p - 1\}$, then $ij \pmod{p}$ runs over $\{1, 2, \dots, p - 1\}$ as well. Therefore, for any $i \in \{1, 2, \dots, p - 1\}$, there are $2k$ sets among T_1, T_2, \dots, T_{p-1} that contain i . According to the Drawer principle, there is i_0 such that $|T_{i_0} \cap S_0| \geq \lfloor |S_1|/3 \rfloor$. Let S be S_{i_0} , we can conclude that $|S \cap S_0| \geq \lfloor |S_1|/3 \rfloor$.

We can also prove that $|S \cap S_0| \geq \lfloor |S_1|/3 \rfloor$ in Theorem 2, with similar proof. By the method based on Theorem 2, however, we can generate cyclic triangle-free graphs more similar to those generated by the CTFP at the distribution of parameters.

The CTFP generates $G_p(S_i)$ and $G_p(S)$ in Theorem 2 with the same probability for any $i \in \{1, \dots, p - 1\}$. Based on a result given by Muzychuk on isomorphic cyclic graphs [22], all cyclic graphs isomorphic to $G_p(S)$ can be obtained in the same manner when the order is an odd prime.

Based on Theorem 2, a novel approach similar to the CTFP can be devised, which can generate cyclic graphs of prime order more effectively. We choose $\lfloor x/3 \rfloor$ integers in S_0 randomly, and generate a cyclic graph of order p by iteratively adding parameters. Firstly, we add parameters chosen uniformly in $\{1, \dots, a - 1\}$ at random and subject to the constraint that no triangle is formed in the cyclic graph obtained, until no more parameter in $\{1, 2, \dots, a - 1\}$ can be added. Then, we iteratively add parameters that are chosen

uniformly in $\{a, a + 1, \dots, b\}$ at random and conformed to the constraint that no triangle is formed in the cyclic graph obtained, until there is no such parameter that is different from those $\lfloor x/3 \rfloor$ integers chosen in S_0 earlier.

This method is similar to the CTFP. Together with other improvement, the new method can generate cyclic triangle-free graphs more quickly than the CTFP. Among cyclic graphs of order p that have x parameters generated using this method, in many cases, the number of parameters in $S \cap S_0$ equals to $\lfloor x/3 \rfloor$ or $\lfloor x/3 \rfloor + 1$.

If we generate 100 cyclic triangle-free graphs of order 313 by this method, of which the number of parameters is at most 20, then the computation spends about 28 seconds in average. This allows to generate more graphs. We have generated more than 10^5 graphs of order 313 whose the number of parameters is at most 20, and found a triangle-free graph of order 313 and independence number 41. We have also generated many cyclic triangle-free graphs of order 317, of which the number of parameters is at most 20, and found a triangle-free graph of order 317 and independence number 41.

From computing results obtained by the new method, we can see that, in studying lower bounds on $R(3, t)$, the new method can achieve results similar to that derived by the CTFP more efficiently. \square

4. New Lower Bounds on Small $R(3, t)$

Take the Ramsey number $R(3, 37)$ as an example, the lower bound $R(3, 37) \geq 255$ was obtained in [11]. If we can improve it into $R(3, 37) \geq 258$ based on a cyclic triangle-free graph G of order 257, then G has at most 18 parameters. It is easy to find a large independent set in some cases, in which the neighborhood of a vertex is a 36-independent set contained in a larger independent set. Hence $\alpha(G) > 36$ in these cases, and graphs obtained can not be used in improving $R(3, 37) \geq 255$.

This method is powerful in computing graphs when the orders of cyclic graphs are large. Although useful, this method is less important in improving the efficiency as the method on generating random cyclic triangle-free graphs, as discussed in the last section.

We have conducted much computation to improve the best known lower bound on $R(3, t)$, based on graphs of prime order p , generated by the CTFP and the new method

TABLE 2: Small independence numbers of some cyclic triangle-free graphs of given prime orders.

p	127	131	137	139	149	151	157	163	167	173
α	22	22	23	23	24	25	25	26	27	27
p	179	181	191	193	197	199	211	223	227	229
α	28	28	29	29	30	30	31	32	33	33
p	233	239	241	251	257	263	269	271	277	281
α	34	34	34	35	36	36	37	37	38	38
p	283	293	307	311	313	317	331	337	347	349
α	38	39	40	41	41	41	42	43	44	44
p	353	359	367	373	379	383	389	397	401	
α	44	45	46	46	46	47	48	48	48	

TABLE 3: Parameter sets of some graphs obtained by the CTFP.

Order	$\alpha(G)$	Parameter set
229	33	3, 31, 40, 41, 49, 50, 51, 58, 63, 67, 73, 79, 84, 86, 88, 97
241	34	6, 10, 11, 31, 43, 44, 45, 58, 70, 72, 73, 91, 93, 100, 108, 109, 112
251	35	1, 3, 8, 17, 19, 21, 26, 33, 58, 60, 64, 70, 80, 95, 104, 108, 119
263	36	3, 5, 7, 11, 20, 33, 39, 49, 57, 58, 67, 81, 89, 102, 108, 110, 118, 131
271	37	8, 29, 36, 39, 48, 54, 63, 66, 79, 80, 81, 86, 97, 98, 101, 103, 123
283	38	2, 3, 7, 16, 17, 21, 27, 31, 36, 56, 60, 68, 86, 97, 109, 110, 119, 123, 132
293	39	5, 12, 18, 35, 43, 46, 49, 50, 56, 72, 82, 83, 103, 110, 114, 116, 124, 135, 143
307	40	24, 31, 32, 43, 49, 54, 59, 65, 69, 70, 72, 99, 105, 106, 107, 111, 116, 120, 125, 145
317	41	2, 20, 33, 42, 43, 59, 64, 70, 73, 80, 83, 87, 88, 98, 119, 124, 127, 133, 138, 145
331	42	10, 13, 17, 22, 24, 42, 49, 50, 57, 61, 69, 75, 80, 95, 115, 142, 146, 148, 151, 153, 162
337	43	31, 32, 36, 52, 57, 59, 66, 74, 76, 94, 100, 115, 117, 119, 121, 127, 129, 138, 141, 145, 162
353	44	2, 3, 15, 32, 39, 52, 53, 63, 72, 79, 80, 86, 97, 117, 123, 130, 134, 141, 153, 154, 163, 174
359	45	9, 10, 12, 13, 27, 50, 52, 74, 75, 80, 82, 98, 103, 106, 117, 123, 131, 145, 151, 152, 159, 170
379	46	6, 9, 14, 19, 30, 35, 42, 64, 67, 68, 69, 71, 95, 108, 112, 115, 146, 148, 151, 158, 171, 174, 187
383	47	8, 14, 18, 39, 41, 42, 61, 70, 74, 76, 91, 97, 98, 107, 110, 126, 141, 142, 153, 157, 162, 164, 190
401	48	11, 18, 25, 34, 42, 55, 56, 79, 85, 91, 114, 117, 124, 126, 129, 130, 131, 150, 162, 174, 177, 178, 194, 200

described in the last section. For each prime p that ranges from 127 to 401, we have generated more than 10000 cyclic graphs of order p , of which the amount of parameters is small enough to enable improving the best known lower bound on $R(3, t)$.

In [11], a question was proposed for cases n between 200 and 230, i.e., whether additional computation using the CTFP can improve the best known lower bounds on $R(3, t)$. We have carried out much computation for any prime p among $\{211, 223, 227, 229\}$, and have found a cyclic triangle-free graph of order 229 and independence number 33. Therefore $R(3, 34) \geq 230$, which improves the previous best known lower bound on $R(3, 34)$ by 1.

Table 2 presents a list of the smallest independence numbers of generated graphs of order p . In most cases, the results were obtained based on graphs generated by the CTFP, and in some cases were obtained based on graphs generated by the new method presented in the last section. Some results of small prime orders were known before, including $R(3, 24) > 139$, $R(3, 25) > 149$, $R(3, 26) > 157$, $R(3, 27) > 163$, $R(3, 28) > 167$, $R(3, 29) > 179$ and $R(3, 29) > 181$. Although improving lower bounds on $R(3, t)$ in Table 2 can be difficult, by the new method on generating random cyclic triangle-free graphs, obtaining the same lower bounds on $R(3, t)$ is easier.

We have also generated more than 10^5 graphs of order 197 of which the number of parameters is at most 14 by the new method, and the independence numbers are all larger

than 29. Note that $R(3, 30) \geq 195$ is the best known lower bound on $R(3, 30)$ given in [16]. We have not found a cyclic triangle-free graph of order 389 and independence number 47, among the 10^5 generated graphs whose the number of parameters is at most 23, generated using the new method.

Some new lower bounds on $R(3, t)$, obtained based on the results in Table 2, are listed in Theorem 3.

Theorem 3. $R(3, 34) \geq 230$, $R(3, 35) \geq 242$, $R(3, 36) \geq 252$, $R(3, 37) \geq 264$, $R(3, 38) \geq 272$, $R(3, 39) \geq 284$, $R(3, 40) \geq 294$, $R(3, 41) \geq 308$, $R(3, 42) \geq 318$, $R(3, 43) \geq 332$, $R(3, 44) \geq 338$, $R(3, 45) \geq 354$, $R(3, 46) \geq 360$, $R(3, 47) \geq 380$, $R(3, 48) \geq 384$ and $R(3, 49) \geq 402$ can be obtained by cyclic triangle-free graphs.

Compared to the results given in [11], except the first one (i.e., $R(3, 34)$), all the best known lower bound are improved by 5 or more. We list the parameter sets and independence numbers of some graphs obtained by the CTFP or the similar method in Table 3, based on which the result in Theorem 3 is obtained.

For the cases in Table 3 where the order $p > 300$, the lower bound is better when the independence number is even. The reason is given below: if the expected independence number α is even, we can generate cyclic triangle-free graphs of which the number of parameters is no larger than $\alpha/2$ quickly, which allows to deal with many graphs; if more random cyclic triangle-free graphs are generated, we can

improve some best known lower bounds of form $R(3, \alpha + 1) > p$ in which cases the expected independence number α is odd.

5. Conclusion and Discussions

In this paper, we have improved the best known lower bound for $R(3, t)$ based on some cyclic graphs of prime order obtained by the CTFP or a similar method. For t that is not small, the works on the lower bound for $R(3, t)$ based on cyclic triangle-free graphs earlier than [11] were not efficient in finding good parameter sets. The CTFP can be used as a good tool in studying the lower bound on $R(3, t)$ for large t .

We propose a problem on cyclic triangle-free graphs of prime orders.

Problem 1. Suppose that n is an integer and $n \geq 10$. Let $f(n)$ be the minimum among the independence numbers of all cyclic triangle-free graphs of order n . Is there an integer $n_0 \geq 120$ such that for any pair primes p_1 and p_2 , $f(p_1) \geq f(p_2)$ when $p_1 > p_2 > n_0$?

We propose this problem based on the data in Table 2. For instance, we know that $f(311) \leq 41$ and $f(313) \leq 41$, while whether $f(313) \geq f(311)$ is unknown. Note that in this paper we have obtained lower bounds on $R(3, t)$ by computing small upper bounds for $f(n)$.

For a positive integer $n \leq 121$, there is no cyclic $(3, t)$ -graphs of order n that can be used to improve the best known lower bound on $R(3, t)$ given in [15] (see [23]). It is likely that for any integer n between 121 and 200, there is not a cyclic triangle-free graph that can be used to improve the best known lower bound on $R(3, t)$.

Data Availability

The data used to support the findings of this study are included within the article.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

Acknowledgments

Research is supported in part by the National Natural Science Foundation of China (11361008).

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