

Research Article

Nonzero Solutions for Nonlinear Systems of Fourth-Order Boundary Value Problems

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This study is devoted to the investigation of nonlinear systems of fourth-order boundary value problems. Namely, using some techniques from matrix analysis and ordinary differential equations, a Lyapunov-type inequality providing a necessary condition for the existence of nonzero solutions is obtained. Next, an estimate involving generalized eigenvalues is derived as an application of our main result.

1. Introduction

In this study, we investigate the system of differential equations

$$\begin{cases} y^{(4)}(t) = \rho(t)\mu(t, y, z), & 0 < t < 1, \\ z^{(4)}(t) = \sigma(t)\xi(t, y, z), & 0 < t < 1, \end{cases} \quad (1)$$

subjected to the boundary conditions

$$\begin{cases} y(0) = y'(0) = y''(1) = y^{(3)}(1) = 0, \\ z(0) = z'(0) = z''(1) = z^{(3)}(1) = 0, \end{cases} \quad (2)$$

where $\rho, \sigma: [0, 1] \rightarrow \mathbb{R}$ and $\mu, \xi: [0, 1] \times C([0, 1]) \times C([0, 1]) \rightarrow \mathbb{R}$ are the continuous functions with $\mu(\cdot, 0, 0) = \xi(\cdot, 0, 0) \equiv 0$. Clearly, $(y, z) \equiv (0, 0)$ is a trivial solution to (1) and (2). The aim of this study is to obtain necessary conditions for the existence of nonzero solutions to the considered problem. Namely, we establish new Lyapunov-type inequalities [1] for (1) and (2) under reasonable conditions on the nonlinearities $\mu(t, y, z)$ and $\xi(t, y, z)$. Our approach is based essentially on matrix analysis and some arguments from ordinary differential equations.

Fourth-order differential equations are useful in modeling many phenomena from physics ([2–7] and the references therein), which makes the study of such equations particularly interesting. In the literature, several contributions have been devoted to the investigation of sufficient conditions ensuring the existence of solutions to fourth-order boundary value problems ([2, 8–19] and the references therein). The study of necessary conditions for the existence of nontrivial solutions to fourth-order differential equations via Lyapunov-type inequalities has been investigated by some authors [20–22]. For instance, in [20], among other results, it was shown that, if y is a nontrivial solution to

$$\begin{cases} y^{(4)}(t) + \lambda(t)y(t) = 0, & a < t < b, \\ y(a) = y'(a) = y''(a) = y'''(b) = 0, \end{cases} \quad (3)$$

where $\lambda: [a, b] \rightarrow \mathbb{R}$ is continuous, then

$$\int_a^b \lambda^+(s) ds > \frac{512}{9(b-a)^3}, \quad (4)$$

where $\lambda^+(t) = \max\{\lambda(t), 0\}$.

For recent contributions related to Lyapunov-type inequalities, see e.g., [23–29] and the references therein.

This study is organized as follows. The next section is devoted to some preliminaries. In Section 3, the obtained results as well as their proofs are presented. Finally, some applications to generalized eigenvalue problems are given in Section 4.

2. Some Preliminaries

First, we fix some notations. We denote by $\leq_{\mathbb{R}^2}$ the partial order in the Euclidean space \mathbb{R}^2 defined as

$$\vec{u} \leq_{\mathbb{R}^2} \vec{v} \Leftrightarrow u_i \leq v_i, \quad i = 1, 2, \tag{5}$$

for every $\vec{u} = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}, \vec{v} = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} \in \mathbb{R}^2$.

We denote by M_2^+ the set of square matrices having nonnegative coefficients, i.e.,

$$M_2^+ = \left\{ (m_{ij})_{1 \leq i, j \leq 2} : m_{ij} \geq 0, \quad 1 \leq i, j \leq 2 \right\}. \tag{6}$$

For $C \in M_2^+$, the trace of C is denoted by $\text{Trace}(C)$, the determinant of C is denoted by $\det(C)$, and the spectral radius of C is denoted by ρ_C , i.e.,

$$\rho_C = \max \{ |\lambda_i(C)| : i = 1, 2 \}, \tag{7}$$

where $\lambda_i(C)$ are the complex eigenvalues of C .

We equip M_2^+ with the norm $\| \cdot \|$ defined as

$$\|C\| = \sup_{\vec{u} \in \mathbb{R}^2, \|\vec{u}\|_2 \neq 0} \frac{\|C\vec{u}\|_2}{\|\vec{u}\|_2}, \quad C \in M_2^+, \tag{8}$$

where $\| \cdot \|_2$ is the Euclidean norm in \mathbb{R}^2 .

The following lemmas will be useful later.

Lemma 1. Let $\vec{u}, \vec{v} \in \mathbb{R}^2$ with

$$\vec{0} \leq_{\mathbb{R}^2} \vec{u} \leq_{\mathbb{R}^2} \vec{v}, \tag{9}$$

where $\vec{0}$ is the zero vector in \mathbb{R}^2 . Then, $\|\vec{u}\|_2 \leq \|\vec{v}\|_2$.

Proof. The result follows from the fact that

$$P = \left\{ \vec{u} \in \mathbb{R}^2 : \vec{u} \geq_{\mathbb{R}^2} \vec{0} \right\} \tag{10}$$

is a normal cone in \mathbb{R}^2 with normal constant equal to 1 (e.g., [30]). \square

Lemma 2 (See e.g., [31]). Let $C \in M_2^+$. If $\rho_C < 1$, then

$$\lim_{n \rightarrow \infty} \|C\|^n = 0. \tag{11}$$

Lemma 3 (See [25]). Let $C \in M_2^+$. Then,

$$\rho_C = \frac{\text{Trace}(C) + \sqrt{[\text{Trace}(C)]^2 - 4\det(C)}}{2}. \tag{12}$$

Lemma 4 (See [12]). Let $x \in C^4((0, 1)) \cap C^1([0, 1]) \cap C^3((0, 1))$ be a solution to

$$\begin{cases} x^{(4)}(t) = f(t), & 0 < t < 1, \\ x(0) = x'(0) = x''(1) = x^{(3)}(1) = 0, \end{cases} \tag{13}$$

where $f \in C([0, 1])$. Then,

$$x(t) = \int_0^1 H(t, \tau) f(\tau) d\tau, \quad 0 \leq t \leq 1, \tag{14}$$

where

$$0 \leq H(t, \tau) = \begin{cases} \frac{(3t - \tau)\tau^2}{6}, & \text{if } 0 \leq \tau \leq t \leq 1, \\ \frac{(3\tau - t)t^2}{6}, & \text{if } 0 \leq t \leq \tau \leq 1. \end{cases} \tag{15}$$

Lemma 5. For all $0 < \tau < 1$,

$$\max_{0 \leq t \leq 1} H(t, \tau) = H(1, \tau) = \frac{(3 - \tau)\tau^2}{6}. \tag{16}$$

Proof. Fix $0 < \tau < 1$. Since $H(\cdot, \tau)$ is nondecreasing in $[\tau, 1]$, we deduce that

$$H(t, \tau) \leq H(1, \tau), \quad \text{for all } t \in [\tau, 1]. \tag{17}$$

Similarly, it can be easily shown that $H(\cdot, \tau)$ is nondecreasing in $[0, \tau]$, which yields

$$H(t, \tau) \leq H(\tau, \tau) = \frac{\tau^3}{3}, \quad \text{for all } t \in [0, \tau]. \tag{18}$$

Combining (17) with (18), for all $0 \leq t \leq 1$, we obtain

$$H(t, \tau) \leq \max \left\{ \frac{(3 - \tau)\tau^2}{6}, \frac{\tau^3}{3} \right\} = \frac{(3 - \tau)\tau^2}{6}. \tag{19}$$

Hence, (16) is proved. \square

Throughout this study, we denote by $\| \cdot \|_{\infty}$ the norm in $C([0, 1])$ defined as

$$\|\vartheta\|_{\infty} = \max_{0 \leq t \leq 1} |\vartheta(t)|, \quad \vartheta \in C([0, 1]). \tag{20}$$

3. Results and Proofs

We investigate (1) and (2) under the following assumptions:

- (A1) $\rho, \sigma: [0, 1] \rightarrow \mathbb{R}$ are continuous
- (A2) $\mu, \xi: [0, 1] \times C([0, 1]) \times C([0, 1]) \rightarrow \mathbb{R}$ are continuous
- (A3) $\mu(\cdot, 0, 0) = \xi(\cdot, 0, 0) \equiv 0$ (where 0 is the zero function)
- (A4) For all $(t, y, z) \in [0, 1] \times C([0, 1]) \times C([0, 1])$,

$$\begin{aligned} |\mu(t, y, z)| &\leq \mu_{11}(t) \|y\|_{\infty} + \mu_{12}(t) \|z\|_{\infty}, \\ |\xi(t, y, z)| &\leq \xi_{21}(t) \|y\|_{\infty} + \xi_{22}(t) \|z\|_{\infty}, \end{aligned} \tag{21}$$

where $\mu_{11}, \mu_{12}, \xi_{21}, \xi_{22}: [0, 1] \rightarrow [0, \infty)$ are the continuous functions.

By solution to (1) and (2), we mean a pair of functions (y, z) , $y, z \in C^4((0, 1)) \cap C^1([0, 1]) \cap C^3((0, 1))$, satisfying (1) and the initial conditions (2). A solution (y, z) to (1) and (2) is said to be nontrivial, if $(y, z) \equiv (0, 0)$.

For $\delta, \eta \in C([0, 1])$, let

$$J_\delta(\eta) = \int_0^1 (3 - \tau)\tau^2 |\delta(\tau)| \eta(\tau) d\tau. \tag{22}$$

Theorem 1. *If (1) and (2) have a nontrivial solution, then*

$$J_\rho(\mu_{11}) + J_\sigma(\xi_{22}) + \sqrt{(J_\rho(\mu_{11}) - J_\sigma(\xi_{22}))^2 + 4J_\sigma(\xi_{21})J_\rho(\mu_{12})} \geq 12. \tag{23}$$

Proof. Let (y, z) be a nontrivial solution to (1) and (2), and suppose that

$$J_\rho(\mu_{11}) + J_\sigma(\xi_{22}) + \sqrt{(J_\rho(\mu_{11}) - J_\sigma(\xi_{22}))^2 + 4J_\sigma(\xi_{21})J_\rho(\mu_{12})} < 12. \tag{24}$$

By Lemma 4, $(y, z) \in C([0, 1]) \times C([0, 1])$ is a nontrivial solution to the system of integral equations:

$$\begin{cases} y(t) = \int_0^1 H(t, \tau)\rho(\tau)\mu(\tau, y, z)d\tau \\ z(t) = \int_0^1 H(t, \tau)\sigma(\tau)\xi(\tau, y, z)d\tau \end{cases}, \quad 0 \leq t \leq 1. \tag{25}$$

Using (A4) and (16), for all $0 \leq t \leq 1$, we obtain

$$\begin{aligned} |y(t)| &\leq \int_0^1 |H(t, \tau)|\rho(\tau)\|\mu(\tau, y, z)\|d\tau \\ &\leq \int_0^1 \frac{(3 - \tau)\tau^2}{6} |\rho(\tau)|(\mu_{11}(\tau)\|y\|_\infty + \mu_{12}(\tau)\|z\|_\infty)d\tau \\ &= \left(\int_0^1 \frac{(3 - \tau)\tau^2}{6} |\rho(\tau)|\mu_{11}(\tau)d\tau\right)\|y\|_\infty + \left(\int_0^1 \frac{(3 - \tau)\tau^2}{6} |\rho(\tau)|\mu_{12}(\tau)d\tau\right)\|z\|_\infty, \end{aligned} \tag{26}$$

which leads to

$$\begin{aligned} \|y\|_\infty &\leq \left(\int_0^1 \frac{(3 - \tau)\tau^2}{6} |\rho(\tau)|\mu_{11}(\tau)d\tau\right)\|y\|_\infty \\ &\quad + \left(\int_0^1 \frac{(3 - \tau)\tau^2}{6} |\rho(\tau)|\mu_{12}(\tau)d\tau\right)\|z\|_\infty. \end{aligned} \tag{27}$$

Similarly, by (A4) and (16), we obtain

$$\begin{aligned} \|z\|_\infty &\leq \left(\int_0^1 \frac{(3 - \tau)\tau^2}{6} |\sigma(\tau)|\xi_{21}(\tau)d\tau\right)\|y\|_\infty \\ &\quad + \left(\int_0^1 \frac{(3 - \tau)\tau^2}{6} |\sigma(\tau)|\xi_{22}(\tau)d\tau\right)\|z\|_\infty. \end{aligned} \tag{28}$$

Combining (27) with (28), we deduce that

$$\vec{0} \leq_{\mathbb{R}^2} \overrightarrow{\phi}_{y,z} \leq_{\mathbb{R}^2} C \overrightarrow{\phi}_{y,z}, \tag{29}$$

where $\overrightarrow{\phi}_{y,z} = \begin{pmatrix} \|y\|_\infty \\ \|z\|_\infty \end{pmatrix}$ and

$$C = \frac{1}{6} \begin{pmatrix} J_\rho(\mu_{11}) & J_\rho(\mu_{12}) \\ J_\sigma(\xi_{21}) & J_\sigma(\xi_{22}) \end{pmatrix}. \tag{30}$$

Next, using Lemma 3 and (24), we deduce that

$$\rho_C < 1. \tag{31}$$

On the other hand, using Lemma 1 and (29), we obtain

$$\|\overrightarrow{\phi}_{y,z}\|_2 \leq \|C \overrightarrow{\phi}_{y,z}\|_2 \leq \|C\| \|\overrightarrow{\phi}_{y,z}\|_2. \tag{32}$$

Since (y, z) is nontrivial, then $\overrightarrow{\phi}_{y,z} \neq \vec{0}$, and the above inequality leads to

$$\|C\| \geq 1. \tag{33}$$

But by Lemma 2 and (31), we know that

$$\lim_{n \rightarrow \infty} \|C\|^n = 0, \tag{34}$$

which contradicts (33). This proves (23). \square

Next, we discuss some particular cases of Theorem 1.

3.1. Nonlinearities Involving Trigonometric Functions.

Consider the system of differential equations

$$\begin{cases} y^{(4)}(t) = \rho(t)\sin(y(t) + z(t)), & 0 < t < 1, \\ z^{(4)}(t) = \sigma(t)\arctan(y(t) + z(t)), & 0 < t < 1, \end{cases} \tag{35}$$

under the boundary conditions (2), where $\rho, \sigma \in C([0, 1])$. Observe that (35) is a particular case of (1) with

$$\begin{aligned} \mu(t, y, z) &= \sin(y(t) + z(t)), \\ (t, y, z) &\in [0, 1] \times C([0, 1]) \times C([0, 1]), \\ \xi(t, y, z) &= \arctan(y(t) + z(t)), \end{aligned} \tag{36}$$

$$(t, y, z) \in [0, 1] \times C([0, 1]) \times C([0, 1]).$$

Obviously, μ and ξ satisfy (A3). Moreover, for all $(t, y, z) \in [0, 1] \times C([0, 1]) \times C([0, 1])$,

$$\begin{aligned} |\mu(t, y, z)| &= |\sin(y(t) + z(t))| \leq |y(t) + z(t)| \leq \|y\|_\infty + \|z\|_\infty, \\ |\xi(t, y, z)| &= |\arctan(y(t) + z(t))| \leq |y(t) + z(t)| \leq \|y\|_\infty + \|z\|_\infty. \end{aligned} \tag{37}$$

Then, (A4) is satisfied with

$$\mu_{11} = \mu_{12} = \xi_{21} = \xi_{22} = 1. \tag{38}$$

Hence, by Theorem 1, we deduce the following.

Corollary 1. *If (35) and (2) have a nontrivial solution, then*

$$J_\rho(1) + J_\sigma(1) + \sqrt{(J_\rho(1) - J_\sigma(1))^2 + 4J_\sigma(1)J_\rho(1)} \geq 12. \tag{39}$$

3.2. Nonlocal Source Terms. Consider the system of differential equations

$$\begin{cases} y^{(4)}(t) = \rho(t) \int_0^t (t-s)^{\alpha-1} (y(s) + z(s)) ds, & 0 < t < 1, \\ z^{(4)}(t) = \sigma(t) \int_0^t (t-s)^{\beta-1} (y(s) + z(s)) ds, & 0 < t < 1, \end{cases} \tag{40}$$

under the boundary condition (16), where $\rho, \sigma \in C([0, 1])$ and $\alpha, \beta > 0$. Problem (40) is a particular case of (1) with

$$\begin{aligned} \mu(t, y, z) &= \begin{cases} \int_0^t (t-s)^{\alpha-1} (y(s) + z(s)) ds, & \text{if } 0 < t \leq 1, \\ 0, & \text{if } t = 0, \end{cases} \\ \xi(t, y, z) &= \begin{cases} \int_0^t (t-s)^{\beta-1} (y(s) + z(s)) ds & \text{if } 0 < t \leq 1, \\ 0, & \text{if } t = 0, \end{cases} \end{aligned} \tag{41}$$

for all $y, z \in C([0, 1])$. Obviously, μ and ξ satisfy (A3). Moreover, for all $(t, y, z) \in [0, 1] \times C([0, 1]) \times C([0, 1])$,

$$\begin{aligned} |\mu(t, y, z)| &\leq \left(\int_0^t (t-s)^{\alpha-1} ds \right) (\|y\|_\infty + \|z\|_\infty) \\ &= \frac{t^\alpha}{\alpha} (\|y\|_\infty + \|z\|_\infty). \end{aligned} \tag{42}$$

and similarly

$$|\xi(t, y, z)| \leq \frac{t^\beta}{\beta} (\|y\|_\infty + \|z\|_\infty). \tag{43}$$

Then, (A4) is satisfied with

$$\begin{aligned} \mu_{11}(t) = \mu_{12}(t) &= \frac{t^\alpha}{\alpha}, & 0 \leq t \leq 1, \\ \xi_{21}(t) = \xi_{22}(t) &= \frac{t^\beta}{\beta}, & 0 \leq t \leq 1. \end{aligned} \tag{44}$$

Then, by Theorem 1, we deduce the following.

Corollary 2. *If (39) and (2) have a nontrivial solution, then*

$$\begin{aligned} &J_\rho\left(\frac{t^\alpha}{\alpha}\right) + J_\sigma\left(\frac{t^\beta}{\beta}\right) \\ &+ \sqrt{\left(J_\rho\left(\frac{t^\alpha}{\alpha}\right) - J_\sigma\left(\frac{t^\beta}{\beta}\right)\right)^2 + 4J_\sigma\left(\frac{t^\beta}{\beta}\right)J_\rho\left(\frac{t^\alpha}{\alpha}\right)} \geq 12. \end{aligned} \tag{45}$$

Example 1. Consider the system of differential equation (35) with $\rho \equiv 1$ and $\sigma(t) = 9/4(3-t)$ for all $t \in [0, 1]$. Elementary calculations show that

$$\begin{aligned} J_\rho(1) = J_\sigma(1) &= \frac{3}{4}, \\ J_\rho(1) + J_\sigma(1) + \sqrt{(J_\rho(1) - J_\sigma(1))^2 + 4J_\sigma(1)J_\rho(1)} &= 3 < 12. \end{aligned} \tag{46}$$

Hence, by Corollary 1, we deduce that (35) and (2) have no nontrivial solution.

4. Generalized Eigenvalues Problems

We say that $e = (e_{ij})_{1 \leq i, j \leq 2}$ is a generalized eigenvalue of the system of differential equations

$$\begin{cases} y^{(4)}(t) = e_{11}y(t) + e_{12}z(t), & 0 < t < 1, \\ z^{(4)}(t) = e_{21}y(t) + e_{22}z(t), & 0 < t < 1, \end{cases} \tag{47}$$

subjected to the boundary condition (2), if (2) and (44) admit a nonzero solution (y_e, z_e) . Notice that (44) is a particular case of (1) with

$$\begin{aligned} \rho(t) &= 1, \mu(t, y, z) = e_{11}y(t) + e_{12}z(t), \\ (t, y, z) &\in [0, 1] \times C([0, 1]) \times C([0, 1]), \\ \sigma(t) &= 1, \xi(t, y, z) = e_{21}y(t) + e_{22}z(t), \\ (t, y, z) &\in [0, 1] \times C([0, 1]) \times C([0, 1]). \end{aligned} \tag{48}$$

Moreover, (A2)–(A4) are satisfied with

$$\begin{aligned}
 \mu_{11} &= e_{11}, \\
 \mu_{12} &= e_{12}, \\
 \xi_{21} &= e_{21}, \\
 \xi_{22} &= e_{22}.
 \end{aligned}
 \tag{49}$$

Hence, by Theorem 1, if e is a generalized eigenvalue of (2) and (44), then

$$J_1(e_{11}) + J_1(e_{22}) + \sqrt{(J_1(e_{11}) - J_1(e_{22}))^2 + 4J_1(e_{21})J_1(e_{12})} \geq 12.
 \tag{50}$$

On the other hand, for $\lambda \in \mathbb{R}$, we have

$$J_1(\lambda) = \lambda \int_0^1 (3 - \tau)\tau^2 d\tau = \frac{3\lambda}{4}.
 \tag{51}$$

Hence, (50) reduces to

$$e_{11} + e_{22} + \sqrt{(e_{11} - e_{22})^2 + 4e_{21}e_{12}} \geq 16.
 \tag{52}$$

Therefore, the following result follows.

Corollary 3. *If $e = (e_{ij})_{1 \leq i, j \leq 2}$ is a generalized eigenvalue of (43) and (2), then*

$$e_{11} + e_{22} + \sqrt{(e_{11} - e_{22})^2 + 4e_{21}e_{12}} \geq 16.
 \tag{53}$$

5. Conclusion

Using some techniques from matrix analysis and ordinary differential equations, a necessary condition for the existence of nonzero solutions to (1) and (2) is obtained (Theorem 1). As particular cases of (1), we discussed nonlinearities involving trigonometric functions (Corollary 1) and nonlocal source terms (Corollary 2). Finally, we applied our main result to obtain an estimate involving generalized eigenvalues (Corollary 3).

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that there are no conflicts of interest.

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