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Research Article

Existence and Uniqueness Results of Volterra–Fredholm Integro-Differential Equations via Caputo Fractional Derivative

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In this paper, we study a Volterra–Fredholm integro-differential equation. The considered problem involves the fractional Caputo derivatives under some conditions on the order. We prove an existence and uniqueness analytic result by application of the Banach principle. Then, another result that deals with the existence of at least one solution is delivered, and some sufficient conditions for this result are established by means of the fixed point theorem of Schaefer. Ulam stability of the solution is discussed before including an example to illustrate the results of the proposal.

1. Introduction

Fractional calculus and differential equations of fractional order are of great importance since they can be used in analyzing and modeling real word phenomena [1–3]. Recently, there has been a very important progress in the study of the theory of differential equations of fractional order. The theory of differential equations of arbitrary order has been recently proved to be an important tool for modeling many physical phenomena. For more details, refer to [4–9].

The fractional integro-differential equations have been recently used as effective tools in the modeling of many phenomena in various fields of applied sciences and engineering such as acoustic control, signal processing, electrochemistry, viscoelasticity, polymer physics, electromagnetics, optics, medicine, economics, chemical engineering, chaotic dynamics, and statistical physics (see [10–16]).

Hattaf in [17] proposed a new definition of fractional derivative that generalizes the fractional derivatives [18, 19] with nonsingular kernel for both Caputo and Riemann–Liouville types.

The efficient numerical method based on a novel shifted piecewise cosine basis for solving Volterra–Fredholm integral equations of the second kind is investigated (see [20]).

Recently, Wang et al. in [21] studied a nonlinear fractional differential equations with Hadamard derivative and Ulam stability in the weighted space of continuous functions. Some sufficient conditions for existence of solutions are given by using fixed point theorems via a prior estimation in the weighted space of the continuous functions. Ahmad et al. in [22] discussed the existence of solutions for an initial value problem of nonlinear hybrid differential equations of Hadamard type.

Ahmed et al. [23] discussed the existence of solutions by means of endpoint theory for initial value problem of

Hadamard and Riemann–Liouville fractional integro-differential inclusion of the form as follows:

$$\begin{cases} D^{\alpha}\left(x(t) - \sum_{i=1}^{m} I^{\rho_i} G_i(t, x(t)) \in F(t, x(t)), \quad t \in J = [1, T], 0 < \alpha \leq 1, \right. \\ x(1) = 0, \end{cases} \tag{1}$$

where D^{α} denotes the Hadamard fractional derivative of order α for $0 < \alpha \leq 1$. I^{φ} is the Riemann–Liouville integral of order $\varphi > 0$, $\varphi \in \{\rho_1, \rho_2, \ldots, \rho_m\}$, $G_i \in C(J \times \mathbb{R}, \mathbb{R})$ with $G_i(1, 0) = 0$ for $i = 1, 2, 3 \ldots, m$.

Very recent work like Hamoud et al. [24] established some new conditions for the existence and uniqueness of solutions for a class of nonlinear Hadamard fractional Volterra–Fredholm integro-differential equations with initial conditions. The homotopy perturbation method has been successfully applied to find the approximate solution of a Caputo fractional Volterra–Fredholm integro-differential equation.

Motivated by the above works, we will study the following problem of fractional integro-differential equations in the context of Caputo fractional derivative called Caputo fractional Volterra–Fredholm integro-differential equations of the form as follows:

$$\begin{cases} D^{\alpha}\left(x(t) - \sum_{i=1}^{m} I^{\rho_i} f_i(t, x(t))\right) = g(t, x(t), Kx(t), Hx(t)), \quad t \in J = [0, 1], \\ x(0) = 0, \\ D^{\alpha} x(0) = \eta, \ 1 < \alpha < 2, \end{cases} \tag{2}$$

where D^{α} is in the sense of Caputo, $f\colon J \times \mathbb{R}^n \times \mathbb{R}^n \longrightarrow \mathbb{R}^n$ is a given function, K and H are linear integral operators defined by $Kx(t) = \int_0^t k(t, \tau)x(t)\mathrm{d}\tau$ and $Hx(t) = \int_0^t h(t, \tau)x(t)\mathrm{d}\tau$, and its called Volterra–Fredholm integro-differential with $\theta_1 = \sup\{|k(t, \tau)|\colon (t, \tau) \in J \times J\}$ and $\theta_2 = \sup\{|h(t, \tau)|\colon (t, \tau) \in J \times J\}$.

The paper is organised as follows. In Section 2, we recall some definitions and lemmas that are used for the proof of our main results. In Section 3, we prove the main theorems of this paper by the existence and uniqueness of the solution which have been proved and some numerical simulation of the solution. A brief conclusion is given in Section 6.

2. Preliminaries

In this section, we introduce some definitions, lemmas, and preliminaries facts which are used throughout this paper (see [7] for more information). Let $|.|$ be a suitable norm in $\mathbb{R}^n$ and $\|.\|$ be the matrix norm. Let $E = C(J, \mathbb{R})$ denote the Banach space of continuous function on J with the norm

$$\|x\| = \sup\{|x|, x \in J\}. \tag{3}$$

Definition 1. The Riemann–Liouville integral of order $\alpha > 0$ for a continuous function $\varphi \in L^1((0, 1], \mathbb{R})$ is given by

$$I^{\alpha}\varphi(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t - \tau)^{\alpha - 1} \varphi(t)\mathrm{d}\tau, \quad \forall t \in (0, 1], \tag{4}$$

with $\Gamma(\alpha)\colon = \int_0^{\infty} e^{-u} u^{\alpha - 1}\mathrm{d}u$.

Definition 2. If $\varphi \in C^n([0, 1], \mathbb{R})$ and $n - 1 < \alpha \leq n$, then the Caputo fractional derivative is given by

$$\begin{aligned} D^{\alpha}\varphi(t) &= I^{n-\alpha}\frac{\mathrm{d}^n}{\mathrm{d}t^n}(\varphi(t)) \\ &= \frac{1}{\Gamma(n - \alpha)} \int_0^t (t - s)^{n-\alpha-1} \varphi^{(n)}(s)\mathrm{d}s, \end{aligned} \tag{5}$$

where the parameter α is the order of the derivative and is allowed to be real or even complex.

Lemma 1. *Let $n \in \mathbb{N}^*$ and $n - 1 < \alpha < n$, then the general solution of $D^{\alpha}u(t) = 0$ is given by*

$$u(t) = \sum_{i=0}^{n-1} c_i t^i, \tag{6}$$

such that $c_i \in \mathbb{R}, i = 0, 1, 2, \ldots, n - 1$.

Lemma 2. *Taking $n \in \mathbb{N}^*$ and $n - 1 < \alpha < n$, then we have*

$$I^{\alpha}D^{\alpha}u(t) = u(t) + \sum_{k=0}^{n-1} \frac{u^{(k)}(0)}{k!} t^k, \tag{7}$$

with $t > 0$, $n - 1 < \alpha < n$.

Definition 3. Let X be a Banach space. Then, a map $T\colon X \longrightarrow X$ is called a contraction mapping on X if there exists $q \in [0, 1)$ such that

$$\|T(x) - T(3y)\| \leq q\|x - y\|, \tag{8}$$

for all $x, y \in X$.

Theorem 1 (Banach's fixed point theorem, see [25]). *Let Ω be a nonempty closed subset of a Banach space X. Then, any* *contraction mapping T of Ω into itself has a unique fixed point.*

Theorem 2 (Schaefer's fixed point theorem, see [25]). *Let X be a Banach space, and let $N\colon X \longrightarrow X$ be a completely continuous operator. If the set $E = \{y \in X\colon\ y = \lambda N y \text{ for some } \lambda \in (0,1)\}$ is bounded, then N has fixed points.*

3. Existence and Uniqueness Results

We begin this section by some result that helps us for solving the problem considered in (2).

Lemma 3. *Let $1 < \alpha < 2$ and $G \in C(J, \mathbb{R}^n)$. Then, we can state that the problem*

$$\begin{cases} D^{\alpha}\left(x(t) - \sum_{i=1}^{m} I_i^{p} f_i(t, x(t))\right) = G(t), \quad t \in J = [0,1], \\ x(0) = 0, D^{\alpha} x(0) = \eta, \end{cases} \tag{9}$$

admits as integral solution the following representation:

$$x(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} G(\tau) d\tau + \sum_{i=1}^{m} I^{\rho_i} f_i(t, x(t)). \tag{10}$$

Proof. Using Lemma 2, we get

$$\begin{aligned} I^{\alpha} D^{\alpha}\left(x(t) - \sum_{i=1}^{m} I_i^{p} f_i(t, x(t))\right) &= I^{\alpha} G(t), \\ x(t) - \sum_{i=1}^{m} I_i^{p} f_i(t, x(t)) &= I^{\alpha} G(t) + c_1 t + c_0, \\ x(t) &= I^{\alpha} G(t) + \sum_{i=1}^{m} I_i^{p} f_i(t, x(t)) + c_1 t + c_0. \end{aligned} \tag{11}$$

Using the initial conditions $x(0) = 0$ and $D^{\alpha} x(0) = \eta$, we get $c_1 = c_2 = 0$ which implies that the proof is completed. □

Let us now transform the above problem to a fixed point one. Consider the nonlinear operator $T\colon E \longrightarrow E$ defined by

$$\begin{aligned} Tx(t) = {} & \frac{1}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} g(\tau, x(\tau), Kx(\tau), Hx(\tau)) d\tau \\ & + \sum_{i=1}^{m} I^{\rho_i} f_i(t, x(t)). \end{aligned} \tag{12}$$

To prove the main results, we need to work with the following hypotheses:

(H1) There exists a constant $L_f > 0$, such that

$$\left| f_i(t, x(t)) - f_i(t, y(t)) \right| \le L_f |x(t) - y(t)|. \tag{13}$$

(H2) There exist functions $c_1(t)$, $c_2(t)$, $c_3(t)$, and $a_i(t) \in C(J, \mathbb{R})$ such that

$$\begin{aligned} |g(t, x, y, z)| &\le c_1(t) + c_2(t)|y| + c_2(t)|z|, \quad \forall (t, x, y, z) \in I \times \mathbb{R}^3, \\ \left| f_i(t, x) \right| &\le a_i(t), \quad \forall (t, x) \in I \times \mathbb{R}. \end{aligned} \tag{14}$$

Set $\sup_{i\in I} |c_1(t)| = \|c_1\|$, $\sup_{i\in I} |c_2(t)| = \|c_2\|$, $\sup_{i\in I} |c_3(t)| = \|c_3\|$, and $\sup_{i\in I} |a_i(t)| = \|a_i\|, i = 1, \ldots, m$.

(H3) There exist constants $L_1, L_2, L_3 > 0$ such that

$$\left| g(t, x_1, y_1, z_1) - g(t, x_2, y_2, z_2) \right| \le L_1 \left| x_1 - x_2 \right| + L_2 \left| y_1 - y_2 \right| + L_3 \left| z_1 - z_2 \right|, \quad \forall t \in J, x_i, y_i, z_i \in \mathbb{R}, i = 1, 2. \tag{15}$$

Also, we consider the quantity:

$$R = \frac{L_1}{\Gamma(\alpha+1)} + \frac{\theta_1 L_2 + \theta_2 L_3}{\Gamma(\alpha)} + \sum_{i=1}^{m} \frac{L_f}{\Gamma(\rho_i + 1)}. \tag{16}$$

Theorem 3. *Assume that the hypothesis (H1)-(H2) are fulfilled, and if*

$$L_f \sum_{i=1}^{m} \frac{1}{\Gamma(\rho_i + 1)} < 1, \tag{17}$$

then there exists at least one solution for the problem (2).

Proof. Consider the ball $B_r = \{x \in E: \|x\| \leq r\}$ with $r > 0$, where

$$r \geq \frac{\sum_{i=1}^{m}\left(\|a_i\|/\Gamma(\rho_i + 1)\right) + \left(\|c_1\|/\Gamma(\alpha + 1)\right)}{1 - (1/\Gamma(\alpha))\left(\|c_2\|\theta_1 + \|c_3\|\theta_2\right)}. \quad (18)$$

We define the operators P and Q such that $T = P + Q$, by

$$Px(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t - \tau)^{\alpha - 1} g(\tau, x(\tau), Kx(\tau), Hx(\tau))\mathrm{d}\tau,$$
$$Qx(t) = \sum_{i=1}^{m} \frac{1}{\Gamma(\rho_i)} \int_0^t (t - \tau)^{\rho_i - 1} f_i(t, x(t))\mathrm{d}\tau. \quad (19)$$

For any $x \in B_r$, we have

$$\begin{aligned} |Tx(t)| &\leq \frac{1}{\Gamma(\alpha)} \int_0^t (t - \tau)^{\alpha - 1} |g(\tau, x(\tau), Kx(\tau), Hx(\tau))|\mathrm{d}\tau + \sum_{i=1}^{m} \frac{1}{\Gamma(\rho_i)} \int_0^t (t - \tau)^{\rho_i - 1} \left|f_i(\tau, x(\tau))\right|\mathrm{d}\tau \\ &\leq \frac{1}{\Gamma(\alpha)} \int_0^t (t - \tau)^{\alpha - 1} \left(\left|c_1(\tau)\right| + \left|c_2(\tau)\right||Kx(\tau)| + \left|c_3(\tau)\right||Hx(\tau)|\right)\mathrm{d}\tau \\ &\quad + \sum_{i=1}^{m} \frac{1}{\Gamma(\rho_i)} \int_0^t (t - \tau)^{\rho_i - 1} \left|a_i(\tau)\right|\mathrm{d}\tau \\ &\leq \frac{\|c_1\|}{\Gamma(\alpha + 1)} + \left(\|c_2\|\theta_1 + \|c_3\|\theta_2\right)\frac{r}{\Gamma(\alpha)} + \sum_{i=1}^{m} \frac{\|a_i\|}{\Gamma(\rho_i + 1)}, |Px(t) + Qx(t)| \leq r. \end{aligned} \quad (20)$$

Now, we will show that P is continuous and compact. The operator P is obviously continuous. Also, P is uniformly bounded on B_r as

$$\|Px\| \leq \frac{\|c_1\|}{\Gamma(\alpha + 1)} + \left(\|c_2\|\theta_1 + \|c_3\|\theta_2\right)\frac{r}{\Gamma(\alpha)}. \quad (21)$$

Let $t_1, t_2 \in J$ with $t_1 < t_2$ and $x \in B_r$. Then, we have

$$\begin{aligned} \left|Px(t_2) - Px(t_1)\right| &= \frac{1}{\Gamma(\alpha)} \int_0^{t_2} (t_2 - \tau)^{\alpha - 1} g(\tau, x(\tau), Kx(\tau), Hx(\tau))\mathrm{d}\tau \\ &\quad - \frac{1}{\Gamma(\alpha)} \int_0^{t_1} (t_1 - \tau)^{\alpha - 1} g(\tau, x(\tau), Kx(\tau), Hx(\tau))\mathrm{d}\tau, \\ \left|Px(t_2) - Px(t_1)\right| &\leq \frac{\|c_1\| + \|c_2\|\theta_1 + \|c_3\|\theta_2}{\gamma(\alpha + 1)} \left(t_2^{\alpha} - t_1^{\alpha}\right). \end{aligned} \quad (22)$$

We remark that when $t_2 \longmapsto t_1$, the quantity $\|Px(t_2) - Px(t_1)\| \longmapsto 0$.

Thus, P is equicontinuous and relatively compact on B_r. Then, we show by the Arzelà–Ascoli theorem that P is compact on B_r. Let us show now that Q is a contraction mapping and consider $x, y \in B_r$. Then, for $t \in J$, we have

$$\begin{aligned}|Qx(t) - Qy(t)| &= \left|\sum_{i=1}^{m} \frac{1}{\Gamma(\rho_i)} \int_0^t (t-\tau)^{\rho_i - 1} f_i(\tau, x(\tau))\mathrm{d}\tau - \sum_{i=1}^{m} \frac{1}{\Gamma(\rho_i)} \int_0^t (t-\tau)^{\rho_i - 1} f_i(\tau, y(\tau))\mathrm{d}\tau\right| \\ &\le \sum_{i=1}^{m} \frac{1}{\Gamma(\rho_i)} \int_0^t (t-\tau)^{\rho_i - 1} \left|f_i(\tau, x(\tau)) - f_i(\tau, y(\tau))\right|\mathrm{d}\tau \\ &\le \sum_{i=1}^{m} \frac{1}{\Gamma(\rho_i)} \int_0^t (t-\tau)^{\rho_i - 1} L_f \|x - y\| \\ &\le L_f \sum_{i=1}^{m} \frac{1}{\Gamma(\rho_i + 1)} \|x - y\|.\end{aligned} \tag{23}$$

We can therefore deduce that T is a contraction map. Since all the assumptions of the Krasnoselskii fixed point theorem are now satisfied, problem (2) then admits at least one solution on J which ends the proof. □

Theorem 4. *Assume that (H1) and (H3) are satisfied. Then, problem (2) has a unique solution, provided that* $R < 1$.

Proof. We show that T has a unique fixed point, which is unique solution of problem (2).

Our objective is to show that $TB_r \subset B_r$.

Let $B_r = \{x \in E\colon \|x\| \le r\}$ with $r > 0$, where

$$r \ge \frac{(\mu/\Gamma(\alpha+1)) + \sum_{i=1}^{m} (\nu_i/\Gamma(\rho_i + 1))}{(L_1/\Gamma(\alpha+1)) + (\theta_1 L_2 + \theta_2 L_3/\gamma(\alpha)) + L_f \sum_{i=1}^{m} (1/\Gamma(\rho_i + 1))}. \tag{24}$$

Let us set now

$$\begin{aligned}\mu &= \sup_{i\in J} |g(t, 0, 0, 0)|, \\ \nu_i &= \sup_{i\in J} \left|f_i(t, 0)\right|.\end{aligned} \tag{25}$$

For $x \in B_r$, we have

$$\begin{aligned}|Tx(t)| &= \left|\frac{1}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} g(\tau, x(\tau), Kx(\tau), Hx(\tau))\mathrm{d}\tau + \sum_{i=1}^{m} \frac{1}{\Gamma(\rho_i)} \int_0^t (t-\tau)^{\rho_i - 1} f_i(\tau, x(\tau))\mathrm{d}\tau\right| \\ &\le \sup_{t\in J}\left[\frac{1}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} |g(\tau, x(\tau), Kx(\tau), Hx(\tau))|\mathrm{d}\tau + \sum_{i=1}^{m} \frac{1}{\Gamma(\rho_i)} \int_0^t (t-\tau)^{\rho_i - 1} \left|f_i(\tau, x(\tau))\right|\mathrm{d}\tau\right] \\ &\le \sup_{t\in J}\left\{\frac{1}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} |g(\tau, x(\tau), Kx(\tau), Hx(\tau)) - g(\tau, 0, 0, 0)|\mathrm{d}\tau\right\} \\ &\quad + \sum_{i=1}^{m} \frac{1}{\Gamma(\rho_i)} \int_0^t (t-\tau)^{\rho_i - 1} \left|f_i(\tau, x(\tau)) - f_i(\tau, 0)\right|\mathrm{d}\tau \\ &\le \frac{1}{\Gamma(\alpha+1)} (L_1 r + \mu) + (\theta_1 L_2 + \theta_2 L_3) \frac{r}{\Gamma(\alpha)} + \sum_{i=1}^{m} \frac{L_f r + \nu_i}{\Gamma(\rho_i + 1)} \le r,\end{aligned} \tag{26}$$

which implies that $TB_r \subset B_r$.

Now, for $x, y \in X$ and for each $t \in J$, we obtain

$$\begin{aligned}|Tx(t) - Ty(t)| &\le \sup_{t\in J}\left\{\frac{1}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} |g(\tau, x(\tau), Kx(\tau), Hx(\tau)) - g(\tau, (\tau), Kx(\tau), Hx(\tau))|\mathrm{d}\tau\right\} \\ &\quad + \sum_{i=1}^{m} \frac{1}{\Gamma(\rho_i)} \int_0^t (t-\tau)^{\rho_i - 1} \left|f_i(\tau, x(\tau)) - f_i(\tau, y(\tau))\right|\mathrm{d}\tau \\ &\le \left(\frac{L_1}{\Gamma(\alpha+1)} + \frac{\theta_1 L_2 + \theta_2 L_3}{\Gamma(\alpha)} + \sum_{i=1}^{m} \frac{L_f}{\Gamma(\rho_i + 1)}\right) \|x - y\| \le R\|x - y\|.\end{aligned} \tag{27}$$

Consequently, we observe that $\|Tx - Ty\| \leq R\|x - y\|$. Since $R < 1$, the operator T is a contracting mapping. Hence, we conclude that the operator T has a unique fixed point $x \in X$. □

4. Ulam Stability Results

In this section, we will study the Ulam stability of problem (2). Let us consider the following inequality:

$$\left| D^{\alpha}\left(x(t) - \sum_{i=1}^{m} I^{\rho_i} f_i(t, x(t)) \right) - g(t, x(t), Kx(t), Hx(t)) \right| \leq \varepsilon. \tag{28}$$

Definition 4. The Equation in (2) is Ulam–Hyers stable if there exists a real number $C_f > 0$ such that for each $\varepsilon > 0$ and for each solution $y \in C(J, \mathbb{R})$ of inequality (28), there exists a solution $x \in C(J, \mathbb{R})$ of equation (2) with

$$|y(t) - x(t)| \leq \varepsilon C_f, \quad t \in J. \tag{29}$$

Theorem 5. *Assume that (H1) and (H3) are fulfilled. Then, problem (2) is Ulam–Hyers stable if $R < 1$.*

Proof. Let $\varepsilon > 0$, and let $y \in C(J, \mathbb{R})$ be a function which satisfies inequality (28), and let $x \in C(J, \mathbb{R})$ be the unique solution of the following problem. Then, we recall that

$$x(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t - \tau)^{\alpha - 1} g(\tau, x(\tau), Kx(\tau), Hx(\tau)) d\tau + \sum_{i=1}^{m} I^{\rho_i} f_i(t, x(t)). \tag{30}$$

Integrating inequality (28) and using the initial condition of problem (2), we get

$$\left| y(t) - \sum_{i=1}^{m} I^{\rho_i} f_i(t, y(t)) - \frac{1}{\Gamma(\alpha)} \int_0^t (t - \tau)^{\alpha - 1} g(\tau, y(\tau), Ky(\tau), Hy(\tau)) d\tau \right| \leq \frac{\varepsilon t^{\alpha}}{\Gamma(\alpha + 1)}. \tag{31}$$

Now, we have

$$\begin{aligned} |y(t) - x(t)| \leq & \left| y(t) - \sum_{i=1}^{m} I^{\rho_i} f_i(t, y(t)) - \frac{1}{\Gamma(\alpha)} \int_0^t (t - \tau)^{\alpha - 1} g(\tau, y(\tau), Ky(\tau), Hy(\tau)) d\tau \right| \\ & + \sum_{i=1}^{m} \frac{1}{\Gamma(\rho_i)} \int_0^t (t - \tau)^{\alpha - 1} \left| f_i(t, y(t)) - f_i(t, x(t)) \right| d\tau + \frac{1}{\Gamma(\alpha)} \int_0^t (t - \tau)^{\alpha - 1} \\ & \times |g(\tau, y(\tau), Ky(\tau), Hy(\tau)) - g(\tau, x(\tau), Kx(\tau), Hx(\tau))| d\tau. \end{aligned} \tag{32}$$

Using hypothesis (H1) and (H3) and inequality (31), we obtain

$$\|y - x\| \leq \frac{\varepsilon}{\Gamma(\alpha + 1)} + \left(\sum_{i=1}^{m} \frac{L_f}{\Gamma(\rho_i + 1)} + \frac{L_1}{\Gamma(\alpha + 1)} + \frac{\theta_1 L_2 + \theta_2 L_3}{\Gamma(\alpha)} \right) \|y - x\|, \tag{33}$$

and consequently we get

$$\|y - x\| \leq \frac{\varepsilon}{\Gamma(\alpha + 1)(1 - R)} = \varepsilon C, \tag{34}$$

where $C = (1/\Gamma(\alpha + 1)(1 - R))$.

Thus, the considered problem (2) has the Ulam–Hyers stability. □

5. Illustrative Example

In this section, an application of the results which have proved is provided. Let us consider Caputo fractional integro-differential equation as follows:

$$\begin{cases} D^{(3/2)}\left(x(t) - \sum_{i=1}^{2} I^{((3i+2)/3)} f_i(t, x(t))\right) = g(t, x(t), Kx(t), Hx(t)), \quad t \in J = [0, 1], \\ x(0) = 0, \\ D^{(3/2)} x(0) = \eta, \quad 1 < \alpha < 2, \end{cases} \tag{35}$$

where

$$f_i(t, x(t)) = \frac{2x(t)}{(2i + t\sqrt{3})(23 + i)},$$
$$g(t, x(t), y(t), z(t)) = \frac{|x(t)|}{100(1 + |x(t)|)} + \frac{2y(t)}{30(1 + y^2(t))} + \frac{2z(t)}{55(1 + z^2(t))} - \cos t, \quad \forall t \in J; x, y, z \in \mathbb{R}. \tag{36}$$

Then, we have

$$\left|f_i(t, x(t)) - f_i(t, y(t))\right| \leq L_f |x - y|. \tag{37}$$

An easy computation gives

$$L_f = \frac{2}{2 + \sqrt{3}},\quad L_1 = \frac{1}{100},\quad L_2 = \frac{1}{30},\quad L_3 = \frac{1}{55},\quad \theta_1 = \frac{\ln 2}{3},\quad \theta_2 = \frac{\ln 2}{5}. \tag{38}$$

Then, we have $R = (L_1/\Gamma(\alpha + 1)) + (\theta_1 L_2 + \theta_2 L_3/\Gamma(\alpha)) + \sum_{i=1}^{m} (L_f/\Gamma(\rho_i + 1)) = 0.398 < 1$.

By Theorem 1, we see that problem (2) has a unique solution and has also the Ulam–Hyers stability.

6. Conclusion

In this work, we have considered a coupled Volterra–Fredholm integro-differential equation, and we have used the Caputo derivative operator. We prove two theorems and an example to illustrate our results. In the first theorem, we prove the existence and uniqueness of the solution, and the second theorem deals with the existence of at least one solution. The methods used here are Banach's fixed point theorem and Schaefer's fixed point theorem. Here, two Caputo derivative operators of different fractional orders were used in the considered equation, and it would be relevant to generalize this idea by considering several Caputo operators of different fractional orders. The example given on this work establishes the precision and efficiency of the proposed technique and shows that the problem has a unique solution. Before that, we have discussed the Ulam stability of the solution of problem (2).

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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