Research Article

General Complex-Valued Grouping Functions

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Grouping function is a special kind of aggregation function which measures the amount of evidence in favor of either of the two choices. Recently, complex fuzzy sets have been successfully used in many fields. This paper extends the concept of grouping functions to the complex-valued setting. We introduce the concepts of complex-valued grouping, complex-valued 0-grouping, complex-valued 1-grouping, and general complex-valued grouping functions. We present some interesting results and construction methods of general complex-valued grouping functions.

1. Introduction

In 2012, Bustince et al. [1] introduced the concept of grouping function as a special type of aggregation functions [2]. A grouping function measures the amount of evidence in favor of either of two choices in decision making. It plays an important role in many aspects of applications such as image processing [1, 3], classification [4, 5], and decision making [6, 7].

In this paper, we consider complex-valued grouping function, which is both a generation of real-valued grouping function and a special kind of complex fuzzy aggregation operator. In the literature, one can find many works on real-valued grouping functions. The concepts of general grouping functions [8] and interval-valued grouping functions [9, 10] have been proposed. Some properties including migrativity, homogeneity, idempotency, and distributivity of grouping functions have been studied [1, 11–14]. The multiplicative generators [15] and additive generators [16] of grouping functions have been investigated. (G, N)-implications [17] and QL-implications [18] derived from grouping functions have been constructed.

Meanwhile, we can also find several works on complex fuzzy aggregation operators and related concepts. In 2002,

Ramot et al. [19, 20] introduced the concepts of complex fuzzy sets and complex fuzzy aggregation operators, which have been successfully used in signal processing [20–22], time series forecasting [23–25], and decision making [26, 27]. Distance and entropy measures of complex fuzzy sets and its extensions have been proposed [28–30]. Moreover, some new concepts such as orthogonality and rotational invariance have been presented for complex fuzzy sets and complex fuzzy aggregation operators [31–33].

Recently, Chen et al. [34] introduced the concept of complex-valued overlap function, which measures the overlapping degree between two objects with complex-valued information. In order to measure the amount of evidence in favor of either of two choices with complex-valued information, this paper extends traditional real-valued grouping functions to complex-valued grouping functions. As mentioned in [34], some features of complex fuzzy sets can lead to special properties for complex-valued overlap functions.

Both real-valued grouping functions and complex fuzzy aggregation operators have gained a rapid development both in application and theory. However, as far as we know, nowadays, there are no corresponding discussions to propose the complex-valued grouping functions. Therefore, in

this paper, we introduce the concepts of complex-valued grouping functions. This paper is organized as follows. In Section 2, we recall the concepts of grouping functions. In Section 3, we introduce complex-valued grouping functions and their properties. In Section 4, we introduce construction methods of general complex-valued grouping functions. Conclusions are given in Section 5.

2. Preliminaries

In this section, we recall the concepts of bivariate grouping functions and n -dimensional grouping functions [1, 8, 12, 16].

2.1. Overlap Functions

Definition 1 (see [1]). A mapping $G: [0, 1]^2 \rightarrow [0, 1]$ is a grouping function if it is symmetric, nondecreasing, continuous, and has the following properties:

- (G1) $G(a, b) = 0$ if and only if $a = b = 0$
- (G2) $G(a, b) = 1$ if and only if $a = 1$ or $b = 1$

As introduced in [16], a mapping $G: [0, 1]^2 \rightarrow [0, 1]$ is a 0-grouping function if we replace the property (G1) by the following:

- (G1') $a = b = 0 \implies G(a, b) = 0$ without changing others.

Similarly, a mapping $G: [0, 1]^2 \rightarrow [0, 1]$ is a 1-grouping function if we replace the property (G2) by the following:

- (G2') $a = 1$ or $b = 1 \implies G(a, b) = 1$ without changing others.

Definition 2 (see [12]). An n -ary mapping $G_n: [0, 1]^n \rightarrow [0, 1]$ is an n -dimensional grouping function if it is commutative, nondecreasing, continuous, and has the following properties:

- (G_n1) $G_n(a_1, \dots, a_n) = 0$ if and only if $a_i = 0$ for all $i = 1, 2, \dots, n$
- (G_n2) $G_n(a_1, \dots, a_n) = 1$ if and only if there exists $i \in \{1, 2, \dots, n\}$ such that $a_i = 1$

Analogously, an n -ary mapping $G_n: [0, 1]^n \rightarrow [0, 1]$ is an n -dimensional 0-grouping function if we replace the property (G_n1) by the following:

- (G_n1') If $a_i = 0$ for all $i = 1, 2, \dots, n$, then $G_n(a_1, \dots, a_n) = 0$ without changing others.

An n -ary mapping $G_n: [0, 1]^n \rightarrow [0, 1]$ is an n -dimensional 1-grouping function if we replace the property (G_n2) by the following:

- (G_n2') If there exists $i \in \{1, 2, \dots, n\}$ such that $a_i = 1$, then $G_n(a_1, \dots, a_n) = 1$ without changing others.

Based on the concepts of n -dimensional 0-grouping and 1-grouping functions, the general grouping functions is defined as follows.

Definition 3 (see [8]). A mapping $G_n: [0, 1]^n \rightarrow [0, 1]$ is an n -dimensional general grouping function if it is commutative, nondecreasing, continuous, and has the following properties:

- (Gg_n1) If $\sum_{i=1}^n a_i = 0$ then $G_n(a_1, \dots, a_n) = 0$
- (Gg_n2) If there exists $i \in \{1, 2, \dots, n\}$ such that $a_i = 1$, then $G_n(a_1, \dots, a_n) = 1$

3. N-Dimensional Complex-Valued Grouping Functions

Let $D = \{\alpha \in \mathbb{C} | |\alpha| \leq 1\}$, each $a \in D$ is of the form $r_a e^{j\theta_a}$, where $j = \sqrt{-1}$, the amplitude term $r_a \in \mathbb{R}$ and the phase term $\theta_a \in [0, 2\pi)$.

We define n -dimensional complex-valued grouping functions.

Definition 4. An n -ary mapping $Cg_n: D^n \rightarrow D$ is an n -dimensional complex-valued grouping function if it is commutative, continuous, and has the following properties:

- (Cg_n1) $Cg_n(a_1, \dots, a_n) = 0$ if and only if $a_i = 0$ for all $i = 1, 2, \dots, n$
- (Cg_n2) $Cg_n(a_1, \dots, a_n) = 1$ if and only if there exists $i \in \{1, 2, \dots, n\}$ such that $a_i = 1$
- (Cg_n3) Cg_n is amplitude monotonic in the first variate: $|Cg_n(a, a_2, \dots, a_n)| \leq |Cg_n(b, a_2, \dots, a_n)|$ when $|a| \leq |b|$

Since Cg_n is commutative, it is also amplitude monotonic in any other variate based on the third property (Cg_n3). Obviously, these properties are analogous to those of Definition 1. When the domain is limited to $[0, 1]$, it reduces to an n -dimensional real-valued grouping function of Definition 2.

Example 1. Nevertheless, there exists mapping such that is a grouping function in the domain $[0, 1]$ but not a complex-valued grouping function. The function given as follows:

$$f(a, b) = 1 - (1 - a)(1 - b), \quad (1)$$

is a grouping function but not a complex-valued grouping function; for example, $f(0.5, -1) = 0$, then (Cg_n1) does not hold.

Traditional real-valued grouping functions are the dual notion of overlap functions. $f(a, b)$ is an overlap function on $[0, 1]$, and $g(a, b) = 1 - f(1 - a, 1 - b)$ is a grouping function. However, this is not true for complex-valued overlap functions. For example, $f(a, b) = ab$ is a complex-valued overlap function on D in [34], but $g(a, b) = 1 - (1 - a)(1 - b)$ is not a complex-valued grouping function.

Similarly, we introduce some types of grouping functions, such as n -dimensional complex-valued 0-overlap and 1-overlap grouping functions.

An n -ary mapping $Cg_n: D^n \rightarrow D$ is an n -dimensional complex-valued 0-grouping function if we replace the property (Cg_n1) by the following:

- (Cg_n1') If $a_i = 0$ for all $i = 1, 2, \dots, n$, then $Cg_n(a_1, \dots, a_n) = 0$ without changing others.

An n -ary mapping $Cg_n: D^n \longrightarrow D$ is an n -dimensional complex-valued 1-grouping function if we replace the property (Cg_n2) by the following:

(Cg_n2') If there exists $i \in \{1, 2, \dots, n\}$ such that $a_i = 1$ then $Cg_n(a_1, \dots, a_n) = 1$ without changing others.

Based on these concepts, we define the concept of n -dimensional general complex-valued grouping functions.

Definition 5. An n -ary mapping $GCg_n: D^n \longrightarrow D$ is an n -dimensional general complex-valued grouping function if it is commutative, continuous, and has the following properties:

(GCg_n1) If $a_i = 0$ for all $i = 1, 2, \dots, n$, then $GCg_n(a_1, \dots, a_n) = 0$

(GCg_n2) If there exists $i \in \{1, 2, \dots, n\}$ such that $a_i = 1$ then $GCg_n(a_1, \dots, a_n) = 1$

(GCg_n3) GCg_n is amplitude monotonic in the first variate: $|GCg_n(a, a_2, \dots, a_n)| \leq |GCg_n(b, a_2, \dots, a_n)|$ when $|a| \leq |b|$

The relations between n -dimensional complex-valued grouping functions, complex-valued 0-grouping functions, complex-valued 1-grouping functions, and general complex-valued grouping functions are given as follows.

Proposition 1. Let $g: D^n \longrightarrow D$ be an n -ary mapping, then

- (1) If it is an n -dimensional complex-valued grouping function, then it is an n -dimensional complex-valued 0-grouping and 1-grouping function
- (2) If it is an n -dimensional complex-valued 0-grouping (or 1-grouping) function, then it is also a general complex-valued grouping function

This relation between complex-valued grouping functions is similar to that between interval-valued overlap functions [9].

Now, we give several examples to demonstrate their relations of complex-valued grouping functions.

Example 2. The binary function $g: D^2 \longrightarrow D$ given by

$$g(a, b) = \max(|a|, |b|) \cdot e^{\theta_a \wedge \theta_b}, \quad (2)$$

is a complex-valued overlap function.

Example 3. The binary function $f: D^2 \longrightarrow D$ given by

$$f(a, b) = |a| \vee |b|, \quad (3)$$

is a general complex-valued grouping function and complex-valued 1-grouping function, but neither a complex-valued grouping function nor a complex-valued 0-grouping function; for example, $f(-1, -1) = 1$, then the property (Cg_n2) does not hold.

Example 4. The binary function $h: D^2 \longrightarrow D$ given by

$$h(a, b) = \begin{cases} 1, & a = 1 \text{ or } b = 1, \\ 0, & \text{otherwise,} \end{cases} \quad (4)$$

is a general complex-valued grouping function and complex-valued 0-grouping function. However, h is not a complex-valued 1-grouping function; for example, $f(0.5, 0.5) = 0$, then the property (Cg_n1) does not hold.

Negative operation $(-)$ is closed on D but not closed on $[0, 1]$. Then, we have following property only for complex-valued function on D .

Definition 6. The n -ary function $g: D^n \longrightarrow D$ is symmetric with respect to the point 0, if

$$g(a_1, \dots, a_n) = g(-a_1, -a_2, \dots, -a_n), \quad (5)$$

holds for any $a_1, \dots, a_n \in D$.

Example 5. The n -ary function $g: D^n \longrightarrow D$ given by

$$g(a_1, \dots, a_n) = \min\left(n - \sum_{i=1}^n (1 - |a_i|), 1\right) \cdot e^{\theta_{a_1} \wedge \dots \wedge \theta_{a_n}}, \quad (6)$$

is a complex-valued grouping function, which is symmetric with respect to the point 0.

4. Construction of General Complex-Valued Overlap Functions

Proposition 2. If an n -ary mapping $\varphi: D^n \longrightarrow D$ is n -dimensional complex-valued grouping (0-grouping, 1-grouping, or general grouping) function which is expressed as

$$\varphi(a_1, \dots, a_n) = \phi(r_{a_1}, r_{a_2}, \dots, r_{a_n}) e^{j\psi(\theta_{a_1}, \theta_{a_2}, \dots, \theta_{a_n})}, \quad (7)$$

then the function ϕ is an n -dimensional grouping (0-grouping, 1-grouping, or general grouping) function on $[0, 1]$.

Theorem 1. If the n -ary function $\phi: [0, 1]^n \longrightarrow [0, 1]$ is an n -dimensional 1-grouping function, the function $\psi: [0, 2\pi)^n \longrightarrow [0, 2\pi)$ satisfies the following properties:

- (i) ψ is commutative
- (ii) There exists $i \in \{1, 2, \dots, n\}$ such that $\theta_{a_i} = 0$ if and only if $\psi(\theta_{a_1}, \dots, \theta_{a_n}) = 0$
- (iii) ψ is continuous

Then, the function $\varphi: D^n \longrightarrow D$ defined by equation (7) is an n -dimensional complex-valued 1-grouping function.

Proof. It is immediate that φ is commutative, amplitude monotonic, and continuous. Now, we prove the properties (Cg_n1) and (Cg_n2') .

$(Cg_n1): (\Rightarrow)$ If $\varphi(a_1, \dots, a_n) = \phi(r_{a_1}, r_{a_2}, \dots, r_{a_n}) e^{j\psi(\theta_{a_1}, \theta_{a_2}, \dots, \theta_{a_n})} = 0$, then $\phi(r_{a_1}, r_{a_2}, \dots, r_{a_n}) = 0$. Then, $r_{a_i} = 0$ for all $i = 1, 2, \dots, n$ since ϕ is a 1-grouping function. Then, $a_i = 0$ for all $i = 1, 2, \dots, n$.

(Cgn2'): ($\Leftarrow$) If there exists $i \in \{1, 2, \dots, n\}$ such that $a_i = 1$, i.e., $r_{a_i} = 1$ and $\theta_{a_i} = 0$, then $\phi(r_{a_1}, r_{a_2}, \dots, r_{a_n}) = 1$ since ϕ is a 1-grouping function, and $\psi(\theta_{a_1}, \theta_{a_2}, \dots, \theta_{a_n}) = 0$ since ψ satisfies (ii). Then, $\varphi(a_1, \dots, a_n) = \phi(r_{a_1}, r_{a_2}, \dots, r_{a_n})e^{j\psi(\theta_{a_1}, \theta_{a_2}, \dots, \theta_{a_n})} = 1$. $e^{j0} = 1$. $\square$

- (i) ψ is commutative
- (ii) There exists $i \in \{1, 2, \dots, n\}$ such that $\theta_{a_i} = 0$ if and only if $\psi(\theta_{a_1}, \dots, \theta_{a_n}) = 0$
- (iii) ψ is continuous

However, this method cannot obtain complex-valued grouping function. For example, we get $r_{a_i} = 1$ from ϕ is a grouping function and $\theta_{a_k} = 0$ from property (ii), but $i \neq k$; thus, we cannot get $a_i = 1$. So, we consider the construction of general complex-valued overlap functions. If the n -dimensional complex-valued function $\varphi: D^n \longrightarrow D$ is defined by equation (7), then we can easily see that it is a key step to construct the function $\psi: [0, 2\pi]^n \longrightarrow [0, 2\pi]$, which satisfies the condition (ii) of Corollary 1.

Example 6. The n -ary function $\psi_1: [0, 2\pi)^n \longrightarrow [0, 2\pi)$ given by

$$\psi_1(\theta_1, \theta_2, \dots, \theta_n) = \frac{\theta_1}{2\pi} \times \frac{\theta_2}{2\pi} \times \dots \times \frac{\theta_n}{2\pi}, \quad (8)$$

The n -ary function $\psi_2: [0, 2\pi)^n \longrightarrow [0, 2\pi)$ given by

$$\psi_2(\theta_1, \theta_2, \dots, \theta_n) = \theta_1 \wedge \theta_2 \wedge \dots \wedge \theta_n, \quad (9)$$

Based on these functions, we give the general complex-valued grouping functions.

$$\varphi_1(a_1, \dots, a_n) = (r_{a_1} \vee r_{a_2} \vee \dots \vee r_{a_n}) \cdot e^{j\psi_1(\theta_1, \theta_2, \dots, \theta_n)}, \quad (10)$$

The function $\varphi_\gamma: D^n \longrightarrow D$ given by

$$\varphi_2(a_1, \dots, a_n) = \min(r_{a_1} + r_{a_2} + \dots + r_{a_n}, 1) \cdot e^{j\psi_2(\theta_1, \theta_2, \dots, \theta_n)}, \quad (11)$$

Now, we consider the way of constructing new complex-valued grouping functions from some given complex-valued grouping functions. In the case of real-valued grouping functions, it is possible to use some aggregation operations to obtain a new grouping function from some given grouping functions. Unfortunately, this approach fails for complex fuzzy arithmetic aggregation because it does not satisfy the property of amplitude monotonicity [35]. This means the weighted sum cannot be used to construct new complex-valued grouping functions.

Example 7. From Examples 2 and 3, $g(a, b) = \max(|a|, |b|) \cdot e^{\theta_a \wedge \theta_b}$ and $f(a, b) = \max(|a|, |b|)$ are complex-valued general grouping functions. However, $h(a, b) = (g(a, b) + f(a, b))/2$ is not a complex-valued general grouping function since it does not satisfy GCg_3 . We give an example such that $a_1 < a_2$ but $|h(a_1, b)| > |h(a_2, b)|$; let $a_1 = 0.5e^{2\pi i/3}$, $a_2 = 0.9e^\pi$, and $b = 0.5e^{2\pi i/3}$, then

$$\left| h(0.5e^{2\pi/3}, 0.5e^{2\pi/3}) \right| = \left| \frac{0.5e^{2\pi/3} + 0.5}{2} \right| = 0.25, \quad (12)$$

$$\left| h(0.9e^\pi, 0.5e^{2\pi/3}) \right| = \left| \frac{0.9e^\pi + 0.5}{2} \right| = 0.2.$$

Thus, $|h(0.5e^{2\pi/3}, 0.5e^{2\pi/3})| > |h(0.9e^\pi, 0.5e^{2\pi/3})|$.

However, it is possible to use product operation to obtain a new grouping function from some given grouping functions. The product of $\psi, \varphi: D^n \longrightarrow D$ is defined informally by

$$(\psi \cdot \varphi)(a_1, \dots, a_n) = \psi(a_1, \dots, a_n) \times \varphi(a_1, \dots, a_n). \quad (13)$$

Theorem 2. Let ψ, φ be two n -dimensional general complex-valued grouping functions, then t th power of $(\psi \cdot \varphi)$ also is an n -dimensional general complex-valued grouping function.

Proof. It is immediate that φ is commutative and continuous. Now, we prove the properties (GCg_n1) , (GCg_n2) , and (GCq_n3) .

(GC g_n): If $a_i = 0$ for all $i = 1, 2, \dots, n$, then $\psi(a_1, \dots, a_n) = 0$ and $\varphi(a_1, \dots, a_n) = 0$ since ϕ, φ are n -dimensional general complex-valued grouping functions. Then, $\psi^f(a_1, \dots, a_n) = 0 \dots 0 = 0$.

(GC g_n): If there exists $i \in \{1, 2, \dots, n\}$ such that $a_i = 1$, then $\phi(a_1, a_2, \dots, a_n) = \varphi(a_1, a_2, \dots, a_n) = 1$ since ϕ, φ are n -dimensional general complex-valued grouping functions. Then, $\psi^t(a_1, \dots, a_n) = 1 \dots 1 = 1$.

(GC_n3): If $|a| \leq |b|$, then $|\psi(a, a_2, \dots, a_n)| \leq |\psi(b, a_2, \dots, a_n)|$ and $|\varphi(a, a_2, \dots, a_n)| \leq |\varphi(b, a_2, \dots, a_n)|$ since ϕ, φ are n -dimensional general complex-valued grouping function. Then, $|\psi(a, a_2, \dots, a_n) \times \varphi(a, a_2, \dots, a_n)| = |\psi(a, a_2, \dots, a_n)| \times |\varphi(a, a_2, \dots, a_n)| \leq |\psi(b, a_2, \dots, a_n) \times \varphi(b, a_2, \dots, a_n)| = |\psi(b, a_2, \dots, a_n)| \times |\varphi(b, a_2, \dots, a_n)|$.

$$(b, a_2, \dots, a_n) \times |\varphi(b, a_2, \dots, a_n)| = |\psi^t(b, a_2, \dots, a_n)| \times \varphi(b, a_2, \dots, a_n).$$

Similarly, we can get the following result. $\square$

Theorem 3. Let ψ, φ be two n -dimensional general complex-valued 1-grouping functions, then their product $\psi \cdot \varphi$ also is an n -dimensional complex-valued 1-grouping function.

Proof. From Theorem 2 and the definition of n -dimensional complex-valued 1-grouping function, we only need to prove the following property:

$$(CG_n1): (\Rightarrow) \text{ If } (\psi \cdot \varphi)(a_1, \dots, a_n) = \psi(a_1, \dots, a_n) \times \varphi(a_1, \dots, a_n) = 0, \text{ then } \varphi(a_1, \dots, a_n) = 0 \text{ or } \psi(a_1, \dots, a_n) = 0. \text{ If } \psi(a_1, \dots, a_n) = 0, \text{ then } a_i = 0 \text{ for all } i = 1, 2, \dots, n \text{ since } \phi \text{ is an } n\text{-dimensional general complex-valued 1-grouping function. We have same result when } \varphi(a_1, \dots, a_n) = 0.$$

Now, we consider the power of complex-valued grouping function. First, for positive integers t , the t th power of $\psi: D^n \rightarrow D$ is defined informally by

$$\psi^t(a_1, \dots, a_n) = \underbrace{\psi(a_1, \dots, a_n) \times \dots \times \psi(a_1, \dots, a_n)}_{t \text{ times}}. \quad (14) \quad \square$$

Theorem 4. Let ψ be an n -dimensional general complex-valued grouping function, then t th power of ψ also is an n -dimensional general complex-valued grouping function.

Proof. It is immediate that φ is commutative and continuous. Now, we prove the properties (GCg_n1) , (GCg_n2) , and (GCg_n3) .

(GCg_n1) : If $a_i = 0$ for all $i = 1, 2, \dots, n$, then $\psi(a_1, \dots, a_n) = 0$ since ϕ is an n -dimensional general complex-valued grouping function. Then, $\psi^t(a_1, \dots, a_n) = 0 \dots 0 = 0$.

(GCg_n2) : If there exists $i \in \{1, 2, \dots, n\}$ such that $a_i = 1$, then $\phi(a_1, a_2, \dots, a_n) = 1$ since ϕ is an n -dimensional general complex-valued grouping function. Then, $\psi^t(a_1, \dots, a_n) = 1 \dots 1 = 1$.

(GCg_n3) : If $|a| \leq |b|$, then $|\psi(a, a_2, \dots, a_n)| \leq |\psi(b, a_2, \dots, a_n)|$ since ϕ is an n -dimensional general complex-valued grouping function. Then, $|\psi^t(a, a_2, \dots, a_n)| = |\psi(a, a_2, \dots, a_n)|^t \leq |\psi(b, a_2, \dots, a_n)|^t = |\psi^t(b, a_2, \dots, a_n)|$.

Similarly, we can get the following result. $\square$

Theorem 5. Let ψ be an n -dimensional general complex-valued 1-grouping function, then t th power of ψ also is an n -dimensional complex-valued 1-grouping function.

Proof. From Theorem 4 and the definition of n -dimensional complex-valued 1-grouping function, we only need to prove the following property:

$$(CG_n1): (\Rightarrow) \text{ If } \psi^t(a_1, \dots, a_n) = (\psi(a_1, \dots, a_n))^t = 0, \text{ then } \psi(a_1, \dots, a_n) = 0. \text{ Then, } a_i = 0 \text{ for all}$$

$i = 1, 2, \dots, n$ since ϕ is an n -dimensional general complex-valued 1-grouping function $\square$

Remark 1. In the unit interval $[0, 1]$, $a \cdot b = 1$ if and only if $a = b = 1$. However, it is not true for the complex unit disk D since $(-1) \cdot (-1) = 1$. We cannot establish a theorem similar to Theorem 2 or Theorem 4 for n -dimensional complex-valued grouping (or 0-grouping) function.

5. Conclusions

This paper introduced the concepts of complex-valued grouping, complex-valued 0-grouping, complex-valued 1-grouping, and general complex-valued grouping functions. Then, we presented some construction methods for general complex-valued grouping functions. Our method includes the construction of a general real-valued grouping function and a continuous, commutative function $\psi: [0, 2\pi)^n \rightarrow [0, 2\pi)$ satisfying the following property:

- (i) There exists $i \in \{1, 2, \dots, n\}$ such that $\theta_{a_i} = 0$ if and only if $\psi(\theta_{a_1}, \dots, \theta_{a_n}) = 0 \bmod 2\pi$

Of course, the application and properties of complex-valued overlap/grouping functions need further investigation.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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