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Research Article

Distance-Based Polynomials and Topological Indices for Hierarchical Hypercube Networks

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Topological indices are the numbers associated with the graphs of chemical compounds/networks that help us to understand their properties. The aim of this paper is to compute topological indices for the hierarchical hypercube networks. We computed Hosoya polynomials, Harary polynomials, Wiener index, modified Wiener index, hyper-Wiener index, Harary index, generalized Harary index, and multiplicative Wiener index for hierarchical hypercube networks. Our results can help to understand topology of hierarchical hypercube networks and are useful to enhance the ability of these networks. Our results can also be used to solve integral equations.

1. Introduction

The field of mathematics which deals with the problems of chemistry mathematically is mathematical chemistry. The topology of chemical structures, for example, topological labels or indices, is investigated in chemical graph theory. Actually, topological indices are real numbers associated with graph of chemical compounds and are useful in quantitative structure-property relationships and quantitative structure-activity relationships. Topological indices predict some important properties of chemical structures even without using lab, for example, boiling point, viscosity, radius of gyration, and so on [1–3] can be obtained from the indices.

Just like topological indices, polynomials also have significant applications in chemistry, for example, Hosoya polynomial [4] plays a vital role in calculation of distance-based topological indices. Like Hosoya polynomial, M-polynomial [5] plays the same role in calculation of many degree-based TIs [6–12].

Wiener defined the first topological index when he was examining boiling point of paraffins [13]. Consequently, Wiener set up the skeleton of topological indices [14–18].

This paper concerns with the topological indices of hierarchical hypercube networks. Interconnection networks have a pivotal role in the execution of parallel systems. This paper studies the interconnection topology that is called the hierarchical hypercube (HHC) [19]. This topology is suitable for extensively parallel systems with many number of processors. An appealing property of this network is the low number of connections per processor, which enhances the VLSI design and fabrication of the system [20, 21]. Other alluring features include symmetry and logarithmic diameter, which imply easy and fast algorithms for communication [22]. Moreover, the HHC is scalable, that is, it can embed HHCs of lower dimensions. Malluhi and Bayoumi [21] introduced the hierarchical hypercube network of order n ($n-$ HHC). The structure of an $n-$ HHC consists of three levels of hierarchy. At the lowest level of hierarchy, there is a pool of $2n$ nodes. These nodes are grouped into clusters of $2m$ nodes each, and the nodes in each cluster are interconnected to form an $m-$ cube called the Son-cube or the S-cube. The set of the S-cubes constitutes the second level of hierarchy [23].

Being a hierarchical structure, the HHC bears the advantages usually gained by hierarchy [24]. In general,

hierarchy is a useful means for modular design. In addition, hierarchical structures are capable of exploiting the locality of reference (communication), and they are fault tolerant [25]. Other attractive properties of the HHC structure are logarithmic diameter and a topology inherited from, and closely related to, the hypercube topology. The former property implies fast communication, and the latter implies easy mapping of operations from HC to HHC. The HHC can emulate the hypercube for a large class of problems (divide and conquer), without a significant increase in processing time. The HHC can embed rings and HHCs of lower dimension. In addition, the HHC embeds the cube connected cycle (CCC) [26–28]. As a result, the performance of HHC is in the worst case equivalent to the performance of the CCC [29–34]. The number of vertices and edges in $(\mathrm{HHC}-1)$ is $16a+16$ and $24a+20$, respectively, where a is a natural number. The number of vertices and edges in $(\mathrm{HHC}-2)$ is $16a+16$ and $32a+28$, respectively. $(\mathrm{HHC}-1)$ and $(\mathrm{HHC}-2)$ are shown in Figures 1 and 2, respectively.

In this paper, we computed Hosoya polynomials, Harary polynomials, Wiener index, modified Wiener index, hyper-Wiener index, Harary index, generalized Harary index, and multiplicative Wiener index for hierarchical hypercube networks.

2. Preliminaries

In this section, we give the definitions and known results that are used in proving main results of this paper. A graph G is simple if it has no loop or multiple edges and is connected if there is a path between every two vertices of it. The distance between any two vertices u and v is denoted by $d(u,v)$ and is the length of the shortest path between u and v. The diameter of a graph is the maximum distance between any two vertices of G. The notions that are used in this paper but not defined can be found in [35, 36].

Definition 1 (Hosoya polynomial [16]).

For a simple connected graph G, the Hosoya polynomial is defined as

$$H(G,x)=\frac{1}{2}\sum_{y\in V(G)}\sum_{z\in V(G)}x^{d(y,z)}, \tag{1}$$

where $d(y,z)$ represents the distance between the vertices y and z.

Definition 2 (Wiener index [37]). The Wiener index for a simple connected graph G is denoted by $W(G)$ and is defined as the sum of distances between all pairs of vertices in G, i.e.,

$$W(G)=\frac{1}{2}\sum_{y\in V(G)}\sum_{z\in V(G)}d(y,z). \tag{2}$$

It can be observed that the Wiener index is the first-order derivative of the Hosoya polynomial at $x=1$.

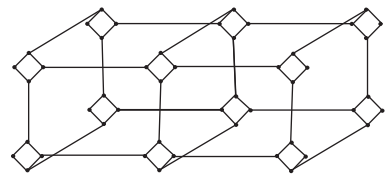

Figure 1: The hierarchical hypercube network $(\mathrm{HHC}-1)_{2\times 2}$.

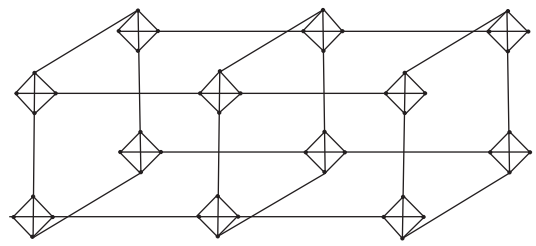

Figure 2: The hierarchical hypercube network $(\mathrm{HHC}-2)_{2\times 2}$.

$$W(G)=\frac{\mathrm{d}H(G)}{\mathrm{d}x}\Big|_{x=1}. \tag{3}$$

Definition 3 (modified Wiener index). For a simple connected graph G, the modified Wiener index is denoted by $W_\lambda(G)$ and is defined as

$$W_{(\lambda G)}=\frac{1}{2}\sum_{y\in V(G)}\sum_{z\in V(G)}d(y,z)^{\lambda}, \tag{4}$$

where λ is any positive integer.

Definition 4 (hyper-Wiener index). For a simple connected graph G, the hyper-Wiener index is denoted by $\mathrm{WW}(G)$ and is defined as

$$\mathrm{WW}(G)=\frac{1}{2}\sum_{y\in V(G)}\sum_{z\in V(G)}\left(d(y,z)+d(y,z)^2\right). \tag{5}$$

The hyper-Wiener index (HWI) was introduced by Randić [38] and is used to forecast physical chemistry characteristics of organic compounds.

Definition 5 (modified hyper-Wiener index). Another variant of Wiener index is hyper-Wiener index (MHWI) which is denoted by $\mathrm{WW}_\lambda(G)$. For a simple connected graph G, the modified Wiener index is defined as

$$\mathrm{WW}_\lambda(G)=\frac{1}{2}\sum_{y\in V(G)}\sum_{z\in V(G)}\left(d(y,z)^{\lambda}+d(y,z)^{2\lambda}\right), \tag{6}$$

where λ is any positive integer.

Definition 6 (Harary polynomial). The Harary polynomial for a simple connected graph G is denoted by $h(G)$ and is defined as

$$h(G)=\sum_{y\in V(G)}\sum_{z\in V(G)}\frac{1}{d(y,z)}x^{d(y,z)}. \tag{7}$$

Definition 7 (generalized Harary index). The generalized Harary index for a simple connected graph G is denoted by $h_t(G)$ and is defined as

$$h_t(G) = \sum_{y \in V(G)} \sum_{z \in V(G)} \frac{1}{d(y,z)+t}, \tag{8}$$

where $t = 1, 2, 3, 4, \ldots$.

For detailed study on Harary polynomial and Harary index, we refer to the readers [39, 40] and references therein.

Definition 8 (multiplicative Wiener index). The multiplicative Wiener index for a simple connected graph G is denoted by $\pi(G)$ and is defined as

$$\pi(G) = \prod_{\{y,z\} \subseteq V(G)} d(y,z). \tag{9}$$

3. Methodology

With the aid of distance matrix of a graph G, we can evaluate Hosoya polynomial. To calculate the Hosoya polynomial, we must calculate the number of pairs of vertices at distance $1, 2, 3, \ldots, \mathrm{dia}(G)$, where $\mathrm{dia}(G) = \max\{d(u,v);\ u, v \in V(G)\}$. Mathematical induction will be used for the above cause. The usual vision of Hosoya polynomial is given below, where d represents the maximum distance in graph.

$$H(G;x) = a_0(n)x^0 + a_1(n)x^1 + a_2(n)x^2 + \cdots + a_d(n)x^d. \tag{10}$$

4. Distance-Based Polynomials and Indices for Hierarchical Hypercube Networks

This section consists of two subsections. In Section 4.1, we present results about HHC-1, and in Section 4.2, we present results about HHC-2.

4.1. Distance-Based Polynomials and Indices for Hierarchical Hypercube Network HHC-1. In Theorem 1, we give Hosoya polynomial of HHC-1.

Theorem 1. *For $n \geq 3$, the Hosoya polynomial of HHC − 1 is*

$$\begin{aligned} H(\mathrm{HHC}-1;x) &= (20+24n)x + (24+40n)x^2 + (24+64n)x^3 + (4+100n)x^4 + (-28+124n)x^5 \\ &\quad + (-80+132n)x^6 + (-152+132n)x^7 + \sum_{8 \leq m \leq 2n+2} ((8(37-8m+16n)))x^m + 108x^{2n+3} + 60x^{2n+4} + 20x^{2n+5}. \end{aligned} \tag{11}$$

Proof. To prove this result, we need to calculate $|a_m(n)|$ = number of pair of vertices at distance m, where $m = 1, 2, 3, \ldots, 2n+5$. Using Figure 1, the number of pair of vertices at different distances is computed and is listed in Tables 1 and 2.

Now, by using Table 1, we have

$$\begin{aligned} |a_1(n)| &= 20+24n, \\ |a_2(n)| &= 24+40n, \\ |a_3(n)| &= 24+64n, \\ |a_4(n)| &= 4+100n, \\ |a_5(n)| &= -28+124n, \\ |a_6(n)| &= -80+132n, \\ |a_7(n)| &= -152+132n. \end{aligned} \tag{12}$$

The remaining proof is divided into the following two main cases.

Case 1. When $m \equiv 0 \pmod 2$ and $8 < m \leq 2n+2$.

It can be observed from Table 2 that

$$\begin{aligned} |a_8(3)| &= 168, \\ |a_8(4)| &= 296, \\ |a_8(5)| &= 424, \\ |a_8(6)| &= 552, \\ |a_8(7)| &= 680, \\ |a_8(8)| &= 808. \end{aligned} \tag{13}$$

Now, we can deduce that

$$|a_8(n)| = 40 + 128(n-2). \tag{14}$$

In a similar fashion, we have

$$\begin{aligned} |a_{10}(4)| &= 168, \\ |a_{10}(5)| &= 296, \\ |a_{10}(6)| &= 424, \\ |a_{10}(7)| &= 552, \\ |a_{10}(8)| &= 680, \\ |a_{10}(9)| &= 808. \end{aligned} \tag{15}$$

Table 1: Pair of vertices in HHC-1 at different distances.

m	n								
	3	4	5	6	7	8	.	.	.
1	92	116	140	164	188	212	.	.	.
2	144	184	224	264	304	344	.	.	.
3	216	280	344	408	472	536	.	.	.
4	304	404	504	604	704	804	.	.	.
5	344	468	592	716	840	964	.	.	.
6	316	448	580	712	844	976	.	.	.
7	244	376	508	640	772	904	.	.	.

Table 2: Pair of vertices in HHC-1 at different distances.

m	n								
	3	4	5	6	7	8	.	.	.
8	168	296	424	552	680	808	.	.	.
9	—	232	360	488	616	744	.	.	.
10	—	168	296	424	552	680	.	.	.
11	—	—	232	360	488	616	.	.	.
12	—	—	168	296	424	552	.	.	.
13	—	—	—	232	360	488	.	.	.
14	—	—	—	168	296	424	.	.	.
15	—	—	—	—	232	360	.	.	.
16	—	—	—	—	168	296	.	.	.
17	—	—	—	—	—	232	.	.	.
18	—	—	—	—	—	168	.	.	.
19	—	—	—	—	—	—	.	.	.
20	—	—	—	—	—	—	—	.	.
21	—	—	—	—	—	—	—	—	.

It implies that

$$\left|a_{10}(n)\right| = 40 + 128(n-3). \tag{16}$$

In a similar fashion, we infer

$$\begin{aligned} \left|a_{12}(n)\right| &= 40 + 128(n-4), \\ \left|a_{14}(n)\right| &= 40 + 128(n-5), \\ \left|a_{16}(n)\right| &= 40 + 128(n-6), \\ &\cdots. \end{aligned} \tag{17}$$

Now, we have

$$\left|a_m(n)\right| = 40 + 128\left(n - \frac{(m-4)}{2}\right) = 8(37 - 8m + 16n). \tag{18}$$

Case 2. When $m \equiv 1 \pmod 2$ and $9 < m \leq 2n+1$.

It can be observed from Table 2 that

$$\begin{aligned} \left|a_9(4)\right| &= 232, \\ \left|a_9(5)\right| &= 360, \\ \left|a_9(6)\right| &= 488, \\ \left|a_9(7)\right| &= 616, \\ \left|a_9(8)\right| &= 744, \\ \left|a_9(9)\right| &= 872. \end{aligned} \tag{19}$$

Now, we can deduce that

$$\left|a_9(n)\right| = 104 + 128(n-3). \tag{20}$$

By means of the same trick, we obtain

$$\begin{aligned} \left|a_{11}(5)\right| &= 232, \\ \left|a_{11}(6)\right| &= 360, \\ \left|a_{11}(7)\right| &= 488, \\ \left|a_{11}(8)\right| &= 616, \\ \left|a_{11}(9)\right| &= 744, \\ \left|a_{11}(10)\right| &= 872, \end{aligned} \tag{21}$$

which reveal that

$$\left|a_{11}(n)\right| = 104 + 128(n-4). \tag{22}$$

With a similar approach, we get

$$\begin{aligned} \left|a_{13}(n)\right| &= 104 + 128(n-5), \\ \left|a_{15}(n)\right| &= 104 + 128(n-6), \\ \left|a_{17}(n)\right| &= 104 + 128(n-7), \\ &\cdots. \end{aligned} \tag{23}$$

Hence, we have

$$\begin{aligned} \left|a_m(n)\right| &= 104 + 128\left(n - \frac{(m-3)}{2}\right)\left|a_m(n)\right| \\ &= 8(37 - 8m + 16n). \end{aligned} \tag{24}$$

One can see that in both cases, we get the same result, so we can write for $8 < m \leq 2n+1$ as

$$\left|a_m(n)\right| = 8(37 - 8m + 16n), \tag{25}$$

and the remaining last three distances having fixed values are

$$\begin{aligned} \left|a_{2n+3}(n)\right| &= 108, \\ \left|a_{2n+4}(n)\right| &= 60, \\ \left|a_{2n+5}(n)\right| &= 20. \end{aligned} \tag{26}$$

By what have been mentioned above and using definition of Hosoya polynomial, we arrive at our desired result. □

In Theorem 2, we give Harary polynomial for HHC-1.

Theorem 2. *For $n \geq 3$, the Harary polynomial of HHC − 1 is*

$$\begin{aligned} h(\mathrm{HHC}-1;x) = {} & (20+24n)x + \frac{(24+40n)}{2}x^2 + \frac{(24+64n)}{3}x^3 + \frac{(4+100n)}{4}x^4 + \frac{(-28+124n)}{5}x^5 \\ & + \frac{(-80+132n)}{6}x^6 + \frac{(-152+132n)}{7}x^7 \\ & + \sum_{8\leq m\leq 2n+2} \frac{(8(37-8m+16n))}{m}x^m + \frac{108}{2n+3}x^{2n+3} + \frac{60}{2n+4}x^{2n+4} + \frac{20}{2n+5}x^{2n+5}. \end{aligned} \tag{27}$$

Proof. The proof of this result is easy to follow by using information given in Theorem 1 and definition of Harary polynomial.

In Theorem 3, we give modified Wiener index, modified hyper-Wiener index, generalized Harary index, and multiplicative Wiener index of HHC-1. □

Theorem 3. *For HHC − 1, we have*

(1) The modified Wiener index:

$$\begin{aligned} W_\lambda(\mathrm{HHC}-1) = {} & (20+24n)1^\lambda + (24+40n)2^\lambda + (24+64n)3^\lambda + (4+100n)4^\lambda + (-28+124n)5^\lambda \\ & + (-80+132n)6^\lambda + (-152+132n)7^\lambda \\ & + \sum_{8\leq m\leq 2n+2} (8(37-8m+16n))\left(m^\lambda\right) + (108)(2n+3)^\lambda + (60)(2n+4)^\lambda + (20)(2n+5)^\lambda. \end{aligned} \tag{28}$$

(2) The modified hyper-Wiener index:

$$\begin{aligned} \mathrm{WW}_\lambda(\mathrm{HHC}-1) = {} & (20+24n)\left(1^\lambda + 1^{2\lambda}\right) \\ & + (24+40n)\left(2^\lambda + 2^{2\lambda}\right) + (24+64n)\left(3^\lambda + 3^{2\lambda}\right) \\ & + (4+100n)\left(4^\lambda + 4^{2\lambda}\right) + (-28+124n)\left(5^\lambda + 5^{2\lambda}\right) + (-80+132n)\left(6^\lambda + 6^{2\lambda}\right) + (-152+132n)\left(7^\lambda + 7^{2\lambda}\right) \\ & + \sum_{8\leq m\leq 2n+2} (8(37-8m+16n))\left(m^\lambda + m^{2\lambda}\right) + (108)\left((2n+3)^\lambda + (2n+3)^{2\lambda}\right) \\ & + (60)\left((2n+4)^\lambda + (2n+4)^{2\lambda}\right) + (20)\left((2n+5)^\lambda + (2n+5)^{2\lambda}\right). \end{aligned} \tag{29}$$

(3) The generalized Harary index:

$$h_t(\mathrm{HHC}-1)=\frac{(20+24n)}{1+t}+\frac{(24+40n)}{2+t}+\frac{(24+64n)}{3+t}+\frac{(4+100n)}{4+t}+\frac{(-28+124n)}{5+t}+\frac{(-80+132n)}{6+t}+\frac{(-152+132n)}{7+t}+\sum_{8\le m\le 2n+2}(8(37-8m+16n))\frac{1}{m+t}+\frac{108}{2n+3+t}+\frac{60}{2n+4+t}+\frac{20}{2n+5+t}. \tag{30}$$

(4) The multiplicative Wiener index:

$$\pi(\mathrm{HHC}-1)=1^{(20+24n)}\times 2^{(24+40n)}\times 3^{(24+64n)}\times 4^{(4+100n)}\times 5^{(-28+124n)}\times 6^{(-80+132n)}\times 7^{(-152+132n)}\times\prod_{8<m\le 2n+2}m^{(8(37-8m+16n))}\times(2n+3)^{108}\times(2n+4)^{60}\times(2n+5)^{20}. \tag{31}$$

From Theorem 3, we get the following results immediately.

Corollary 1. *For HHC − 1, we have*

$$W(\mathrm{HHC}-1)=\frac{256n^3}{3}+592n^2+\frac{2696n}{3}+376. \tag{32}$$

Proof. This result can be easily established by taking $\lambda=1$ in (1) of Theorem 3. □

Corollary 2. *For HHC − 1, we have*

$$\mathrm{WW}(\mathrm{HHC}-1)=\frac{256n^4}{3}+\frac{2624n^3}{3}+\frac{10544n^2}{3}+\frac{14056n}{3}+1816. \tag{33}$$

Proof. This result can be easily established by taking $\lambda=1$ in (2) of Theorem 3. □

Corollary 3. *For HHC − 1, we have*

$$H(\mathrm{HHC}-1)=\frac{15599n}{100}+\frac{6347931761169771}{18014398509481984}+\sum_{8\le m\le 2n+2}\frac{(8(37-8m+16n))}{m}+\frac{108}{2n+3}+\frac{20}{2n+5}+\frac{30}{n+2}. \tag{34}$$

Proof. This result can be easily established by taking $t = 1$ in (3) of Theorem 3. □

4.2. Distance-Based Polynomials and Indices for Hierarchical Hypercube Network HHC-2. In Theorem 4, we give Hosoya polynomial for HHC-2.

Theorem 4. *For HHC − 2, the Hosoya polynomial is*

$$\begin{aligned} H(\mathrm{HHC}-2;x) &= (28+32n)x + (24+48n)x^2 + (36+96n)x^3 + (-4+124n)x^4 + (-60+128n)x^5 \\ &+ (-120+132n)x^6 + \sum_{m\equiv 0\,(\mathrm{mod}2),8\le m\le 2n+2} (8(33-8m+16n))x^m \\ &+ \sum_{m\equiv 1\,(\mathrm{mod}2),7\le m\le 2n+1} (-64(m-2(2+n)))x^m + 68x^{2n+3} + 32x^{2n+4} + 4x^{2n+5}. \end{aligned} \tag{35}$$

Proof. To prove this result, we have to calculate $|a_m(n)|$ where $m = 1, 2, 3, \ldots, 2n+5$. Here $|a_m(n)|$ is the same as in Theorem 1. From Figure 2, we can compute the number of pair of vertices in HHC-2 at different distances, which is given in Tables 3 and 4.

Now from Table 3, we have

$$\begin{aligned} |a_1(n)| &= 28+32n, \\ |a_2(n)| &= 24+48n, \\ |a_3(n)| &= 36+96n, \\ |a_4(n)| &= -4+124n, \\ |a_5(n)| &= -60+128n, \\ |a_6(n)| &= -120+132n. \end{aligned} \tag{36}$$

The remaining proof is divided into the following two main cases.

Case 1. When $m \equiv 0\,(\mathrm{mod}2)$ and $8 < m \le 2n+2$.

It can be observed from Table 4 that

$$\begin{aligned} |a_8(3)| &= 136, \\ |a_8(4)| &= 264, \\ |a_8(5)| &= 392, \\ |a_8(6)| &= 520, \\ |a_8(7)| &= 648, \\ |a_8(8)| &= 776. \end{aligned} \tag{37}$$

Now, we can deduce that

$$|a_8(n)| = 8 + 128(n-2). \tag{38}$$

In a similar fashion, we have

$$\begin{aligned} |a_{10}(4)| &= 136, \\ |a_{10}(5)| &= 264, \\ |a_{10}(6)| &= 392, \\ |a_{10}(7)| &= 520, \\ |a_{10}(8)| &= 648, \\ |a_{10}(9)| &= 776. \end{aligned} \tag{39}$$

It implies that

$$|a_{10}(n)| = 8 + 128(n-3). \tag{40}$$

In a similar fashion, we infer

$$\begin{aligned} |a_{12}(n)| &= 8+128(n-4), \\ |a_{14}(n)| &= 8+128(n-5), \\ |a_{16}(n)| &= 8+128(n-6), \\ &\cdots, \end{aligned} \tag{41}$$

which yield

$$|a_m(n)| = 8 + 128\left(n - \frac{(m-4)}{2}\right) = 8(33-8m+16n). \tag{42}$$

Case 2. When $m \equiv 1\,(\mathrm{mod}2)$ and $7 < m \le 2n+1$.

It can be observed from Table 4 that

$$\begin{aligned} |a_7(3)| &= 192, \\ |a_7(4)| &= 320, \\ |a_7(5)| &= 448, \\ |a_7(6)| &= 576, \\ |a_7(7)| &= 704, \\ |a_7(8)| &= 832. \end{aligned} \tag{43}$$

Now, we can deduce that

Table 3: Number of pair of vertices in HHC-2 at different distances.

m	n								
	2	3	4	5	6	7	.	.	.
1	92	124	156	188	220	252	.	.	.
2	120	168	216	264	312	360	.	.	.
3	228	324	420	516	612	708	.	.	.
4	244	368	492	616	740	864	.	.	.
5	196	324	452	580	708	836	.	.	.
6	144	276	408	540	672	804	.	.	.

Table 4: Number of pair of vertices of HHC-2 at different distances.

m	n								
	3	4	5	6	7	8	.	.	.
7	192	320	448	576	704	832	.	.	.
8	136	264	392	520	648	776	.	.	.
9	—	192	320	448	576	704	.	.	.
10	—	136	264	392	520	648	.	.	.
11	—	—	192	320	448	576	.	.	.
12	—	—	136	264	392	520	.	.	.
13	—	—	—	192	320	448	.	.	.
14	—	—	—	136	264	392	.	.	.
15	—	—	—	—	192	320	.	.	.
16	—	—	—	—	136	264	.	.	.
17	—	—	—	—	—	192	.	.	.
18	—	—	—	—	—	136	.	.	.
19	—	—	—	—	—	—	.	.	.
20	—	—	—	—	—	—	—	.	.
21	—	—	—	—	—	—	—	—	.

$$\left|a_7(n)\right| = 64 + 128(n-2). \tag{44}$$

So, we obtain

$$\begin{aligned}\left|a_9(4)\right| &= 192,\\ \left|a_9(5)\right| &= 320,\\ \left|a_9(6)\right| &= 448,\\ \left|a_9(7)\right| &= 576,\\ \left|a_9(8)\right| &= 704,\\ \left|a_9(9)\right| &= 832,\end{aligned} \tag{45}$$

which reveal that

$$\left|a_9(n)\right| = 64 + 128(n-3). \tag{46}$$

Also, we get

$$\begin{aligned}\left|a_{13}(n)\right| &= 64 + 128(n-4),\\ \left|a_{15}(n)\right| &= 64 + 128(n-5),\\ \left|a_{17}(n)\right| &= 64 + 128(n-6),\\ &\cdots.\end{aligned} \tag{47}$$

Hence, we have

$$\begin{aligned}\left|a_m(n)\right| &= 64 + 128\left(n - \frac{(m-3)}{2}\right)\\ &= -64(m - 2(2+n)),\end{aligned} \tag{48}$$

and the remaining last three distances having fixed values are

$$\begin{aligned}\left|a_{2n+3}(n)\right| &= 68,\\ \left|a_{2n+4}(n)\right| &= 32,\\ \left|a_{2n+5}(n)\right| &= 4.\end{aligned} \tag{49}$$

By what have been mentioned above and using the definition of Hosoya polynomial, we arrive at our desired result. □

In Theorem 5, we give Harary polynomial for HHC-2.

Theorem 5. *For HHC − 2, the Harary polynomial is*

$$h(\mathrm{HHC}-2;x)=(28+32n)x+\frac{(24+48n)}{2}x^2+\frac{(36+96n)}{3}x^3+\frac{(36+96n)}{4}x^4+\frac{(-60+128n)}{5}x^5$$
$$+\frac{(-120+132n)}{6}x^6+\sum_{8\le m\le 2n+2}\frac{(8(33-8m+16n))}{m}x^m \tag{50}$$
$$+\sum_{7\le m\le 2n+1}\frac{(-64(m-2(2+n)))}{m}x^m+\frac{68}{2n+3}x^{2n+3}+\frac{32}{2n+4}x^{2n+4}+\frac{4}{2n+5}x^{2n+5}.$$

Proof. The proof of this theorem is straightforward from the facts specified in Theorem 4 and by definition of Harary polynomial. □

In Theorem 6, we give modified Wiener index, modified hyper-Wiener index, generalized Harary index, and multiplicative Wiener index for HHC-2.

Theorem 6. *For HHC − 2, we have*

(1) The modified Wiener index:

$$W_\lambda(\mathrm{HHC}-2)=(28+32n)1^\lambda+(24+48n)2^\lambda+(36+96n)3^\lambda+(36+96n)4^\lambda+(-60+128n)5^\lambda$$
$$+(-120+132n)6^\lambda+\sum_{m\equiv 0\,(\mathrm{mod}2),8\le m\le 2n+2}(8(33-8m+16n))\left(m^\lambda\right) \tag{51}$$
$$+\sum_{m\equiv 1\,(\mathrm{mod}2),7\le m\le 2n+1}(-64(m-2(2+n)))\left(m^\lambda\right)+(68)(2n+3)^\lambda+(32)(2n+4)^\lambda+(4)(2n+5)^\lambda.$$

(2) The modified hyper-Wiener index:

$$\mathrm{WW}_\lambda(\mathrm{HHC}-2)=(28+32n)\left(1^\lambda+1^{2\lambda}\right)+(24+48n)\left(2^\lambda+2^{2\lambda}\right)+(36+96n)\left(3^\lambda+3^{2\lambda}\right)$$
$$+(36+96n)\left(4^\lambda+4^{2\lambda}\right)+(-60+128n)\left(5^\lambda+5^{2\lambda}\right)+(-120+132n)\left(6^\lambda+6^{2\lambda}\right)$$
$$+\sum_{m\equiv 0\,(\mathrm{mod}2),8\le m\le 2n+2}(8(33-8m+16n))\left(m^\lambda+m^{2\lambda}\right)$$
$$+\sum_{m\equiv 1\,(\mathrm{mod}2),7\le m\le 2n+1}\begin{array}{l}(-64(m-2(2+n)))\left(m^\lambda+m^{2\lambda}\right)\\+(68)\left((2n+3)^\lambda+(2n+3)^{2\lambda}\right)+(32)\left((2n+4)^\lambda+(2n+4)^{2\lambda}\right)+(4)\left((2n+5)^\lambda+(2n+5)^{2\lambda}\right)\end{array}. \tag{52}$$

(3) The generalized Harary index:

$$h_t(\mathrm{HHC}-2)=\frac{(28+32n)}{1+t}+\frac{(24+48n)}{2+t}+\frac{(36+96n)}{3+t}+\frac{(36+96n)}{4+t}+\frac{(-60+128n)}{5+t}+\frac{(-120+132n)}{6+t}$$
$$+\sum_{m\equiv 0\,(\mathrm{mod}2),8\le m\le 2n+2}(8(33-8m+16n))\frac{1}{m+t} \tag{53}$$
$$+\sum_{m\equiv 1\,(\mathrm{mod}2),7\le m\le 2n+1}(-64(m-2(2+n)))\frac{1}{m+t}+\frac{108}{2n+3+t}+\frac{60}{2n+4+t}+\frac{20}{2n+5+t}.$$

(4) The multiplicative Wiener index:

$$\pi(\mathrm{HHC}-2) = 1^{(28+32n)} \times 2^{(24+48n)} \times 3^{(36+96n)} \times 4^{(36+96n)} \times 5^{(-60+128n)} \times 6^{(-120+132n)} \times \prod_{m\equiv 0\,(\mathrm{mod}2),8<m\leq 2n+2} m^{(8(33-8m+16n))} \times \prod_{m\equiv 1\,(\mathrm{mod}2),7<m\leq 2n+1} m^{(-64(m-2(2+n)))} (2n+3)^{68} \times (2n+4)^{32} \times (2n+5)^{4}. \tag{54}$$

From Theorem 6, we get the following results immediately.

Corollary 4. *For HHC − 2, we have*

$$W(\mathrm{HHC}-2) = \frac{256n^3}{3} + 520n^2 + \frac{2288n}{3} + 316. \tag{55}$$

Proof. This result can be easily established by taking $\lambda = 1$ in (1) of Theorem 6. □

Corollary 5. *For HHC − 2, we have*

$$\mathrm{WW}(\mathrm{HHC}-2) = \frac{256n^4}{3} + \frac{2336n^3}{3} + \frac{8264n^2}{3} + \frac{10480n}{3} + 1320. \tag{56}$$

Proof. This result can be easily established by taking $\lambda = 1$ in (2) of Theorem 6. □

Corollary 6. *For HHC − 2, we have*

$$H(\mathrm{HHC}-2) = \frac{798n}{5} + 29 + \sum_{m\equiv 0\,(\mathrm{mod}2),8\leq m\leq 2n+2} (8(33-8m+16n))\frac{1}{m} + \sum_{m\equiv 1\,(\mathrm{mod}2),7\leq m\leq 2n+1} (-64(m-2(2+n)))\frac{1}{m} + \frac{108}{2n+3} + \frac{20}{2n+5} + \frac{30}{n+2}. \tag{57}$$

Proof. This result can be easily established by taking $t = 1$ in (3) of Theorem 6. □

5. Conclusion

Topological indices can be applied in different fields of science, such as material science, arithmetic, informatics, biology, and so on. However, the most critical use of topological indices to date is in the nonexact quantitative structure-property relationships and quantitative structure-activity relationships. In this paper, we studied hierarchical hypercube networks. We computed distance-based polynomials and distance-based indices for these networks. In fact, we computed Hosoya polynomials, Harary polynomials, Wiener index, and different variants of Wiener indices for the studied networks. Our results can help in understanding the topology of hierarchical hypercube networks and can be used to solve integral equations.

Data Availability

The data used to support the findings of this study are included within the article.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

Authors' Contributions

Tingmei Gao wrote the final version of this paper, analyzed the results, and arranged funding for this study. Iftikhar Ahmed proved the results and wrote the first version of this paper.

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