Hindawi
Journal of Mathematics
Volume 2021, Article ID 5878006, 10 pages
https://doi.org/10.1155/2021/5878006

Research Article

A Note on $\mathscr{LP}$-Sasakian Manifolds with Almost Quasi-Yamabe Solitons

Sunil Kumar Yadav,[1] **D. L. Suthar**,[2] **and Biniyam Shimelis**[2]

[1]*Department of Applied Science and Humanities, United College of Engineering & Research, A-31, UPSIDC Industrial Area, Naini, Prayagraj-211010, Uttar Prasdesh, India*
[2]*Department of Mathematics, Wollo University, P. O. Box 1145, Dessie, Ethiopia*

Correspondence should be addressed to Biniyam Shimelis; biniyam.shimelis@wu.edu.et

Received 27 August 2021; Revised 16 October 2021; Accepted 9 November 2021; Published 10 December 2021

Academic Editor: Ram Jiwari

We categorize almost quasi-Yamabe solitons on $\mathscr{LP}$-Sasakian manifolds and their $\mathscr{CR}$-submanifolds whose potential vector field is torse-forming, admitting a generalized symmetric metric connection of type (α, β). Finally, a nontrivial example is provided to confirm some of our results.

1. Introduction

Yamabe solitons (YS) are ideas that generate self-similar Yamabe flow (YF) solutions [1]:

$$\frac{\partial}{\partial \check{t}} \ddot{g}(\check{t}) = -\widehat{\delta}(\check{t}) \ddot{g}(\check{t}). \qquad (1)$$

Di Cerbo and Disconzi were the first to notice them in [2]. Chen and Deshmukh proposed the concept of quasi-Yamabe soliton (QYS) in [3], which we will explore in this study for a more broader situation when the constants are functions.

Definition 1. An almost quasi-Yamabe solitons (AQYS) on Remannian manifold $(\Theta^n, \ddot{g})$ is a set of data $(\ddot{g}, \mathscr{K}, \mu, \widehat{\sigma})$ that fulfill the following equation:

$$\frac{1}{2}\mathfrak{L}_{\mathscr{K}} \ddot{g} = (\widehat{\delta} - \widehat{\sigma})\ddot{g} + \mu \mathscr{K}^{\diamond} \otimes \mathscr{K}^{\diamond}, \qquad (2)$$

where $\mathfrak{L}_{\mathscr{K}}$ is operator of the Lie derivative in view of $\mathscr{K}$, $\widehat{\sigma}$ and μ are smooth functions on $(\widehat{\Theta^n}, \ddot{g})$, $\mathscr{K}^{\diamond}$ is the 1-form of $\mathscr{K}$, and $\widehat{\delta}$ is the scalar curvature. If $\widehat{\sigma} < 0$, $\widehat{\sigma} = 0$, or $\widehat{\sigma} > 0$, we call an AQYS shrinking, stable, or growing, respectively, using Yamabe solitons nomenclature.

If $\mathscr{K} = \nabla \widetilde{f}$ is a gradient type, (2) yields

$$\operatorname{Hess}(\widehat{f}) = (\widehat{\delta} - \widehat{\sigma})\ddot{g} + \mu \mathrm{d}\widehat{f} \otimes \mathrm{d}\widehat{f}. \qquad (3)$$

It is nothing more than a generalized quasi-Yamabe gradient soliton (GQYGS) (see [4, 5]). Several authors have thoroughly investigated AQYS and QYGS in [6–13].

Definition 2. A vector field $\mathscr{K}$ on $(\widehat{\Theta^n}, \ddot{g})$ meets the following conditions and is known as a torse-forming vector field [14]:

$$\nabla_{\mathscr{P}^*} \mathscr{K} = \psi \mathscr{P}^* + \pi \mathscr{P}^* \mathscr{K}, \quad \forall \mathscr{P}^* \in \chi(\Theta), \qquad (4)$$

where π is a 1-form and ψ is in $C^{\infty}(\Theta)$.

We classify such vector field as

(i) It is concircular if the 1-form π vanishes identically [15]
(ii) For concurrent, $\psi = 1$ and $\pi = 0$ [16]
(iii) It is recurrent if $\psi = 0$
(iv) Parallel if $\psi = \pi = 0$

If the vector field $\mathscr{K}$ fulfills (4) with $\pi(\mathscr{K}) = 0$, it called as torqued vector field [17].

The content of the paper is as follows. After the opening remark, Section 2 contains the fundamental result of an

$\mathscr{LP}$-Sasakian manifold. We discuss the concept of a generalized symmetric metric connection-(α,β) (GSMC-(α,β)) in Section 3. With regard to GSMC-(α,β), Section 4 is devoted to $\mathscr{CR}$-submanifolds of an $\mathscr{LP}$-Sasakian manifold. With respect to such a connection, we examine QYS in view of a torse-forming vector field on an $\mathscr{LP}$-Sasakian manifold in Section 5. The study of QYS with a torse-forming vector field on $\mathscr{CR}$-submanifolds of an $\mathscr{LP}$-Sasakian manifolds is also covered in Section 6. Finally, in Section 7, we look at AQYGS with a torse-forming vector field by considering the tangential $\mathscr{K}^t$ and normal $\mathscr{K}^n$ components of such a vector field on $\mathscr{CR}$-submanifolds.

2. *LP*-Sasakian Manifolds

If a $(1,1)$ tensor field ϕ_*, a contravariant vector field $\widehat{\zeta}$, a 1-form η_*, and the Lorentzian metric $\ddot{g}$ are admitted to a differentiable manifold $\Theta^{\widehat{n}}$; it is termed as an $\mathscr{LP}$-Sasakian manifold $\Theta^{\widehat{n}}(\phi_*,\widehat{\zeta},\eta_*,\ddot{g})$ (see [18, 19]); then,

$$\begin{aligned} \eta_*(\widehat{\zeta}) &= -1,\\ \phi_*^2(\mathscr{P}^*) &= \mathscr{P}^* + \eta_*(\mathscr{P}^*)\widehat{\zeta},\\ \ddot{g}(\phi_*\mathscr{P}^*,\phi_*\mathscr{Q}^*) &= \ddot{g}(\mathscr{P}^*,\mathscr{Q}^*) + \eta_*(\mathscr{P}^*)\eta_*(\mathscr{Q}^*),\\ \ddot{g}(\mathscr{P}^*,\widehat{\zeta}) &= \eta_*(\mathscr{P}^*),\\ \nabla_{\mathscr{P}^*}\widehat{\zeta} &= \phi_*\mathscr{P}^*,\\ (\nabla_{\mathscr{P}^*}\phi_*)(\mathscr{Q}^*) &= \ddot{g}(\mathscr{P}^*,\mathscr{Q}^*)\widehat{\zeta} + \eta_*(\mathscr{Q}^*)\mathscr{P}^*\\ &\quad + 2\eta_*(\mathscr{P}^*)\eta_*(\mathscr{Q}^*), \end{aligned} \tag{5}$$

where ∇ be the Levi-Civita connection along the metric $\ddot{g}$. In an $\mathscr{LP}$-Sasakian manifold, we yields

$$\begin{aligned} \phi_*\widehat{\zeta} &= 0,\\ \eta_*(\phi_*\mathscr{P}^*) &= 0,\\ \operatorname{rank}(\phi_*) &= \widehat{n}-1. \end{aligned} \tag{6}$$

If we write

$$\widehat{\Phi}(\mathscr{P}^*,\mathscr{Q}^*) = \ddot{g}(\phi_*\mathscr{P}^*,\mathscr{Q}^*), \quad \mathscr{P}^*,\mathscr{Q}^* \in \Theta^{\widehat{n}}, \tag{7}$$

then $\widehat{\Phi}(\mathscr{P}^*,\mathscr{Q}^*)$ is a symmetric $(0,2)$ tensor field. Now, η_* is closed on $\Theta^{\widehat{n}}(\phi_*,\widehat{\zeta},\eta_*,\ddot{g})$ (see [18, 20]); then,

$$(\nabla_{\mathscr{P}^*}\eta)(\mathscr{Q}^*) = \widehat{\Phi}(\mathscr{P}^*,\mathscr{Q}^*), \quad \widehat{\Phi}(\mathscr{P}^*,\widehat{\zeta}) = 0. \tag{8}$$

In an $\mathscr{LP}$-Sasakian manifold $(\Theta^{\widehat{n}},\ddot{g})$, the following relationships are maintained (see [20, 21]):

$$\begin{aligned} \ddot{g}(\mathscr{R}(\mathscr{P}^*,\mathscr{Q}^*)\mathscr{Z}^*,\widehat{\zeta}) &= \ddot{g}(\mathscr{Q}^*,\mathscr{Z}^*)\eta_*(\mathscr{P}^*) - \ddot{g}(\mathscr{P}^*,\mathscr{Z}^*)\eta_*(\mathscr{Q}),\\ \ddot{g}(\mathscr{R}(\widehat{\zeta},\mathscr{P}^*)\mathscr{Q}^*) &= \ddot{g}(\mathscr{P}^*,\mathscr{Q}^*)\widehat{\zeta} - \eta_*(\mathscr{Q}^*)\mathscr{P}^*,\\ \mathscr{R}(\mathscr{P}^*,\mathscr{Q}^*)\widehat{\zeta} &= \eta_*(\mathscr{Q}^*)\mathscr{P}^* - \eta_*(\mathscr{P}^*)\mathscr{Q}^*,\\ \mathscr{R}(\widehat{\zeta},\mathscr{P}^*)\widehat{\zeta} &= \mathscr{P}^* + \eta_*(\mathscr{P}^*)\widehat{\zeta},\\ \mathscr{S}(\mathscr{P}^*,\widehat{\zeta}) &= (\widehat{n}-1)\eta_*(\mathscr{P}^*),\\ \mathscr{S}(\phi_*\mathscr{P}^*,\phi_*\mathscr{Q}^*) &= \mathscr{S}(\mathscr{P}^*,\mathscr{Q}^*) + (\widehat{n}-1)\eta_*(\mathscr{P}^*)\eta_*(\mathscr{Q}^*), \end{aligned} \tag{9}$$

for any vector fields $\mathscr{P}^*$, $\mathscr{Q}^*$, and $\mathscr{Z}^*$ on $\Theta^{\widehat{n}}$, where $\mathscr{R}$ and $\mathscr{S}$ are the curvature tensor and Ricci tensor of $\Theta^{\widehat{n}}$, simultaneously.

Let Θ be a submanifold of an $\mathscr{LP}$-Sasakian manifold. The Gauss and Weingarten formulas are given by

$$\nabla_{\mathscr{P}^*}\mathscr{Q}^* = \dot{\nabla}_{\mathscr{P}^*}\mathscr{Q}^* + h(\mathscr{P}^*,\mathscr{Q}^*), \quad \forall \mathscr{P}^*,\mathscr{Q}^* \in \Gamma(T^*\dot{\Theta}), \tag{10}$$

$$\nabla_{\mathscr{P}^*}\mathscr{N}^* = -A_{\mathscr{N}^*}\mathscr{P}^* + \nabla^{\perp}_{\mathscr{P}^*}\mathscr{N}^*, \quad \forall \mathscr{N}^* \in \Gamma(T^{*\perp}\dot{\Theta}), \tag{11}$$

where $\nabla_{\mathscr{P}^*}\mathscr{Q}^*$ and $\{h(\mathscr{P}^*,\mathscr{Q}^*),\nabla^{\perp}_{\mathscr{P}^*}\mathscr{N}^*\}$ belong to $\Gamma(T'\Theta)$ and $\Gamma(T^{*\perp}\Theta)$, respectively.

3. Generalized Symmetric Metric Connection of Type (α,β)

Let $\widetilde{\nabla}$ and ∇ be a linear and Levi-Civita connection on an $\mathscr{LP}$-Sasakian manifold. Now, we will go through the results that will be used.

Lemma 1 (see [22]). *In an $\mathscr{LP}$-Sasakian manifold $\Theta^{\widehat{n}}(\phi_*,\widehat{\zeta},\eta_*,\ddot{g})$, the GSMC $\widetilde{\nabla}$ of type (α,β) is given by*

$$\begin{aligned} \widetilde{\nabla}_{\mathscr{P}^*}\mathscr{Q}^* &= \nabla_{\mathscr{P}^*}\mathscr{Q}^{**} + \alpha\{\eta_*(\mathscr{Q}^*)\mathscr{P}^* - \ddot{g}(\mathscr{P}^*P^*,\mathscr{Q}^*)\widehat{\zeta}\}\\ &\quad + \beta\{\eta_*(\mathscr{Q}^*)\phi_*\mathscr{P}^* - \ddot{g}(\phi_*\mathscr{P}^*,\mathscr{Q}^*)\widehat{\zeta}\}, \end{aligned} \tag{12}$$

for all $\mathscr{P}^*$ and $\mathscr{Q}^*$ on $\Theta^{\widehat{n}}$.

Lemma 2 (see [22]). *The following relations hold on an $\mathscr{LP}$-Sasakian manifold in light of GSMC-(α,β):*

$$\begin{aligned} (\widetilde{\nabla}_{\mathscr{P}^*}\phi_*)(\mathscr{Q}^*) &= [(1-\beta)\{\ddot{g}(\mathscr{P}^*,\mathscr{Q}^*) + 2\eta_*(\mathscr{P}^*)\eta_*(\mathscr{Q}^*)\}\\ &\quad - \alpha\widehat{\Phi}(\mathscr{P}^*,\mathscr{Q}^*)]\widehat{\zeta}\\ &\quad + (1-\beta)\eta_*(\mathscr{Q}^*)\mathscr{P}^* - \alpha\eta_*(\mathscr{Q}^*)\phi_*\mathscr{P}^*,\\ \widetilde{\nabla}_{\mathscr{P}^*}\widehat{\zeta} &= (1-\beta)\phi_*\mathscr{P}^* - \alpha\mathscr{P}^* - \alpha\eta_*(\mathscr{P}^*)\widehat{\zeta},\\ (\widetilde{\nabla}_{\mathscr{P}^*}\eta_*)(\mathscr{Q}^*) &= (1-\beta)\widehat{\Phi}(\mathscr{P}^*,\mathscr{Q}^*) - \alpha\ddot{g}(\phi_*\mathscr{P}^*,\phi_*\mathscr{Q}^*),\\ \widetilde{\mathscr{R}}(\mathscr{P}^*,\mathscr{Q}^*)\widehat{\zeta} &= (1-\beta+\beta^2)\{\eta_*(\mathscr{Q}^*)\mathscr{P}^* - \eta_*(\mathscr{P}^*)\mathscr{Q}^*\}\\ &\quad + \alpha(1-\beta)\{\eta_*(\mathscr{P}^*)\phi_*\mathscr{Q}^* - \eta_*(\mathscr{Q}^*)\phi_*\mathscr{P}\},\\ \widetilde{\mathscr{R}}(\widehat{\zeta},\mathscr{Q}^*)\widehat{\zeta} &= (1-\beta+\beta^2)\{\eta_*(\mathscr{Q}^*)\widehat{\zeta} + \mathscr{Q}^{**}\} + \alpha(\beta-1)\phi_*\mathscr{Q}^*,\\ \widetilde{\mathscr{S}}(\mathscr{P}^*,\mathscr{Q}^*) &= \mathscr{S}(\mathscr{P}^*,\mathscr{Q}^*) + \{-\alpha\beta + (\widehat{n}-2)(\alpha\beta-\alpha)\\ &\quad + (\beta^2-2\beta)\widehat{\Phi}\}\widehat{\Phi}(\mathscr{P}^*,\mathscr{Q}^*)\\ &\quad + \{-2\alpha^2 + \beta - \beta^2 + \widehat{n}\alpha^2 + (\alpha\beta-\alpha)\widehat{\Phi}\}\ddot{g}(\mathscr{P}^*,\mathscr{Q}^*)\\ &\quad + \{-2\alpha^2 + \widehat{n}(\alpha^2+\beta-\beta^2)\}\eta_*(\mathscr{P}^*)\eta_*(\mathscr{Q}^*), \end{aligned} \tag{13}$$

for any $\mathscr{P}^*$, $\mathscr{Q}^* \in (T\Theta)$.

4. CR-Submanifolds of an $\mathscr{L}\mathscr{P}$-Sasakian Manifold with GSMC-(α, β)

Here, we have recall the well-known definition in the following manner.

Definition 3 (see [23]). A Riemannian manifold $(\widehat{\Theta^n}, \ddot{g})$ of an $\mathscr{L}\mathscr{P}$-Sasakian manifold $\widehat{\Theta^n}(\phi_*, \widehat{\zeta}, \eta, \ddot{g})$ is called a $\mathscr{C}\mathscr{R}$-submanifold $(\Theta, \ddot{g})$ if $\widehat{\zeta}$ is tangent to Θ and there exists on Θ a differentiable distribution $\mathscr{D}^*: x \longrightarrow \mathscr{D}_x^* \subset T_x^*(\Theta)$ such that

(i) $\mathscr{D}^*$ is invariant under ϕ_*, i.e., $\phi_*\mathscr{D}^* \subset \mathscr{D}^*$

(ii) The orthogonal complement distribution $\mathscr{D}^*\perp: x \longrightarrow \mathscr{D}_x^{*\perp} \subset T_x^*\Theta$ of the distribution $\mathscr{D}^*$ on Θ is totally real, i.e., $\phi_*\mathscr{D}^{*\perp} \subset T^{*\perp}\Theta$

Definition 4 (see [23]). If the distribution $\mathscr{D}^*$ (resp., $\mathscr{D}^{*\perp}$) is horizontal (resp., vertical), then the pair $(\mathscr{D}^*, \mathscr{D}^{*\perp})$ is called $\widehat{\zeta}$-horizontal (resp., $\widehat{\zeta}$-vertical) if $\widehat{\zeta} \in \Gamma(\mathscr{D}^*)$ (resp., $\widehat{\zeta} \in \Gamma(\mathscr{D}^{*\perp})$). The $\mathscr{C}\mathscr{R}$-submanifold is also called $\widehat{\zeta}$-horizontal (resp., $\widehat{\zeta}$-vertical) if $\widehat{\zeta} \in \Gamma(\mathscr{D}^*)$ (resp., $\widehat{\zeta} \in \Gamma(\mathscr{D}^{*\perp})$).

The orthogonal complement $\phi_*\mathscr{D}^{*\perp} \in T^{*\perp}\Theta$ is given by

$$\begin{aligned} T^*\Theta &= \mathscr{D}^* \oplus \mathscr{D}^{*\perp}, \\ T^{*\perp}\Theta &= \phi_*\mathscr{D}^{*\perp} \oplus \varrho, \end{aligned} \tag{14}$$

where $\phi_*\varrho = \varrho$.

Let Θ be a $\mathscr{C}\mathscr{R}$-submanifold of an $\mathscr{L}\mathscr{P}$-Sasakian manifold with a GSMC-(α, β). For any $\mathscr{P}^* \in \Gamma(T^*\Theta)$ and $\mathscr{N}^* \in \Gamma(T^{*\perp}\Theta)$, we can write

$$\mathscr{P}^* = U\mathscr{P}^* + V\mathscr{P}^*, \quad U\mathscr{P}^* \in \Gamma(\mathscr{D}^*), V\mathscr{P}^* \in \Gamma(\mathscr{D}^{*\perp}), \tag{15}$$

$$\phi_*\mathscr{N}^* = B\mathscr{N}^* + C\mathscr{N}^*, \quad B\mathscr{N}^* \in \Gamma(\mathscr{D}^{*\perp}), C\mathscr{N}^* \in \Gamma(\varrho). \tag{16}$$

The Gauss and Weingarten formulas with respect to $\widetilde{\nabla}$ are, respectively, given by

$$\widetilde{\nabla}_{\mathscr{P}^*}\mathscr{Q}^* = \widetilde{\nabla}'_{\mathscr{P}^*}\mathscr{Q}^* + \widetilde{h}(\mathscr{P}^*, \mathscr{Q}^*), \tag{17}$$

$$\widetilde{\nabla}_{\mathscr{P}^*}\mathscr{N}^* = -\widetilde{A}_{\mathscr{N}^*}\mathscr{P}^* + \widetilde{\nabla}^{\perp}_{\mathscr{P}^*}\mathscr{N}^*, \tag{18}$$

for any $\mathscr{P}^*, \mathscr{Q}^* \in \Gamma(T^*\Theta)$, where $\widetilde{\nabla}_{\mathscr{P}^*}\mathscr{Q}^*$, $\widetilde{A}_{\mathscr{N}^*}\mathscr{P}^* \in \Gamma(T^*\Theta)$. Here, $\widetilde{\nabla}$, $\widetilde{h}$, and $\widetilde{A}_{\mathscr{N}^*}$ are called the induced connection on Θ, the second fundamental form, and the Weingarten mapping with respect to $\widetilde{\nabla}$, respectively. In view of (10), (12), and (17), we obtain

$$\begin{aligned} \widetilde{\nabla}'_{\mathscr{P}^*}\mathscr{Q}^* + \widetilde{h}(\mathscr{P}^*, \mathscr{Q}^*) = \nabla'_{\mathscr{P}^*}\mathscr{Q}^* + h(\mathscr{P}^*, \mathscr{Q}^*) \\ + \alpha\{\eta_*(\mathscr{Q}^*)\mathscr{P}^* - \ddot{g}(\mathscr{P}^*, \mathscr{Q}^*)\widehat{\zeta}\} \\ + \beta\{\eta_*(\mathscr{Q}^*)\phi_*\mathscr{P}^* - \ddot{g}(\phi_*\mathscr{P}^*, \mathscr{Q}^*)\widehat{\zeta}\}. \end{aligned} \tag{19}$$

Using (15) and (16) in (19), we obtain

$$\begin{aligned} U\widetilde{\nabla}'_{\mathscr{P}^*}\mathscr{Q}^* = U\nabla'_{\mathscr{P}^*}\mathscr{Q}^* + \alpha\eta_*(\mathscr{Q}^*)U\mathscr{P}^* - \alpha\ddot{g}(\mathscr{P}^*, \mathscr{Q}^*)U\widehat{\zeta} \\ + \beta\eta_*(\mathscr{Q}^*)\phi_*U\mathscr{P}^* - \beta\ddot{g}(\phi_*\mathscr{P}^*, \mathscr{Q}^*)U\widehat{\zeta}, \end{aligned} \tag{20}$$

$$\widetilde{h}(\mathscr{P}^*, \mathscr{Q}^*) = h(\mathscr{P}^*, \mathscr{Q}^*) + \beta\eta_*(\mathscr{Q}^*)\phi_*\mathscr{Q}^*\mathscr{P}^*, \tag{21}$$

$$\begin{aligned} V\widetilde{\nabla}'_{\mathscr{P}^*}\mathscr{Q}^* = V\nabla'_{\mathscr{P}^*}\mathscr{Q}^* + \alpha\eta_*(\mathscr{Q}^*)V\mathscr{P}^* - \alpha\ddot{g}(\mathscr{P}^*, \mathscr{Q}^*)V\widehat{\zeta} \\ - \beta\ddot{g}(\phi_*\mathscr{P}^*, \mathscr{Q}^*)V\widehat{\zeta}, \end{aligned} \tag{22}$$

for any $\mathscr{P}^*, \mathscr{Q}^* \in (T\Theta)$.

5. Quasi-Yamabe Solitons (QYS) with Torse-Forming Vector Field

We classify QYS with torse-forming vector fields on an $\mathscr{L}\mathscr{P}$-Sasakian manifold admitting a GSMC-(α, β) in this section. As a result, we can prove the theorem below.

Theorem 1. *An $\mathscr{L}\mathscr{P}$-Sasakian manifold $\widehat{\Theta^n}(\phi_*, \widehat{\zeta}, \eta_*, \ddot{g})$, $n > 1$, with respect to GSMC-(α, β) admitting QYS. If $\mathscr{K}$ is a torse-forming vector field, then the data $(\ddot{g}, \mathscr{K}, \widehat{\sigma}, \mu)$ is growing, steadying, and contracting in accordance with $\psi - \widehat{\delta} - (1/\widehat{n})\{\pi(\mathscr{K}) + \alpha(\widehat{n} - 1)\eta_*(\mathscr{K}) - \mu\tau\} \lesseqgtr 0$, unless $\psi - \widehat{\delta} - (1/\widehat{n})\{\pi(\mathscr{K}) + \alpha(\widehat{n} - 1)\eta_*(\mathscr{K}) - \mu\tau\}$ is constant.*

Proof. Let the data $(\ddot{g}, \mathscr{K}, \widehat{\sigma}, \mu)$ be a QYS on $\widehat{\Theta^n}(\phi_*, \widehat{\zeta}, \eta_*, \ddot{g})$ in terms of GSMC-(α, β). From (3), we have

$$\begin{aligned} \frac{1}{2}\left(\widetilde{\mathfrak{L}}_{\mathscr{K}}\ddot{g}\right)(\mathscr{P}^*, \mathscr{Q}^*) = (\widetilde{\widehat{\delta}} - \widehat{\sigma})\ddot{g}(\mathscr{P}^*, \mathscr{Q}^*) \\ + \mu\ddot{g}\left(\mathscr{K}^{\diamond}, \mathscr{P}^*\right)\ddot{g}\left(\mathscr{K}^{\diamond}, \mathscr{Q}^*\right). \end{aligned} \tag{23}$$

From (3) and (12), we obtain

$$\begin{aligned} \left(\widetilde{\mathfrak{L}}_{\mathscr{K}}\ddot{g}\right)(\mathscr{P}^*, \mathscr{Q}^*) &= \ddot{g}(\widetilde{\nabla}_{\mathscr{P}^*}\mathscr{K}, \mathscr{Q}^*) + \ddot{g}(\mathscr{P}^*, \widetilde{\nabla}_{\mathscr{Q}^*}\mathscr{K}) \\ &= 2\psi\ddot{g}(\mathscr{P}^*, Y) + \pi(\mathscr{P}^*)\ddot{g}(\mathscr{K}, \mathscr{Q}^*) + \pi(\mathscr{Q}^*)\ddot{g}(\mathscr{K}, \mathscr{P}^*) \\ &\quad + \alpha\{2\eta_*(\mathscr{K})\ddot{g}(\mathscr{P}^*, \mathscr{Q}^*) - \ddot{g}(\mathscr{P}^*, \mathscr{K})\eta_*(\mathscr{Q}^*) - \ddot{g}(\mathscr{Q}^*, \mathscr{K})\eta_*(\mathscr{P}^*)\} \\ &\quad + \beta\{2\eta_*(\mathscr{K})\ddot{g}(\phi_*\mathscr{P}^*, \mathscr{Q}^*) - \ddot{g}(\phi\mathscr{P}^*, \mathscr{K})\eta_*(\mathscr{Q}^*) - \ddot{g}(\phi_*\mathscr{Q}^*, \mathscr{K})\eta_*(\mathscr{P}^*)\}, \end{aligned} \tag{24}$$

for all $\mathscr{P}^*, \mathscr{Q}^* \in \chi(\Theta)$.

With the help of (23) and (24), we obtain

$$\begin{aligned}(\psi-\tilde{\tilde{\delta}}-\widehat{\sigma})\ddot{g}(\mathscr{P}^*,\mathscr{Q}^*)&=\frac{1}{2}\left\{\pi(\mathscr{P}^*)\ddot{g}(\mathscr{K},\mathscr{Q}^*)+\pi(\mathscr{Q}^*)\ddot{g}(\mathscr{K},\mathscr{P}^*)\right\}\\&+\alpha\eta_*(\mathscr{K})\ddot{g}(\mathscr{P}^*,\mathscr{Q}^*)+\beta\eta_*(\mathscr{K})\ddot{g}(\phi_*\mathscr{P}^*,\mathscr{Q}^*)\\&-\frac{\alpha}{2}\left\{\ddot{g}(\mathscr{P}^*,\mathscr{K})\eta_*(\mathscr{Q}^*)+\ddot{g}(\mathscr{Q}^*,\mathscr{K})\eta_*(\mathscr{P}^*)\right\}\\&-\frac{\beta}{2}\left\{\ddot{g}(\phi_*\mathscr{P}^*,\mathscr{K})\eta_*(\mathscr{Q}^*)+\ddot{g}(\phi_*\mathscr{Q}^*,\mathscr{K})\eta_*(\mathscr{P}^*)\right\}\\&-\mu\ddot{g}\left(\mathscr{K}^{\diamond},\mathscr{P}^*\right)\ddot{g}\left(\mathscr{K}^{\diamond},\mathscr{Q}^*\right).\end{aligned}\tag{25}$$

On contracting (25), we find

$$\widehat{\sigma}=\psi-\tilde{\tilde{\delta}}-\frac{1}{\widehat{n}}\{\pi(\mathscr{K})+\alpha(\widehat{n}-1)\eta_*(\mathscr{K})-\mu\tau\},\tag{26}$$

where $\tau=g(\mathscr{K}^{\diamond},\mathscr{K}^{\diamond})=|\mathscr{K}^{\diamond}|^2$.

As a result, Theorem 1 is proven.

In this sequel, the corollaries are as follows. □

Corollary 1. *If (2) defines a QYS on an $\mathscr{LP}$-Sasakian manifold $\Theta^{\widehat{n}}(\phi_*,\widehat{\zeta},\eta_*,\ddot{g})$, $\widehat{n}>1$, admitting a GSMC-(α,β), then there are the existing relationships representing in Table 1.*

In Table 1, $\Omega_1=\psi-\tilde{\tilde{\delta}}-(1/\widehat{n})\{\pi(\mathscr{K})+\alpha(\widehat{n}-1)\eta_*(\mathscr{K})-\mu\tau\}$, $\Omega_2=\psi-\tilde{\tilde{\delta}}-(1/\widehat{n})\{\alpha(\widehat{n}-1)\eta_*(\mathscr{K})-\mu\tau\}$, $\Omega_3=1-\tilde{\tilde{\delta}}-(1/\widehat{n})\{\alpha(\widehat{n}-1)\eta_*(\mathscr{K})-\mu\tau\}$, $\Omega_4=\tilde{\tilde{\delta}}-(1/\widehat{n})\{\pi(\mathscr{K})+\alpha(\widehat{n}-1)\eta_*(\mathscr{K})-\mu\tau\}$, $\Omega_5=\tilde{\tilde{\delta}}-(1/n)\{\alpha(\widehat{n}-1)\eta_*(\mathscr{K})-\mu\tau\}$, and $\Omega_6=\psi-\tilde{\tilde{\delta}}-(1/\widehat{n})\{\alpha(\widehat{n}-1)\eta_*(\mathscr{K})-\mu\tau\}$.

Corollary 2. *Let data $(\ddot{g},\mathscr{K},\widehat{\sigma},\mu)$ be a QYS on an $\mathscr{LP}$-Sasakian manifold $M^{\widehat{n}}(\phi_*,\widehat{\zeta},\eta_*,\ddot{g})$, $\widehat{n}>1$, with respect to a GSMC-$(\alpha,\beta)=(1,0)$. If $\mathscr{K}$ is torse-forming vector field, then $(\ddot{g},\mathscr{K},\widehat{\sigma},\mu)$ is growing, steadying, and contracting according to $\psi-\tilde{\tilde{\delta}}-(1/\widehat{n})\{\pi(\mathscr{K})+(\widehat{n}-1)\eta_*(\mathscr{K})-\mu\tau\}\lesseqgtr 0$, unless $\psi-\tilde{\tilde{\delta}}-(1/\widehat{n})\{\pi(\mathscr{K})+(\widehat{n}-1)\eta_*(\mathscr{K})-\mu\tau\}$ is constant.*

Corollary 3. *Let $\Theta^{\widehat{n}}(\phi_*,\widehat{\zeta},\eta_*,\ddot{g})$, $\widehat{n}>1$, be an $\mathscr{LP}$-Sasakian manifold endowed with a GSMC-$(\alpha,\beta)=(0,1)$. If a data $(\ddot{g},\mathscr{K},\widehat{\sigma},\mu)$ be a QYS on $\Theta^{\widehat{n}}$ and $\mathscr{K}$ is a torse-forming vector field, then $(\ddot{g},\mathscr{K},\widehat{\sigma},\mu)$ is growing, steadying, and contracting according to $\psi-\tilde{\tilde{\delta}}-(1/\widehat{n})\{\pi(\mathscr{K})-\mu\tau\}\lesseqgtr 0$, unless $\psi-\tilde{\tilde{\delta}}-(1/\widehat{n})\{\pi(\mathscr{K})-\mu\tau\}$ is constant.*

Corollary 4. *If (2) defines a QYS on an $\mathscr{LP}$-Sasakian manifold $\Theta^{\widehat{n}}(\phi_*,\widehat{\zeta},\eta_*,\ddot{g})$, $\widehat{n}>1$, with respect to a GSMC-$(\alpha,\beta)=(0,1)$, then we obtain the relationship in Table 2.*

Corollary 5. *Let a data $(\ddot{g},\mathscr{K},\widehat{\sigma},\mu)$ be a QYS on an $\mathscr{LP}$-Sasakian manifold $\Theta^{\widehat{n}}(\phi_*,\widehat{\zeta},\eta_*,\ddot{g})$, $\widehat{n}>1$, with respect to a GSMC-$(\alpha,\beta)=(1,0)$. Then, the following relationships are maintained in Table 3.*

In Table 3, $\Omega_7=\psi-\tilde{\tilde{\delta}}-(1/\widehat{n})\{\pi(\mathscr{K})+(\widehat{n}-1)\eta_*(\mathscr{K})-\mu\tau\}$, $\Omega_8=\psi-\tilde{\tilde{\delta}}-(1/\widehat{n})\{(\widehat{n}-1)\eta_*(\mathscr{K})-\mu\tau\}$, $\Omega_9=1-\tilde{\tilde{\delta}}-(1/\widehat{n})\{(\widehat{n}-1)\eta_*(\mathscr{K})-\mu\tau\}$, $\Omega_{10}=\tilde{\tilde{\delta}}-(1/\widehat{n})\{\pi(\mathscr{K})+(\widehat{n}-1)\eta_*(\mathscr{K})-\mu\tau\}$, $\Omega_{11}=\tilde{\tilde{\delta}}-(1/\widehat{n})\{(\widehat{n}-1)\eta_*(\mathscr{K})-\mu\tau\}$, and $\Omega_{12}=\psi-\tilde{\tilde{\delta}}-(1/\widehat{n})\{(\widehat{n}-1)\eta_*(\mathscr{K})-\mu\tau\}$.

6. Quasi-Yamabe Solitons (QYS) with Potential Vector Field is Torse-Forming on $\mathscr{CR}$-Submanifold

We investigate QYS in relating to a torse-forming vector field on $\mathscr{CR}$-submanifolds of an $\mathscr{LP}$-Sasakian manifold with regard to the induced connection $\tilde{\nabla}$ of type (α,β) in this section. The following is our theorem.

Theorem 2. *Let a $\mathscr{CR}$-submanifold Θ' of an $\mathscr{LP}$-Sasakian manifold be $\Theta^{\widehat{n}}(\phi_*,\widehat{\zeta},\eta_*,\ddot{g})$, $\widehat{n}>1$, admitting a GSMC $\tilde{\nabla}'$ is $\widehat{\zeta}$-horizontal (resp. $\widehat{\zeta}$-vertical) and $\mathscr{D}^*$ is parallel with respect to $\tilde{\nabla}'$. If data $(\ddot{g},\mathscr{K},\widehat{\sigma},\mu)$ is a QYS on Θ' and $\mathscr{K}$ is a torse-forming vector field, then $(\ddot{g},\mathscr{K},\widehat{\sigma},\mu)$ is growing, steadying, or contracting according to $\psi-\tilde{\tilde{\delta}}-(1/\widehat{n})\{\pi(\mathscr{K})+\alpha(\widehat{n}-1)\eta_*(\mathscr{K})-\mu\tau\}\lesseqgtr 0$, unless $\psi-\tilde{\tilde{\delta}}-(1/\widehat{n})\{\pi(\mathscr{K})+\alpha(\widehat{n}-1)\eta_*(\mathscr{K})-\mu\tau\}$ is constant.*

Proof. If Θ is $\widehat{\zeta}$-horizontal for all $\mathscr{P}^*,\mathscr{Q}^*\in\Gamma(\mathscr{D}^*)$ and $\mathscr{D}^*$ is parallel in relation to $\tilde{\nabla}$, then there is, from (20),

$$\begin{aligned}\tilde{\nabla}'_{\mathscr{P}^*}\mathscr{Q}^*&=\nabla'_{\mathscr{P}^*}\mathscr{Q}^*+\alpha\left\{\eta(\mathscr{Q}^*)\mathscr{P}^*-g(\mathscr{P}^*,\mathscr{Q}^*)\widehat{\zeta}\right\}\\&+\beta\left\{\eta_*(\mathscr{Q}^*)\phi_*\mathscr{P}^*-g(\phi_*\mathscr{P}^*,\mathscr{Q}^*)\widehat{\zeta}\right\}.\end{aligned}\tag{27}$$

With the help of Lemma 1, the induced connection $\tilde{\nabla}'$ is also a GSMC-(α,β). This leads to the statement of Theorem 2.

Table 1: Relationship between $\mathscr{K}$, existence condition, and nature of solitons.

$\mathscr{K}$	Existence condition	Nature of solitons
Torse-forming	Ω_1 = constant	$\Omega_1 \overset{\le}{\underset{>}{=}} 0$
Concircular	Ω_2 = constant	$\Omega_2 \overset{\le}{\underset{>}{=}} 0$
Concurrent	Ω_3 = constant	$\Omega_3 \overset{\le}{\underset{>}{=}} 0$
Recurrent	Ω_4 = constant	$\Omega_4 \overset{\le}{\underset{>}{=}} 0$
Parallel	Ω_5 = constant	$\Omega_5 \overset{\le}{\underset{>}{=}} 0$
Torqued	Ω_6 = constant	$\Omega_6 \overset{\le}{\underset{>}{=}} 0$

Table 2: Relationship between $\mathscr{K}$, existence condition, and nature of solitons.

$\mathscr{K}$	Existence condition	Nature of solitons
Torse-forming	$\psi - \tilde{\tilde{\delta}} - (1/\widehat{n})\{\pi(\mathscr{K}) - \mu\tau\}$ = constant	$\psi - \tilde{\tilde{\delta}} - (1/\widehat{n})\{\pi(\mathscr{K}) - \mu\tau\} \overset{\le}{\underset{>}{=}} \overline{0}$
Concircular	$\psi - \tilde{\tilde{\delta}} - \mu\tau$ = constant	$\psi - \tilde{\tilde{\delta}} - \mu\tau \overset{\le}{\underset{>}{=}} 0$
Concurrent	$1 - \tilde{\tilde{\delta}} - \mu\tau$ = constant	$\tilde{\tilde{\delta}} - \mu\tau \overset{\le}{\underset{>}{=}} 1$
Recurrent	$\tilde{\tilde{\delta}} - (1/\widehat{n})\{\pi(\mathscr{K}) - \mu\tau\}$ = constant	$\tilde{\tilde{\delta}} - (1/\widehat{n})\{\pi(\mathscr{K}) - \mu\tau\} \overset{\le}{\underset{>}{=}} 0$
Parallel	$\tilde{\tilde{\delta}} - \mu\tau$ = constant	$\tilde{\tilde{\delta}} - \mu\tau \overset{\le}{\underset{>}{=}} 0$
Torqued	$\psi - \tilde{\tilde{\delta}} - \mu\tau$ = constant	$\psi - \tilde{\tilde{\delta}} - \mu\tau \overset{\le}{\underset{>}{=}} 0$

Table 3: Relationship between $\mathscr{K}$, existence condition, and nature of solitons.

$\mathscr{K}$	Existence condition	Nature of solitons
Torse-forming	Ω_7 = constant	$\Omega_7 \overset{\le}{\underset{>}{=}} 0$
Concircular	Ω_8 = constant	$\Omega_8 \overset{\le}{\underset{>}{=}} 0$
Concurrent	Ω_9 = constant	$\Omega_9 \overset{\le}{\underset{>}{=}} 0$
Recurrent	Ω_{10} = constant	$\Omega_{10} \overset{\le}{\underset{>}{=}} 0$
Parallel	Ω_{11} = constant	$\Omega_{11} \overset{\le}{\underset{>}{=}} 0$
Torqued	Ω_{12} = constant	$\Omega_{12} \overset{\le}{\underset{>}{=}} 0$

In this sequel, we write the following corollaries. □

Corollary 6. *Let a $\mathscr{C}\mathscr{R}$-submanifold $\overset{\prime}{\Theta}$ of an $\mathscr{L}\mathscr{P}$-Sasakian manifold be $\Theta^{\widehat{n}}(\phi_*, \widehat{\zeta}, \eta_*, \ddot{g})$, $\widehat{n} > 1$, admitting a GSMC $\overset{\prime}{\tilde{\nabla}}$ is $\widehat{\zeta}$-horizontal (resp. $\widehat{\zeta}$-vertical) and $\mathscr{D}^*$ is parallel in terms of $\overset{\prime}{\nabla}$. If (2) defines a QYS on $\overset{\prime}{\Theta}$ and $\mathscr{K}$ is a torse-forming vector field, then the results hold in Table 4.*

Corollary 7. *Let a $\mathscr{C}\mathscr{R}$-submanifold $\overset{\prime}{\Theta}$ of an $\mathscr{L}\mathscr{P}$-Sasakian manifold be $\Theta^{\widehat{n}}(\phi_*, \widehat{\zeta}, \eta_*, \ddot{g})$, $\widehat{n} > 1$, admitting a GSMC $\overset{\prime}{\tilde{\nabla}}$ is $\widehat{\zeta}$-horizontal (resp. $\widehat{\zeta}$-vertical) and $\mathscr{D}^*$ is parallel in term of $\overset{\prime}{\tilde{\nabla}}$ of type $(\alpha, \beta) = (0, 1)$. If data $(\ddot{g}, \mathscr{K}, \widehat{\sigma}, \mu)$ is a QYS on $\overset{\prime}{\Theta}$ and $\mathscr{K}$ is a torse-forming vector field, then, in Table 5, relationships must be true.*

Corollary 8. *Let a $\mathscr{C}\mathscr{R}$-submanifold $\overset{\prime}{\Theta}$ of an $\mathscr{L}\mathscr{P}$-Sasakian manifold be $\Theta^{\widehat{n}}(\phi_*, \widehat{\zeta}, \eta_*, \ddot{g})$, $\widehat{n} > 1$, admitting a GSMC $\overset{\prime}{\tilde{\nabla}}$ of type $(\alpha, \beta) = (1, 0)$ is $\widehat{\zeta}$-horizontal (resp. $\widehat{\zeta}$-vertical) and $\mathscr{D}^*$ is parallel with respect to $\overset{\prime}{\nabla}$. If $(\ddot{g}, \mathscr{K}, \widehat{\sigma}, \mu)$ is a QYS on $\overset{\prime}{\Theta}$ and $\mathscr{K}$ is a torse-forming vector field, then $(\ddot{g}, \mathscr{K}, \widehat{\sigma}, \mu)$ is growing, steadying, or contracting according to $\psi - \tilde{\tilde{\delta}} - (1/\widehat{n})\{\pi(\mathscr{K}) + (\widehat{n} - 1)\eta_*(\mathscr{K}) - \mu\tau\} \overset{\le}{\underset{>}{=}} 0$, unless $\psi - \tilde{\tilde{\delta}} - (1/\widehat{n})\{\pi(K) + (\widehat{n} - 1)\eta_*(\mathscr{K}) - \mu\tau\}$ is constant.*

Table 4: Relationship between $\mathscr{K}$, existence condition, and nature of solitons.

$\mathscr{K}$	Existence condition	Nature of solitons
Torse-forming	Ω_1 = constant	$\Omega_1 \lesseqgtr 0$
Concircular	Ω_2 = constant	$\Omega_2 \lesseqgtr 0$
Concurrent	Ω_3 = constant	$\Omega_3 \lesseqgtr 0$
Recurrent	Ω_4 = constant	$\Omega_4 \lesseqgtr 0$
Parallel	Ω_5 = constant	$\Omega_5 \lesseqgtr 0$
Torqued	Ω_6 = constant	$\Omega_6 \lesseqgtr 0$

Table 5: Relationship between $\mathscr{K}$, existence condition, and nature of solitons.

$\mathscr{K}$	Existence condition	Nature of solitons
Torse-forming	Ω_7 = constant	$\Omega_7 \lesseqgtr 0$
Concircular	Ω_8 = constant	$\Omega_8 \lesseqgtr 0$
Concurrent	Ω_9 = constant	$\Omega_9 \lesseqgtr 0$
Recurrent	Ω_{10} = constant	$\Omega_{10} \lesseqgtr 0$
Parallel	Ω_{11} = constant	$\Omega_{11} \lesseqgtr 0$
Torqued	Ω_{12} = constant	$\Omega_{12} \lesseqgtr 0$

Corollary 9. *Let a $\mathscr{CR}$-submanifold Θ' of an $\mathscr{LP}$-Sasakian manifold be $\Theta^{\widehat{n}}(\phi_*, \widehat{\zeta}, \eta_*, \ddot{g})$, $\widehat{n} > 1$, in relation to a GSMC-$(\alpha, \beta) = (0, 1)$ which is $\widehat{\zeta}$-horizontal (resp. $\widehat{\zeta}$-vertical) and $\mathscr{D}^*$ which is parallel in view og $\widetilde{\nabla}'$. If $(\ddot{g}, \mathscr{K}, \widehat{\sigma}, \mu)$ is a QYS on Θ' and $\mathscr{K}$ is a torse-forming vector field, then $(\ddot{g}, \mathscr{K}, \widehat{\sigma}, \mu)$ is growing, steadying, or shrinking according as $\psi - \widetilde{\widehat{\delta}} - (1/\widehat{n})\{\pi(\mathscr{K}) - \mu\tau\} \lesseqgtr 0$, unless $\psi - \widetilde{\widehat{\delta}} - (1/\widehat{n})\{\pi(\mathscr{K}) - \mu\tau\}$ is constant.*

Corollary 10. *If (2) defines a QYS on an $\mathscr{LP}$-Sasakian manifold $\Theta^{\widehat{n}}(\phi_*, \widehat{\zeta}, \eta_*, \ddot{g})$, $\widehat{n} > 1$, concerning a GSMC-$(\alpha, \beta) = (0, 1)$, then, in Table 6, relationships are true.*

7. Almost Quasi-Yamabe Solitons (AQYS) whose Potential Vector Field is Torse-Forming on $\mathscr{CR}$-Submanifold

In this section, we classify AQYS whose potential field is torse-forming on $\mathscr{CR}$-submanifold of an $\mathscr{LP}$-Sasakian manifold with respect to a GSMC-(α, β). At this stage, we denote $\mathscr{K}^t$ and $\mathscr{K}^n$ as tangential and normal components of such vector field. To begin, we will prove the outcome.

Theorem 3. *An almost quasi-Yamabe soliton $(\ddot{g}, \mathscr{K}^t, \widehat{\sigma}, \mu)$ on $\mathscr{CR}$-submanifold of an $\mathscr{LP}$-Sasakian manifold in relation to a GSMC-(α, β) satisfies*

$$\begin{aligned}&\left(\widehat{\delta}-\widehat{\sigma}-\psi+\eta_*(\mathscr{K}^n)\right)\ddot{g}(\mathscr{P}^*,\mathscr{Q}^*)+\mu\ddot{g}\left(\mathscr{K}^{\diamond},\mathscr{P}^*\right)\ddot{g}\left(\mathscr{K}^{\diamond},\mathscr{Q}^*\right)\\&=\ddot{g}\left(A^n_{\mathscr{K}}\mathscr{P}^*,\mathscr{Q}^*\right)\\&\quad+\frac{1}{2}\left\{\pi(\mathscr{P}^*)\ddot{g}(\mathscr{K},\mathscr{Q}^*)+\pi(\mathscr{Q}^*)\ddot{g}(\mathscr{P}^*,\mathscr{K})\right\}\\&\quad+\frac{\beta}{2}\left\{\ddot{g}(\mathscr{K}^n,\phi_*\mathscr{P}^*)\eta_*(\mathscr{Q}^*)+\ddot{g}(\phi_*\mathscr{Q}^*,\mathscr{K}^n)\eta_*(\mathscr{P}^*)\right\},\end{aligned} \tag{28}$$

any type of vector field $\mathscr{P}^*, \mathscr{Q}$ on Θ'.

Proof. In light of (3), (12), (17), and (18), we have

$$\begin{aligned}\psi\mathscr{P}^*+\pi(P^*)\mathscr{K}&=\widetilde{\nabla}_{\mathscr{P}^*}\mathscr{K}=\widetilde{\nabla}_{\mathscr{P}^*}\left(\mathscr{K}^t+\mathscr{K}^n\right)=\nabla'_{\mathscr{P}^*}\mathscr{K}^t+h\left(\mathscr{P}^*,\mathscr{K}^t\right)+\beta\eta_*\left(\mathscr{K}^t\right)\phi_*\mathscr{Q}^*\mathscr{P}^*\\&\quad-A_{\mathscr{K}^n}\mathscr{P}^*+\nabla^{\perp}_{\mathscr{P}^*}\mathscr{K}^n+\alpha\eta_*(\mathscr{K}^n)\mathscr{P}^*+\beta\eta_*(\mathscr{K}^n)\phi_*\mathscr{P}^*-\beta\ddot{g}(\phi_*\mathscr{P}^*,\mathscr{K}^n)\widehat{\zeta}.\end{aligned} \tag{29}$$

We obtain the following equation when we compare the tangential and normal components of (29):

$$\nabla'_{\mathscr{P}^*}\mathscr{K}^t=\psi\mathscr{P}^*+\pi(\mathscr{P}^*)K+A_{\mathscr{K}^n}\mathscr{P}^*-\alpha\eta(\mathscr{K}^n)\mathscr{P}^*-\beta\eta_*(\mathscr{K}^n)\phi_*\mathscr{P}^*+\beta\ddot{g}(\phi_*\mathscr{P}^*,\mathscr{K}^n)\widehat{\zeta}, \tag{30}$$

$$h\left(\mathscr{P}^*,\mathscr{K}^t\right)=-\nabla^{\perp}_{\mathscr{P}^*}\mathscr{K}^n-\beta\eta_*(\mathscr{K}^n)\phi_*\mathscr{Q}^*\mathscr{P}^*. \tag{31}$$

We may deduce from the concept of Lie derivative and (31) that

Table 6: Relationship between $\mathscr{K}$, existence condition, and nature of solitons.

$\mathscr{K}$	Existence condition	Nature of solitons
Torse-forming	$\psi - \tilde{\tilde{\delta}} - (1/\widehat{n})\{\pi(\mathscr{K}) - \mu\tau\} = \text{constant}$	$\psi - \tilde{\tilde{\delta}} - (1/\widehat{n})\{\pi(\mathscr{K}) - \mu\tau\} \lesseqgtr 0$
Concircular	$\psi - \tilde{\tilde{\delta}} - \mu\tau = \text{constant}$	$\psi - \tilde{\tilde{\delta}} - \mu\tau \lesseqgtr 0$
Concurrent	$1 - \tilde{\tilde{\delta}} - \mu\tau = \text{constant}$	$\tilde{\tilde{\delta}} - \mu\tau \lesseqgtr 1$
Recurrent	$\tilde{\tilde{\delta}} - (1/\widehat{n})\{\pi(\mathscr{K}) - \mu\tau\} = \text{constant}$	$\tilde{\tilde{\delta}} - (1/\widehat{n})\{\pi(\mathscr{K}) - \mu\tau\} \lesseqgtr 0$
Parallel	$\tilde{\tilde{\delta}} - \mu\tau = \text{constant}$	$\tilde{\tilde{\delta}} - \mu\tau \lesseqgtr 0$
Torqued	$\psi - \tilde{\tilde{\delta}} - \mu\tau = \text{constant}$	$\psi - \tilde{\tilde{\delta}} - \mu\tau \lesseqgtr 0$

$$\begin{aligned}\mathfrak{L}_{\mathscr{K}^t}\ddot{g}(\mathscr{P}^*,\mathscr{Q}^*) = {} & 2\psi\ddot{g}(\mathscr{P}^*,\mathscr{Q}^*) + 2\ddot{g}(A^n_{\mathscr{K}}\mathscr{P}^*,\mathscr{Q}^*) - 2\eta_*(\mathscr{K}^n)\ddot{g}(\mathscr{P}^*,\mathscr{Q}^*)\\ & + \pi(\mathscr{P}^*)\ddot{g}(\mathscr{K},\mathscr{Q}^*) + \pi(\mathscr{Q}^*)\ddot{g}(\mathscr{P}^*,\mathscr{K}) + \beta\{\ddot{g}(\mathscr{K}^n,\phi_*\mathscr{P}^*)\eta_*(\mathscr{Q}^*)\\ & + \ddot{g}(\phi\mathscr{Q}^*,\mathscr{K}^n)\eta_*(\mathscr{P}^*)\}.\end{aligned} \tag{32}$$

Using (32) in (2), we yield

$$\begin{aligned}\left(\widehat{\delta} - \widehat{\sigma} - \psi + \eta_*(\mathscr{K}^n)\right)\ddot{g}(\mathscr{P}^*,\mathscr{Q}^*) + \mu\widehat{g}\left(\mathscr{K}^\diamond,\mathscr{P}^*\right)\ddot{g}\left(\mathscr{K}^\diamond,\mathscr{Q}^*\right) = {} & \ddot{g}(A^n_{\mathscr{K}}\mathscr{P}^*,\mathscr{Q}^*)\\ & + \frac{1}{2}\{\pi(\mathscr{P}^*)\ddot{g}(\mathscr{K},\mathscr{Q}^*) + \pi(\mathscr{Q}^*)\ddot{g}(\mathscr{P}^*,\mathscr{K})\}\\ & + \frac{\beta}{2}\{\ddot{g}(\mathscr{K}^n,\phi_*\mathscr{P}^*)\eta_*(\mathscr{Q}^*) + \ddot{g}(\phi_*\mathscr{Q}^*,\mathscr{K}^n)\eta_*(\mathscr{P}^*)\}.\end{aligned} \tag{33}$$

This completed our assertion.

As a result, the following corollaries are stated. □

Corollary 11. *If (2) defines AQYS on $\mathscr{CR}$-submanifold of an $\mathscr{LP}$-Sasakian manifold in relation to a GSMC-(α,β) which is minimal, consequently, the following relationship holds:*

$$\left(\widehat{\delta} - \widehat{\sigma} - \psi + \eta_*(\mathscr{K}^n)\right)\widehat{n} + \mu\tau = \pi(\mathscr{K}). \tag{34}$$

Corollary 12. *If (2) defines AQYS on $\mathscr{CR}$-submanifold of an $\mathscr{LP}$-Sasakian manifold and the distribution is $\widehat{\zeta}$-horizontal (resp. $\widehat{\zeta}$-vertical), $\mathscr{P}^*, \mathscr{Q}^* \in \Gamma(\mathscr{D}^*)$, where $\mathscr{D}^*$ is parallel with induced connection ∇ of type (α,β), then we have*

$$\begin{aligned}\left(\widehat{\delta} - \widehat{\sigma} - \psi + \eta_*(\mathscr{K}^n)\right)\ddot{g}(\mathscr{P}^*,\mathscr{Q}^*) + \mu\ddot{g}\left(\mathscr{K}^\diamond,\mathscr{P}^*\right)\ddot{g}\left(\mathscr{K}^\diamond,\mathscr{Q}^*\right) = {} & \ddot{g}(A^n_{\mathscr{K}}\mathscr{P}^*,\mathscr{Q}^*)\\ & + \frac{1}{2}\{\pi(\mathscr{P}^*)\ddot{g}(\mathscr{K},\mathscr{Q}^*) + \pi(\mathscr{Q}^*)\ddot{g}(\mathscr{P}^*,\mathscr{K})\}\\ & + \frac{\beta}{2}\{\ddot{g}(\mathscr{K}^n,\phi_*\mathscr{P}^*)\eta_*(\mathscr{Q}^*) + \ddot{g}(\phi_*\mathscr{Q}^*,\mathscr{K}^n)\eta_*(\mathscr{P}^*)\},\end{aligned} \tag{35}$$

in all vector fields $\mathscr{P}^*, \mathscr{Q}^*$ on Θ'.

Corollary 13. *If data $(\ddot{g},\mathscr{K}^t,\widehat{\sigma},\mu)$ is a AQYS on $\mathscr{CR}$-submanifold of an $\mathscr{LP}$-Sasakian manifold and the distribution is $\widehat{\zeta}$-horizontal (resp. $\widehat{\zeta}$-vertical), $\mathscr{P}^*, \mathscr{Q}^* \in \Gamma(\mathscr{D}^*)$, $\mathscr{D}^*$ is parallel with induced connection ∇ of type (α,β) is minimal, then the relation holds:*

$$\left(\widehat{\delta} - \widehat{\sigma} - \psi + \eta_*(\mathscr{K}^n)\right)\widehat{n} + \mu\tau = \pi(\mathscr{K}). \tag{36}$$

8. Example

A 4-dimensional differentiable manifold is taken into consideration, that is, $\Theta^4 = \{(p,q,z,t) \in \mathfrak{R}^4 : (p,q,z,t) \neq 0\}$, where (p,q,z,t) is the standard coordinate in $\mathfrak{R}^4$. At each point along Θ^4, $(\breve{e}_1, \breve{e}_2, \breve{e}_3, \breve{e}_4)$ is a set of linearly independent vector fields and is described as

$$\begin{aligned}\breve{e}_1 &= e^{p-t}\frac{\partial}{\partial p},\\ \breve{e}_2 &= e^{q-t}\frac{\partial}{\partial q},\\ \breve{e}_3 &= e^{z-t}\frac{\partial}{\partial z},\\ \breve{e}_4 &= \frac{\partial}{\partial t}.\end{aligned} \tag{37}$$

Also, the Lie bracket's nonvanishing components are as follows:

$$\begin{aligned}\left[\breve{e}_1, \breve{e}_4\right] &= \breve{e}_1,\\ \left[\breve{e}_2, \breve{e}_4\right] &= \breve{e}_2,\\ \left[\breve{e}_3, \breve{e}_4\right] &= \breve{e}_3.\end{aligned} \tag{38}$$

Let $\ddot{g}$ on Θ be the Lorentzian metric as

$$\ddot{g}_{ij} = \ddot{g}\left(\breve{e}_i, \breve{e}_j\right) = \begin{cases} 0, & \text{if } i \neq j,\\ -1, & \text{if } i = j = 4,\\ 1, & \text{otherwise.}\end{cases} \tag{39}$$

Let η_* be the 1-form corresponding to the Lorentzian metric $\ddot{g}$:

$$\eta_*(\mathscr{P}^*) = \ddot{g}\left(\mathscr{P}^*, \breve{e}_4\right), \tag{40}$$

for any $\mathscr{P}^* \in \Gamma(\Theta)$. If ϕ_* is defined as the $(1,1)$-tensor field,

$$\begin{aligned}\phi_*\left(\breve{e}_1\right) &= \breve{e}_1,\\ \phi_*\left(\breve{e}_2\right) &= \breve{e}_2,\\ \phi_*\left(\breve{e}_3\right) &= \breve{e}_3,\\ \phi_*\left(\breve{e}_4\right) &= 0.\end{aligned} \tag{41}$$

We can readily prove this using the linearity characteristics of ϕ_* and $\ddot{g}$:

$$\begin{aligned}\eta_*\left(\breve{e}_4\right) &= -1,\\ \phi_*^2(\mathscr{P}^*) &= \mathscr{P}^* + \eta_*(\mathscr{P}^*)\breve{e}_4,\\ \ddot{g}(\phi_*\mathscr{P}^*, \phi_*\mathscr{Q}^*) &= \ddot{g}(\mathscr{P}^*, \mathscr{Q}^*) + \eta_*(\mathscr{P}^*)\eta_*(\mathscr{Q}^*),\end{aligned} \tag{42}$$

for any $\mathscr{P}^*, \mathscr{Q}^* \in \Gamma(T\Theta)$. Thus, for $\breve{e}_4 = \widehat{\zeta}$, the frame $(\phi_*, \widehat{\zeta}, \eta_*, \ddot{g})$ leads to an $\mathscr{LP}$-contact skeleton, which is known as the $\mathscr{LP}$-contact manifold of dimension 4. Now, for $\breve{e}_4 = \widehat{\zeta}$, Koszul's formula gives the nonvanishing component:

$$\begin{aligned}\nabla_{\breve{e}_1}\breve{e}_1 &= \breve{e}_4,\\ \nabla_{\breve{e}_1}\breve{e}_4 &= \breve{e}_1,\\ \nabla_{\breve{e}_2}\breve{e}_2 &= \breve{e}_4,\\ \nabla_{\breve{e}_2}\breve{e}_4 &= \breve{e}_2,\\ \nabla_{\breve{e}_3}\breve{e}_3 &= \breve{e}_4,\\ \nabla_{\breve{e}_3}\breve{e}_4 &= e_3.\end{aligned} \tag{43}$$

Using the above equation, it can be easily verified that $\nabla_{\mathscr{P}^*}\breve{e}_4 = \phi_*\mathscr{P}^*$ holds for each $\mathscr{P}^* \in \chi(\Theta)$. Thus, an $\mathscr{LP}$-contact manifold is a 4-dimensional $\mathscr{LP}$-Sasakian manifold. From (12), we calculate as follows:

$$\begin{aligned}\widetilde{\nabla}_{\breve{e}_1}\breve{e}_1 &= (1-\alpha-\beta)\breve{e}_4,\\ \widetilde{\nabla}_{\breve{e}_1}\breve{e}_4 &= (1-\alpha-\beta)\breve{e}_1,\\ \widetilde{\nabla}_{\breve{e}_2}\breve{e}_2 &= (1-\alpha-\beta)\breve{e}_4,\\ \widetilde{\nabla}_{\breve{e}_2}\breve{e}_4 &= (1-\alpha-\beta)\breve{e}_2,\\ \widetilde{\nabla}_{\breve{e}_3}\breve{e}_3 &= (1-\alpha-\beta)\breve{e}_4,\\ \widetilde{\nabla}_{\breve{e}_3}\breve{e}_4 &= (1-\alpha-\beta)\breve{e}_3.\end{aligned} \tag{44}$$

It is clear from (12) that $\widetilde{\nabla}_{\mathscr{P}^*}\breve{e}_4 = (1-\beta)\phi_*\mathscr{P}^* - \alpha\mathscr{P}^* - \alpha\eta_*(\mathscr{P}^*)\breve{e}_4$ holds for each $\mathscr{P}^* \in \chi(\Theta)$. So, an $\mathscr{LP}$-Sasakian manifold admitted a GSMC-(α,β).

The nonvanishing components of the curvature tensor using the aforementioned formulas are

$$\begin{aligned}\widetilde{\mathscr{R}}\left(\breve{e}_1, \breve{e}_2\right)\breve{e}_1 &= -(1-\alpha-\beta)^2\breve{e}_2,\\ \widetilde{\mathscr{R}}\left(\breve{e}_1, \breve{e}_3\right)\breve{e}_1 &= -(1-\alpha-\beta)^2\breve{e}_3,\\ \widetilde{\mathscr{R}}\left(\breve{e}_1, \breve{e}_4\right)\breve{e}_1 &= -(1-\alpha-\beta)\breve{e}_4,\\ \widetilde{\mathscr{R}}\left(\breve{e}_1, \breve{e}_2\right)\breve{e}_2 &= (1-\alpha-\beta)^2\breve{e}_1,\\ \widetilde{\mathscr{R}}\left(\breve{e}_2, \breve{e}_3\right)\breve{e}_2 &= -(1-\alpha-\beta)^2\breve{e}_3,\\ \widetilde{\mathscr{R}}\left(\breve{e}_2, \breve{e}_4\right)\breve{e}_2 &= -(1-\alpha-\beta)\breve{e}_4,\\ \widetilde{\mathscr{R}}\left(\breve{e}_1, \breve{e}_3\right)\breve{e}_3 &= (1-\alpha-\beta)^2\breve{e}_1,\\ \widetilde{\mathscr{R}}\left(\breve{e}_2, \breve{e}_3\right)\breve{e}_3 &= (1-\alpha-\beta)^2\breve{e}_2,\\ \widetilde{\mathscr{R}}\left(\breve{e}_3, \breve{e}_4\right)\breve{e}_3 &= -(1-\alpha-\beta)\breve{e}_4,\\ \widetilde{\mathscr{R}}\left(\breve{e}_1, \breve{e}_4\right)\breve{e}_4 &= -(1-\alpha-\beta)\breve{e}_1,\\ \widetilde{\mathscr{R}}\left(\breve{e}_2, \breve{e}_4\right)\breve{e}_4 &= -(1-\alpha-\beta)\breve{e}_2,\\ \widetilde{\mathscr{R}}\left(\breve{e}_3, \breve{e}_4\right)\breve{e}_4 &= -(1-\alpha-\beta)\breve{e}_3.\end{aligned} \tag{45}$$

The Ricci tensor of Θ is defined as $\widetilde{\mathscr{S}}(\mathscr{P}^*, \mathscr{Q}^*) = \sum_{i=1}^4 \varepsilon_i \ddot{g}(\widetilde{\mathscr{R}}(\breve{e}_i, \mathscr{P}^*)\mathscr{Q}^*, \breve{e}_i)$, where $\varepsilon_i = \ddot{g}(\breve{e}_i, \breve{e}_i)$, and is given by

$$\widetilde{\mathcal{S}}\left(\breve{e}_i,\breve{e}_j\right)=\begin{bmatrix}3(1-\alpha-\beta)^2 & 0 & 0 & 0\\ 0 & 3(1-\alpha-\beta)^2 & 0 & 0\\ 0 & 0 & 3(1-\alpha-\beta)^2 & 0\\ 0 & 0 & 0 & -3(1-\alpha-\beta)^2\end{bmatrix}. \tag{46}$$

Also, the scalar curvature $\widetilde{\widetilde{\delta}}=\sum_{i=1}^{4}\widetilde{\mathcal{S}}(\breve{e}_i,\breve{e}_j)=6(1-\alpha-\beta)^2$.

Let any vector fields $\mathcal{P}^*,\mathcal{Q}^*$, and $\mathcal{Z}^*\in\chi(\Theta^4)$; it is possible to write

$$\begin{aligned}\mathcal{P}^* &= a_1\breve{e}_1+b_1\breve{e}_2+c_1\breve{e}_3+d_1\breve{e}_4,\\ \mathcal{Q}^* &= a_2\breve{e}_1+b_2\breve{e}_2+c_2\breve{e}_3+d_2\breve{e}_4,\\ \mathcal{Z} &= a_3\breve{e}_1+b_3\breve{e}_2+c_3\breve{e}_3+d_3\breve{e}_4,\end{aligned} \tag{47}$$

where $a_i,b_i,c_i,d_i\in\mathfrak{R}^+$, $i=1,2,3,4$, in order for

$$\frac{\beta}{2}\left[\frac{2(c_2-d_2)(a_1a_3+b_1b_3+c_1c_3)-(c_3-d_3)(a_1a_2+b_1b_2+c_1c_2)-(c_1-d_1)(a_2a_3+b_2b_3+c_2c_3)}{a_1a_3+b_1b_3+c_1c_3-d_1d_3}\right]\neq 0. \tag{48}$$

If we consider the 1-form π by $\pi(\mathcal{P}^*)=\ddot{g}(\mathcal{P}^*,(1-\alpha-\beta)\breve{e}_4)$, for any $\mathcal{P}^*\in\chi(\Theta)$ and considering $\psi\in C^\infty(\Theta)$ as

$$\psi=\frac{\beta}{2}\left[\frac{2(c_2-d_2)(a_1a_3+b_1b_3+c_1c_3)-(c_3-d_3)(a_1a_2+b_1b_2+c_1c_2)-(c_1-d_1)(a_2a_3+b_2b_3+c_2c_3)}{a_1a_3+b_1b_3+c_1c_3-d_1d_3}\right], \tag{49}$$

therefore, the relation,

$$\nabla_{\mathcal{P}^*}\mathcal{Q}^*=\psi\mathcal{P}^*+\pi(\mathcal{P}^*)\mathcal{Q}^*, \tag{50}$$

holds. As per these consequences, from (24), we obtain

$$\begin{aligned}\left(\mathfrak{L}_{\mathcal{Q}^*}\ddot{g}\right)(\mathcal{P}^*,\mathcal{Z}^*) &= \ddot{g}(\breve{\nabla}_{\mathcal{P}^*}\mathcal{Q}^*,\mathcal{Z}^*)+\ddot{g}(\mathcal{P}^*,\breve{\nabla}_{\mathcal{Z}^*}\mathcal{Q}^*)\\ &= 2\psi\ddot{g}(\mathcal{P}^*,\mathcal{Z}^*)+\pi(\mathcal{P}^*)\ddot{g}(\mathcal{Q}^*,\mathcal{Z}^*)+\pi(\mathcal{Z}^*)\ddot{g}(\mathcal{Q}^*,\mathcal{P}^*)\\ &\quad+\alpha\left\{2\eta_*(\mathcal{Q}^*)\ddot{g}(\mathcal{P}^*,\mathcal{Z}^*)-\ddot{g}(\mathcal{P}^*,\mathcal{Q}^*)\eta_*(\mathcal{Z}^*)-\ddot{g}(\mathcal{Z}^*,\mathcal{Q}^*)\eta_*(\mathcal{P}^*)\right\}\\ &\quad+\beta\left\{2\eta_*(\mathcal{Q}^*)\ddot{g}(\phi_*\mathcal{P}^*,\mathcal{Z}^*)-\ddot{g}(\phi_*\mathcal{P}^*,\mathcal{Q}^*)\eta_*(\mathcal{Z}^*)-\ddot{g}(\phi_*\mathcal{Z}^*,\mathcal{Q}^*)\eta_*(\mathcal{P}^*)\right\}.\end{aligned} \tag{51}$$

Also, we calculate

$$\begin{aligned}\ddot{g}(\mathcal{P}^*,\mathcal{Z}^*) &= a_1a_3+b_1b_3+c_1c_3-d_1d_3,\\ \ddot{g}(\mathcal{Q}^*,\mathcal{Z}^*) &= a_2a_3+b_2b_3+c_2c_3-d_2d_3,\\ \ddot{g}(\mathcal{P}^*,\mathcal{Q}^*) &= a_1a_2+b_1b_2+c_1c_2-d_1d_2.\end{aligned} \tag{52}$$

Also,

$$\begin{aligned}\pi(\mathcal{P}^*) &= -(1-\alpha-\beta)d_1,\\ \pi(\mathcal{Q}^*) &= -(1-\alpha-\beta)d_2,\\ \pi(\mathcal{Z}^*) &= -(1-\alpha-\beta)d_3.\end{aligned} \tag{53}$$

With the help of (52) and (53), equation (51) reduces

$$\begin{aligned}\frac{1}{2}\left(\mathfrak{L}_{\mathcal{Q}^*}\ddot{g}\right)(\mathcal{P}^*,\mathcal{Z}^*) &= \psi\{a_1a_3+b_1b_3+c_1c_3-d_1d_3\}-\frac{1}{2}(1-\alpha-\beta)\{d_1(a_2a_3\\ &\quad+b_2b_3+c_2c_3-d_2d_3)+d_3(a_1a_2+b_1b_2+c_1c_2-d_1d_2)\}+\frac{\alpha}{2}\{2(c_2-d_2)(a_1a_3\\ &\quad+b_1b_3+c_1c_3-d_1d_3)-(c_3-d_3)(a_1a_2+b_1b_2+c_1c_2-d_1d_2)\\ &\quad-(c_1-d_1)(a_2a_3+b_2b_3+c_2c_3-d_2d_3)\}+\frac{\beta}{2}\{2(c_2-d_2)(a_1a_3+b_1b_3+c_1c_3)\\ &\quad-(c_3-d_3)(a_1a_2+b_1b_2+c_1c_2)-(c_1-d_1)(a_2a_3+b_2b_3+c_2c_3)\}.\end{aligned} \tag{54}$$

Also,

$$(\tilde{\tilde{\delta}} - \hat{\sigma})\ddot{g}(\mathscr{P}^*, \mathscr{X}^*) = \left(6(1-\alpha-\beta)^2\right) - \hat{\sigma})) \cdot \{a_1a_3 + b_1b_3 + c_1c_3 - d_1d_3\}. \tag{55}$$

We consider that $a_1a_3 + b_1b_3 + c_1c_3 - d_1d_3 \neq 0$, $2d_1(a_2a_3 + b_2b_3 + c_2c_3 - d_2d_3) + 2d_3\ (a_1a_2 + b_1b_2 + c_1c_2 - d_1d_2) + d_2(a_1a_3 + b_1b_3 + c_1c_3 - d_1d_3) = 0$, and $-2(c_3 - d_3)(a_1a_2 + b_1b_2 + c_1c_2 - d_1d_2) - 2(c_1 - d_1)(a_2a_3 + b_2b_3 + c_2c_3 - d_2d_3) - 3(c_2 - d_2)\ (a_1a_3 + b_1b_3 + c_1c_3 - d_1d_3) = 0$.

Thus, we get $(\ddot{g}, \mathscr{Q}^*, \hat{\sigma})$ is a Yamabe soliton, that is, $(1/2)\mathfrak{L}_{\mathscr{Q}^*}\ddot{g}(\mathscr{P}^*, \mathscr{X}^*) = (\tilde{\tilde{\delta}} - \hat{\sigma})\ddot{g}(\mathscr{P}^*, \mathscr{X}^*)$ holds, unless

$$\begin{aligned}\hat{\sigma} &= \psi - 6(1-\alpha-\beta)^2 - \frac{1}{4}\{-(1-\alpha-\beta)d_2 + 3\alpha(c_2 - d_2)\}\\ &= \psi - \tilde{\tilde{\delta}} - \frac{1}{4}\{\pi(\mathscr{Q}^*) + 3\alpha\eta_*(\mathscr{Q}^*)\} = \text{constant}.\end{aligned} \tag{56}$$

As a result, the existence of the YS $(\ddot{g}, \mathscr{Q}^*, \hat{\sigma})$ on a 4-dimensional $\mathscr{LP}$-Sasakian manifold admitting a GSMC-(α, β) with potential vector field $\mathscr{Q}^*$ as torse-forming is justified. Then, Theorems 1 and 2 are verified.

Data Availability

No data were used to support this study.

Conflicts of Interest

There are no conflicts of interest regarding the publication of this article.

References

[1] R. S. Hamilton, "The Ricci flow on surfaces," *Contemporary Mathematics*, vol. 71, pp. 237–262, 1998.

[2] L. F. Di Cerbo and M. M. Disconzi, "Yamabe solitons, determinant of the laplacian and the uniformization theorem for riemann surfaces," *Letters in Mathematical Physics*, vol. 83, no. 1, pp. 13–18, 2008.

[3] B.-Y. Chen and S. Deshmukh, "Yamabe and quasi-Yamabe solitons on Euclidean submanifolds," *Mediterranean Journal of Mathematics*, vol. 15, no. 5, p. 194, 2018.

[4] G. Huang and H. Li, "On a classification of the quasi Yamabe gradient solitons," *Methods and Applications of Analysis*, vol. 21, no. 3, pp. 379–390, 2014.

[5] B. L. Neto and H. P. De Oliveira, "Generalized quasi Yamabe gradient solitons," *Differential Geometry and its Applications*, vol. 49, pp. 167–175, 2016.

[6] X. Chen, "Almost quasi-Yamabe solitons on almost cosymplectic manifolds," *International Journal of Geometric Methods in Modern Physics*, vol. 17, no. 5, p. 2050070, 2020.

[7] V. Pirhadi and A. Razavi, "On the almost quasi-Yamabe solitons," *International Journal of Geometric Methods in Modern Physics*, vol. 14, p. 1750161, 2017.

[8] K. De and U. C. De, "Almost quasi-Yamabe solitons and gradient almost quasi-yamabe solitons in paracontact geometry," *Quaestiones Mathematicae*, pp. 1–12, 2020.

[9] S. Ghosh, U. C. De, and A. Yildiz, "A note on almost quasi Yamabe solitons and gradient almost quasi Yamabe solitons," *Hacettepe Journal of Mathematics and Statistics*, vol. 50, no. 3, pp. 770–777, 2021.

[10] J.-B. Jun and M. Danish Siddiqi, "Almost quasi-Yamabe soliton on $(\mathrm{LCS})_n$-manifolds," *Honam Mathematical Journal*, vol. 42, no. 3, pp. 521–536, 2020.

[11] K. De and U. C. De, "Almost quasi-Yamabe and gradient almost quasi-Yamabe soliton on quasi-Sasakian manifolds," *Acta Mathematica Universitatis Comenianae*, vol. 90, no. 3, pp. 333–340, 2021.

[12] S. K. Yadav, O. Bahadir, and S. K. Chaubey, "Almost Yamabe solitons on LP-Sasakian manifolds with generalized symmetric metric connection of type (α, β)," *Balkan Journal of Geometry and its Applications*, vol. 25, no. 2, pp. 124–139, 2020.

[13] A. M. Blagaa, "A note onWarped product almost quasi-yamabe solitons," *Filomat*, vol. 33, no. 7, pp. 2009–2016, 2019.

[14] K. Yano, "On torse-forming direction in a Riemannian space," *Proceedings of the Imperial Academy Tokyo*, vol. 20, pp. 340–345, 1944.

[15] K. Yano, "On semi-symmetric metric connection," *Revue Roumaine de Mathématique Pures et Appliquées*, vol. 15, pp. 1579–1586, 1970.

[16] K. Yano and B. Y. Chen, "On concurrent vector fields of immersed manifolds," *Kodai Mathematical Seminar Reports*, vol. 23, pp. 343–350, 1971.

[17] B.-Y. Chen, "Classification of torqued vector fields and its applications to Ricci solitons," *Kragujevac Journal of Mathematics*, vol. 41, no. 2, pp. 239–250, 2017.

[18] K. Matsumoto, "On lorentzian paracontact manifolds," *Bulletin of the Yamagata University. Natural science*, vol. 12, pp. 151–156, 1989.

[19] I. Mihai and R. Rosca, *On Lorentzian P-Sasakian Manifolds, Clssical Analysis*, pp. 155–169, World Scientific Publishing, Signapore, 1992.

[20] I. Mihai, A. A. Shaikh, and U. C. De, "On Lorentzian para-Sasakian manifolds, rendiconti del seminario matematico di Messina, serie II," *IOSR Journal of Mathematics*, vol. 9, no. 6, 1999.

[21] K. Matsumoto and I. Mihai, "On a certain transformation in a Lorentzian para Sasakian manifold," *Tensor (N.S.)*, vol. 47, pp. 189–197, 1988.

[22] O. Bahadir and S. K. Chaubey, "Some result on *LP*-Sasakian manifolds with generalized symmetric metric connection," *Honam Mathematical Journal*, vol. 42, no. 3, pp. 461–476, 2020.

[23] U. C. De and A. K. Sengupta, "*CR*-submanifolds of a Lorentzian para-Sasakian manifold," *Bulletin of the Malaysian Mathematical Sciences Society*, vol. 23, pp. 99–106, 2000.