Research Article

The Leading Coefficients for an Affine Weyl Group of Type $\widetilde{G}_2$: The Lowest Two-Sided Cell Case

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This study focusses on the leading coefficients $\mu(u, w)$ of the Kazhdan–Lusztig polynomials $P_{u,w}$ for the lowest cell c_0 of an affine Weyl group of type $\widetilde{G}_2$ and gives an estimation $\mu(u, w) \leq 3$ for $u, w \in c_0$.

1. Introduction

The Hecke algebra $\mathcal{H}$ of the Coxeter group W over $\mathbb{Z}[q^{(1/2)}, q^{-(1/2)}]$ is a free $\mathcal{A}$ -module and has a basis $\{T_w\}_{w \in \widetilde{W}}$. Its multiplication is defined by the relations $(T_s - q)(T_s + 1) = 0$ if $s \in S$ and $T_w T_u = T_{wu}$ if $l(wu) = l(w) + l(u)$. David Kazhdan and George Lusztig proved that there exist a unique basis C_w in $\mathcal{H}$, such that $C_w = q^{-(l(w)/2)} \sum_{u \leq w} P_{u,w} T_u$ and $\overline{C_w} = C_w$ [1] for each $w \in W$, where $P_{u,w} \in \mathbb{Z}[q]$ of degree less than or equal to $(1/2)(l(w) - l(u) - 1)$ for $u \leq w$ and $P_{w,w} = 1$ in [1]. It is called Kazhdan–Lusztig basis. The corresponding polynomial $P_{u,w}$ is called Kazhdan–Lusztig polynomial.

The leading coefficient $\mu(u, w)$ of $P_{u,w}$ is the coefficient of the term $q^{(1/2)(l(w) - l(u) - 1)}$ in $P_{u,w}$. It plays an important role in Kazhdan–Lusztig theory, Lie theory, and representation theory. It can be used to estimate the dimensions of the first extension groups for irreducible modules of algebraic groups and finite groups of Lie type [2]. By the recursive formula of Kazhdan–Lusztig polynomials in [1], we see that some $\mu(u, w)$ play crucial roles in understanding these polynomials.

However, there are few results for the leading coefficients in general. In [3], in order to understand the local finite problem of $\widetilde{B}_2$ type, George Lusztig computed some special leading coefficients for the affine Weyl group of this type. In

[4], for an affine Weyl group of type A_5 , some nontrivial leading coefficients are found. Timothy McLarnan and Gregory Warrington have shown that $\mu(u, w)$ can be greater than 1 for a symmetric group which is a counter example for $(0, 1)$ conjecture [5]. In [6], Nanhua Xi showed that when $u \leq w$ and $a(u) \leq a(w)$, $\mu(u, w) \leq 1$ if W is a symmetric group or an affine Weyl group of type A_n . In [2], Leonard Scott and Nanhua Xi showed that the leading coefficient $\mu(u, w)$ of Kazhdan–Lusztig polynomial $P_{u,w}$ for an affine Weyl group of type A_n is $n + 2$ if $n \geq 4$. Recently, Brant Jones and Richard Green [7, 8] showed that the coefficients are ≤ 1 for some special pairs of elements in the symmetric group and other Weyl groups. In [9, 10], Liping Wang computed most of the leading coefficients for an affine Weyl group of type $\widetilde{B}_2$ and $\widetilde{A}_2$.

In order to use the conclusions in [5] to calculate $\mu(u, w)$ for $u, w \in c_0$, we need to compute $C_x C_y$ for some $x, y \in c_0$. It is difficult to compute these multiplications $C_x C_y$, especially for $l(x), l(y)$ large enough, since we need to compute the amount of $\mu(u, v)$ for $l(u), l(v)$ large enough. The calculation is so complicated that we cannot do it manually. Luckily, it can be carried out by a powerful computer. Although there are some programs to compute $P_{u,v}$, such as Mark Goresky's Pascal and Fokko du Cloux's Coxeter 3.0, these algorithms are not efficient for our problem. So we work out a new algorithm to compute $P_{u,v}$ and $C_s C_y$ for the affine Weyl

group of type $\widetilde{G}_2$. This algorithm is realized by visual C++ program, and this program should run on the main types of workstation under Windows. By this algorithm, we take some time to get all $C_s C_y$ which we need. Moreover, we obtain all $\delta_{x,y,z}$ defined in [11] to prove our main result. We have a bound of $\mu(u, v)$ for $u, w \in c_0$ by some conclusions of representation of algebraic groups [7].

We now outline the structure of this study. In Section 2, we recall the definitions and properties Kazhdan–Lusztig basis, Kazhdan–Lusztig polynomial, Kazhdan–Lusztig cell, and Kazhdan–Lusztig coefficient. We compute all elements of c_0 in Section 2.3 by the method mentioned in [2]. In Section 3, we give an explicit description of the algorithm which is used to compute Kazhdan–Lusztig polynomials and multiplications of Kazhdan–Lusztig basis. In Section 4, we compute all multiplications of Kazhdan–Lusztig basis which we need. From these multiplications, we get the main theorem of this study that $\mu(u, v) \leq 3$ for $u, w \in c_0$.

2. Preliminaries

In this section, we review some basic facts for Kazhdan–Lusztig theory, properties of Kazhdan–Lusztig polynomial and Kazhdan–Lusztig coefficient, lowest two-sided cell, and the formula of Kazhdan–Lusztig coefficient for the lowest two-sided cell.

2.1. Basic Definitions and Conventions. Let R be a root system, and W_0 be its Weyl group. The semidirect product $W = W_0 \ltimes \Lambda_r$ is the affine Weyl group, and $\widetilde{W} = W_0 \ltimes \Lambda$ is the extended affine Weyl group, where Λ is the weight lattice of R , and $\Lambda_r = \mathbb{Z}R$ is its corresponding root lattice.

There is an abelian subgroup Ω of $\widetilde{W}$, such that $\omega S = S\omega$ for any $\omega \in \Omega$, where S is the set of simple reflections of W , and $\widetilde{W} = W \ltimes \Omega$. The length function l of W and the Bruhat order $\leq$ on W can be extended to $\widetilde{W}$ by setting $l(\omega w) = l(w)$ and $\omega w \leq \omega' u$ if and only if $\omega = \omega'$ and $w \leq u$, where ω, ω' are in Ω and w, u are in W . The strict Bruhat order is similar defined as above.

Let $\mathcal{H}$ be the Hecke algebra of $(\widetilde{W}, S)$ over Laurent polynomial ring $\mathcal{A} = \mathbb{Z}[q^{(1/2)}, q^{-(1/2)}]$ (q is an indeterminate). $\mathcal{H}$ is a free $\mathcal{A}$ -module and has the basis $\{T_w\}_{w \in \widetilde{W}}$. Its multiplication is defined by the relations $(T_s - q)(T_s + 1) = 0$ if $s \in S$ and $T_w T_u = T_{wu}$ if $l(wu) = l(w) + l(u)$. The combination $C_w = q^{-(l(w)/2)} \sum_{u \leq w} P_{u,w} T_u$, $w \in \widetilde{W}$ be its Kazhdan–Lusztig basis, where $P_{u,w} \in \mathbb{Z}[q](u \leq w)$ are the Kazhdan–Lusztig polynomials (here C_w is just C'_w in [1]). The degree of $P_{u,w}$ is less than or equal to $(1/2)(l(w) - l(u) - 1)$ if $u < w$ and $P_{w,w} = 1$. The subalgebra $\mathcal{H}'$ of $\mathcal{H}$ generated by all T_s ($s \in S$) is the Hecke algebra of the Coxeter system (W, S) . For ω in Ω and w in W , we have $C_{\omega w} = T_\omega C_w$. Thus, we have $P_{\omega u, \omega w} = P_{u,w}$ and $P_{\omega' u, \omega w} = 0$ for any different ω', ω in Ω and u, w in W . It is known that the coefficients of these polynomials are all nonnegative [12].

The Kazhdan–Lusztig polynomials can be written as $P_{u,w} = \mu(u, w)q^{(1/2)(l(w)-l(u)-1)}$ + lower degree terms. The coefficient $\mu(u, w)$ is very interesting, which can be seen even from the recursive formula for Kazhdan–Lusztig

polynomials [1]. We call $\mu(u, w)$ the leading coefficient of $P_{u,w}$. Write $u < w$ if $u \leq w$ and $\mu(u, w) \neq 0$. Define $\tilde{\mu}(u, w) = \mu(u, w)$ if $u \leq w$, $(-1)^{l(w)-l(u)} < 0$ and $\tilde{\mu}(u, w) = \mu(w, u)$ if $w \leq u$.

For $\mu(u, w)$, there are the following conclusions [1].

- (1) Let $u, w \in W, s \in S$, such that $u < w, su > u, sw < w$. Then, $u < w$ if and only if $w = su$. Moreover, $\mu(u, w) = 1$ in this case.
- (2) Let $u, w \in W, s \in S$, such that $u < w, us > u, ws < w$. Then, $u < w$ if and only if $w = us$. Moreover, $\mu(u, w) = 1$ in this case.

The elements C_w have the following properties [1]. For $s \in S$, we have

$$\begin{aligned} C_s C_w &= \begin{cases} (q^{(1/2)} + q^{-(1/2)})C_w, & sw \leq w, \\ C_{sw} + \sum_{y < w, sy \leq y} C_y, & sw \geq w, \end{cases} \\ C_w C_s &= \begin{cases} (q^{(1/2)} + q^{-(1/2)})C_w, & ws \leq w, \\ C_{ws} + \sum_{y < w, ys \leq y} C_y, & ws \geq w. \end{cases} \end{aligned} \quad (1)$$

Given $x, y \in W$, $C_x C_y = \sum_{z \in W} h_{x,y,z} C_z$, $h_{x,y,z} \in \mathcal{A}$. For any $z \in W$, define $a(z) = \min\{i \in \mathbb{N} | q^{-(i/2)} h_{x,y,z} \in \mathbb{Z}[q^{-(1/2)}]\}$, for any $x, y \in W$. Denote $h_{x,y,z} = \gamma_{x,y,z} q^{a(z)/2} + \delta_{x,y,z} q^{(a(z)-1)/2}$ + lower degree terms, where $\gamma_{x,y,z}$ and $\delta_{x,y,z}$ are all nonnegative.

2.2. The Lowest Two-Sided Cell. It is known that $c_0 = \{w \in \widetilde{W} | a(w) = l(w_0)\}$ is a two-sided cell, which is the lowest one with respect to the partial order $\leq_{LR}$, where w_0 is the longest element of W_0 [13–15]. The set c_0 is called the lowest two-sided cell of $\widetilde{W}$.

In [2], Leonard Scott and Nanhua Xi gave a description of c_0 . Let R^+ (respectively, R^- and Δ) be the set of positive (respectively, negative and simple) roots in the root system R of W_0 . The set of dominant weights Λ^+ is the set $\{z \in \Lambda | l(zw_0) = l(z) + l(w_0)\}$. For each simple root α , s_α is the corresponding simple reflection in W_0 and x_α is the corresponding fundamental weight. For each $w \in W_0$, set $d_w = w \prod_{\{\alpha \in \Delta, w(\alpha) \in R^-\}} x_\alpha$. Then,

$$c_0 = \{d_w z w_0 d_u^{-1} | w, u \in W_0, z \in \Lambda^+\}. \quad (2)$$

Moreover, the set $c'_{0,w} = \{d_w z w_0 d_u^{-1} | u \in W_0, z \in \Lambda^+\}$ is a right cell of $\widetilde{W}$ and $c_{0,u} = \{d_w z w_0 d_u^{-1} | w \in W_0, z \in \Lambda^+\}$ is a left cell of $\widetilde{W}$. The distinguished involutions of c_0 are $d_w w_0 d_w^{-1}$, $w \in W_0$.

For $x \in \Lambda^+$, let $V(x)$ be a rational irreducible G -module of the highest weight x and S_x be the corresponding element defined in [16]. For any $x \in \Lambda$, Bernstein Joseph gave an element $\theta_x \in \mathcal{H}$, defined as

$$\theta_x = \left(q^{-l(x')} T_{x'} \right) \left(q^{-l(x'')} T_{x''} \right)^{-1}, \quad (3)$$

where $x', x'' \in \Lambda^+$ satisfies $x = x' - x''$. For $x \in \Lambda^+$, $(x)W_0$ is the W_0 -orbit in Λ which contains x . Set $z_x = \sum_{x' \in (x)W_0} \theta_{x'}$. For $x \in \Lambda^+$, define

$$S_x = \sum_{\{x' \in \Lambda^+, x' \leq x\}} d_{x'}(x) z_{x'}, \quad (4)$$

where $d_{x'}(x)$ is the multiplicity of x' in $V(x)$. Then, $\{S_x | x \in \Lambda^+\}$ is the $\mathcal{A}$ -base of the center of $\mathcal{H}$.

For every $w \in W_0$, define two cell-related combinations:

$$\begin{aligned} E_{d_w} &= q^{-(l(d_w)/2)} \sum_{\{y \leq d_w l(yw_0) = l(y) + l(w_0)\}} P_{yw_0, d_w w_0} T_y, \\ F_{d_w} &= q^{-(l(d_w)/2)} \sum_{\{y \leq d_w l(yw_0) = l(y) + l(w_0)\}} P_{yw_0, d_w w_0} T_{y^{-1}}, \end{aligned} \quad (5)$$

Then, the following facts were concluded in [16]:

- (a) $E_{d_w} S_x C_{w_0} F_{d_u} = C_{d_w x w_0 d_u^{-1}}$ for any $w, u \in W$ and $x \in \Lambda^+$
- (b) $S_x S_y = \sum_{z \in \Lambda^+} m_{x,y,z} S_z$ for any $x, y \in \Lambda^+$. Here, $m_{x,y,z}$ is defined to be the multiplicity of $V(z)$ in the tensor product $V(x) \otimes V(y)$;
- (c) For any $y, w \in c_0$, if $\mu(y, w) \neq 0$, then we have $y \sim_L w$ and $y \not\sim_R w$ or $y \sim_R w$ and $y \not\sim_L w$;
- (d) Let $w, w', u \in W_0$, $y, z \in \Lambda^+$ and let $d_w y w_0 d_u^{-1}$, $d_{w'} z w_0 d_u^{-1}$ be elements of the left cell $c_{0,u}$ of the extended affine Weyl group $\tilde{W}$. Then, $\mu(d_w y w_0 d_u^{-1}, d_{w'} z w_0 d_u^{-1}) = \mu(d_w y w_0, d_{w'} z w_0)$;
- (e) Let $x = d_u x w_0$, $y = d_{u'} x' w_0$, where $u, u', v \in W_0$ and $x, x' \in \Lambda^+$. Then, the set $\{z \in W | h_{w_0 d_u^{-1}, d_{u'} w_0, z} \neq 0\}$ is finite, and we know that $z = z_1 w_0$ for some $z_1 \in \Lambda^+$ by $h_{w_0 d_u^{-1}, d_{u'} w_0, z} \neq 0$. Moreover, $\mu(y, w) = \sum_{z \in \Lambda^+} m_{x^*, x, z^*} \delta_{w_0 d_u^{-1}, d_{u'} w_0, z w_0}$, where $x^* = w_0 x^{-1} w_0$.

2.3. The Lowest Two-Sided Cell of $\tilde{G}_2$. For the affine Weyl group of type $\tilde{G}_2$, $W_0 = \{e, s, t, st, ts, sts, tst, stst, tsts, ststs, tstst, ststst\}$, and the simple reflection set $\Delta = \{\alpha, \beta\}$.

The fundamental weights of $\tilde{G}_2$ are $x_\alpha = 2\alpha + \beta = ststsr$ and $x_\beta = 3\alpha + 2\beta = (tstsr)^2$ in W .

As above subsection mentioned, for each $w \in W_0$, $d_w = w \prod_{\{\alpha \in \Delta, w(\alpha) \in R^-\}} x_\alpha$. Then, we get the following results.

$$d_e = e,$$

$$d_s = s x_\alpha = s(ststsr) = tsts,$$

$$d_t = t x_\beta = s(tstsr)^2 = stststsr,$$

$$d_{st} = st x_\beta = st(tstsr)^2 = tstrstsr,$$

$$d_{ts} = nmtsx_\alpha = ts(ststsr) = stsr,$$

$$d_{stst} = nmstsx_\alpha = stst(ststsr) = tstr,$$

$$d_{tst} = ntstx_\beta = tst(tstsr)^2 = strstsr,$$

$$d_{stst} = nststx_\beta = stst(tstsr)^2 = trstsr,$$

$$d_{tsts} = nmtstsx_\alpha = tsts(ststsr) = sr,$$

$$d_{ststs} = nststsx_\alpha = ststs(ststsr) = r,$$

$$d_{tstst} = tststx_\beta = tstst(tstsr)^2 = rstsr,$$

$$d_{ststst} = stststx_\alpha x_\beta = rstststsr.$$

From these results and $c_0 = \{d_w z w_0 d_u^{-1} | w, u \in W_0, z \in \Lambda^+\}$, the two-sided cell c_0 consists of the following elements:

$$v(i, j, a, b) = v_i ststst(x)^a (y)^b v_j^{-1}, \quad (7)$$

where $1 \leq i, j \leq 12$, $a, b \in \mathbb{N}$, and $v_1 = e, v_2 = r, v_3 = sr, v_4 = tsr, v_5 = stsr, v_6 = tsts, v_7 = rstsr, v_8 = trstsr, v_9 = strstsr, v_{10} = tstrstsr, v_{11} = ststrstsr, v_{12} = rststrstsr, x = rststs, y = (rstst)^2$.

3. Algorithm for Computing $C_s C_w$

We describe our algorithm in this section in order to compute Kazhdan–Lusztig polynomials for the affine Weyl group and some special multiplications of Kazhdan–Lusztig basis. Before doing this, we give some lemmas for preparation.

The following lemma makes sure whether two subwords w_1 and w_2 is equal in $\tilde{W}$.

Lemma 1. Given a point $v_0 \in A_0^+$, where $A_0^+ = \{x \in \mathfrak{S}^* | \langle x, \check{\alpha} \rangle > 0, \alpha \in \Pi, \langle x, \check{\alpha}_0 \rangle < 1\}$, then $w_1 v_0 = w_2 v_0$ if and only if $w_1 = w_2$.

Proof. If $w_1 = w_2$, it is easy to see that $w_1 v_0 = w_2 v_0$. If $w_1 v_0 = w_2 v_0$, then $w_1 v_0 = w_2 v_0 \in w_1 A^+ \cap w_2 A^+ \neq \emptyset$ and

$w_1 A^+ = w_2 A^+$. Since W_a acts on all alcoves simply transitively, we get $w_1 = w_2$. $\square$

Lemma 2 (see [17]). *Let (W, S) be a Coxeter group and $w \in W$. Fix any reduced expression $w = s_1, \dots, s_n$. Then, $v \leq w$ if and only if there exists $1 \leq j_1 < j_2 < \dots < j_p \leq n$, such that $v = s_1, \dots, \widehat{s}_{j_1}, \dots, \widehat{s}_{j_p}, \dots, s_n$. In fact, for $v \leq w$, $j = (j_1 < j_2 < \dots < j_p)$ can be choosed, such that the above expression is a reduced expression of v .*

The following lemma makes sure whether w is a reduced form.

Lemma 3 (see [18]). *Assume that $w = s_1, s_2, \dots, s_q$ with $s_i \in S$. We can find a subsequence $i_1 < i_2 < \dots < i_r$ of $1, 2, \dots, r$, such that $w = s_{i_1}, s_{i_2}, \dots, s_{i_r}$ is a reduced expression.*

The following lemma increases the speed of the algorithm.

Lemma 4. *Let $x, y \in W$ and $l(x) - l(y) \geq 3$. If there is a $s \in S$, such that $sx < x$ and $sy > y$ (respectively, $xs < x$ and $ys > y$), then $\mu(x, y) = 0$.*

Proof. If there is $s \in S$, such that $sx < x$ and $sy > y$, then $P_{y,x} = P_{sy,x}$. Since $l(x) - l(y) \geq 3$, the degree of $P_{y,x}$ is equal to the degree of $P_{sy,x} \leq (l(x) - l(sy) - 1)/2 = (l(x) - l(y) - 2)/2 < (l(x) - l(y) - 1)/2$, then $\mu(x, y) = 0$.

We describe our algorithm as follows.

First step: all subwords of w are obtained by following two cases

$$\begin{aligned} & \{e, s_0, s_1, s_2, s_0 s_1, s_0 s_2, s_1 s_0, s_1 s_2, s_2 s_1, s_0 s_1 s_0, s_0 s_2 s_1, s_1 s_2 s_1, s_1 s_2 s_0, s_2 s_1 s_0, \\ & s_0 s_1 s_2 s_1, s_0 s_1 s_2 s_0, s_0 s_2 s_1 s_0, s_1 s_2 s_1 s_0, s_0 s_1 s_2 s_1 s_0, s_0 s_0, s_1 s_0, s_2 s_0, s_1 s_2 s_0, s_2 s_1 s_0, s_0 s_1 s_0 s_0, s_0 s_2 s_1 s_0, \\ & s_1 s_2 s_1 s_0, s_1 s_2 s_0 s_0, s_2 s_1 s_0 s_0, s_0 s_1 s_2 s_1 s_0, s_0 s_1 s_2 s_0 s_0, s_0 s_2 s_1 s_0 s_0, s_1 s_2 s_1 s_0 s_0, s_0 s_1 s_2 s_1 s_0 s_0, s_0 s_2, s_1 s_2, s_2 s_2, \\ & s_0 s_1 s_2, s_0 s_2 s_2, s_1 s_0 s_2, s_1 s_2 s_2, s_2 s_1 s_2, s_0 s_1 s_0 s_2, s_0 s_2 s_1 s_2, s_1 s_2 s_1 s_2, s_1 s_2 s_0 s_2, s_2 s_1 s_0 s_2, s_0 s_1 s_2 s_1 s_2, \\ & s_0 s_1 s_2 s_0 s_2, s_0 s_2 s_1 s_0 s_2, s_1 s_2 s_1 s_0 s_2, s_0 s_1 s_2 s_1 s_0 s_2, s_1 s_2, s_0 s_1 s_2, s_1 s_1 s_2, s_2 s_1 s_2, s_0 s_1 s_1 s_2, s_0 s_2 s_1 s_2, s_1 s_0 s_1 s_2, \\ & s_1 s_2 s_1 s_2, s_2 s_1 s_1 s_2, s_0 s_1 s_0 s_1 s_2, s_0 s_2 s_1 s_1 s_2, s_1 s_2 s_1 s_1 s_2, s_1 s_2 s_0 s_1 s_2, s_2 s_1 s_0 s_1 s_2, s_0 s_1 s_2 s_1 s_1 s_2, s_0 s_1 s_2 s_0 s_1 s_2, \\ & s_0 s_2 s_1 s_0 s_1 s_2, s_1 s_2 s_1 s_0 s_1 s_2, s_0 s_1 s_2 s_1 s_0 s_1 s_2, s_0 s_1 s_0, s_0 s_2 s_0, s_1 s_0 s_0\}. \end{aligned} \quad (10)$$

Second step: get all unique and reduced subwords of w . All subwords of w are denoted by M . For any $m \in M$, $|m|$ is the length of m . Then, an increasing order of all elements of M is given with respect to $|\cdot|$. The elements of M can be reordered as $w_1^1, w_1^2, \dots, w_1^r$, where $|w_1^1| \leq |w_1^2| \leq \dots \leq |w_1^r|$. There is a correspondence sequence $w_1^1 v_0, w_1^2 v_0, \dots, w_1^r v_0$, where v_0 is a rational point of A_0^+ . By the lemma above, we delete w_1^i if there exists $j < i$, such that $w_1^j v_0 = w_1^i v_0$. Then, all unique and reduced subwords of w are obtained by this way.

Third step: compute Kazhdan–Lusztig polynomial $P_{x,y}$ for $x, y \in W$ by following cases

- (1) If the length of w is small ($l(w) \leq 10$), all subwords are obtained by the algorithm of binary tree. For example, in $\widetilde{G}_2$, if $w = s_0 s_1 s_2 s_1 s_0$, then

$$\begin{aligned} A = \{ & e, s_0, s_1, s_2, s_0 s_1, s_0 s_2, s_0 s_0, s_1 s_0, s_1 s_2, \\ & s_1 s_1, s_2 s_1, s_2 s_0, s_0 s_1 s_2, s_0 s_1 s_1, s_0 s_1 s_0, s_0 s_2 s_1, \\ & s_0 s_2 s_0, s_1 s_2 s_1, s_1 s_2 s_0, s_1 s_1 s_0, s_2 s_1 s_0, s_0 s_1 s_2 s_1, \\ & s_0 s_1 s_2 s_0, s_0 s_1 s_1 s_0, s_0 s_2 s_1 s_0, s_1 s_2 s_1 s_0, s_0 s_1 s_2 s_1 s_0 \} \end{aligned} \quad (8)$$

is the set of all subwords of w .

- (2) If the length of w is large enough ($l(w) > 10$), w is split into two smaller subwords w_1, w_2 . We obtain the set A (respectively, B) of all subwords of w_1 (respectively, w_2) by the algorithm of binary tree. The generated relations of the Coxeter group can be used to do some preliminary reduces for A and B . The results are denoted by A' and B' . All subwords of w are given by gluing the elements of A' and B' . For example, in $\widetilde{G}_2$, $w = s_0 s_1 s_2 s_1 s_0 s_1 s_2 s_1 s_2 s_1$. We split w into $w_1 = s_0 s_1 s_2 s_1 s_0$ and $w_2 = s_1 s_2$. Then,

$$\begin{aligned} A' = \{ & e, s_0, s_1, s_2, s_0 s_1, s_0 s_2, s_1 s_0, s_1 s_2, s_2 s_1, s_0 s_1 s_0, \\ & s_0 s_2 s_1, s_1 s_2 s_1, s_1 s_2 s_0, s_2 s_1 s_0, s_0 s_1 s_2 s_1, s_0 s_1 s_2 s_0, \\ & s_0 s_2 s_1 s_0, s_1 s_2 s_1 s_0, s_0 s_1 s_2 s_1 s_0 \}, \\ B' = \{ & e, s_1, s_2, s_1 s_2 \}. \end{aligned} \quad (9)$$

We get all subwords of w ,

- (i) If $l(y) < l(x)$, $P_{x,y} = 0$
- (ii) For $l(y) = l(x)$, if $yv_0 = xv_0$, then $y = x$, $P_{x,y} = 1$. Otherwise, $P_{x,y} = 0$.
- (iii) For $l(x) < l(y)$, $y = sw$ and $l(y) = l(w) + 1$. We use the following recursive formula to compute the corresponding Kazhdan–Lusztig polynomial:
 - (ii) For $l(y) = l(x)$, if $yv_0 = xv_0$, then $y = x$, $P_{x,y} = 1$. Otherwise, $P_{x,y} = 0$.
 - (iii) For $l(x) < l(y)$, $y = sw$ and $l(y) = l(w) + 1$. We use the following recursive formula to compute the corresponding Kazhdan–Lusztig polynomial:

Where $a = 1$ if $sx < x$, $a = 0$ if $sx > x$.

By Lemma 2, if $xv_0 = x_i^1 v_0$, where x_i^1 is a subword of sx , then $sx > x$ and $a = 0$; otherwise, $sx < x$, and sx is not a reduced form; then, $a = 1$.

In our algorithm, we compute $P_{x,y}$ for reduced words x, y . So we get the reduced form of sx by following proceeding. By the second step, we get the reordered subwords set M and the corresponding sequence $x_1^1 v_0, x_2^1 v_0, \dots, x_n^1 v_0$ of sx . The by Lemma 3, the reduced form of sx is x_i^1 , where x_i^1 is the first subword of sx that satisfies the condition $x_i^1 v_0 = sxv_0$.

The z in the above recursive formula is obtained by following proceedings.

- (1) All unique reduced subwords of w are obtained by the second step. The results are denoted by the set A ;
- (2) We search all $u \in A$ satisfying the condition $l(w) - l(u)$ is odd. The results are denoted by the set B ;
- (3) We search all $u_1 \in B$ satisfying the condition $su_1 v_0 = u_1' v_0$, where u_1' is a reduced subword of u_1 . The results are denoted by the set C ;
- (4) We search all $u_2 \in C$ satisfying the condition $xv_0 = u_2' v_0$, where u_2' is a reduced subword of u_2 . The results are denoted by the set D ;
- (5) We search all $u_3 \in D$ which do not satisfy the condition Lemma 4. The results are denoted by the set E ;
- (6) We search all $u_4 \in E$ satisfying the condition that the term $\mu(w, w)q^{(l(w)-l(w_i)+1)/2}$ of $P_{w_i, w}$ is nonzero. The results are denoted by the set Z . The elements of Z are all z in the above recursive formula.

If $P_{sx, w}$, $P_{x, w}$, $P_{z, w}$, and $P_{x, z}$ are known, $P_{x, y}$ are computed by the above induction formula. Otherwise, we compute $P_{x, y}$ through the recursive algorithm.

Fourth step: compute $C_s C_x$ for $s \in S, x \in W$ by following induction formula:

$$C_{sw} = C_s C_w - \sum_{y < w, sy \leq y} \mu(y, w) C_y, \quad sw \geq w. \quad (12)$$

□

4. Computation $\mu(y, w)$ for $y, w \in c_0$

In this section, we use the algorithm mentioned in Section 3 to calculate $C_s C_x$ for some $s \in S, x \in W$. These multiplications are used to manually compute $C_{w_0 d_u^{-1} d_{u'}^{-1} w_0}$ for $u, u' \in W_0$, in order to get $\delta_{w_0 d_u^{-1} d_{u'}^{-1} w_0, z_1 w_0}$, which is the coefficient of the term $q^{(1/2)(a(z_1 w_0)-1)}$ of $h_{w_0 d_u^{-1} d_{u'}^{-1} w_0, z_1 w_0}$. By (e) in Section 2.2, $\mu(y, w) = \sum_{z \in \Lambda^+} m_{x^*, x'} \delta_{w_0 d_u^{-1} d_{u'}^{-1} w_0, zw_0}$, where $x^* = w_0 x^{-1} w_0$. Our main result Theorem 1 is obtained by Lemmas 5 and 6.

We use an algebra antiautomorphism of $\mathcal{H}$ which is defined by George Lusztig in [11] to prove the following lemma.

Lemma 5. Let $x = d_u x w_0, y = d_{u'} x' w_0$, where $u, u', v \in W_0$ and x, x' belong to Λ^+ . Then, $C_x C_y = C_{y^{-1} C_x^{-1}}$.

Proof. There is a unique algebra antiautomorphism of H which is denoted by φ , such that $\varphi(T_x) = T_{x^{-1}}$, for all $x \in W$; it maps Kazhdan–Lusztig basis element $C_x \rightarrow C_{x^{-1}}$. For any $x, y \in W$, $\varphi(C_x C_y) = \varphi(C_y) \varphi(C_x) = C_{y^{-1}} C_{x^{-1}}$, that is, $\varphi(C_x C_y) = \varphi(\sum_{z \in W} h_{x, y, z} C_z) = \sum_{z \in W} h_{x, y, z} C_{z^{-1}} = \sum_{z \in W} h_{y^{-1}, x^{-1}, z} C_z$, and we get $h_{x, y, z} = h_{y^{-1}, x^{-1}, z^{-1}}$. By (d) in Section 2.2, we know that $h_{x, y, z} \neq 0, z = z_1 w_0$, for some $z_1 \in \Lambda^+$. Then, if $h_{x, y, z} \neq 0$, the relation $h_{x, y, z} = h_{x, y, z_1 w_0} = h_{y^{-1}, x^{-1}, z^{-1}} = h_{y^{-1}, x^{-1}, w_0 z^{-1}}$ holds. In the case of $\bar{G}_2$, we know that $z_1 w_0 = w_0 z^{-1}$, that is, $h_{x, y, z} = h_{y^{-1}, x^{-1}, z}$, for all $z \in W$.

In the following, for any $u, u' \in W_0$, all $C_{w_0 d_u^{-1} d_{u'}^{-1} w_0}$ for $u, u' \in W_0$ are computed to obtain $\delta_{w_0 d_u^{-1} d_{u'}^{-1} w_0, z_1 w_0}$, which is the coefficient of the term $q^{(1/2)(a(z_1 w_0)-1)}$ of the multiplication polynomial $h_{w_0 d_u^{-1} d_{u'}^{-1} w_0, z_1 w_0}$. □

Proposition 1. For any $u, u' \in W_0$, if $\delta_{w_0 d_u^{-1} d_{u'}^{-1} w_0, z_1 w_0}$ is not equal to zero, we must have $\delta_{w_0 d_u^{-1} d_{u'}^{-1} w_0, z_1 w_0} = 1$ and $z_1 \in \{0, x_\alpha, x_\beta\}$, where $x_\alpha, x_\beta \in \Lambda^+$ are fundamental weights.

Proof. We compute $C_x C_y$ through the following steps.

- (a) We use the formula $C_w C_s = C_x + \sum_{u < w, us < u} \mu(u, w) C_u$ for $x = ws$ and $l(x) = l(w) + 1$ [1] to get a decomposition of C_x
- (b) For the reflection s in step (a), we compute $C_s C_y$ by the formula $C_s C_y = C_{sy} + \sum_{v < y, sv < y} \mu(v, y) C_v$ for $sy > y$ and $C_s C_y = (q^{(1/2)} + q^{-(1/2)}) C_y$ for $sy < y$ [1]
- (c) We get $C_x C_y = \sum_{u, v \in W} C_u C_v$, where $l(u) < l(x)$ by step (a) and step (b). Repeat this proceeding until $C_x C_y = \sum_{u', v' \in W} C_{u'} C_{v'}$, where $l(u') = 0$; then, we obtain $C_x C_y$.

It is known that $P_{w, v} = 1$ for any $w, v \in W_0$. By this conclusion, the complete decomposition $C_{w_0} = C_1 C_2 C_1 C_2 C_1 C_2 - 4C_1 C_2 C_1 C_2 + 3C_1 C_2$ is obtained. This complete decomposition plays a crucial role in the proof of the following proposition. $C_{w_0 d_u^{-1} d_{u'}^{-1} w_0}$ for $u, u' \in W_0$ will be calculated with the increase of $l(w_0 d_u^{-1})$ by following cases.

- (i) If $C_{w_0 d_u^{-1}} = C_{w_0}$, we get $C_{w_0 d_u^{-1} d_{u'}^{-1} w_0} = C_{w_0} C_{d_{u'}^{-1} w_0} = (C_1 C_2 C_1 C_2 C_1 C_2 - 4C_1 C_2 C_1 C_2 + 3C_1 C_2) C_{d_{u'}^{-1} w_0}$. The proposed algorithm is applied step by step to compute $C_1 C_2 C_1 C_2 C_1 C_2 C_{d_{u'}^{-1} w_0}$, $C_1 C_2 C_1 C_2 C_{d_{u'}^{-1} w_0}$, and $C_1 C_2 C_{d_{u'}^{-1} w_0}$. Then, the decomposition of $C_{w_0} C_{d_{u'}^{-1} w_0}$ is obtained;
- (ii) If $l(w_0 d_u^{-1}) > l(w_0)$, there are two types of calculation of $C_{w_0 d_u^{-1} d_{u'}^{-1} w_0}$
 - (1) Step (a) and step (b) mentioned above will be repeatedly used to get linear combination $C_{w_0 d_u^{-1} d_{u'}^{-1} w_0} = \sum_{u, v \in W} C_u C_v$, where $C_u C_v$ can be computed by recursion from case (i)
 - (2) If $C_{w_0 d_u^{-1} d_{u'}^{-1} w_0}$ cannot be computed by (1), step (a) and step (b) mentioned above will be repeatedly used to get the linear combination $C_{w_0 d_u^{-1} d_{u'}^{-1} w_0} = \sum_{u, v \in W} C_u C_v$, where $C_u = C_{w_0}$. All $C_{w_0} C_v$ can be computed by (i).

We set $[2] \triangleq q^{(1/2)} + q^{-(1/2)}$, $1 \triangleq s$, $2 \triangleq t$, and $0 \triangleq r$.
 Compute $C_{121212}C_{121201210121212}$ as follows:

$$\begin{aligned}
 C_{121212} &= C_2C_1C_2C_1C_2C_1 - 4C_2C_1C_2C_1 + 3C_2C_1, \\
 C_1C_{121201210121212} &= [2]C_{121201210121212}, \\
 C_2C_{121201210121212} &= C_{2121201210121212} + C_{21201210121212} + C_{w_0}, \\
 C_2C_1C_{121201210121212} &= [2](C_{x_\beta w_0} + C_{21201210121212} + C_{w_0}), \\
 C_1C_{21201210121212} &= C_{121201210121212} + C_{1201210121212}, \\
 C_1C_2C_1C_{121201210121212} &= [2](C_{x_\beta w_0} + C_{121201210121212} + C_{1201210121212} + [2]C_{w_0}), \\
 C_2C_{1201210121212} &= C_{21201210121212} + C_{201210121212} + C_{x_\alpha w_0}, \\
 C_2C_1C_2C_1C_{121201210121212} &= [2]\left\{([2]^2 + 1)C_{x_\beta w_0} + 2C_{21201210121212} + C_{201210121212} + C_{x_\alpha w_0} + ([2]^2 + 1)C_{w_0}\right\}, \\
 C_1C_{201210121212} &= C_{1201210121212} + C_{01210121212}, \\
 C_1C_2C_1C_2C_1C_{121201210121212} &= [2]\left\{([2]^3 + [2])C_{x_\beta w_0} + 2C_{121201210121212} + 3C_{1201210121212} + C_{01210121212} + [2]C_{x_\alpha w_0} + ([2]^3 + [2])C_{w_0}\right\}, \\
 C_2C_{01210121212} &= C_{201210121212}, \\
 C_2C_1C_2C_1C_2C_1C_{121201210121212} &= [2]\left\{([2]^4 + [2]^2 + 2)C_{x_\beta w_0} + 5C_{21201210121212} + 4C_{201210121212} + [2]^2C_{x_\alpha w_0} + ([2]^4 + [2]^2 + 2)C_{w_0}\right\}, \\
 C_{121212}C_{121201210121212} &= [2]\left\{([2]^4 + [2]^2 + 2)C_{x_\beta w_0} + 5C_{21201210121212} + 4C_{201210121212} + [2]^2C_{x_\alpha w_0} + ([2]^4 + [2]^2 + 2)C_{w_0}\right\} - 4[2] \\
 &\quad \left\{([2]^2 + 1)C_{x_\beta w_0} + 2C_{21201210121212} + C_{201210121212} + C_{x_\alpha w_0} + ([2]^2 + 1)C_{w_0}\right\} \\
 &\quad + 3[2](C_{x_\beta w_0} + C_{21201210121212} + C_{w_0}) \\
 &= ([2]^5 - 3[2]^3 + [2])C_{x_\beta w_0} + ([2]^3 - [2])C_{x_\alpha w_0} + ([2]^5 - 3[2]^3 + [2])C_{w_0}.
 \end{aligned} \tag{13}$$

We obtain the following Kazhdan–Lusztig basis multiplication results through the proposed the Kazhdan–Lusztig polynomials and the Kazhdan–Lusztig basis multiplication program. All Kazhdan–Lusztig basis multiplication of the

lowest cell of the affine Weyl group of type $\tilde{G}_2$ are computed, and we gave the step by step computation details in reference [19].

$$\begin{aligned}
 C_{w_0}C_{w_0} &= ([2]^6 - 4[2]^4 + 3[2]^2)C_{w_0}, \\
 C_{121212}C_{0121212} &= C_{1212120}C_{121212} = [2]C_{x_\alpha w_0} + ([2]^5 - 3[2]^3 + [2])C_{w_0}, \\
 C_{121212}C_{10121212} &= C_{12121201}C_{121212} = [2]^2C_{x_\alpha w_0} + ([2]^4 - 2[2]^2)C_{w_0}, \\
 C_{121212}C_{210121212} &= C_{121212012}C_{121212} = ([2]^3 - [2])C_{x_\alpha w_0} + ([2]^3 - [2])C_{w_0}, \\
 C_{121212}C_{1210121212} &= C_{1212120121}C_{121212} = ([2]^4 - 2[2]^2)C_{x_\alpha w_0} + [2]^2C_{w_0}, \\
 C_{121212}C_{21210121212} &= C_{12121201212}C_{121212} = ([2]^5 - 3[2]^3 + [2])C_{x_\alpha w_0} + [2]C_{w_0}, \\
 C_{121212}C_{01210121212} &= C_{12121201210}C_{121212} = [2]C_{x_\beta w_0} + ([2]^3 - [2])C_{x_\alpha w_0} + [2]C_{w_0}, \\
 C_{121212}C_{201210121212} &= C_{121212012102}C_{121212} = [2]^2C_{x_\beta w_0} + ([2]^4 - [2]^2)C_{x_\alpha w_0} + [2]^2C_{w_0}, \\
 C_{121212}C_{1201210121212} &= C_{1212120121021}C_{121212} = ([2]^3 - [2])C_{x_\beta w_0} + ([2]^3 - [2])C_{w_0} \\
 &\quad + ([2]^5 - 2[2]^3 + [2])C_{x_\alpha w_0},
 \end{aligned}$$

$$\begin{aligned}
C_{121212}C_{21201210121212} &= C_{12121201210212}C_{121212} = ([2]^4 - 2[2]^2)C_{x_\beta w_0} \\
&\quad + ([2]^4 - [2]^2)C_{x_\alpha w_0} + ([2]^4 - 2[2]^2)C_{w_0}, \\
C_{121212}C_{121201210121212} &= C_{121212012102121}C_{121212} = ([2]^5 - 3[2]^3 + [2])C_{x_\beta w_0} + ([2]^3 - [2])C_{x_\alpha w_0} \\
&\quad + ([2]^5 - 3[2]^3 + [2])C_{w_0}, \\
C_{121212}C_{0121201210121212} &= C_{121212120121021210}C_{121212} = C_{x_\alpha x_\beta w_0} + [2]^2 C_{x_\alpha^2 w_0} + (2[2]^2)C_{x_\alpha w_0} \\
&\quad + ([2]^4 - 2[2]^2)C_{x_\beta w_0} + ([2]^4 - 2[2]^2)C_{w_0}, \\
C_{1212120}C_{121201210121212} &= C_{121212012102121}C_{0121212} = C_{x_\alpha x_\beta w_0} + [2]^2 C_{x_\alpha^2 w_0} + 3[2]^2 C_{x_\alpha w_0} \\
&\quad + ([2]^4 - 2[2]^2)C_{x_\beta w_0} + (2[2]^4 - 4[2]^2)C_{w_0}, \\
C_{1212120}C_{0121201210121212} &= C_{1212120121021210}C_{0121212} = [2]C_{121212}C_{0121201210121212}, \\
C_{12121201}C_{10121212} &= ([2]^4 - [2]^2)C_{x_\alpha w_0} + ([2]^6 - 3[2]^4 + 2[2]^2)C_{w_0}, \\
C_{12121201}C_{210121212} &= C_{121212012}C_{10121212} = ([2]^5 - 2[2]^3 + [2])C_{w_0} + [2]C_{x_\beta w_0} + (2[2]^3 - [2])C_{x_\alpha w_0}, \\
C_{12121201}C_{1210121212} &= C_{1212120121}C_{10121212} = [2]^4 C_{x_\alpha w_0} + [2]^2 C_{x_\beta w_0} + ([2]^4 - [2]^2)C_{w_0}, \\
C_{12121201}C_{21210121212} &= C_{12121201212}C_{10121212} = [2]C_{x_\alpha^2 w_0} + (2[2]^3 - [2])C_{x_\alpha w_0} \\
&\quad + 2[2]C_{x_\beta w_0} + [2]^3 C_{w_0}, \\
C_{12121201}C_{01210121212} &= C_{12121201210}C_{10121212} = ([2]^3 - [2])C_{w_0} + ([2]^3 - [2])C_{x_\beta w_0} \\
&\quad + ([2]^5 - 2[2]^3 + [2])C_{x_\alpha w_0}, \\
C_{12121201}C_{201210121212} &= C_{121212012102}C_{10121212} = [2]^2 C_{x_\alpha^2 w_0} + (2[2]^4 - [2]^2)C_{x_\alpha w_0} \\
&\quad + 2[2]^2 C_{x_\beta w_0} + [2]^4 C_{w_0}, \\
C_{12121201}C_{1201210121212} &= C_{1212120121021}C_{10121212} = [2]^3 C_{x_\alpha^2 w_0} + ([2]^5 + [2]^3 - [2])C_{x_\alpha w_0} \\
&\quad + ([2]^3 + [2])C_{x_\beta w_0} + ([2]^5 - [2]^3 + [2])C_{w_0}, \\
C_{12121201}C_{21201210121212} &= C_{12121201210212}C_{10121212} = C_{x_\alpha x_\beta w_0} + 2[2]^2 C_{x_\alpha^2 w_0} + 2[2]^2 C_{x_\beta w_0} \\
&\quad + 2([2]^4 - [2]^2)C_{w_0} + ([2]^4 + 3[2]^2)C_{x_\alpha w_0}, \\
C_{12121201}C_{121201210121212} &= C_{121212012102121}C_{10121212} = [2]C_{x_\alpha x_\beta 121212} + [2]^3 C_{x_\alpha^2 121212} \\
&\quad + ([2]^5 - [2]^3 - [2])C_{w_0} + (2[2]^3 + [2])C_{x_\alpha w_0} + ([2]^3 - [2])C_{x_\beta w_0}, \\
C_{12121201}C_{0121201210121212} &= C_{1212120121021210}C_{10121212} = [2]^2 C_{x_\alpha x_\beta w_0} + (2[2]^4 - 3[2]^2)C_{w_0} \\
&\quad + ([2]^4 + 2[2]^2)C_{x_\alpha w_0} + ([2]^4 - [2]^2)C_{x_\beta w_0} + [2]^4 C_{x_\alpha^2 w_0}, \\
C_{121212012}C_{210121212} &= [2]^2 C_{x_\beta w_0} + [2]^4 C_{x_\alpha w_0} + ([2]^6 - 3[2]^4 + 2[2]^2)C_{w_0},
\end{aligned}$$

$$\begin{aligned}
C_{121212012}C_{1210121212} &= C_{1212120121}C_{210121212} = [2]C_{x_a^2w_0} + 2[2]C_{x_\beta w_0} + 2[2]^3C_{x_a w_0} \\
&\quad + ([2]^5 - 2[2]^3 + [2])C_{w_0}, \\
C_{121212012}C_{21210121212} &= C_{12121201212}C_{210121212} = C_{121212012}C_{01210121212} = C_{12121201210}C_{210121212} \\
&= [2]^2C_{x_a^2w_0} + [2]^4C_{x_a w_0} + [2]^2C_{x_\beta w_0} + ([2]^4 - [2]^2)C_{w_0}, \\
C_{121212012}C_{201210121212} &= C_{121212012102}C_{210121212} = [2]^3C_{x_a^2w_0} + [2]^5C_{x_a w_0} + [2]^3C_{x_\beta w_0} + ([2]^5 - [2]^3)C_{w_0}, \\
C_{121212012}C_{1201210121212} &= C_{1212120121021}C_{210121212} = C_{x_a x_\beta w_0} + 3[2]^2C_{x_a^2w_0} + 3[2]^2C_{x_\beta w_0} \\
&\quad + 2([2]^4 + [2]^2)C_{x_a w_0} + (2[2]^4 - [2]^2)C_{w_0}, \\
C_{121212012}C_{21201210121212} &= C_{12121201210212}C_{210121212} = [2]C_{x_a x_\beta w_0} + ([2]^3 + [2])C_{x_a^2w_0} \\
&\quad + ([2]^5 + 2[2])C_{x_a w_0} + ([2]^3 + [2])C_{x_\beta w_0} + ([2]^5 - [2]^3 + [2])C_{w_0}, \\
C_{121212012}C_{121201210121212} &= C_{121212012102121}C_{210121212} = [2]^2C_{x_a x_\beta w_0} + 2[2]^2C_{x_\beta w_0} + [2]^4C_{w_0} \\
&\quad + ([2]^4 + 2[2]^2)C_{x_a w_0} + 2[2]^2C_{x_a^2w_0}, \\
C_{121212012}C_{0121201210121212} &= C_{1212120121021210}C_{210121212} = [2]^3C_{x_a x_\beta w_0} + (2[2]^3 - [2])C_{x_\beta w_0} \\
&\quad + ([2]^5 + [2])C_{x_a w_0} + 2[2]^3C_{x_a^2w_0} + (2[2]^3 - [2])C_{w_0}, \\
C_{1212120121}C_{1210121212} &= [2]^2C_{x_a^2w_0} + [2]^4C_{x_a w_0} + [2]^4C_{x_\beta w_0} + ([2]^6 - 3[2]^4 + 2[2]^2)C_{w_0}, \\
C_{1212120121}C_{21210121212} &= C_{12121201212}C_{1210121212} = ([2]^3 - [2])C_{x_a^2w_0} + (2[2]^3 - [2])C_{x_a w_0} \\
&\quad + 2([2]^3 - [2])C_{x_\beta w_0} + ([2]^5 - 2[2]^3)C_{w_0}, \\
C_{1212120121}C_{01210121212} &= C_{12121201210}C_{1210121212} = [2]^3C_{x_a^2w_0} + (2[2]^3 - [2])C_{x_a w_0} + [2]C_{x_\beta w_0} \\
&\quad + ([2]^5 - 2[2]^3 + [2])C_{w_0}, \\
C_{1212120}C_{0121212} &= [2]^2C_{x_a w_0} + ([2]^6 - 3[2]^4 + [2]^2)C_{w_0}, \\
C_{1212120}C_{10121212} &= C_{12121201}C_{0121212} = [2]^3C_{x_a w_0} + ([2]^5 - 2[2]^3)C_{w_0}, \\
C_{1212120}C_{210121212} &= C_{121212012}C_{0121212} = ([2]^4 - [2]^2)C_{x_a w_0} + ([2]^4 - [2]^2)C_{w_0}, \\
C_{1212120}C_{1210121212} &= C_{1212120}C_{1210121212} = [2]C_{x_\beta w_0} + 2([2]^3 - [2])C_{x_a w_0} + [2]^3C_{w_0}, \\
C_{1212120}C_{21210121212} &= C_{12121201212}C_{0121212} = C_{1212120}C_{01210121212} = C_{12121201210}C_{0121212} \\
&= [2]^2C_{x_\beta w_0} + ([2]^4 - [2]^2)C_{x_a w_0} + [2]^2C_{w_0}, \\
C_{1212120}C_{201210121212} &= C_{121212012102}C_{0121212} = [2]^3C_{x_\beta w_0} + ([2]^5 - [2]^3)C_{x_a w_0} + [2]^3C_{w_0}, \\
C_{1212120}C_{1201210121212} &= C_{1212120121021}C_{0121212} = [2]^2C_{x_a^2121212} + (2[2]^4 - [2]^2)C_{x_a121212} + 2[2]^2C_{x_\beta121212} + [2]^4C_{w_0}, \\
C_{1212120}C_{21201210121212} &= C_{12121201210212}C_{0121212} = ([2]^3 - [2])C_{x_a^2w_0} + (3[2]^3 - 2[2])C_{x_a w_0} \\
&\quad + ([2]^3 - [2])C_{x_\beta w_0} + ([2]^5 - [2]^3 - [2])C_{w_0},
\end{aligned}$$

$$\begin{aligned}
C_{12121201212}C_{01210121212} &= C_{12121201210}C_{21210121212} = C_{x_\alpha x_\beta w_0} + 2[2]^2 C_{x_\alpha^2 w_0} + 3[2]^2 C_{x_\alpha w_0} \\
&\quad + [2]^2 C_{x_\beta w_0} + ([2]^4 - [2]^2) C_{w_0}, \\
C_{12121201212}C_{201210121212} &= C_{121212012102}C_{21210121212} \\
&\quad + [2]^3 C_{x_\beta w_0} + ([2]^5 - [2]^3) C_{w_0}, \\
C_{12121201212}C_{1201210121212} &= C_{1212120121021}C_{21210121212} = [2]^2 C_{x_\alpha x_\beta w_0} + ([2]^4 + [2]^2) C_{x_\alpha^2 w_0} \\
&\quad + (2[2]^4 - 2[2]^2) C_{w_0} + (2[2]^4 + [2]^2) C_{x_\alpha w_0} + [2]^4 C_{x_\beta w_0}, \\
C_{12121201212}C_{21201210121212} &= C_{12121201210212}C_{21210121212} = ([2]^3 - [2]) C_{x_\alpha x_\beta w_0} + (2[2]^3 - [2]) C_{x_\alpha^2 w_0} \\
&\quad + (2[2]^3 - [2]) C_{w_0} + ([2]^5 + [2]^3 - 2[2]) C_{x_\alpha w_0} + (2[2]^3 - [2]) C_{x_\beta w_0}, \\
C_{12121201212}C_{121201210121212} &= C_{121212012102121}C_{21210121212} = ([2]^4 - 2[2]^2) C_{x_\alpha x_\beta w_0} + 2[2]^2 C_{x_\beta w_0} \\
&\quad + (3[2]^4 - 5[2]^2) C_{x_\alpha w_0} + ([2]^4 - [2]^2) C_{x_\alpha^2 w_0} + 2[2]^2 C_{w_0}, \\
C_{12121201212}C_{0121201210121212} &= C_{1212120121021210}C_{21210121212} = ([2]^3 + [2]) C_{x_\alpha x_\beta w_0} + 2[2] C_{x_\alpha^3 w_0} \\
&\quad + (2[2]^3 + [2]) C_{x_\alpha^2 w_0} + (4[2]^3 - [2]) C_{x_\alpha w_0} + ([2]^3 + 2[2]) C_{w_0} + ([2]^3 + 4[2]) C_{x_\beta w_0}, \\
C_{12121201210}C_{0121012121212} &= ([2]^4 - [2]^2) C_{x_\alpha^2 w_0} + ([2]^4 - [2]^2) C_{x_\alpha w_0} + ([2]^6 - 3[2]^4 + 2[2]^2) C_{w_0}, \\
C_{12121201210}C_{20121012121212} &= C_{121212012102}C_{01210121212} = [2] C_{x_\alpha x_\beta w_0} + 2[2]^3 C_{x_\alpha^2 w_0} + [2]^3 C_{x_\beta w_0} \\
&\quad + ([2]^5 - [2]^3) C_{w_0} + 3[2]^3 C_{x_\alpha w_0}, \\
C_{12121201210}C_{120121012121212} &= C_{1212120121021}C_{01210121212} = [2]^2 C_{x_\alpha x_\beta w_0} + 2([2]^4 - [2]^2) C_{w_0} \\
&\quad + (2[2]^4 + [2]^2) C_{x_\alpha w_0} + [2]^4 C_{x_\beta w_0} + ([2]^4 + [2]^2) C_{x_\alpha^2 w_0}, \\
C_{12121201210}C_{2120121012121212} &= C_{12121201210212}C_{01210121212} = [2] C_{x_\alpha^3 w_0} + 4[2]^3 C_{x_\alpha w_0} + 2[2]^3 C_{w_0} \\
&\quad + ([2]^3 + 3[2]) C_{x_\beta w_0} + 2[2] C_{x_\alpha x_\beta w_0} + ([2]^3 + 3[2]) C_{x_\alpha^2 w_0}, \\
C_{12121201210}C_{12120121012121212} &= C_{121212012102121}C_{01210121212} = [2]^2 C_{x_\alpha^3 w_0} + 2([2]^4 - [2]^2) C_{x_\alpha w_0} \\
&\quad + 3[2]^2 C_{x_\beta w_0} + [2]^2 C_{x_\alpha x_\beta w_0} + 2[2]^2 C_{x_\alpha^2 w_0} + 2[2]^2 C_{w_0}, \\
C_{12121201210}C_{012120121012121212} &= C_{1212120121021210}C_{01210121212} = [2]^3 C_{x_\alpha^3 w_0} + [2]^3 C_{x_\alpha x_\beta w_0} \\
&\quad + (2[2]^3 + [2]) C_{x_\beta w_0} + ([2]^5 - [2]) C_{x_\alpha w_0} + 2[2]^3 C_{x_\alpha^2 w_0} + ([2]^3 + [2]) C_{w_0}, \\
C_{121212012102}C_{12012101212121212} &= C_{1212120121021}C_{201210121212} = [2] C_{x_\alpha^3 w_0} + (8[2]^3 + [2]) C_{x_\alpha w_0} \\
&\quad + (3[2]^3 + 2[2]) C_{x_\beta w_0} + 4[2] C_{x_\alpha x_\beta w_0} \\
&\quad + (4[2]^3 + 3[2]) C_{x_\alpha^2 w_0} + ([2]^5 + 2[2]^3 - [2]) C_{w_0}, \\
C_{121212012102}C_{2120121012121212} &= C_{12121201210212}C_{201210121212} = [2]^2 C_{x_\alpha^3 w_0} + 4[2]^4 C_{x_\alpha w_0} \\
&\quad + ([2]^4 + 3[2]^2) C_{x_\beta w_0} + 2[2]^2 C_{x_\alpha x_\beta w_0} + ([2]^4 + 3[2]^2) C_{x_\alpha^2 w_0} + 2[2]^4 C_{w_0},
\end{aligned}$$

$$\begin{aligned}
C_{121212012102}C_{121201210121212} &= C_{121212012102121}C_{201210121212} = [2]C_{x_\beta^2w_0} + ([2]^3 + [2])C_{x_\alpha x_\beta w_0} \\
&\quad + (2[2]^3 + 3[2])C_{x_\alpha^2w_0} + 2[2]C_{x_\alpha^3w_0} + (6[2]^3 - 2[2])C_{x_\alpha w_0} \\
&\quad + ([2]^3 + 6[2])C_{x_\beta w_0} + (2[2]^3 + 2[2])C_{w_0}, \\
C_{121212012102}C_{0121201210121212} &= C_{1212120121021210}C_{201210121212} = [2]^2C_{x_\beta^2w_0} + ([2]^4 + [2]^2)C_{x_\alpha x_\beta w_0} \\
&\quad + (2[2]^4 + 2[2]^2)C_{x_\alpha^2w_0} + (4[2]^4 - [2]^2)C_{x_\alpha w_0} \\
&\quad + ([2]^4 + 4[2]^2)C_{x_\beta w_0} + ([2]^4 + [2]^2)C_{w_0} + 2[2]^2C_{x_\alpha^3w_0}, \\
C_{1212120121021}C_{1201210121212} &= [2]^2C_{x_\alpha^3w_0} + 6[2]^4C_{x_\alpha w_0} + (2[2]^4 + 2[2]^2)C_{x_\beta w_0} + 3[2]^2C_{x_\alpha x_\beta w_0} \\
&\quad + (3[2]^4 + 2[2]^2)C_{x_\alpha^2w_0} + ([2]^6 + [2]^2)C_{w_0}, \\
C_{1212120121021}C_{21201210121212} &= C_{12121201210212}C_{1201210121212} = [2]C_{x_\beta^2w_0} + ([2]^3 + 3[2])C_{x_\alpha x_\beta w_0} \\
&\quad + (5[2]^3 + 3[2])C_{x_\alpha^2w_0} + (10[2]^3 - [2])C_{x_\alpha w_0} \\
&\quad + (3[2]^3 + 5[2])C_{x_\beta w_0} + ([2]^5 + 2[2]^3 + [2])C_{w_0} + 2[2]C_{x_\alpha^3w_0}, \\
C_{1212120121021}C_{121201210121212} &= C_{121212012102121}C_{1201210121212} = [2]^2C_{x_\beta^2w_0} + [2]^4C_{x_\alpha x_\beta w_0} + 2[2]^4C_{w_0} \\
&\quad + (2[2]^4 + [2]^2)C_{x_\alpha^2w_0} + 4[2]^4C_{x_\alpha w_0} + ([2]^4 + 3[2]^2)C_{x_\beta w_0} + [2]^2C_{x_\alpha^3w_0}, \\
C_{1212120121021}C_{0121201210121212} &= C_{1212120121021210}C_{1201210121212} = [2]C_{x_\alpha^2x_\beta w_0} + 2[2]C_{x_\beta^2w_0} \\
&\quad + (4[2]^3 + [2])C_{w_0} + (5[2]^3 + 5[2])C_{x_\beta w_0} + 9[2]^3C_{x_\alpha w_0} \\
&\quad + (5[2]^3 + 5[2])C_{x_\alpha^2w_0} + (3[2]^3 + 2[2])C_{x_\alpha x_\beta w_0} + ([2]^3 + 3[2])C_{x_\alpha^3w_0}, \\
C_{12121201210212}C_{21201210121212} &= [2]^2C_{x_\beta^2w_0} + ([2]^6 - 3[2]^4 + 2[2]^2)C_{w_0} + (2[2]^4 + 3[2]^2)C_{x_\alpha^2w_0} \\
&\quad + (3[2]^4 + 4[2]^2)C_{x_\alpha w_0} + ([2]^4 + 3[2]^2)C_{x_\beta w_0} + 3[2]^2C_{x_\alpha x_\beta w_0}, \\
C_{1212120121}C_{201210121212} &= C_{121212012102}C_{1210121212} = C_{x_\alpha x_\beta w_0} + 3[2]^2C_{x_\alpha^2w_0} + 2[2]^2C_{x_\beta w_0} \\
&\quad + 2([2]^4 - [2]^2)C_{w_0} + ([2]^4 + 3[2]^2)C_{x_\alpha w_0}, \\
C_{1212120121}C_{1201210121212} &= C_{1212120121021}C_{1210121212} = [2]C_{x_\alpha x_\beta w_0} + 2[2]^3C_{x_\alpha^2w_0} + ([2]^5 - [2])C_{w_0} \\
&\quad + ([2]^5 + [2]^3 + [2])C_{x_\alpha w_0} + (2[2]^3 - [2])C_{x_\beta w_0}, \\
C_{1212120121}C_{21201210121212} &= C_{12121201210212}C_{1210121212} = [2]^2C_{x_\alpha x_\beta w_0} + 3[2]^2C_{x_\alpha^2w_0} + 3[2]^2C_{x_\beta w_0} \\
&\quad + ([2]^4 + [2]^2)C_{w_0} + (2[2]^4 + [2]^2)C_{x_\alpha w_0}, \\
C_{1212120121}C_{121201210121212} &= C_{121212012102121}C_{1210121212} = ([2]^3 - [2])C_{x_\alpha x_\beta w_0} + ([2]^3 + [2])C_{x_\beta w_0} \\
&\quad + ([2]^3 + [2])C_{w_0} + ([2]^5 - [2])C_{x_\alpha w_0} + [2]^3C_{x_\alpha^2w_0}, \\
C_{1212120121}C_{0121201210121212} &= C_{1212120121021210}C_{1210121212} = [2]^2C_{x_\alpha^3w_0} + 2[2]^4C_{x_\alpha w_0} + 4[2]^2C_{x_\beta w_0}
\end{aligned}$$

$$\begin{aligned}
& + 2[2]^2 C_{x_\alpha x_\beta w_0} + 4[2]^2 C_{x_\alpha^2 w_0} + 3[2]^2 C_{w_0}, \\
C_{12121201212} C_{21210121212} &= ([2]^4 - 2[2]^2) C_{x_\alpha^2 w_0} + ([2]^4 - [2]^2) C_{x_\alpha w_0} + (2[2]^4 - 3[2]^2) C_{x_\beta w_0} \\
&+ ([2]^6 - 3[2]^4 + [2]^2) C_{w_0}, \\
C_{12121201210212} C_{121201210121212} &= C_{121212012102121} C_{21201210121212} = ([2]^3 + [2]) C_{x_\alpha x_\beta w_0} + 4[2]^3 C_{x_\alpha^2 w_0} \\
&+ (4[2]^3 + 2[2]) C_{x_\alpha w_0} + (4[2]^3 - 3[2]) C_{x_\beta w_0} + ([2]^3 - [2]) C_{x_\beta^2 w_0} \\
&+ ([2]^5 + [2]^3 - 2[2]) C_{w_0} + ([2]^3 - [2]) C_{x_\alpha^3 w_0}, \\
C_{12121201210212} C_{0121201210121212} &= C_{1212120121021210} C_{21201210121212} = [2]^2 C_{x_\alpha^2 x_\beta w_0} + (2[2]^4 + 3[2]^2) C_{x_\beta w_0} \\
&+ 6[2]^2 C_{x_\alpha^2 w_0} + 3[2]^2 C_{x_\alpha^3 w_0} + [2]^2 C_{x_\beta^2 w_0} + 4[2]^2 C_{x_\alpha x_\beta w_0} \\
&+ (2[2]^4 - [2]^2) C_{w_0} + (3[2]^4 + 6[2]^2) C_{x_\alpha w_0}, \\
C_{121212012102121} C_{121201210121212} &= C_{121212012102121} C_{121201210121212} = [2]^4 C_{x_\alpha x_\beta w_0} + (2[2]^4 - [2]^2) C_{x_\alpha^2 w_0} \\
&+ 2[2]^2 C_{x_\alpha w_0} + ([2]^6 - [2]^4 - 2[2]^2) C_{w_0} + ([2]^4 - 3[2]^2) C_{x_\beta^2 w_0} \\
&+ (3[2]^4 - 6[2]^2) C_{x_\beta w_0} + ([2]^4 - 3[2]^2) C_{x_\alpha^3 w_0}, \\
C_{121212012102121} C_{0121201210121212} &= C_{1212120121021210} C_{121201210121212} = [2] C_{x_\alpha^4 w_0} + 4[2] C_{x_\alpha^2 x_\beta w_0} \\
&+ (7[2]^3 + 7[2]) C_{x_\alpha^2 w_0} + ([2]^3 + 8[2]) C_{x_\alpha x_\beta w_0} + [2]^3 C_{x_\beta^2 w_0} + ([2]^5 + 4[2]^3 + [2]) C_{x_\beta w_0} \\
&+ ([2]^3 + 4[2]) C_{x_\alpha^3 w_0} + ([2]^5 + 2[2]^3 + 11[2]) C_{x_\alpha w_0} + ([2]^5 + [2]^3 + 2[2]) C_{w_0}, \\
C_{1212120121021210} C_{0121201210121212} &= [2]^2 C_{x_\alpha^4 w_0} + 4[2]^2 C_{x_\alpha^2 x_\beta w_0} + (6[2]^4 + 7[2]^2) C_{x_\alpha^2 w_0} + [2]^4 C_{x_\beta^2 w_0} \\
&+ ([2]^4 + 7[2]^2) C_{x_\alpha x_\beta w_0} + ([2]^6 + 3[2]^4 + 2[2]^2) C_{x_\beta w_0} + ([2]^6 + [2]^4 + 9[2]^2) C_{x_\alpha w_0} \\
&+ ([2]^6 - [2]^4 + 5[2]^2) C_{w_0} + ([2]^4 + 4[2]^2) C_{x_\alpha^3 w_0}.
\end{aligned} \tag{14}$$

We know that $\delta_{w_0 d_u^{-1}, d_{u'}, w_0, z_1} w_0$ is the coefficient of $q^{(5/2)}$ of $h w_0 d_u^{-1}, d_{u'}, w_0, z_1 w_0$. By the above process of calculation, we get the conclusion.

Define the following corresponding sets of nontrivial coefficients of the multiplications polynomials:

$$\begin{aligned}
\mathcal{U}_1 &= \left\{ (d_u, d_{u'}, z) | u, u' \in W_0, \exists! z \in \{0, x_\alpha, x_\beta\}, \delta_{w_0 d_u^{-1}, d_{u'}, w_0, z w_0} = 1 \right\}, \\
\mathcal{U}_2 &= \left\{ (d_u, d_{u'}) | ((d_u^{-1}, d_{u'}, z_1), (d_u^{-1}, d_{u'}, z_2)), u, u' \in W_0, (z_1, z_2) = (0, x_\alpha), \delta_{w_0 d_u^{-1}, d_{u'}, w_0, z_1 w_0} = \delta_{w_0 d_u^{-1}, d_{u'}, w_0, z_2 w_0} = 1 \right\}, \\
\mathcal{U}_3 &= \left\{ (d_u, d_{u'}) | ((d_u^{-1}, d_{u'}, z_1), (d_u^{-1}, d_{u'}, z_2)), u, u' \in W_0, (z_1, z_2) = (0, x_\beta), \delta_{w_0 d_u^{-1}, d_{u'}, w_0, z_1 w_0} = \delta_{w_0 d_u^{-1}, d_{u'}, w_0, z_2 w_0} = 1 \right\}, \\
\mathcal{U}_4 &= \left\{ (d_u, d_{u'}) | ((d_u^{-1}, d_{u'}, z_1), (d_u^{-1}, d_{u'}, z_2), (d_u^{-1}, d_{u'}, z_3)), u, u' \in W_0, \right. \\
&\left. (z_1, z_2, z_3) = (0, x_\alpha, x_\beta), \delta_{w_0 d_u^{-1}, d_{u'}, w_0, z_1 w_0} = 1 = \delta_{w_0 d_u^{-1}, d_{u'}, w_0, z_2 w_0} = \delta_{w_0 d_u^{-1}, d_{u'}, w_0, z_3 w_0} \right\}.
\end{aligned} \tag{15}$$

Using the conclusion of Proposition 1, we can list the corresponding elements of nontrivial coefficients of the multiplications polynomials as follows:

$$\begin{aligned}
\mathcal{U}_1 = & \{(e, r, 0), (r, e, 0), (e, tstrsr, x_\alpha), (tstrsr, e, x_\alpha), \\
& (e, strtsr, x_\alpha), (strtsr, e, x_\alpha), (sr, r, 0), (r, trtsr, x_\alpha), (trtsr, r, x_\alpha), (r, tstrtsr, 0), \\
& (tstrtsr, r, 0), (r, trtsr, x_\alpha), (trtsr, r, x_\alpha), (sr, tsr, 0), (sr, rtsr, x_\alpha), (rtsr, sr, x_\alpha), \\
& (sr, strtsr, 0), (strtsr, sr, 0), (tsr, tsr, 0), (tsr, rstrtsr, x_\alpha), (r, sr, 0), \\
& (tsr, sr, 0), (rstrtsr, tsr, x_\alpha), (tsr, tstr, 0), (tstr, tsr, 0), (tsr, rtsr, 0), \\
& (rtsr, tsr, 0), (tsr, strtsr, x_\alpha), (strtsr, tsr, x_\alpha), (tstr, trtsr, 0), \\
& (tsr, tsr, 0), (tstr, tstrtsr, x_\alpha), (trtsr, tsr, 0), (tstrtsr, tsr, x_\alpha), \\
& (rtsr, trtsr, 0), (trtsr, rtsr, 0), (rtsr, rstrtsr, x_\alpha), (rstrtsr, rtsr, x_\alpha), \\
& (trtsr, strtsr, 0), (strtsr, trtsr, 0), (rstrts, tstrtsr, 0), (tstrtsr, strtsr, 0), \\
& (strtsr, tstrtsr, 0), (tstrtsr, strtsr, 0), (tstrtsr, strtsr, 0), (strtsr, tstrtsr, 0)\}, \\
\mathcal{U}_2 = & \{(sr, strtsr), (strtsr, sr), (tsr, trtsr), \\
& (trtsr, tsr), (tsr, tstrtsr), (tstrtsr, tsr), \\
& (tsr, strtsr), (strtsr, tsr)\}, \\
\mathcal{U}_3 = & \{(e, strtsr), (strtsr, e)\}, \\
\mathcal{U}_4 = & \{(strtsr, rstrtsr), (rstrtsr, strtsr)\}.
\end{aligned} \tag{16}$$

The following lemma is a well-known result in representation theory [18]. $\square$

Lemma 6. Assume that L is a semisimple complex Lie algebra, and λ and λ' are the dominant weights. Let $K_{\lambda, \gamma}$ be the weight multiplicity of weight γ in the irreducible L -module $V(\lambda)$, and $m_{\lambda, \lambda', \nu}$ be the tensor product multiplicity of irreducible module $V(\nu)$ in $V(\lambda) \otimes V(\lambda')$. Then, we have that $0 \leq m_{\lambda, \lambda', \nu} \leq K_{\lambda, \nu - \lambda'}$.

Theorem 1. For any $y, w \in c_0$, if $y \sim_L w$, that is, $y = d_u x w_0 d_v$, $w = d_{u'} x' w_0 d_{v'}$, where $u, u', v \in W_0$, and $x, x' \in \Lambda^+$. Then, we must have the following:

- (i) If there is $z \in \{0, x_\alpha, x_\beta\}$, such that $(d_u, d_{u'}, z) \in \mathcal{U}_1$, then we have $\mu(y, w) = m_{x, x', z}$
- (ii) If there is $(d_u, d_{u'}) \in \mathcal{U}_2$, then we have $\mu(y, w) = m_{x, x', 0} + m_{x, x', x_\alpha}$
- (iii) If there is $(d_u, d_{u'}) \in \mathcal{U}_3$, then we have $\mu(y, w) = m_{x, x', 0} + m_{x, x', x_\beta}$
- (iv) If there is $(d_u, d_{u'}) \in \mathcal{U}_4$, then we have $\mu(y, w) = m_{x, x', 0} + m_{x, x', x_\alpha} + m_{x, x', x_\beta}$

Otherwise, $\mu(y, w) = 0$. Moreover, for any $y, w \in c_0$, we get $\mu(y, w) \leq 3$.

Proof. For any $y, w \in c_0$, if $\mu(y, w) \neq 0$; then, we know that $y \sim_L w$ or $y \sim_R w$. If $y \sim_R w$, then $y^{-1} \sim_L w^{-1}$. Since we have $\mu(y, w) = \mu(y^{-1}, w^{-1})$, we can assume $y \sim_L w$.

If $y = d_u x w_0 d_v$, $w = d_{u'} x' w_0 d_{v'}$, for some $u, u', v \in W_0$, and $x, x' \in \Lambda^+$. By (c) in Section 2.2, we also can assume $y = d_u x w_0$, $w = d_{u'} x' w_0$. By (d) in Section 2.2, we know that

$$\mu(y, w) = \sum_{z \in \Lambda^+} m_{x^*, x', z^*} \delta_{w_0 d_u^{-1} d_{u'} w_0 z w_0}, \tag{17}$$

where $x^* = w_0 x^{-1} w_0$, $z^* = w_0 z^{-1} w_0 \in \Lambda^+$.

From this, we get the following:

- (1) If there is $z \in \{0, x_\alpha, x_\beta\}$, such that $(d_u, d_{u'}, z) \in \mathcal{U}_1$, then $\mu(y, w) = m_{x, x', z}$
- (2) If there is $(d_u, d_{u'}) \in \mathcal{U}_2$, then $\mu(y, w) = m_{x, x', 0} + m_{x, x', x_\alpha}$
- (3) If there is $(d_u, d_{u'}) \in \mathcal{U}_3$, then $\mu(y, w) = m_{x, x', 0} + m_{x, x', x_\beta}$
- (4) If there is $(d_u, d_{u'}) \in \mathcal{U}_4$, then $\mu(y, w) = m_{x, x', 0} + m_{x, x', x_\alpha} + m_{x, x', x_\beta}$

Since, we have

$$\begin{aligned}
m_{x^*, x', z^*} &= \dim \text{Hom}_G(V(x^*) \otimes V(x'), V(z^*)) \\
&= \dim \text{Hom}_G(V(z^*) \otimes V(x'), V(x^*)) \\
&= m_{z, x', x^*}.
\end{aligned} \tag{18}$$

By the Freudenthal formula in Lie algebra and Lemma 6, we can get if $z \in \{0, x_\alpha, x_\beta\}$, then we have $\dim V(z)_\lambda$ for any weight of $V(z)$, then by the fact (e) in Section 2.2, we get $m_{x^*, x', z^*} = m_{z, x', x^*} \leq 1$ for $x, x' \in \Lambda^+$ and $z \in \{0, x_\alpha, x_\beta\}$. Then, we get the theorem. $\square$

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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