Research Article

On the Number of Conjugate Classes of Derangements

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The number of conjugate classes of derangements of order n is the same as the number $h(n)$ of the restricted partitions with every portion greater than 1. It is also equal to the number of isotopy classes of $2 \times n$ Latin rectangles. Sometimes the exact value is necessary, while sometimes we need the approximation value. In this paper, a recursion formula of $h(n)$ will be obtained and also will some elementary approximation formulae with high accuracy for $h(n)$ be presented. Although we may obtain the value of $h(n)$ in some computer algebra system, it is still meaningful to find an efficient way to calculate the approximate value, especially in engineering, since most people are familiar with neither programming nor CAS software. This paper is mainly for the readers who need a simple and practical formula to obtain the approximate value (without writing a program) with more accuracy, such as to compute the value in a pocket science calculator without programming function. Some methods used here can also be applied to find the fitting functions for some types of data obtained in experiments.

1. Introduction

Below n is a positive integer greater than 1.

On some occasions, it is necessary to know the number of conjugate classes of derangements.

When generating the representatives of all the isotopy classes of Latin rectangles of order n by some method, we need to know the number of the isotopy classes of $2 \times n$ Latin rectangles for verification. In some cases, we need the approximate value in a simple and efficient method. (When writing a C program to generate the representatives of all the isotopy classes of Latin rectangles of order n , we need to prepare some space in the memory module (RAM) to store the cycle structures of derangements so as to make the program more efficient; otherwise, we have to allocate memory dynamically, which will cost more time in memory addressing when writing and reading data frequently in the particular position in the memory module. So, we need to know the number of the isotopy classes of $2 \times n$ Latin rectangles for verification.)

Let S_n be the symmetry group of the set $X = \{1, 2, \dots, n\}$, i.e., the set (together with the operation of combination) of

the bijections from X to itself. An element σ in the symmetry group S_n is also called a permutation (of order n). If $\sigma \in S_n$, $\sigma(i) \neq i$ ($\forall i \in X$), σ will be called a derangement of order n . If a permutation σ transforms any element in X to a distinct element, then the sequence $[\sigma(1), \sigma(2), \dots, \sigma(n)]$ will also be called a derangement. The number of derangements of order n is denoted by D_n (or $!n$ in some literatures). It is mentioned in nearly every combinatorics textbook that

$$\begin{aligned} D_n &= (n-1)(D_{n-1} + D_{n-2}) \\ &= n! \sum_{i=0}^n \frac{(-1)^i}{i!} \doteq \left\lfloor \frac{n!}{e} + \frac{1}{2} \right\rfloor, \quad n \geq 1, \end{aligned} \quad (1)$$

where $\lfloor x \rfloor$ is the floor function, which stands for the maximum integer that is less than or equal to the real x .

For $x, y \in S_n$, if $\exists z \in S_n$, s.t. $x = zyz^{-1}$, then x and y will be called conjugate and y is called the conjugation of x . Of course the conjugacy relation is an equivalence relation. So, the set of derangements of order n can be divided into some conjugate classes. This paper is mainly concerned on the number of conjugate classes of

derangements of order n . The main method is similar to that described in reference [1].

A matrix of size $k \times n$ ($1 \leq k \leq n-1$) with every row being a reordering of a fixed set of n elements and every column being a part of a reordering of the same set of n elements is called a Latin rectangle. Usually, the set of the n elements is assumed to be $\{1, 2, 3, \dots, n\}$. (In some literatures, the members in a Latin rectangle is assumed in the set $\{0, 1, 2, \dots, n-1\}$.)

A $2 \times n$ Latin rectangle with the first row in increasing order could be considered as a derangement. An isotopy class of $2 \times n$ Latin rectangles will correspond to a unique conjugate class of derangements. So, the number of isotopy classes of $2 \times n$ Latin rectangles is the same as the number of conjugate classes of derangements of order n .

All the members in a conjugate class of derangements in S_n share the same cycle structure. Here, we define the cycle structure of a derangement as the sequence in nondecreasing order of the lengths (with duplicate entries) of all the cycles in the cycle decomposition of the derangement. A cycle structure of a derangement of order n could be considered as an integer solution of the equation as follows:

$$s_1 + s_2 + \dots + s_q = n, \quad (2 \leq s_1 \leq s_2 \leq \dots \leq s_q), \quad (2)$$

where $s_1, s_2, \dots, s_q$ are unknowns.

For a fixed q , designate the number of integer solutions of equation (2) as $H_q(n)$ where q is less than $\lfloor n/2 \rfloor + 1$ (otherwise, $H_q(n)$ is defined by 0), and denote $h(n)$ the number of all the integer solutions of equation (1) for all possible q , i.e.,

$$h(n) = \sum_{q=1}^{\lfloor n/2 \rfloor} H_q(n). \quad (3)$$

So, the number of conjugate classes of derangements of order n is $h(n)$. Since $h(n)$ is the number of a type of restricted partitions, it is tightly connected with the partition number $p(n)$.

Following the notation of [2], denote by $P_q(n)$ the number of integer solutions of equation

$$s_1 + s_2 + \dots + s_q = n, \quad (1 \leq s_1 \leq s_2 \leq \dots \leq s_q), \quad (4)$$

for a fixed q , where $1 \leq q \leq n$, and by $p(n)$ the number of all the (unrestricted) partitions of n . It is clear that (in a lot of articles, $p(n, q)$ is used instead of $P_q(n)$, but in some other literatures, $p(n, q)$ stands for some other number.)

$$p(n) = \sum_{q=1}^n P_q(n). \quad (5)$$

There is a brief introduction of the important results on the partition number (or partition function) $p(n)$ and $P_q(n)$ in reference [2], such as the recursion formula of $p(n)$ and $P_q(n)$. More information about the partition number $p(n)$ may be found in reference [3]. There is a list of some important papers and book chapters on the partition number in [4] (including the "LINKS" and "REFERENCES") and [5]. Reference [1] presented some estimation formulae with high

accuracy for $p(n)$, which are revised from Hardy-Ramanujan's asymptotic formula.

There are also a lot of literatures on the number of some types of restricted partitions of n (such as [6–13]) or on the congruence properties of (restricted) partition function (such as [14–21]).

In [22], we can find many cases of restricted partitions (some of them are introduced in [23, 24] or [25]). One class is concerned on the restriction of the sizes of portions, such as portions restricted to Fibonacci numbers, powers (of 2 or 3), unit, primes, nonprimes, composites, or noncomposites; another class is related to the restriction of the number of portions, such as the cases that the number of parts will not exceed k ; the third class is about the restrictions for both, for example, the cases that the number of parts is restricted while the parts restricted to powers or primes. However, the author has not found too much information on the number $h(n)$, especially on the approximate calculation, although we can find a lot of information on other restricted partition numbers.

The basic method of function fitting (curve fitting) could be found in any textbook of numerical analysis, such as [26, 27]. Some tricks used here may be found in some books of data analysis such as [28]. They may be helpful for understanding the methods described in Section 3.

Section 2 will deduce the recursion formula for $h(n)$ and will show the relation of $h(n)$ and $p(n)$. Subsection 3.1 will deduce the asymptotic formula of $h(n)$ from Hardy-Ramanujan's asymptotic formula of $p(n)$ (mentioned in [1]). This new asymptotic formula $I_g(n)$ coincides with Ingham's result (refer [29, 30]). By bringing in two parameters $C_1(n)$ and $C_2(n)$ in the new asymptotic formula $I_g(n)$, we have reached several estimation formulae for $h(n)$ with high accuracy in Subsection 3.3, using the same ideas described in [1]. By fitting $h(n) - I_g(n)$, we have another two estimation formulae for $h(n)$ in Subsection 3.4. When $n < 100$, we have a more accurate estimation formula for $h(n)$ in Subsection 3.5. The relative errors of these estimation formulae will be presented to shown the accuracy.

2. Some Formulae for $h(n)$

In this section, we will derive a recursive formula from the method mentioned in reference ([2], p. 5355).

By definition, $h(k) = 0$ when $k < 2$, but here we assume that $h(k) = 0$ when $k < 0$, $h(0) = 1$, and $h(1) = 0$, for convenience.

It is mentioned in reference ([2], p. 52) that in 1942, Auluck gave an estimation of $P_q(n)$ by

$$P_q(n) \approx (1/q!) \binom{n-1}{q-1} \text{ when } n \text{ is large enough.}$$

By the same method shown in reference ([2], p. 53, 57), we can obtain the generation function of $h(n)$:

$$\begin{aligned} G(x) &= \sum_{n=0}^{\infty} h(n)x^n = \prod_{i=2}^{\infty} (1-x^i)^{-1} \\ &= \frac{1}{(1-x^2)} \frac{1}{(1-x^3)} \frac{1}{(1-x^4)} \cdots \frac{1}{(1-x^i)} \cdots, \end{aligned} \quad (6)$$

and a formula

$$h(n) = \frac{1}{2\pi i} \oint_C \frac{G(x)}{x^{n+1}} dx, \quad (7)$$

where $h(0) = 1$, $h(1) = 0$, and C is a contour around the original point.

It is difficult to get a simple formula to count the solutions of equation (2) in general. However, for a fixed integer q , the number $H_q(n)$ of solutions is 0 (when $q > \lfloor n/2 \rfloor$) or

$$\begin{aligned} & \sum_{s_1=2}^{\lfloor n/q \rfloor} \sum_{s_2=s_1}^{\lfloor n-s_1/q-1 \rfloor} \cdots \sum_{s_{q-1}=s_{q-2}}^{\lfloor (n-s_1-s_2-\cdots-s_{q-2})/2 \rfloor} 1 \\ &= \sum_{s_1=2}^{\lfloor n/q \rfloor} \sum_{s_2=s_1}^{\lfloor n-s_1/q-1 \rfloor} \cdots \sum_{s_{q-2}=s_{q-3}}^{\lfloor (n-s_1-s_2-\cdots-s_{q-3})/3 \rfloor} \\ & \quad \left(\frac{n-s_1-s_2-\cdots-s_{q-2}}{2} - s_{q-2} + 1 \right) \\ &= P_q(n-q) \left(\text{when } \left\lfloor q \leq \left\lfloor \frac{n}{2} \right\rfloor \right\rfloor \right). \end{aligned} \quad (8)$$

It is not difficult to find out that

$$h(n) = \sum_{q=1}^{\lfloor n/2 \rfloor} H_q(n) = \sum_{q=1}^{\lfloor n/2 \rfloor} P_q(n-q). \quad (9)$$

and there is a recursion for $P_q(n)$ in reference ([2], p. 51) as follows:

$$P_q(n) = \sum_{j=1}^t P_j(n-q), \quad (10)$$

where $t = \min\{q, n-q\}$, so there is no difficulty to obtain the values of $P_q(n)$ and $h(n)$ when n is small.

For the value of $p(n)$, there is a recursion as follows:

$$\begin{aligned} p(n) &= p(n-1) + p(n-2) - p(n-5) \\ &\quad - p(n-7) + \cdots + (-1)^{k-1} p\left(n - \frac{3k^2 \pm k}{2}\right) + \cdots, \\ &= \sum_{k=1}^{k_1} (-1)^{k-1} p\left(n - \frac{3k^2 + k}{2}\right) \\ &\quad + \sum_{k=1}^{k_2} (-1)^{k-1} p\left(n - \frac{3k^2 - k}{2}\right), \end{aligned} \quad (11)$$

where

$$k_1 = \left\lfloor \frac{\sqrt{24n+1} - 1}{6} \right\rfloor, k_2 = \left\lfloor \frac{\sqrt{24n+1} + 1}{6} \right\rfloor, \quad (12)$$

and assume that $p(0) = 1$ (refer [2], p. 55). Here, we assume that $p(x) = 0$ when $x < 0$.

We can obtain the same recursion for $h(n)$ as follows:

$$\begin{aligned} h(n) &= h(n-1) + h(n-2) - h(n-5) - h(n-7) \\ &\quad + \cdots + (-1)^{k-1} h\left(n - \frac{3k^2 \pm k}{2}\right) + \cdots \\ &= \sum_{k=1}^{k_1} (-1)^{k-1} h\left(n - \frac{3k^2 + k}{2}\right) \\ &\quad + \sum_{k=1}^{k_2} (-1)^{k-1} h\left(n - \frac{3k^2 - k}{2}\right), \end{aligned} \quad (13)$$

where k_1 and k_2 are determined by equation (12), and assume that $h(0) = 1$ and $h(k) = 0$ when $k < 0$.

The proof of equation (13) is easy to understand.

By equation (6), we have

$$\left(\sum_{n=0}^{\infty} h(n)x^n \right) \left(\prod_{i=2}^{\infty} (1-x^i) \right) = 1. \quad (14)$$

Since $F(x) = \sum_{n=0}^{\infty} p(n)x^n = \prod_{i=1}^{\infty} (1-x^i)^{-1}$ (where $p(0) = 1$), $(\sum_{n=0}^{\infty} p(n)x^n) (\prod_{i=1}^{\infty} (1-x^i)) = 1$ or

$$\left(\sum_{n=0}^{\infty} p(n)x^n \right) (1-x) \left(\prod_{i=2}^{\infty} (1-x^i) \right) = 1. \quad (15)$$

Comparing equations (14) and (15), we have

$$\begin{aligned} \sum_{n=0}^{\infty} h(n)x^n &= \left(\sum_{n=0}^{\infty} p(n)x^n \right) (1-x) \\ &= \sum_{n=0}^{\infty} (p(n) - p(n-1))x^n, \end{aligned} \quad (16)$$

by assumption $h(k) = p(k) = 0$ when $k < 0$. Hence,

$$h(n) = p(n) - p(n-1), \quad (n = 0, 1, 2, \dots). \quad (17)$$

By equation (17),

$$\begin{aligned} h(n) &= h(n-1) + h(n-2) - h(n-5) - h(n-7) \\ &\quad + \cdots + (-1)^{k-1} h\left(n - \frac{3k^2 \pm k}{2}\right) + \cdots \\ &= \sum_{k=1}^{k_1} (-1)^{k-1} h\left(n - \frac{3k^2 + k}{2}\right) \\ &\quad + \sum_{k=1}^{k_2} (-1)^{k-1} h\left(n - \frac{3k^2 - k}{2}\right). \end{aligned} \quad (18)$$

We can easily obtain the solutions of equation (2) by hand when $n < 13$. By equation (13), we can obtain the number $h(n)$ of solutions of equation (2) by writing a small program in some Computer Algebra System (CAS) software such as “maple,” “maxima,” and “axiom” or some other software likewise (be aware of that 0 is not a valid index value in some software such like maple).

The value of $h(n)$ when $n < 250$ is shown in Table 1 (on page 5) and Table 2 (on page 5). Some values of $H_q(n)$ are shown on Table 3 (on page 5).

Obviously, $h(n) < p(n)$ holds by definition (when $n > 1$). As $p(n)$ grows much more slowly than exponential functions, i.e., for any $r > 1$, $p(n) < r^n$ will hold when n is large enough, which means we cannot estimate $p(n)$ and $h(n)$ by an exponential function. As $p(n)$ grows faster than any power of n , it means we cannot estimate $p(n)$ by a polynomial function (refer [2], p. 53). So, $h(n)$ cannot be estimated by a polynomial function, either.

3. The Estimation of $h(n)$

The recursion formula equation (13) for $h(n)$ is not convenient in practical for a lot of people who do not want to write programs. Sometimes we need the approximation value, such as the cases mentioned in [1], so an estimation formula is necessary.

The graph of data $(n, \ln(h(n)))$ ($n = 60 + 20k, k = 1, 2, \dots, 397$) is shown in Figure 1 on page 5. Here, the data points are displayed by small hollow circles, and the circles are very crowded that we may believe that the circles themselves be a very thick curve if we notice only the right-hand part. In this figure, the data points in the lower left part are sparse (compared with the points in the upper right part), and we may find some hollow circles easily. If there is a curve passes through these hollow circles, we will notice it (as shown in Figure 2 on page 14). However, later in Figure 3, the circles distribute uniformly on a curve, it will be difficult to distinguish the circles, and the curve passes through the centers of them.

The author has not found a practical estimation formula with high accuracy of the number $h(n)$ before.

Actually, it is very difficult to find directly a simple function to fit the data on Figure 1 with high accuracy. The main reason is that the fitting functions obtained by the methods used frequently could not reach satisfying accuracy.

We have several accurate estimation formula of $p(n)$ (refer [1]), such as

$$R'_{h2}(n) = \left[\frac{\exp(\sqrt{2/3} \pi \sqrt{n})}{4\sqrt{3} (n + a_2 \sqrt{n + c_2} + b_2)} + \frac{1}{2} \right], \quad (n \geq 80), \quad (19)$$

and

TABLE 1: The value of $h(n)$ when $1 \leq n \leq 100$.

n	$h(n)$	n	$h(n)$	n	$h(n)$	n	$h(n)$	n	$h(n)$
1	0	21	165	41	7245	61	155038	81	2207851
2	1	22	210	42	8591	62	178651	82	2501928
3	1	23	253	43	10087	63	205343	83	2832214
4	2	24	320	44	11914	64	236131	84	3205191
5	2	25	383	45	13959	65	270928	85	3623697
6	4	26	478	46	16424	66	310962	86	4095605
7	4	27	574	47	19196	67	356169	87	4624711
8	7	28	708	48	22519	68	408046	88	5220436
9	8	29	847	49	26252	69	466610	89	5887816
10	12	30	1039	50	30701	70	533623	90	6638248
11	14	31	1238	51	35717	71	609237	91	7478186
12	21	32	1507	52	41646	72	695578	92	8421448
13	24	33	1794	53	48342	73	792906	93	9476370
14	34	34	2167	54	56224	74	903811	94	10659543
15	41	35	2573	55	65121	75	1028764	95	11981699
16	55	36	3094	56	75547	76	1170827	96	13462885
17	66	37	3660	57	87331	77	1330772	97	15116626
18	88	38	4378	58	101066	78	1512301	98	16967206
19	105	39	5170	59	116600	79	1716486	99	19031739
20	137	40	6153	60	134647	80	1947826	100	21339417

$$R'_{h0}(n) = \left[\frac{\exp(\sqrt{2/3} \pi \sqrt{n})}{4\sqrt{3} (n + C'_2(n))} + \frac{1}{2} \right], \quad 1 \leq n \leq 100, \quad (20)$$

where $a_2 = 0.4432884566$, $b_2 = 0.1325096085$, $c_2 = 0.274078$, and

$$C'_2(n) = \begin{cases} 0.4527092482 \times \sqrt{n + 4.35278} - \\ 0.05498719946, & n = 3, 5, 7, \dots, 99; \\ 0.4412187317 \times \sqrt{n - 2.01699} + \\ 0.2102618735, & n = 4, 6, 8, \dots, 100. \end{cases} \quad (21)$$

By (17), we can obtain $h(n)$ by

$$h_1(n) = \begin{cases} R'_{h0}(n) - R'_{h0}(n-1), & 2 \leq n \leq 80, \\ R'_{h2}(n) - R'_{h2}(n-1), & n > 80. \end{cases} \quad (22)$$

and the error of this formula will not exceed twice of the error of $R'_{h2}(n)$ or $R'_{h0}(n)$. Of course, this formula is not as simple as we want, but the accuracy is very good.

3.1. Asymptotic Formula. As $h(n) = p(n) - p(n-1)$, by Hardy–Ramanujan’s asymptotic formula,

$$p(n) \sim \frac{1}{4n\sqrt{3}} \exp\left(\sqrt{\frac{2}{3}} \pi \sqrt{n}\right), \quad (23)$$

(refer [1, 3, 31–35]), we assume that, when $n \gg 1$, $h(n) \sim \exp(\sqrt{2/3} \pi \sqrt{n})/4\sqrt{3}n - \exp(\sqrt{2/3} \pi \sqrt{n-1})/4\sqrt{3}(n-1)$. So,

TABLE 2: The value of $h(n)$ when $101 \leq n \leq 250$.

n	$h(n)$	n	$h(n)$	n	$h(n)$	n	$h(n)$
101	23911834	116	124763797	131	593224104	146	2608194590
102	26784253	117	138801828	132	656291385	147	2871619379
103	29983571	118	154364067	133	725798623	148	3160747519
104	33552415	119	171594522	134	802411183	149	3477935703
105	37524344	120	190680895	135	886795381	150	3825880113
106	41950627	121	211798491	136	979745604	160	9775430911
107	46873053	122	235172861	137	1082063336	170	24329692015
108	52353455	123	261017329	138	1194696815	180	59110637816
109	58443396	124	289602259	139	1318608064	190	140453804468
110	65217506	125	321186852	140	1454928240	200	326926597263
111	72739457	126	356095340	141	1604811073	210	746521272980
112	81098953	127	394641603	142	1769604112	220	1674422848222
113	90374472	128	437214305	143	1950689437	230	3693304861665
114	100674037	129	484193270	144	2149671688	240	8019313019148
115	112093786	130	536043530	145	2368203564	250	17156634544056

TABLE 3: The number of solutions of equation (1) for different q .

n	$h(n)$	$H_1(n)$	$H_2(n)$	$H_3(n)$	$H_4(n)$	$H_5(n)$	$H_6(n)$	$H_7(n)$
4	2	1	1					
5	2	1	1					
6	4	1	2	1				
7	4	1	2	1				
8	7	1	3	2	1			
9	8	1	3	3	1			
10	12	1	4	4	2	1		
11	14	1	4	5	3	1		
12	21	1	5	7	5	2	1	
13	24	1	5	8	6	3	1	
14	34	1	6	10	9	5	2	1
15	41	1	6	12	11	7	3	1

FIGURE 1: The graph of the data $(n, \ln h(n))$. FIGURE 2: The graph of the data $(n, \ln h(n))$ and the fitting curve $\ln(I_{g1}(n))$.

FIGURE 3: The graph of the data $(n, \pi \exp(\sqrt{2/3} \sqrt{n}) / 12 \sqrt{2h(n)})$ and the fitting curve.

$$\begin{aligned}
 h(n) &\sim \frac{\exp(\sqrt{2/3} \pi \sqrt{n-1})}{4\sqrt{3}} \left(\frac{\exp(\pi \sqrt{2/3} (\sqrt{n} - \sqrt{n-1}))}{n} - \frac{1}{n-1} \right) \\
 &= \frac{\exp(\sqrt{2/3} \pi \sqrt{n-1})}{4\sqrt{3}} \left(\frac{\exp(\pi \sqrt{2/3} / \sqrt{n} + \sqrt{n-1})}{n} - \frac{1}{n-1} \right) \\
 &\sim \frac{\exp(\sqrt{2/3} \pi \sqrt{n})}{4\sqrt{3}} \left(\frac{\exp(\pi \sqrt{2/3} / 2\sqrt{n})}{n} - \frac{1}{n-1} \right) \sim \frac{\exp(\sqrt{2/3} \pi \sqrt{n})}{4\sqrt{3}} \left(\frac{1 + \pi/\sqrt{6n}}{n} - \frac{1}{n-1} \right) \\
 &\sim \frac{\exp(\sqrt{2/3} \pi \sqrt{n})}{4\sqrt{3}} \left(\frac{\pi}{\sqrt{6n^3}} \right) = \frac{\pi \exp(\sqrt{2/3} \pi \sqrt{n})}{12\sqrt{2n^3}}.
 \end{aligned} \tag{24}$$

So,

$$h(n) \sim \frac{\pi}{12\sqrt{2n^3}} \exp(\sqrt{2/3} \pi \sqrt{n}). \tag{25}$$

In coincidence, half a year after the main results were obtained in this paper, the author found an asymptotic formula as follows:

$$P_{a,b}(n) \sim \Gamma(b/a) \pi^{b/a-1} 2^{-(3/2)-(b/2a)} 3^{-(b/2a)} a^{-(1/2)+(b/2a)} n^{-a+b/2a} e^{\pi \sqrt{2n/3a}}, \tag{26}$$

in [30]. When $a=1$ and $b=2$, we will have

$$P_{1,2}(n) \sim \frac{\pi}{12\sqrt{2n^3}} \exp(\pi \sqrt{2/3n}), \tag{27}$$

which coincides with the asymptotic formula obtained here.

Formula (25) will be called the Ingham–Meinardus asymptotic formula in this paper since Daniel mentioned in [30] that this general asymptotic formula (26) was first given by A. E. Ingham in [29] and the proof was refined by G. Meinardus later in another two papers written in German.

Later in this paper, $(\pi/12\sqrt{2n^3}) \exp(\sqrt{2/3} \pi \sqrt{n})$ will be denoted by $I_g(n)$ for short.

It is not satisfying to estimate $h(n)$ by $I_g(n)$ when n is small. The relative error of $I_g(n)$ to $h(n)$ is greater than

6% as shown in Table 4 (on page 6). The round approximation

$$I'_g(n) = \left\lfloor I_g(n) + \frac{1}{2} \right\rfloor, \tag{28}$$

will not change the accuracy distinctly, as shown in Table 5 (on page 6). So, it is necessary to modify the asymptotic formula for better accuracy.

3.2. Method A: Modifying the Exponent. In this subsection, we consider fitting $h(n)$ by $I_{ga} = (\pi/12\sqrt{2n^3}) \exp(\sqrt{2/3} \pi \sqrt{n + C_1(n)})$ or fitting $(n, (3/2\pi^2) (\ln(12\sqrt{2n^3} h(n)/\pi))^2 - n)$ ($n=60+20k$, $k=1, 2, \dots, 397$) by a function

$$f_1(x) \doteq \frac{a_1}{(x+c_1)^{e_1}} + b_1. \tag{29}$$

Let $C_1(n) = (3/2\pi^2) (\ln(12\sqrt{2n^3} h(n)/\pi))^2 - n$. The reason that we fit $C_1(x)$ by the function in the form displayed in (29) is the same as that described in Section 3 of [1] (although the data differ distinctly). Many other types of functions have been tested, but they cannot fit these data very well.

However, here it is not valid to obtain the constants in $f_1(n)$ by the iteration method described in reference [1].

The graph of the data $(n, (3/2\pi^2) (\ln(12\sqrt{2n^3} h(n)/\pi))^2 - n)$ ($n=60+20k$, $k=1, 2, \dots, 397$) is shown in Figure 4 on page 7.

TABLE 4: The relative error of $I_g(n)$ to $h(n)$ when $n \leq 1000$.

n	Rel-Err (%)	n	Rel-Err (%)	n	Rel-Err (%)	n	Rel-Err (%)	n	Rel-Err (%)
1		16	50.30	40	32.10	220	13.10	520	8.39
2	146.24	17	56.82	50	28.60	240	12.50	540	8.23
3	202.89	18	46.69	60	25.90	260	12.00	560	8.08
4	95.59	19	52.75	70	23.90	280	11.50	580	7.93
5	156.43	20	44.94	80	22.30	300	11.10	600	7.79
6	68.62	21	48.48	90	20.90	320	10.80	640	7.54
7	121.38	22	43.47	100	19.80	340	10.40	680	7.31
8	65.43	23	46.00	110	18.80	360	10.10	720	7.10
9	88.38	24	41.09	120	18.00	380	9.86	760	6.91
10	62.58	25	43.68	130	17.20	400	9.60	800	6.73
11	79.47	26	39.93	140	16.60	420	9.36	840	6.56
12	53.29	27	41.27	150	16.00	440	9.14	880	6.41
13	70.98	28	38.50	160	15.40	460	8.93	920	6.27
14	53.12	29	39.70	180	14.50	480	8.74	960	6.13
15	60.35	30	37.00	200	13.70	500	8.56	1000	6.01

TABLE 5: The relative error of $[I_g(n) + 1/2]$ to $h(n)$ when $n \leq 30$.

n	Rel-Err	n	Rel-Err	n	Rel-Err	n	Rel-Err	n	Rel-Err
1	—	7	125%	13	70.83%	19	52.38%	25	43.60%
2	100%	8	71.43%	14	52.94%	20	45.26%	26	39.96%
3	200%	9	87.50%	15	60.98%	21	48.48%	27	41.29%
4	100%	10	66.67%	16	50.91%	22	43.33%	28	38.56%
5	150%	11	78.57%	17	57.58%	23	45.85%	29	39.67%
6	75%	12	52.38%	18	46.59%	24	40.94%	30	37.05%

FIGURE 4: The graph of the data $(n, (3/2\pi^2)(\ln(12\sqrt{2n^3}h(n)/\pi))^2 - n)$.

First, we try to fit $(n, (3/2\pi^2)(\ln(12\sqrt{2n^3}h(n)/\pi))^2 - n)$ ($n = 60 + 20k$, $k = 1, 2, \dots, 397$) by a function in the form as follows:

$$f_1(x) \doteq \frac{a_1}{\sqrt{x+c_1}} + b_1. \quad (30)$$

That means we have assumed that $e_1 = 1/2$, temporarily. We will explain the reason in Subsection 3.2.3.

The average error of $f_1(x)$ is

$$\begin{aligned} E_1 &= \sqrt{\frac{1}{K_1} \sum_n (C_1(n) - f_1(n))^2} \\ &= \sqrt{\frac{1}{K_1} \sum_n \left(C_1(n) - \frac{a_1}{\sqrt{n+c_1}} - b_1 \right)^2} \\ E_1 &= \sqrt{\frac{1}{K_1} \sum_{k=1}^{K_1} \left(C_1(60+20k) - \frac{a_1}{\sqrt{60+20k+c_1}} - b_1 \right)^2}, \end{aligned} \quad (31)$$

where $K_1 = 397$ and n ranges from 80 to 8000, by step 20. Here, only a_1 , b_1 , and c_1 are unknown, so we can consider E_1 as a function of the variables (a_1, b_1, c_1) .

We want to find a triple (a_1, b_1, c_1) such that E_1 reaches its minimum or to make E_1 as small as possible.

Since a lot of functions have several local minimum points, it is necessary to find out whether $E_1 = E_1(a_1, b_1, c_1)$ has more than one local minimum before we start to calculate the minimum point by the numeral method. However, $E_1 = E_1(a_1, b_1, c_1)$ is too complicate, and it is very difficult to know all the critical points in the range we are considering.

3.2.1. Preparation Work. For a given pair (a_1, c_1) , by the property of the arithmetic mean (for some given data x_i ($i=1, 2, \dots, k$; $x_i \in \mathbb{R}$), the function $s(t) = \sqrt{(1/k) \sum_{i=1}^k (x_i - t)^2}$ reaches its minimum at $t = (1/k) \sum_{i=1}^k x_i$), it is clear that E_1 reaches its minimum when

$$\begin{aligned} b_1 &= \frac{1}{K_1} \sum_n \left(C_1(n) - \frac{a_1}{\sqrt{n+c_1}} \right), \\ &= \overline{C_1} - \frac{1}{K_1} \sum_{n'} \frac{a_1}{\sqrt{n'+c_1}}, \end{aligned} \quad (32)$$

where $\overline{C_1} = (1/K_1) \sum_{n'} C_1(n')$.

Let $E_1 = \sqrt{(1/K_1)\sum_n (C_1(n) - a_1/\sqrt{n+c_1} - \overline{C_1} + (1/K_1)\sum_{n'} a_1/\sqrt{n'+c_1})^2}$ be the average error of the fitting function $f_1(x) \doteq a_1/\sqrt{x+c_1} + \overline{C_1} - (1/K_1)\sum_{n'} a_1/\sqrt{n'+c_1}$. (Here, a_1 and b_1 are undetermined coefficients.)

Let $G_1 = E_1^2 = (1/K_1)\sum_n (C_1(n) - a_1/\sqrt{n+c_1} - \overline{C_1} + (1/K_1)\sum_{n'} a_1/\sqrt{n'+c_1})^2$. Here, only a_1 and b_1 are unknowns. G_1 could be considered as a function of (a_1, c_1) . In order to find the minimum point of G_1 , we can draw the figure of the function $G_1 = G_1(a_1, c_1)$ (in a cube coordinate system with axis a_1, c_1 , and G_1) as shown in Figures 5–7. Figures 8 and 9 are the projection of the graph of (a_1, c_1, G_1) (when $-100 \leq a_1 \leq 100$ and $-50 \leq c_1 \leq 100$, which is a part of a surface) on the $a_1 - G_1$ plane (spanned by the axis a_1 and G_1) and $c_1 - G_1$ plane, respectively.

From these figures, we can find out that the influence of c_1 to G_1 is much less than that of a_1 . In Figure 10, we find that when G_1 reaches its minimum, a_1 is between 0.50 and 0.53, but there is not a definite range for c_1 .

It is possible that for different range of c_1 , the range of a_1 when G_1 reaches its minimum will be different. However, considering that $a_1/\sqrt{n+c_1} + b_1$ is a real, c_1 should be greater than -1 in theory. (For the fitting data used here, c_1 should be greater than -80 .) From Figure 8, we can see that G_1 touches its bottom when $-15 \leq a_1 \leq 15$. Although we cannot see clearly the exact value of a_1 in the minimum points, we can draw another figure of (a_1, c_1, G_1) when $-15 \leq a_1 \leq 15$ and $-1 \leq c_1 \leq 100$ to observe more details (the figure is not presented here), and then we will find that the more detailed range of a_1 for the minimum points is $[-3, 3]$ in the new figure (not presented), and next we will draw the figures of (a_1, c_1, G_1) when $-3 \leq a_1 \leq 3$, $0 \leq a_1 \leq 1$, $0.2 \leq a_1 \leq 0.8$, or $0.45 \leq a_1 \leq 0.6$, respectively, while $-1 \leq c_1 \leq 100$, we will find the range of a_1 of the minimum points of G_1 is about $[0.50, 0.53]$. The projections of Figure 7 of (a_1, c_1, G_1) when $0.45 \leq a_1 \leq 0.6$ and $-1 \leq c_1 \leq 100$ are shown in Figures 11 and 10.

In Figure 9, for the curves on the bottom, G_1 decreases with c_1 at first then increases with c_1 , but it is difficult to find the critical point in the figure since different curves have different critical points.

Although we cannot find a satisfying value of a_1 or c_1 from the figures to construct a fitting function f_1 , these pictures show that the figure of $G_1 = G_1(a_1, c_1, G_1)$ has only one bottom in the domain we are considering, unlike the figure of another function shown in Figure 12, so the existence of the minimum point is almost confirmed; therefore, we are confident to find the value of a_1 or c_1 in the minimum point by the numerical method. This guarantees the validity of the numerical calculation by loop in the next step.

3.2.2. Find c_1 . On the other hand, by the least square method, to fit the data (x_k, y_k) ($k=1, 2, \dots, K_1$) by a linear function $y = a \times x + b$, the result is that

FIGURE 5: The graph of the data (a_1, c_1, G_1) when $-100 \leq a_1 \leq 100$ and $-50 \leq c_1 \leq 100$.

FIGURE 6: The graph of the data (a_1, c_1, G_1) when $-100 \leq a_1 \leq 100$ and $-50 \leq c_1 \leq 100$ version 2.

$$a = \frac{\overline{x_n \cdot y_n} - \overline{x_n} \cdot \overline{y_n}}{\overline{x_n^2} - (\overline{x_n})^2}, \quad (33)$$

$$b = \frac{\overline{x_n \cdot x_n \cdot y_n} - \overline{x_n^2} \cdot \overline{y_n}}{(\overline{x_n})^2 - \overline{x_n^2}} = \overline{y_n} - \overline{x_n} \cdot a, \quad (34)$$

FIGURE 7: The graph of the data (a_1, c_1, G_1) when $0.45 \leq a_1 \leq 0.60$ and $-1 \leq c_1 \leq 100$.

FIGURE 9: The projection of the graph of the data (a_1, c_1, G_1) on the $c_1 - G_1$ plane when $-100 \leq a_1 \leq 100$ and $-50 \leq c_1 \leq 100$.

FIGURE 8: The projection of the graph of the data (a_1, c_1, G_1) on the $a_1 - G_1$ plane when $-100 \leq a_1 \leq 100$ and $-50 \leq c_1 \leq 100$.

FIGURE 10: The projection of the graph of the data (a_1, c_1, G_1) on the $c_1 - G_1$ plane when $0.45 \leq a_1 \leq 0.60$ and $-1 \leq c_1 \leq 100$.

where $\overline{x_n} = (1/K_1) \sum_{i=1}^{K_1} x_i$ is the average value of x_i , $\overline{y_n} = (1/K_1) \sum_{i=1}^{K_1} y_i$, $\overline{x_i^2} = (1/K_1) \sum_{i=1}^{K_1} x_i^2$, and $\overline{x_n \cdot y_n} = (1/K_1) \sum_{i=1}^{K_1} x_i \cdot y_i$. By definition, $(\overline{x_n})^2 = ((1/K_1) \sum_{i=1}^{K_1} x_i)^2$ is the square of

the average value of x_n . So, by the least square method, a and b are uniquely determined by the given data (x_k, y_k) ($k=1, 2, \dots, K_1$).

FIGURE 11: The projection of the graph of the data (a_1, c_1, G_1) on the $a_1 - G_1$ plane when $0.45 \leq a_1 \leq 0.60$ and $-1 \leq c_1 \leq 100$.

FIGURE 12: Example of a surface with more than one bottom.

For every given value of c_1 (greater than -80), we can fit $(n, C_1(n))$ ($n = 60 + 20k$, $k = 1, 2, \dots, 397$) by a function $f_1(x) \doteq a_1/\sqrt{x+c_1} + b_1$ by the least square method if we consider $1/\sqrt{60+20k+c_1}$ and $C_1(60+20k)$ as x_k and y_k , respectively. Then,

$$\begin{aligned}\overline{x_n} &= \frac{1}{K_1} \sum_{k=1}^{K_1} x_k = \frac{1}{K_1} \sum_{k=1}^{K_1} \frac{1}{\sqrt{60+20k+c_1}}, \\ \overline{y_n} &= \frac{1}{K_1} \sum_{k=1}^{K_1} y_k = \frac{1}{K_1} \sum_{k=1}^{K_1} C_1(60+20k), \\ \overline{x_n^2} &= \frac{1}{K_1} \sum_{k=1}^{K_1} x_k^2 = \frac{1}{K_1} \sum_{k=1}^{K_1} \frac{1}{60+20k+c_1}, \\ \overline{x_n \cdot y_n} &= \frac{1}{K_1} \sum_{k=1}^{K_1} x_k \cdot y_k = \frac{1}{K_1} \sum_{k=1}^{K_1} \frac{C_1(60+20k)}{\sqrt{60+20k+c_1}}, \\ a_1 &= \frac{\overline{x_n \cdot y_n} - \overline{x_n} \cdot \overline{y_n}}{\overline{x_n^2} - (\overline{x_n})^2}, \\ b_1 &= \overline{y_n} - \overline{x_n} \cdot a_1.\end{aligned}\tag{35}$$

So, a_1 and b_1 could both be considered as functions of c_1 , denoted by $a_1 = a_1(c_1)$ and $b_1 = b_1(c_1)$, since they are uniquely determined by c_1 with the given data.

Then, $G_2 = E_1^2 = (1/K_1) \sum_{k=1}^{K_1} (C_1(60+20k) - a_1(c_1)/\sqrt{60+20k+c_1} - b_1(c_1))^2$ is a function of c_1 .

It will cost some time to plot the figure of the function $G_2 = G_2(c_1)$ in a CAS software.

If we plot the figure of the function $G_2 = G_2(c_1)$ on the coordinates (as shown in Figures 13–15), we will find that G_2 reaches its minimum when $c_1 \approx -3.2594807$. In Figure 15, we find that the curve of $G_2 = G_2(c_1)$ is not so smooth. The reason is that we hold up 18 significant digits in the process. If we compute more significant digits in the process, the curve on Figure 15 will be more smooth, at the cost of much more time. By writing a small program (since the default function to find the minimum provided by the software Maple 18 is unable to deal with such a complicated function $G_2 = G_2(c_1)$ involving so much data), we can obtain a more accurate value of the critical point as follows:

$$c_1 = -3.259480684.\tag{36}$$

When the value of c_1 is obtained, we can find the value of a_1 and b_1 by the least square method without difficulty, i.e.,

$$\begin{aligned}a_1 &= 0.5097429624, \\ b_1 &= -1.453552800.\end{aligned}\tag{37}$$

However, here c_1 is less than -1 , so the estimation formula for $h(n)$ constructed from these coefficients is invalid when $n < 4$.

FIGURE 13: The graph of the function $G_2 = G_2(c_1)$ when $-50 \leq c_1 \leq 100$.

FIGURE 14: The graph of the function $G_2 = G_2(c_1)$ when $-4 \leq c_1 \leq -2$.

3.2.3. Confirm e_1 . In [1], we fit $C_1(n) = 3/2 \cdot (\ln(4n\sqrt{3}p(n)))^2/\pi^2 - n$ by a function $f_1(x) \doteq a_1/(n+c_1)^{e_1} + b_1$ when estimating $p(n)$ and found that $e_1 \approx 0.50$ by iteration. Here, the iteration method does not work well, so we fit $C_1(n) = (3/2\pi^2)(\ln(12\sqrt{2n^3}h(n)/\pi))^2 - n$ by a function $f_1(x) \doteq a_1/\sqrt{n+c_1} + b_1$ directly, which means that we have assumed that $e_1 = 1/2$. Here, we may doubt that whether $e_1 = 0.5$ is the best option for us?

Here, we use the same idea described in Subsection 3.2.1.

For every pair (e_1, c_1) , we can obtain corresponding a_1 and b_1 by the least square method, just like (33) and (34), except that here $x_k = 1/(60 + 20k + c_1)^{e_1}$.

So, the square of the average error

$$G_3 = E_1^2 = \frac{1}{K_1} \sum_n \left(C_1(n) - \frac{a_1(e_1, c_1)}{(n+c_1)^{e_1}} - b_1(e_1, c_1) \right)^2, \quad (38)$$

could be considered as a function of e_1 and c_1 as both $a_1 = a_1(e_1, c_1)$ and $b_1 = b_1(e_1, c_1)$ could be expressed by certain elementary functions of e_1 and c_1 .

If we draw the figure of the function $G_3 = G_3(e_1, c_1)$, we will find that the surface has only one bottom when $0.1 \leq e_1 \leq 0.9$ and $-50 \leq c_1 \leq 100$, as shown in Figure 16. However, the process to draw the figure is time-consuming. It costs more than 5 hours on a notebook (ThinkPad E40 Edge, with 6 GB RAM and AMD P360 Dual-Core Processor 2.30 GHz) by Maple 18 in Ubuntu 14.04.1 system.

After that, by writing another program, we can obtain the approximate value of (e_1, c_1) where G_3 touches the bottom, i.e., $e_1 \approx 0.494$ and $c_1 \approx -4.85$, when 18 significant digits are involved in the process, which still costs tens of minutes. Considering that we have used only a small part of data, we cannot afford the time for computing more significant digits in process, and the computing is so complicated; hence, error accumulation effect is considerable, so we choose $e_1 = 0.50$ while it differs very little with 0.494. Another reason is that we prefer simple exponent, as the time spent on computing a square root is much less than that to compute a power with exponent 0.494 in general. Here, the value of $c_1 \approx -4.85$ is obviously different from the value obtained at the end of Subsection 3.2.2 because of the little difference on e_1 . Therefore, it will be fine to use the result in Subsection 3.2.2).

In this figure, the points are shown as small circles which are very close to each other. These crowded circles seem like a thick curve. A fitting curve passes through the center of these circles. The fitting curve might not be found in reduce printing. That means the curve fits the points (displayed as circles) very well.

3.2.4. The Result. By the value $a_1 = 0.5097429624$, $b_1 = -1.453552800$, and $c_1 = -3.259480684$ obtained in Subsection 3.2.2, we will have a fitting function $f_1(x) \doteq a_1/\sqrt{x+c_1} + b_1$.

The graph of $f_1(x)$ (when $-80 \leq x \leq 8000$) and the graph of the data $(n, (3/2\pi^2)(\ln(12\sqrt{2n^3}h(n)/\pi))^2 - n)$ ($n = 60 + 20k$, $k = 1, 2, \dots, 397$) are shown in Figure 17. It shows that $f_1(n)$ fits $(3/2\pi^2)(\ln(12\sqrt{2n^3}h(n)/\pi))^2 - n$ very well.

Then, we could fit $h(n)$ by

$$I_{ga}(n) = \frac{\pi \exp(\sqrt{2/3} \pi \sqrt{n+a_1/\sqrt{x+c_1}+b_1})}{12\sqrt{2n^3}}, \quad n \geq 4. \quad (39)$$

The graph of the function $\ln(I_{ga}(x))$ and the graph of the data $(n, \ln(h(n)))$ ($n = 60 + 20k$, $k = 1, 2, \dots, 397$) is shown in Figure 18. It seems that $I_{ga}(n)$ fits $h(n)$ very well.

The relative error of I_{ga} is shown in Table 6 (when $n \leq 1000$) and Figure 19 (when $1000 < n \leq 10000$).

When $n < 20$, the relative error of I_{ga} is still greater than 2%. Although it is much better than the error of I_g , it is not as good as expected when $n < 40$. If we take the round approximation by

$$I'_{ga}(n) = \left[\frac{\pi \exp(\sqrt{2/3} \pi \sqrt{n+a_1/\sqrt{x+c_1}+b_1})}{12\sqrt{2n^3}} + \frac{1}{2} \right], \quad n \geq 4, \quad (40)$$

FIGURE 15: The graph of the function $G_2 = G_2(c_1)$ when $-3.259483 \leq c_1 \leq -3.259478$.

FIGURE 16: The graph of the function $G_3 = G_3(e_1, c_1)$ when $0.1 \leq e_1 \leq 0.9$ and $-50 \leq c_1 \leq 100$.

the relative error will be obviously smaller with a few exceptions, as shown in Table 7.

Later, we will find out that it is obviously greater than the relative error of I_{g1} and I_{g2} obtained in the next subsection by modifying the denominator part; when $4000 < n < 10000$, the relative error of I_{ga} is about 1000 times of that of I_{g2} .

When $30 < n \leq 1000$, the relative error of $[I_{ga}(n) + 1/2]$ is close to that of $I_{ga}(n)$.

3.3. Method B: Modifying the Denominator. Since $h(n) \sim (\pi/12\sqrt{2n^3})\exp(\sqrt{2/3}\pi\sqrt{n})$, we consider estimating $h(n)$ by $(\pi/12\sqrt{2C_3(n)})\exp(\sqrt{2/3}\pi\sqrt{n})$ (i.e., fit $\pi^2 \exp(2\pi\sqrt{2/3}\sqrt{n})/288h^2(n)$ by a function $C_3(n)$), where $C_3(x)$ is a cubic function or a function like

$$ax^3 + bx^{2.5} + cx^2 + dx^{1.5} + ex + fx^{0.5} + g. \quad (41)$$

FIGURE 17: The graph of the data $(n, (3/2\pi^2)(\ln(12\sqrt{2n^3}h(n)/\pi))^2 - n)$ and the fitting curve.

FIGURE 18: The graph of the data $(n, \ln(h(n)))$ and the fitting curve.

TABLE 6: The relative error of $I_{ga}(n)$ to $h(n)$ when $n \leq 1000$.

n	Rel-Err	n	Rel-Err	n	Rel-Err	n	Rel-Err	n	Rel-Err
1	—	16	-2.16%	40	-0.18%	220	$2.62E-06$	520	$-8.93E-07$
2	—	17	3.29%	50	$-7.12E-04$	240	$1.60E-06$	540	$-9.04E-07$
3	—	18	-2.34%	60	$-3.12E-04$	260	$8.76E-07$	560	$-8.75E-07$
4	8.89%	19	2.71%	70	$-1.01E-04$	280	$3.54E-07$	580	$-8.50E-07$
5	34.03%	20	-1.64%	80	$-3.21E-05$	300	$3.00E-09$	600	$-8.46E-07$
6	-10.34%	21	1.64%	90	$-2.28E-06$	320	$-2.87E-07$	640	$-8.04E-07$
7	21.05%	22	-1.00%	100	$1.16E-05$	340	$-4.83E-07$	680	$-7.54E-07$
8	-6.99%	23	1.51%	110	$1.51E-05$	360	$-6.16E-07$	720	$-7.43E-07$
9	8.67%	24	-1.20%	120	$1.51E-05$	380	$-7.26E-07$	760	$-6.98E-07$
10	-4.02%	25	1.29%	130	$1.40E-05$	400	$-8.05E-07$	800	$-6.69E-07$
11	8.17%	26	-0.72%	140	$1.22E-05$	420	$-8.47E-07$	840	$-6.06E-07$
12	-5.88%	27	0.83%	150	$1.04E-05$	440	$-8.98E-07$	880	$-5.64E-07$
13	6.76%	28	-0.58%	160	$8.81E-06$	460	$-8.97E-07$	920	$-5.65E-07$
14	-2.93%	29	0.80%	180	$6.04E-06$	480	$-8.94E-07$	960	$-5.17E-07$
15	3.07%	30	-0.59%	200	$4.02E-06$	500	$-9.00E-07$	1000	$-4.75E-07$

FIGURE 19: The relative error of $I_{ga}(n)$ when $1000 \leq n \leq 10000$, step 300.TABLE 7: The relative error of $[I_{ga}(n) + 1/2]$ to $h(n)$ when $n \leq 30$.

n	Rel-Err	n	Rel-Err	n	Rel-Err	n	Rel-Err	n	Rel-Err
1	—	7	25%	13	8.33%	19	2.86%	25	1.31%
2	—	8	0	14	-2.94%	20	-1.46%	26	-0.63%
3	—	9	12.5%	15	2.44%	21	1.82%	27	0.87%
4	0	10	0	16	-1.82%	22	-0.95%	28	-0.56%
5	50%	11	7.14%	17	3.03%	23	1.58%	29	0.83%
6	0	12	-4.76%	18	-2.27%	24	-1.25%	30	-0.58%

However, the results are worse as the relative errors are obviously much greater than the relative error of $I_g(n)$ when $n < 350$.

Then, we consider estimating $h(n)$ by $(\pi/12\sqrt{2}C_4(n))\exp(\sqrt{2/3}\pi\sqrt{n})$ or fit $\pi\exp(\pi\sqrt{2/3}\sqrt{n})/12\sqrt{2}h(n)$ by a function as follows:

$$C_4(n) = a_4n^{1.5} + b_4n + c_4n^{0.5} + d_4. \quad (42)$$

The result is very good. The graph of the data $(n, \pi\exp(\pi\sqrt{2/3}\sqrt{n})/12\sqrt{2}h(n))$ and the fitting curve $C_4(n)$ is shown in Figure 3. Here, the fitting curve is displayed by a thick continuous curve, which lies in the middle of the area the circles occupied. Since the circles are

too crowded, the circles themselves look like a very thick curve.

The values of the coefficients in the expression of $C_4(n)$ are as follows:

$$\begin{aligned} a_4 &= 1.000010809, \\ b_4 &= 1.862505234, \\ c_4 &= 1.169930087, \\ d_4 &= -0.7005460222. \end{aligned} \quad (43)$$

The value of a_4 is very close to 1, which means that this fitting function coincides with the Ingham–Meinardus asymptotic formula very well.

So, we have an estimation formula as follows:

$$h(n) \sim I_{g1}(n) = \frac{\pi}{12\sqrt{2}C_4(n)} \exp\left(\sqrt{\frac{2}{3}}\pi\sqrt{n}\right). \quad (44)$$

We may call it the Ingham–Meinardus revised estimation formula (1). The graph of $\ln(I_{g1}(n))$ is shown in Figure 2, together with the data points of $(n, \ln h(n))$. This revised estimation formula is much more accurate than the asymptotic formula. The relative error is less than 1×10^{-6} when $n > 2000$ (as shown in Figure 20 on page 14) and less than 3 when $n \geq 30$ (as shown in Table 8 on page 14). The relative error of the round approximation $I_{g1}'(n) = \lfloor I_{g1}(n) + 1/2 \rfloor$ is shown in Table 9 on page 13.

However, equation (44) is not so satisfying when $n < 30$, especially when $n < 15$ as the relative error is not negligible for some value of n .

As we already know that $h(n) \sim (\pi/12\sqrt{2}n^3)\exp(\sqrt{2/3}\pi\sqrt{n})$, or $n^{3/2} \sim (\pi/12\sqrt{2}h(n))\exp(\sqrt{2/3}\pi\sqrt{n})$, which means that when fitting $\pi \exp(\pi\sqrt{2/3}\sqrt{n})/12\sqrt{2}h(n)$ by a function $C_4(n)$ shown in equation (42), the coefficient a_4 should be exactly 1; hence, we should fit $\pi \exp(\pi\sqrt{2/3}\sqrt{n})/12\sqrt{2}h(n)$ by a function $C_4'(n) = n^{3/2} + b_5n + c_5n^{1/2} + d_5$ or fit $\pi \exp(\pi\sqrt{2/3}\sqrt{n})/12\sqrt{2}h(n) - n^{3/2}$ by a function

$$C_5(n) = b_5n + c_5n^{1/2} + d_5. \quad (45)$$

The graph of the data $(n, \pi \exp(\pi\sqrt{2/3}\sqrt{n})/12\sqrt{2}h(n) - n^{3/2})$ is shown in Figure 21 on page 14 (together with the figure of the fitting function $C_5(n)$ generated by the least square method).

The values of the coefficients in equation (45) are as follows:

$$\begin{aligned} b_5 &= 1.864260743, \\ c_5 &= 1.084436400, \\ d_5 &= 0.4754177757. \end{aligned} \quad (46)$$

So, we have another estimation formula for $h(n)$ as follows:

$$h(n) \sim I_{g2}(n) = \frac{\pi \exp(\sqrt{2/3}\pi\sqrt{n})}{12\sqrt{2}(n^{3/2} + C_5(n))}. \quad (47)$$

We may call it the Ingham–Meinardus revised estimation formula (2). The graph of $\ln(I_{g2}(n))$ is nearly the same as that of $\ln(I_{g1}(n))$ shown in Figure 2. The second revised estimation formula is much more accurate than the first one. The relative error is less than 2×10^{-9} when $n > 3000$ (as shown in Figure 22 on page 14), about 1/500 of the relative error of $I_{g1}(n)$. When $n < 10$, the relative error is also distinctly less than that of $I_{g1}(n)$ (as shown in Table 10 on page 15). The relative error of the round approximation $I_{g2}'(n) = \lfloor I_{g2}(n) + 1/2 \rfloor$ is shown in Table 11 (on page 15).

It should be mentioned that in Figure 21 on page 14, the graph of the data points lies in a line, so we might be willing to fit this line by a first-order equation. The result is

$$C_5'(n) = 1.873818457 \times n + 27.08318017. \quad (48)$$

If we use this fitting function instead of $C_5(n)$ generated before, the relative error to fit $h(n)$ will be about 10000 times more, that is, about 20 times more than that of $I_{g1}(n)$. So, we did not use linear function to fit the data $(n, \pi \exp(\pi\sqrt{2/3}\sqrt{n})/12\sqrt{2}h(n) - n^{3/2})$ before.

3.4. Method C: Fitting $I_g(n) - h(n)$. We wonder whether we can fit $I_g(n) - h(n)$ by a function $r(n)$ and then estimate $h(n)$ by $I_g(n) - r(n)$ which may be believed more accurate than $I_{g2}(n)$ at the price of being more complicated.

By the same tricks used at the beginning of this subsection, we will have

$$I_g(n) - I_g(n-t) \sim \frac{t\pi^2}{24\sqrt{3}n^2} \exp\left(\sqrt{\frac{2}{3}}\pi\sqrt{n}\right). \quad (t \ll n), \quad (49)$$

So, we may fit $I_g(n) - h(n)$ by $(\pi^2/24\sqrt{3}C_6(n))\exp(\sqrt{2/3}\pi\sqrt{n})$ where $C_6(n)$ is a quadratic function or a function like

$$an^2 + bn^{1.5} + cn + dn^{0.5} + e. \quad (50)$$

That means, we can fit $\pi^2 \exp(\sqrt{2/3}\pi\sqrt{n})/24\sqrt{3}(I_g(n) - h(n))$ by a function $C_6(n)$. However, the result is useless. Although $C_6(n)$ will fit the data $\pi^2 \exp(\sqrt{2/3}\pi\sqrt{n})/24\sqrt{3}(I_g(n) - h(n))$ very well, the relative error of $I_g(n) - (\pi^2/24\sqrt{3}C_6(n))\exp(\sqrt{2/3}\pi\sqrt{n})$ to $h(n)$ is much greater than that of $I_{g1}(n)$ or $I_{g2}(n)$, and the relative error differs very little with that of $I_g(n)$ when n is small. Besides, the formula $I_g(n) - (\pi^2/24\sqrt{3}C_6(n))\exp(\sqrt{2/3}\pi\sqrt{n})$ is much more complicated than $I_{g1}(n)$ and $I_{g2}(n)$.

If we fit $I_g(n) - h(n)$ by $(\pi^2/24\sqrt{3}(n^2 + E_6(n)))\exp(\sqrt{2/3}\pi\sqrt{n})$ with the trick described in subsection 3.3, where $E_6(n)$ is a function in the form as follows:

$$bn^{1.5} + cn + dn^{0.5} + e, \quad (51)$$

as we already know the coefficient of n^2 should be 1 in theory. The result will be a little better, but useless too. The accuracy is not as good as that of $I_{g0}(n)$.

Then, we consider fitting $\pi^2 \exp(\sqrt{2/3}\pi\sqrt{n})/24\sqrt{3}n^2(I_g(n) - h(n))$ by a function $C_7(n)$. If $C_7(n)$ is in the form $a/n + b$ or $a/n + b/n^2 + c$, the result is useless, too. If $C_7(n)$ is in the form $a/n^{0.5} + b$, it will be barely satisfactory. If $C_7(n)$ is in the form $a/n^{0.5} + b/n + c/n^{1.5} + d/n^2 + e$ or $a/n^{0.5} + b/n + c/n^{1.5} + e$, the result will be much better than the previous forms, but the accuracy (when estimating $h(n)$) is not as good as that of $I_{g1}(n)$ and $I_{g2}(n)$.

The result of $C_7(n)$ is

$$C_{7a}(n) = \frac{0.8782296151}{n^{0.5}} + \frac{0.2567016063}{n} - \frac{3.580442785}{n^{1.5}} + \frac{21.28305831}{n^2} + 0.6879945549, \quad (52)$$

FIGURE 20: The relative error of $I_{g_1}(n)$ when $1000 \leq n \leq 10000$, step 300.TABLE 8: The relative error of $I_{g_1}(n)$ to $h(n)$ when $n \leq 1000$.

n	Rel-Err (%)	n	Rel-Err (%)	n	Rel-Err	n	Rel-Err	n	Rel-Err
1	—	16	-1.63	40	$-2.21E-05$	220	$7.23E-05$	520	$1.04E-06$
2	-7.23	17	3.82	50	$5.35E-04$	240	$5.74E-05$	540	$3.01E-07$
3	29.97	18	-1.87	60	$6.16E-04$	260	$4.59E-05$	560	$-3.53E-07$
4	-8.44	19	3.18	70	$6.15E-04$	280	$3.68E-05$	580	$-8.95E-07$
5	27.94	20	-1.21	80	$5.35E-04$	300	$2.97E-05$	600	$-1.37E-06$
6	-11.61	21	2.06	90	$4.56E-04$	320	$2.40E-05$	640	$-2.10E-06$
7	20.76	22	-0.61	100	$3.89E-04$	340	$1.93E-05$	680	$-2.64E-06$
8	-6.74	23	1.89	110	$3.30E-04$	360	$1.55E-05$	720	$-2.97E-06$
9	9.21	24	-0.85	120	$2.80E-04$	380	$1.24E-05$	760	$-3.20E-06$
10	-3.44	25	1.63	130	$2.40E-04$	400	$9.79E-06$	800	$-3.36E-06$
11	8.85	26	-0.40	140	$2.06E-04$	420	$7.63E-06$	840	$-3.43E-06$
12	-5.28	27	1.14	150	$1.78E-04$	440	$5.82E-06$	880	$-3.51E-06$
13	7.42	28	-0.29	160	$1.55E-04$	460	$4.32E-06$	920	$-3.49E-06$
14	-2.35	29	1.08	180	$1.18E-04$	480	$3.04E-06$	960	$-3.43E-06$
15	3.66	30	-0.32	200	$9.20E-05$	500	$1.97E-06$	1000	$-3.37E-06$

TABLE 9: The relative error of $[I_{g_1}(n) + 1/2]$ to $h(n)$ when $n \leq 30$.

n	Rel-Err	n	Rel-Err	n	Rel-Err	n	Rel-Err	n	Rel-Err
1	—	7	25%	13	8.33%	19	2.86%	25	1.57%
2	0	8	0	14	-2.94%	20	-1.46%	26	-0.42%
3	0	9	12.5%	15	4.88%	21	1.82%	27	1.22%
4	0	10	0	16	-1.82%	22	-0.48%	28	-0.28%
5	50%	11	7.14%	17	4.55%	23	1.98%	29	1.06%
6	0	12	-4.76%	18	-2.27%	24	-0.94%	30	-0.29%

FIGURE 21: The graph of the data $(n, \pi \exp(\pi\sqrt{2/3} \sqrt{n})/12\sqrt{2} h(n) - n^{3/2})$ and the fitting curve $C_5(n)$.

FIGURE 22: The relative error of $I_{g2}(n)$ when $1000 \leq n \leq 10000$, step 300.TABLE 10: The relative error of $I_{g2}(n)$ to $h(n)$ when $n \leq 1000$.

N	Rel-Err (%)	n	Rel-Err (%)	n	Rel-Err	n	Rel-Err	n	Rel-Err
1		16	-2.49	40	-0.21%	220	$2.15E-06$	520	$1.78E-07$
2	-18.69	17	2.99	50	$-9.06E-04$	240	$1.76E-06$	540	$1.83E-07$
3	19.75	18	-2.59	60	$-4.33E-04$	260	$1.46E-06$	560	$1.59E-07$
4	-13.56	19	2.48	70	$-1.80E-04$	280	$1.20E-06$	580	$1.51E-07$
5	22.51	20	-1.83	80	$-8.70E-05$	300	$9.94E-07$	600	$1.32E-07$
6	-14.58	21	1.46	90	$-4.13E-05$	320	$8.51E-07$	640	$1.03E-07$
7	17.43	22	-1.15	100	$-1.68E-05$	340	$7.18E-07$	680	$5.80E-08$
8	-8.89	23	1.37	110	$-5.98E-06$	360	$6.14E-07$	720	$6.80E-08$
9	7.06	24	-1.32	120	$-7.10E-07$	380	$5.23E-07$	760	$6.70E-08$
10	-5.09	25	1.18	130	$2.07E-06$	400	$4.61E-07$	800	$5.10E-08$
11	7.23	26	-0.82	140	$3.17E-06$	420	$3.90E-07$	840	$5.40E-08$
12	-6.53	27	0.74	150	$3.54E-06$	440	$3.34E-07$	880	$-4.30E-09$
13	6.16	28	-0.66	160	$3.59E-06$	460	$2.96E-07$	920	$7.00E-09$
14	-3.38	29	0.72	180	$3.16E-06$	480	$2.50E-07$	960	$2.80E-08$
15	2.67	30	-0.66	200	$2.64E-06$	500	$2.22E-07$	1000	$3.30E-08$

or

$$C_{7b}(n) = \frac{0.8861039149}{n^{0.5}} - \frac{0.05719053203}{n} + \frac{0.9843423289}{n^{1.5}} + 0.6879343652. \quad (53)$$

The relative errors of

$$F_{7a}(n) = I_g(n) - \frac{\pi^2 \exp(\sqrt{2/3} \pi \sqrt{n})}{24\sqrt{3}n^2 C_{7a}(n)}, \quad (54)$$

and

$$F_{7b}(n) = I_g(n) - \frac{\pi^2 \exp(\sqrt{2/3} \pi \sqrt{n})}{24\sqrt{3}n^2 C_{7b}(n)}. \quad (55)$$

to $h(n)$ when $1000 \leq n \leq 10000$ are shown in Figures 23 and 24 (page 16), respectively. In this interval $(1000, 10000)$, $F_{7a}(n)$ is obviously more accurate than $F_{7b}(n)$. When $n \leq 1000$, the relative error of $\lfloor F_{7a}(n) + 1/2 \rfloor$ and $\lfloor F_{7b}(n) + 1/2 \rfloor$

is shown in Table 12 (page 15) and Table 13 (page 15). In this case, $F_{7b}(n)$ is better than $F_{7a}(n)$. However, neither of them is as good as $I_{g1}(n)$ or $I_{g2}(n)$ although they are more complicated than $I_{g1}(n)$ and $I_{g2}(n)$.

3.5. Estimate $h(n)$ When $n \leq 100$. All the estimation functions for $h(n)$ found now are with very good accuracy when n is greater than 1000, but they are not so accurate when $n < 50$, especially when $n < 25$. Although $I_{g1}'(n)$ and $I_{g2}'(n)$ are better than others, the relative error is still greater than 1 when $n < 40$.

When $n < 40$, it is too difficult to fit $\pi \exp(\pi\sqrt{2/3} \sqrt{n})/12\sqrt{2}h(n) - n^{3/2}$ by a simple smooth function with high accuracy, as shown in Figure 25 (on page 16). The points $(n, \pi \exp(\pi\sqrt{2/3} \sqrt{n})/12\sqrt{2}h(n) - n^{3/2})$ ($n=3, 4, \dots, 100$) are not so complicated (as shown in Figure 25). It seems that we can fit them by a simple piecewise function with 2 pieces, as the even points (where n is even) lie roughly on a smooth curve, so do the odd points. If we try to fit them, respectively, we will have the fitting function as follows:

TABLE 11: The relative error of $[I_{g^2}(n) + 1/2]$ to $h(n)$ when $n \leq 30$.

n	Rel-Err (%)	n	Rel-Err (%)	n	Rel-Err (%)	n	Rel-Err (%)	n	Rel-Err (%)
1		16	50.30	40	32.10	220	13.10	520	8.39
2	146.24	17	56.82	50	28.60	240	12.50	540	8.23
3	202.89	18	46.69	60	25.90	260	12.00	560	8.08
4	95.59	19	52.75	70	23.90	280	11.50	580	7.93
5	156.43	20	44.94	80	22.30	300	11.10	600	7.79
6	68.62	21	48.48	90	20.90	320	10.80	640	7.54
7	121.38	22	43.47	100	19.80	340	10.40	680	7.31
8	65.43	23	46.00	110	18.80	360	10.10	720	7.10
9	88.38	24	41.09	120	18.00	380	9.86	760	6.91
10	62.58	25	43.68	130	17.20	400	9.60	800	6.73
11	79.47	26	39.93	140	16.60	420	9.36	840	6.56
12	53.29	27	41.27	150	16.00	440	9.14	880	6.41
13	70.98	28	38.50	160	15.40	460	8.93	920	6.27
14	53.12	29	39.70	180	14.50	480	8.74	960	6.13
15	60.35	30	37.00	200	13.70	500	8.56	1000	6.01

FIGURE 23: The relative error of $F_{7a}(n)$ when $1000 \leq n \leq 10000$, step 300. FIGURE 24: The relative error of $F_{7b}(n)$ when $1000 \leq n \leq 10000$, step 300.

$$C_8(x) = \begin{cases} 1.942141112 \times x - 0.4796781366 \times \sqrt{x} + 8.291226268, & n = 3, 5, 7, \dots, 99; \\ 1.803056782 \times x + 2.356539877 \times \sqrt{x} - 6.043824511, & n = 4, 6, 8, \dots, 100. \end{cases} \quad (56)$$

Hence, we can calculate $h(n)$ ($3 \leq n \leq 100$) by

$$h(n) \sim I_{g^0}(n) = \frac{\pi \exp(\sqrt{2/3} \pi \sqrt{n})}{12\sqrt{2}(n^{3/2} + C_8(n))}, \quad 3 \leq n \leq 100. \quad (57)$$

Considering that $h(n)$ is an integer, we can take the round approximation of equation (57):

$$I'_{g^0}(n) = \left\lfloor \frac{\pi \exp(\sqrt{2/3} \pi \sqrt{n})}{12\sqrt{2}(n^{3/2} + C_8(n))} + \frac{1}{2} \right\rfloor, \quad 3 \leq n \leq 100. \quad (58)$$

TABLE 12: The relative error of $[F_{7a}(n) + 1/2]$ to $h(n)$ when $n \leq 1000$.

n	Rel-Err	n	Rel-Err	n	Rel-Err	n	Rel-Err	n	Rel-Err
1	—	16	0	40	$-8.13E-04$	220	$-1.74E-06$	520	$3.10E-07$
2	100%	17	4.55%	50	$-3.26E-04$	240	$-1.51E-06$	540	$3.29E-07$
3	100%	18	-1.14%	60	$-1.49E-04$	260	$-1.25E-06$	560	$3.42E-07$
4	50%	19	3.81%	70	$-2.81E-05$	280	$-9.91E-07$	580	$3.51E-07$
5	50%	20	-0.73%	80	$-4.62E-06$	300	$-7.56E-07$	600	$3.56E-07$
6	0	21	2.42%	90	$3.46E-06$	320	$-5.49E-07$	640	$3.57E-07$
7	50%	22	-0.48%	100	$6.70E-06$	340	$-3.72E-07$	680	$3.49E-07$
8	0	23	1.98%	110	$5.27E-06$	360	$-2.23E-07$	720	$3.36E-07$
9	12.5%	24	-0.63%	120	$3.37E-06$	380	$-9.87E-08$	760	$3.19E-07$
10	0	25	1.83%	130	$1.93E-06$	400	$3.54E-09$	800	$3.00E-07$
11	14.29%	26	-0.21%	140	$5.77E-07$	420	$8.70E-08$	840	$2.80E-07$
12	0	27	1.22%	150	$-4.01E-07$	440	$1.55E-07$	880	$2.59E-07$
13	8.33%	28	-0.28%	160	$-1.04E-06$	460	$2.09E-07$	920	$2.39E-07$
14	0	29	1.06%	180	$-1.72E-06$	480	$2.51E-07$	960	$2.19E-07$
15	4.88%	30	-0.29%	200	$-1.86E-06$	500	$2.85E-07$	1000	$1.99E-07$

TABLE 13: The relative error of $[F_{7b}(n) + (1/2)]$ to $h(n)$ when $n \leq 1000$.

n	Rel-Err	n	Rel-Err (%)	n	Rel-Err	n	Rel-Err	n	Rel-Err
1	—	16	-1.82	40	-0.16%	220	$-1.21E-06$	520	$-1.11E-06$
2	0	17	3.03	50	$-6.19E-04$	240	$-1.77E-06$	540	$-1.01E-06$
3	0	18	-2.27	60	$-2.60E-04$	260	$-2.08E-06$	560	$-9.24E-07$
4	0	19	2.86	70	$-7.50E-05$	280	$-2.21E-06$	580	$-8.41E-07$
5	50%	20	-1.46	80	$-1.85E-05$	300	$-2.24E-06$	600	$-7.64E-07$
6	0	21	1.82	90	$3.62E-06$	320	$-2.20E-06$	640	$-6.25E-07$
7	25%	22	-0.95	100	$1.30E-05$	340	$-2.12E-06$	680	$-5.05E-07$
8	0	23	1.58	110	$1.37E-05$	360	$-2.02E-06$	720	$-4.00E-07$
9	12.5%	24	-1.25	120	$1.21E-05$	380	$-1.90E-06$	760	$-3.10E-07$
10	0	25	1.31	130	$1.01E-05$	400	$-1.78E-06$	800	$-2.31E-07$
11	7.14%	26	-0.63	140	$7.74E-06$	420	$-1.66E-06$	840	$-1.63E-07$
12	-4.76%	27	0.87	150	$5.68E-06$	440	$-1.54E-06$	880	$-1.04E-07$
13	8.33%	28	-0.56	160	$3.97E-06$	460	$-1.42E-06$	920	$-5.24E-08$
14	-2.94%	29	0.83	180	$1.40E-06$	480	$-1.31E-06$	960	$-7.83E-09$
15	2.44%	30	-0.58	200	$-2.22E-07$	500	$-1.21E-06$	1000	$3.08E-08$

FIGURE 25: The graph of the data $(n, C_8(n))$.

Here, n begins from 3, not 1 or 2, because $(I'_{g0}(1) - h(1))/h(1)$ is meaningless since $h(1) = 0$, and $I'_{g0}(2)$ differs from $h(2)$ a lot. Besides, the value of $h(1)$ and $h(2)$ is clear by definition, so there is no need to use a complicated formula to estimate them.

The relative error of $I_{g0}(n)$ (or $I'_{g0}(n)$) to $h(n)$ is shown in Table 14 (or Table 15) on page 17. Compared them with Table 11 on page 15, we will find that when $n \geq 80$, $I'_{g2}(n)$ is more accurate than $I'_{g0}(n)$; when $n < 80$, $I'_{g0}(n)$ is better.

TABLE 14: The relative error of $I_{g_0}(n)$ to $h(n)$ when $n \leq 100$.

n	Rel-Err (%)	n	Rel-Err	n	Rel-Err	n	Rel-Err	n	Rel-Err
1	—	21	-0.16%	41	-5.72E-04	61	-1.04E-04	81	7.19E-05
2	—	22	0.12%	42	3.75E-04	62	1.43E-04	82	-5.18E-05
3	-14.85	23	0.05%	43	-4.68E-04	63	-1.04E-04	83	8.12E-05
4	12.72	24	-0.28%	44	4.91E-04	64	1.18E-04	84	-6.50E-05
5	1.99	25	8.37E-04	45	-6.29E-04	65	-5.63E-05	85	8.69E-05
6	-1.83	26	4.75E-04	46	5.45E-04	66	7.10E-05	86	-6.90E-05
7	4.76	27	-0.17%	47	-4.45E-04	67	-2.40E-05	87	8.65E-05
8	-0.64	28	6.69E-04	48	3.17E-04	68	5.74E-05	88	-7.40E-05
9	-0.92	29	-3.78E-04	49	-3.42E-04	69	-1.67E-05	89	9.01E-05
10	0.69	30	-4.43E-04	50	3.79E-04	70	4.05E-05	90	-8.07E-05
11	1.44	31	-1.98E-04	51	-3.98E-04	71	1.30E-05	91	9.09E-05
12	-2.46	32	4.21E-04	52	3.55E-04	72	6.81E-06	92	-8.24E-05
13	1.86	33	-0.12%	53	-2.81E-04	73	3.41E-05	93	8.80E-05
14	-0.24	34	9.27E-04	54	2.47E-04	74	-1.74E-06	94	-8.37E-05
15	-0.54	35	-7.86E-04	55	-2.21E-04	75	3.84E-05	95	8.70E-05
16	-0.04	36	1.82E-04	56	2.44E-04	76	-1.56E-05	96	-8.65E-05
17	0.46	37	-4.80E-04	57	-2.25E-04	77	5.70E-05	97	8.44E-05
18	-0.67	38	6.53E-04	58	2.29E-04	78	-3.58E-05	98	-8.56E-05
19	0.47	39	-9.11E-04	59	-1.55E-04	79	6.90E-05	99	7.92E-05
20	-0.28	40	6.34E-04	60	1.44E-04	80	-4.41E-05	100	-8.48E-05

TABLE 15: The relative error of $I'_{g_0}(n)$ to $h(n)$ when $n \leq 100$.

n	Rel-Err	n	Rel-Err	n	Rel-Err	n	Rel-Err	n	Rel-Err
1	—	9	0	17	0	25	0	33	-0.11%
2	—	10	0	18	-1.14%	26	0	34	9.23E-04
3	0	11	0	19	0	27	-0.17%	35	-7.77E-04
4	0	12	-4.76%	20	0	28	0	36	3.23E-04
5	0	13	0	21	0	29	0	37	-5.46E-04
6	0	14	0	22	0	30	0	38	6.85E-04
7	0	15	0	23	0	31	0	39	-9.67E-04
8	0	16	0	24	-0.31%	32	0.07%	40	6.50E-04

4. Conclusion

We have presented a recursion formula and several practical estimation formulae with high accuracy to calculate the number $h(n)$ of conjugate classes of derangements of order n or the number of isotopy classes of $2 \times n$ Latin rectangles.

If we want to obtain the accurate value of $h(n)$, we can use the recursion formula (13) and write a program based on it, while sometimes we need to know the estimation value in a program for technique reason especially when we use a general programming language.

If we want to obtain the approximation value of $h(n)$ with high accuracy, we can use formulae (29), (31), (36), and so on.

When $2 \leq n \leq 80$, we can use $I'_{g_0}(n)$ (equation (58)), with a relative error less than 0.11% (while $32 \leq n \leq 80$) or mainly 0 with very few exceptions (while $2 \leq n \leq 31$); when $n > 80$, we can use $I'_{g_2}(n)$ (equation (47)).

When $n \geq 100$, formulae $I'_{ga}(n)$ (equation (40)), $I'_{g_1}(n)$ (equation (44)), $F_{7a}(n)$ (equation (54)), and $F_{7b}(n)$ (equation (55)) are also very accurate although they are not as good as equation (47).

With the asymptotic formula (26) described in [30], we can obtain some estimation formulae with high accuracy for some other types of restricted partition numbers by the

methods mentioned in this paper or in [1]. The updated version of this paper is shown on <https://arxiv.org/abs/1612.08186>.

Data Availability

The figures and tables used to support this study are included within the article. The other data used to support this study are obtained from a program made by the first author.

Conflicts of Interest

The authors declare that there are no conflicts of interest regarding the publication of this paper.

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