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Research Article

Pseudo-Parallel Characteristic Jacobi Operators on Contact Metric 3 Manifolds

Wenjie Wang [1] and Xinxin Dai [2]

[1] *School of Mathematics, Zhengzhou University of Aeronautics, Zhengzhou 450046, Henan, China*
[2] *School of Mathematics and Information Science, Henan Normal University, Xinxiang 453007, Henan, China*

Correspondence should be addressed to Wenjie Wang; wangwj072@163.com

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We prove that the characteristic Jacobi operator on a contact metric three manifold is semiparallel if and only if it vanishes. We determine Lie groups of dimension three admitting left invariant contact metric structures such that the characteristic Jacobi operators are pseudoparallel.

1. Introduction

Let $(M^{2n+1}, \phi, \xi, \eta, g)$ be a contact metric manifold and $l: = R(\cdot, \xi)\xi$ be the characteristic Jacobi operator associated with the characteristic (or Reeb) vector field ξ, where R denotes the curvature tensor. In history, the characteristic Jacobi operators were investigated by many authors and played important roles in the study of contact metric manifolds. Here, we refer the reader to [1–7], for more detailed results in this framework. Among others, Koufogiorgos and Tsichlias in [8] classified all contact metric three manifolds with vanishing characteristic Jacobi operators. Cho and Inoguchi in [9] studied model spaces for contact metric three manifolds with vanishing characteristic Jacobi operators and constant $|Q\xi|$. Recently, Cho and Inoguchi in [9] classified all contact metric three manifolds such that ξ is an eigenvector field of the Ricci operator and the characteristic Jacobi operator is invariant along the Reeb flow, namely, $\mathscr{L}_\xi l = 0$, where $\mathscr{L}$ denotes the Lie differentiation. In particular, Cho and Inoguchi in pp. 11 of [9] proposed the following open question.

Classify contact Riemannian three manifolds or unit tangent sphere bundles with semiparallel (*i.e.*, $R\cdot l = 0$) or, more generally, pseudoparallel characteristic Jacobi operator (*i.e.*, $R(X,Y)\cdot l = L(X\wedge Y)\cdot l$ for certain function L).

The second question was solved by Cho and Chun in [10] and the first one has not yet been studied as far as we know. In this paper, we aim to investigate such problem and present that the characteristic Jacobi operators on contact metric three manifolds are semiparallel if and only if they are vanishing. We classify all left invariant contact metric structures on unimodular or nonunimodular Lie groups of dimension three such that the characteristic Jacobi operators are pseudoparallel. This shows that there exist no nontrivial semiparallel characteristic Jacobi operators, but there are nontrivial pseudoparallel characteristic Jacobi operators on contact metric three manifolds.

2. Contact Metric Three Manifolds

Let M^{2n+1} be a differentiable manifold of dimension $2n+1$ equipped with an almost contact metric structure (ϕ, ξ, η, g) which is defined by

$$\eta(\xi) = 1, \eta \circ \phi = 0, \phi^2 X = -X + \eta(X)\xi, \quad (1)$$

$$g(\phi X, \phi Y) = g(X,Y) - \eta(X)\eta(Y), \quad (2)$$

for any $X, Y \in \mathfrak{X}(M)$ (defined as the Lie algebra of all vector fields on M^{2n+1}), where η denotes a global 1 form, Ω is a 2 form, $\xi \in \mathfrak{X}(M)$, and ϕ denotes a $(1,1)$-type tensor filed. M^{2n+1}, equipped with a (ϕ, ξ, η, g)-structure, is called an almost contact metric manifold, and on such manifold, we may define a 2 form by $\Phi(X, Y) = g(X, \phi Y)$, for any $X, Y \in \mathfrak{X}(M)$. An almost contact metric manifold is called contact metric (Riemannian) manifold if $d\eta = \Phi$.

On the product $M^{2n+1} \times \mathbb{R}$ of an almost contact metric manifold M^{2n+1} and $\mathbb{R}$, we define an almost complex structure J by

$$J\left(X, f\frac{\mathrm{d}}{\mathrm{d}t}\right) = \left(\phi X - f\xi, \eta(X)\frac{\mathrm{d}}{\mathrm{d}t}\right), \tag{3}$$

where $X \in \mathfrak{X}(M)$, t is the coordinate of $\mathbb{R}$, and f is a $\mathscr{C}^{\infty}$-function on $M^{2n+1} \times \mathbb{R}$. We denote by $[\phi, \phi]$ the Nijenhuis tensor of ϕ. If

$$[\phi, \phi] = -2d\eta \otimes \xi, \tag{4}$$

is true, then the almost contact metric structure is said to be normal (see [2]). A normal contact metric manifold is said to be a Sasakian manifold. It is well known that an almost contact metric manifold is Sasakian if and only if

$$(\nabla_X \phi)Y = g(X, Y)\xi - \eta(Y)X, \tag{5}$$

for any $X, Y \in \mathfrak{X}(M)$, and this is equivalent to $R(X, Y)\xi = \eta(Y)X - \eta(X)Y$ (see [2]).

On an almost contact metric manifold, we define $h: = 1/2\mathscr{L}_{\xi}\phi$ (where $\mathscr{L}$ denotes the usual Lie differentiation). It is easy to check that h is a symmetric operator and satisfies

$$h\xi = 0, h\phi + \phi h = 0, trh = 0, \nabla\xi = -\phi - \phi h, \tag{6}$$

where ∇ denotes the Levi-Civita connection associated with the metric g. All manifolds are assumed to be connected.

3. Semiparallel Characteristic Jacobi Operators

Definition 1. On a contact metric three manifold, the characteristic Jacobi operator is said to be semiparallel if it satisfies

$$R(X, Y) \cdot l = 0, \tag{7}$$

for any vector fields X, Y and $\cdot$ denotes the derivative action.

In this section, we aim to determine all contact metric three manifolds having semiparallel characteristic Jacobi operators. In general, for an operator defined on a Riemannian manifold, semiparallelism is much weaker than vanishing, but it is not necessarily true on some special manifolds. First, we consider the Sasakian case.

Proposition 1. *The characteristic Jacobi operator on Sasakian three manifolds cannot be semiparallel.*

Proof. Let M^3 be a Sasakian three manifold, and then, we have

$$R(X, Y)\xi = \eta(Y)X - \eta(X)Y, \tag{8}$$

for any vector fields X, Y (see Proposition 7.3 in [2]). It follows directly that $l = -\phi^2$. Suppose that the characteristic Jacobi operator is semiparallel, and we have

$$R(X, Y)lZ = l(R(X, Y)Z), \tag{9}$$

for any vector fields X, Y, Z. The application of $l = -\phi^2$ in this relation gives

$$\eta(Z)R(X, Y)\xi = g(R(X, Y)Z, \xi)\xi. \tag{10}$$

Taking the inner product of the above relation with ξ implies that $R(X, Y)Z$ is orthogonal to ξ for any vector fields X, Y, and Z. Applying this in the above relation yields that $R(X, Y)\xi = 0$ for any vector fields X, Y, and hence, $l = 0$, contradicting $l = -\phi^2$. This completes the proof. □

Next, we consider the non-Sasakian case. Let M^3 be a non-Sasakian contact metric three manifold, and let U_1 be the open subset of M^3, where $h \neq 0$, and U_2 be the open subset of M^3 consisting of point $p \in M$ such that $h = 0$ in a neighborhood of p. Then, $U_1 \cup U_2$ is an open dense subset of M^3. On M^3, there exists a local orthonormal basis of type $\{\xi, e, \phi e\}$, and on U_1, we may set $he = \lambda e$, and hence, $h\phi e = -\lambda\phi e$, where λ is a positive eigenvalue function (λ is continuous on M^3 and smooth on $U_1 \cup U_2$). In this paper, we denote by Ric the Ricci tensor and define $\sigma(X) = \mathrm{Ric}(\xi, X)$ for any $X \in \mathfrak{X}(M)$. Applying some basics (relations (5) and (6)) on contact metric manifolds shown in Section 2, we have the following.

Lemma 1 (see Lemma 2.1 in [11]). *On U_1, we have*

$$\begin{aligned}
&\nabla_{\xi}\xi = 0, \nabla_{\xi}e = -a\phi e, \nabla_{\xi}\phi e = ae,\\
&\nabla_{e}\xi = -(\lambda+1)\phi e, \nabla_{e}e = \frac{1}{2\lambda}(\phi e(\lambda) + \sigma(e))\phi e,\\
&\nabla_{e}\phi e = -\frac{1}{2\lambda}(\phi e(\lambda) + \sigma(e))e + (\lambda+1)\xi,\\
&\nabla_{\phi e}\xi = -(\lambda-1)e, \nabla_{\phi e}\phi e = \frac{1}{2\lambda}(e(\lambda) + \sigma(\phi e))e,\\
&\nabla_{\phi e}e = -\frac{1}{2\lambda}(e(\lambda) + \sigma(\phi e))\phi e + (\lambda-1)\xi,
\end{aligned} \tag{11}$$

where a is a smooth function.

Proposition 2. *The characteristic Jacobi operator on a contact metric three manifold is semiparallel if and only if it is vanishing.*

Proof. In view of Proposition 1, next, we need to only consider the non-Sasakian case. Applying Lemma 1, by a direct calculation, we have

$$le = R(e, \xi)\xi = \left(1 - \lambda^2 + 2a\lambda\right)e + \xi(\lambda)\phi e, \tag{12}$$

$$l\phi e = R(\phi e, \xi)\xi = \xi(\lambda)e + \left(1 - \lambda^2 - 2a\lambda\right)\phi e. \tag{13}$$

Suppose that the characteristic Jacobi operator is semiparallel; from Definition 1, we have $R(\xi, e)l\xi - lR(\xi, e)\xi = 0$, and this is also equivalent to

$$l^2 e = 0. \tag{14}$$

The application of (12) on the above equation yields

$$1 - \lambda^2 + 2a\lambda = 0 \text{ and } \xi(\lambda) = 0. \tag{15}$$

Similarly, because the characteristic Jacobi operator is semiparallel, from Definition 1, we have $R(\xi, \phi e)l\xi - lR(\xi, \phi e)\xi = 0$, and this is also equivalent to

$$l^2 \phi e = 0. \tag{16}$$

With the aid of the second term of (15), the application of (13) on the above equation yields

$$1 - \lambda^2 - 2a\lambda = 0. \tag{17}$$

Comparing (17) with the first term of (15), we obtain $\lambda = 1$, and hence, $a = 0$. Now, from (12) and (13), it is clear to see that the characteristic Jacobi operator vanishes. The converse is trivial. □

The set of all contact metric three manifolds having vanishing characteristic Jacobi operators is rather large, and it has been characterized in [5, 8]. Locally, homogeneous examples of contact metric three manifolds are either locally symmetric or locally isometric to a Lie group provided that the Ricci curvature of the Reeb vector field ξ is a constant (see Theorem 4.5 in [8]).

4. Pseudoparallel Characteristic Jacobi Operators

Because semiparallel characteristic Jacobi operator on a contact metric three manifold must be trivial, in this section, we consider a condition weaker than semiparallelism.

Definition 2. On a contact metric three manifold, the characteristic Jacobi operator is said to be pseudoparallel if it satisfies

$$R(X, Y) \cdot l = L(X \wedge Y) \cdot l, \tag{18}$$

for any vector fields X, Y and $\cdot$ denotes the derivative action for certain function L, where $X \wedge Y$ denotes the wedge operator defined by $(X \wedge Y)Z = g(Y, Z)X - g(X, Z)Y$.

Clearly, pseudoparallelism reduces to semiparallelism when $L = 0$. However, unlike the case of semiparallel characteristic Jacobi operator (in this case, the characteristic Jacobi operator vanishes), on a Sasakian three manifold, we have the following.

Proposition 3. *The characteristic Jacobi operator on Sasakian three manifolds is pseudoparallel with $L = 1$.*

Proof. Following Definition 2, the characteristic Jacobi operator on a Sasakian three manifold is pseudoparallel if and only if

$$R(X, Y)lZ - l(R(X, Y)Z) = L[g(Y, lZ)X - g(X, lZ)Y - l((X \wedge Y)Z)], \tag{19}$$

for any vector fields X, Y, Z. On any Sasakian three manifold, we have $l = -\phi^2$, and making use of it, we see that the above relation is also equivalent to

$$-\eta(Z)R(X, Y)\xi + \eta(R(X, Y)Z)\xi = L[g(\phi Y, \phi Z)X - g(\phi X, \phi Z)Y - g(Y, Z)(X - \eta(X)\xi) + g(X, Z)(Y - \eta(Y)\xi)], \tag{20}$$

for any vector fields X, Y, Z. Now, in the previous relation, replacing Y by ξ and using again $l = -\phi^2$, we obtain

$$\eta(Z)(X - \eta(X)\xi) + [g(X, Z) - \eta(X)\eta(Z)]\xi = L[g(\phi X, \phi Z)\xi + \eta(Z)(X - \eta(X)\xi)]. \tag{21}$$

In the above equation, let $X = Z$ be two unit vector fields orthogonal to ξ; we obtain $L = 1$. Moreover, on any Sasakian three manifold, we have $R(X, Y)\xi = \eta(Y)X - \eta(X)Y$. In view of this, we check that the characteristic Jacobi operator is always pseudoparallel with $L = 1$. □

Next, we show that there also exist some nontrivial pseudoparallel characteristic Jacobi operators on certain non-Sasakian contact metric three manifolds. As seen in [8], there exist many nonhomogeneous contact metric three manifolds with semiparallel and, hence, pseudoparallel characteristic Jacobi operators (even if they are trivial). Because such class is rather large, in this section, we consider only pseudoparallel characteristic Jacobi operators on homogeneous contact metric three manifolds. Applying Milnor's classification (see [12]), Perrone [13] proved that a homogeneous contact metric three manifold is locally isometric to a Lie group equipped with left invariant contact metric structures.

Let G be a Lie group of dimension three and $\mathfrak{g}$ be the corresponding Lie algebra. It is known that a Lie group is unimodular if every adjoint transformation $\mathrm{ad}_X: Y \mapsto [X, Y]$ has traceless for any $X \in \mathfrak{g}$. In view of this, next, we discuss two situations. First, we have the following.

Case (i) ($c_2 = c_3$): from (25), it is clear to see that $L = 1$, and in this case, we have $le_1 = e_1$ and $le_2 = e_2$. According to Milnor's classification, we observe that M^3 is locally isometric to the 3-sphere group $\mathrm{SU}(2)$ (or $\mathrm{SO}(3)$) if $c_2 = c_3 > 0$, or $\mathrm{SL}(2, \mathbb{R})$ (or $O(1, 2)$), i.e., group of 2×2 real matrices of determinant 1 if $c_2 = c_3 < 0$, or the Heisenberg group Nil_3 if $c_2 = c_3 = 0$.

Case (ii) ($c_2 \neq c_3$ and $c_2 + c_3 = 2$): applying this in (25) or (26), we obtain

$$c_2(2 - c_2)(c_2(2 - c_2) - L) = 0. \quad (28)$$

If $c_2 = 0$, we obtain $c_3 = 2$. However, in this case, from the previous curvature tensors, we obtain $le_2 = le_3 = 0$. Thus, the characteristic Jacobi operator vanishes identically, and hence, pseudoparallelism is meaningless. If $c_2 = 2$, then $c_3 = 0$, and thus, the nonexistence proof for such case is the same as the previous one. Finally, it follows that $L = c_2(2 - c_2) \neq 0$. Moreover, if $0 < c_2 < 2$, we have $c_3 = 2 - c_2 > 0$ and the signature of the Lie group is $(+, +, +)$. The manifold is now locally isometric to the 3-sphere group $\mathrm{SU}(2)$ (or $\mathrm{SO}(3)$). If $c_2 < 0$ or $c_2 > 2$, the signature of the Lie group is $(+, +, -)$, and according to Minor's classification, the manifold is locally isometric to the special linear group $\mathrm{SL}(2, \mathbb{R})$ (or $O(1, 2)$).

Case (iii) ($c_2 \neq c_3$ and $c_2 + c_3 \neq 2$): note that, in this case, the Jacobi operator is nonvanishing. Now, (27) becomes

$$L = \frac{1}{4}(c_3 - c_2)^2 - 3 + c_3 + c_2. \quad (29)$$

Subtracting (25) from (26) yields

$$\left(2 - \frac{1}{2}(c_3 - c_2)^2 - L\right)(c_2 + c_3 - 2)(c_2 - c_3) = 0. \quad (30)$$

With the aid of the assumption, putting (29) into (30) gives

$$20 - 3(c_2 - c_3)^2 - 4c_2 - 4c_3 = 0. \quad (31)$$

Following this algebraic equation, if $c_2 = 0$ and $c_3 = -10/3$ (or equivalently, $c_3 = 0$ and $c_2 = -10/3$), then the signature of Lie group is $(+, -, 0)$, and hence, the manifold is locally isometric to group of rigid motions of Minkowski 2-space $E(1, 1)$. In particular, putting (31) into (29), we observe that, in this case, $L = 1$. This completes the proof.

Theorem 1. *Let M^3 be a three-dimensional unimodular Lie groups admitting a left invariant contact metric structure such that the characteristic Jacobi operator is pseudo-parallel. Then, M^3 is locally isometric to $SU(2)$ (or $SO(3)$), $SL(2, \mathbb{R})$ (or $SO(1, 2)$), $E(1, 1)$, or Heisenberg group Nil_3.*

Proof. Let M^3 be a 3-dimensional unimodular Lie group equipped with a left invariant contact metric structure; then, there exists an orthonormal basis $\{e_1, e_2 = \phi e_1, e_3 = \xi\}$ such that

$$[e_1, e_2] = 2e_3,\ [e_2, e_3] = c_2 e_1,\ [e_3, e_1] = c_3 e_2, \quad (22)$$

where c_2 and c_3 are two constants (see Perrone [13]). According to Cho and Chun [14], the curvature tensor of M^3 is described by

$$\begin{aligned}
R(e_1, e_2)e_3 &= R(e_1, e_3)e_2 = R(e_2, e_3)e_1 = 0,\\
R(e_1, e_2)e_2 &= \left(\frac{1}{4}(c_3 - c_2)^2 - 3 + c_3 + c_2\right)e_1,\\
R(e_2, e_1)e_1 &= \left(\frac{1}{4}(c_3 - c_2)^2 - 3 + c_3 + c_2\right)e_2,\\
R(e_2, e_3)e_3 &= \left(\frac{1}{4}(c_3 + c_2)^2 - c_2^2 + 1 + c_2 - c_3\right)e_2,\\
R(e_3, e_2)e_2 &= \left(\frac{1}{4}(c_3 + c_2)^2 - c_2^2 + 1 + c_2 - c_3\right)e_1,\\
R(e_1, e_3)e_3 &= -\left(\frac{1}{4}(c_3 - c_2)^2 + \frac{1}{2}\left(c_3^2 - c_2^2\right) - 1 + c_2 - c_3\right)e_1,\\
R(e_3, e_1)e_1 &= -\left(\frac{1}{4}(c_3 - c_2)^2 + \frac{1}{2}\left(c_3^2 - c_2^2\right) - 1 + c_2 - c_3\right)e_3.
\end{aligned} \quad (23)$$

Suppose that the characteristic Jacobi operator is pseudoparallel; from Definition 2 we have

$$R(X, Y)lZ - l(R(X, Y)Z) = L[g(Y, lZ)X - g(X, lZ)Y - g(Y, Z)lX + g(X, Z)lY], \quad (24)$$

for any vector fields X, Y, Z.

In (24), considering $X = Z = \xi$ and $Y = e_1$ (or equivalently, $X = \xi$, $Y = Z = e_1$) and with the aid of the curvature tensors, we have

$$L\left(-\frac{1}{4}(c_3 - c_2)^2 - \frac{1}{2}\left(c_3^2 - c_2^2\right) + 1 - c_2 + c_3\right) = \left(-\frac{1}{4}(c_3 - c_2)^2 - \frac{1}{2}\left(c_3^2 - c_2^2\right) + 1 - c_2 + c_3\right)^2. \quad (25)$$

Again, in (24), applying $X = Z = \xi$ and $Y = e_2$ (or equivalently, $X = \xi$ and $Y = Z = e_2$), we obtain

$$\begin{aligned} &L\left(\frac{1}{4}(c_3 + c_2)^2 - c_2^2 + 1 + c_2 - c_3\right) \\ &= \left(\frac{1}{4}(c_3 + c_2)^2 - c_2^2 + 1 + c_2 - c_3\right)^2. \end{aligned} \quad (26)$$

Again, in (24), applying $X = Z = e_1$ and $Y = e_2$ (or equivalently, $X = e_1$ and $Y = Z = e_2$), we obtain

$$\begin{aligned} &L\left(c_3^2 - c_2^2 + 2c_2 - 2c_3\right) \\ &= \left(\frac{1}{4}(c_3 - c_2)^2 - 3 + c_3 + c_2\right)\left(c_3^2 - c_2^2 + 2c_2 - 2c_3\right). \end{aligned} \quad (27)$$

One can check that there are no other useful information contained in (24) except for (25)–(27) because (24) is always true when we set $X = \xi$, $Y = e_1$, and $Z = e_2$, $X = \xi$, $Y = e_2$, and $Z = e_1$, or $X = e_1$, $Y = e_2$, and $Z = \xi$. In other words, the characteristic Jacobi operator is pseudoparallel if and only if (25)–(27) are true. With regard to (27), we shall discuss the following several cases. □

The nonunimodular classification theorem is given as follows.

Theorem 2. *On a nonunimodular Lie groups of dimension 3 admitting left invariant contact metric structure, the characteristic Jacobi operator is pseudoparallel if and only if the corresponding Lie algebra is isometric to*

$$[e_1, e_2] = \alpha e_2 + 2e_3, [e_2, e_3] = 0, [e_1, e_3] = 0, \alpha \in \mathbb{R} - \{0\}, \quad (32)$$

and the structure is Sasakian.

Proof. Let M^3 be a three-dimensional nonunimodular Lie group equipped with a left invariant contact metric structure; then, there exists an orthonormal basis $\{e_1, e_2 = \phi e_1, e_3 = \xi\}$ such that

$$[e_1, e_2] = \alpha e_2 + 2e_3, [e_2, e_3] = 0, [e_1, e_3] = \gamma e_2, \quad (33)$$

where α is nonzero constant and $\gamma \in \mathbb{R}$ (see [13]). According to Cho [14], the curvature tensor of M^3 is described by

$$\begin{aligned} &R(e_2, e_3)e_3 = \frac{1}{4}(\gamma + 2)^2 e_2, \\ &R(e_3, e_2)e_2 = \frac{1}{4}(\gamma + 2)^2 e_3, \\ &R(e_1, e_3)e_3 = -\frac{1}{4}\left(3\gamma^2 + 4\gamma - 4\right)e_1, \\ &R(e_3, e_1)e_1 = -\alpha\gamma e_2 - \frac{1}{4}\left(3\gamma^2 + 4\gamma - 4\right)e_3, \\ &R(e_1, e_2)e_2 = \left(\frac{1}{4}\left(\gamma^2 - 4\gamma - 12\right) - \alpha^2\right)e_1, \\ &R(e_1, e_2)e_2 = \left(\frac{1}{4}\left(\gamma^2 - 4\gamma - 12\right) - \alpha^2\right)e_1, \\ &R(e_2, e_1)e_1 = \left(\frac{1}{4}\left(\gamma^2 - 4\gamma - 12\right) - \alpha^2\right)e_2 - \alpha\gamma e_3, \\ &R(e_1, e_2)e_3 = R(e_1, e_3)e_2 = -\alpha\gamma e_1, R(e_2, e_3)e_1 = 0. \end{aligned} \quad (34)$$

Suppose that the characteristic Jacobi operator is pseudoparallel; from Definition 2, we have (24). Replacing $X = Z$ by ξ and Y by e_1 in (24), respectively, we obtain

$$\left(3\gamma^2 + 4\gamma - 4\right)\left(4L + 3\gamma^2 + 4\gamma - 4\right) = 0. \quad (35)$$

Putting $X = \xi$ and $Y = Z = e_1$ in (24), we obtain

$$16\alpha\gamma^2(\gamma + 2) = 0 \text{ and } \left(3\gamma^2 + 4\gamma - 4\right)\left(4L + 3\gamma^2 + 4\gamma - 4\right) = 0. \quad (36)$$

Putting $X = \xi$, $Y = e_1$, and $Z = e_2$ in (24), we obtain

$$\alpha\gamma^2(\gamma + 2) = 0. \quad (37)$$

Putting $X = Z = \xi$ and $Y = e_2$ in (24), we obtain

$$(\gamma + 2)^2\left((\gamma + 2)^2 - 4L\right) = 0. \quad (38)$$

Putting $X = \xi$ and $Y = Z = e_2$ in (24), we obtain again (38). Putting $X = e_1$, $Y = e_2$, and $Z = \xi$ in (24), we obtain

$$\alpha\gamma\left(3\gamma^2 + 4\gamma - 4\right) = 0. \quad (39)$$

Putting $X = Z = e_1$ and $Y = e_2$ in (24), we obtain

$$\gamma(\gamma + 2)\left(\gamma^2 - 4\gamma - 12 - 4\alpha^2 - 4L\right) = 0 \text{ and } \alpha\gamma\left(3\gamma^2 + 4\gamma - 4\right) = 0. \quad (40)$$

Putting $X = e_1$ and $Y = Z = e_2$ in (24), we obtain

$$\gamma(\gamma + 2)\left(\gamma^2 - 4\gamma - 12 - 4\alpha^2 - 4L\right) = 0. \quad (41)$$

Finally, putting $X = \xi$, $Y = e_2$, and $Z = e_1$ in (24), we obtain an identity. From (32), we remark that $\gamma = 0$ if and only if the structure is Sasakian. Since we have already considered Sasakian case in Proposition 3, next we assume that $\gamma \neq 0$. Therefore, according to (37) and $\alpha \neq 0$, we obtain $\gamma = -2$. However, in this context, it is clear to observe that the characteristic Jacobi operator vanishes identically, and hence, pseudoparallelism is meaningless although (35)–(41) are all true. The proof follows from Proposition 3. □

Contact metric structures have also been investigated in Lorentzian settings (cf. [15, 16]). The Lorentzian counterpart and other types of almost contact three manifolds (cf. [17–19]) of our result (cf. Theorems 1 and 2) will be studied in our future work.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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