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Research Article

A New Gauss Sum and Its Recursion Properties

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In this paper, we introduce a new Gauss sum, and then we use the elementary and analytic methods to study its various properties and prove several interesting three-order linear recursion formulae for it.

1. Introduction

Let $q > 1$ be an integer. For any Dirichlet character χ modulo q, the classical Gauss sums $G(m, \chi; q)$ is defined as follows:

$$G(m, \chi; q) = \sum_{a=1}^{q} \chi(a) e\left(\frac{\mathrm{ma}}{q}\right), \tag{1}$$

where m is any integer, $e(y) = e^{2\pi i y}$, and $i^2 = -1$.

For convenience, we write $\tau(\chi) = G(1, \chi; q)$. The Gauss sum plays a very important role in the study of elementary number theory and analytic number theory, and many number theory problems are closely related to it. Because of this, many scholars have studied its various properties and obtained a series of important results. For example, if $(m, q) = 1$, then we have the identity (see [1, 2])

$$G(m, \chi; q) = \overline{\chi}(m) G(1, \chi; q) = \overline{\chi}(m) \tau(\chi). \tag{2}$$

If χ is any primitive character modulo q, then one has also $G(m, \chi; q) = \overline{\chi}(m)\tau(\chi)$ and the identity $|\tau(\chi)| = \sqrt{q}$.

In addition, Zhang and Hu [3] (or Berndt and Evans [4]) studied the properties of some special Gauss sums and obtained the following interesting results: let p be a prime with $p \equiv 1 \bmod 3$. Then for any third-order character λ modulo p, one has the identity

$$\tau^3(\lambda) + \tau^3(\overline{\lambda}) = \mathrm{d}p, \tag{3}$$

where d is uniquely determined by $4p = d^2 + 27b^2$ and $d \equiv 1 \bmod 3$.

Chen and Zhang [5] studied the case of the fourth-order character modulo p and obtained the following conclusion: let p be a prime with $p \equiv 1 \bmod 4$. Then for any four-order character χ_4 modulo p, we have the identity

$$\tau^2(\chi_4) + \tau^2(\overline{\chi}_4) = 2\sqrt{p} \cdot \alpha, \alpha = \frac{1}{2} \sum_{a=1}^{p-1} \left(\frac{a + \overline{a}}{p}\right), \tag{4}$$

where $(*/p) = \chi_2$ denotes Legendre's symbol modulo p.

The constant $\alpha = \alpha(p)$ in (4) has a special meaning. In fact, we have the identity (For this, see Theorems 4–11 in [6])

$$p = \alpha^2 + \beta^2 \equiv \left(\frac{1}{2} \sum_{a=1}^{p-1} \left(\frac{a + \overline{a}}{p}\right)\right)^2 + \left(\frac{1}{2} \sum_{a=1}^{p-1} \left(\frac{a + r\overline{a}}{p}\right)\right)^2, \tag{5}$$

where r is any quadratic nonresidue modulo p. That is, $\chi_2(r) = -1$.

Some other results related to various Gauss sums and their recursion properties can also be found in references [7–10], and we will not list them all here.

In this paper, we introduce a new Gauss sum $A(m) = A(m, p)$, which is defined as follows: let p be an odd prime. For any integer m with $(m, p) = 1$, we define

$$A(m) = \sum_{a=1}^{p-1} \chi_2(a) e\left(\frac{ma^3}{p}\right), G_n(p) = \sum_{m=1}^{p-1} A^{2n}(m), \tag{6}$$

where $\chi_2 = (*/p)$ denotes the Legendre's symbol modulo p.

It is clear that if $(p-1,3)=1$, then note that $\chi_2^3=\chi_2$; from the properties of the reduced residue system modulo p, we have

$$\begin{aligned} A(m) &= \sum_{a=1}^{p-1} \chi_2(a^3) e\left(\frac{ma^3}{p}\right) \\ &= \sum_{a=1}^{p-1} \chi_2(a) e\left(\frac{ma}{p}\right) = \chi_2(m)\cdot\tau(\chi_2). \end{aligned} \tag{7}$$

So this time, $A(m)=G(m,\chi_2;p)=\chi_2(m)\tau(\chi_2)$ becomes the classical Gauss sum.

If $p\equiv 1 \bmod 3$, then we only knew that $A(m)$ is a real number, if $p\equiv 1 \bmod 12$; and $A(m)$ is a pure imaginary number, if $p\equiv 7 \bmod 12$. In fact if $p\equiv 1 \bmod 12$, then note that $\chi_2(-1)=1$, and this time we have

$$\begin{aligned} \overline{A(m)} &= \sum_{a=1}^{p-1} \chi_2(a) e\left(\frac{-ma^3}{p}\right) \\ &= \sum_{a=1}^{p-1} \chi_2(-a) e\left(\frac{m(-a)^3}{p}\right) = A(m). \end{aligned} \tag{8}$$

If $p\equiv 7 \bmod 12$, then note that $\chi_2(-1)=-1$, and this time we have

$$\begin{aligned} \overline{A(m)} &= \sum_{a=1}^{p-1} \chi_2(a) e\left(\frac{-ma^3}{p}\right) \\ &= -\sum_{a=1}^{p-1} \chi_2(-a) e\left(\frac{m(-a)^3}{p}\right) = -A(m). \end{aligned} \tag{9}$$

But beyond these relatively simple properties, we do not know anything else. In this paper, we shall focus on the calculating problems of $G_n(p)$. We shall use the analytic methods to give an interesting three-order linear recursion formula for $G_n(p)$. That is, we shall prove the following two results.

Theorem 1. *Let p be an odd prime with $p\equiv 7 \bmod 12$. Then for any integer $n\geq 3$, we have the recursion formula*

$$\begin{aligned} G_n(p) = &-9pG_{n-1}(p) - 6p\left(4p-d^2\right)G_{n-2}(p) \\ &- p\left(4p-d^2\right)^2 G_{n-3}(p), \end{aligned} \tag{10}$$

where d is uniquely determined by $4p=d^2+27b^2$ and $d\equiv 1 \bmod 3$, and the three initial values $G_0(p)=p-1$, $G_1(p)=-3(p-1)p$, and $G_2(p)=(p-1)p(11p+4d^2)$.

Theorem 2. *Let p be an odd prime with $p\equiv 1 \bmod 12$. Then for any integer $n\geq 3$, we have the recursion formula*

$$\begin{aligned} G_n(p) = &\, 9pG_{n-1}(p) - 6p\left(4p-d^2\right)G_{n-2}(p) \\ &+ p\left(4p-d^2\right)^2 G_{n-3}(p), \end{aligned} \tag{11}$$

where d is the same as in Theorem 1, and the three initial values $G_0(p)=p-1$, $G_1(p)=3(p-1)p$, and $G_2(p)=(p-1)p(11p+4d^2)$.

Of course, our theorems are also true for all integers $n<0$. In particular, we have the following conclusions:

Theorem 3. *For any prime p with $p\equiv 1 \bmod 3$, we have the identities*

$$\begin{aligned} \sum_{m=1}^{p-1} |A(m)|^4 &= (p-1)\cdot p\cdot\left(11p+4d^2\right), \\ \sum_{m=1}^{p-1} \frac{1}{|A(m)|^4} &= \frac{2}{243}\cdot\frac{p-1}{b^4}, \end{aligned} \tag{12}$$

where b is the same as defined in (3), i.e., $4p=d^2+27b^2$.

2. Several Lemmas

In this section, we first give several simple lemmas. Of course, the proofs of these lemmas and theorems need some knowledge of character sums and analytic number theory. They can be found in many number theory books, such as [1, 2, 6], here we do not need to list.

Lemma 1. *Let p be a prime with $p\equiv 1 \bmod 6$. Then for any six-order character $\psi \bmod p$, we have the identity*

$$\tau^3(\psi)+\tau^3(\overline{\psi}) = \begin{cases} p^{1/2}\cdot\left(d^2-2p\right), & \text{if } p\equiv 1 \bmod 12, \\ -i\cdot p^{1/2}\cdot\left(d^2-2p\right), & \text{if } p\equiv 7 \bmod 12, \end{cases} \tag{13}$$

where $i^2=-1$, d is uniquely determined by $4p=d^2+27b^2$, and $d\equiv 1 \bmod 3$.

Proof. For this, refer the study of Chen [11]. □

Lemma 2. *Let p be a prime with $p\equiv 7 \bmod 12$, χ_2 denote Legendre's symbol modulo p, and λ denote any three-order Dirichlet character modulo p. Then for any integer m with $(m,p)=1$, we have the identities*

$$\begin{aligned} A(m) &= \chi_2(m)\tau(\chi_2) + \chi_2(m)\cdot\left(\overline{\lambda}(m)\tau(\chi_2\lambda) + \lambda(m)\tau\left(\chi_2\overline{\lambda}\right)\right), \\ A^2(m) &= -p + 2\chi_2(m)\tau(\chi_2)A(m) + \lambda(m)\tau^2(\chi_2\lambda) + \overline{\lambda}(m)\tau^2\left(\chi_2\overline{\lambda}\right), \\ A^3(m) &= 3\chi_2(m)\tau(\chi_2)A^2(m) + \chi_2(m)\tau(\chi_2)\left(4p-d^2\right). \end{aligned} \tag{14}$$

Proof. It is clear that for any integer r with $(r, p) = 1$, from the properties of the three-order character modulo p, we have

$$1 + \lambda(r) + \overline{\lambda}(r) = \begin{cases} 3, & \text{if } r \text{ is a 3rd residue modulo } p, \\ 0, & \text{otherwise.} \end{cases} \tag{15}$$

It is clear that $\chi_2 = \overline{\chi}_2$ is a real character modulo p; from (15) and the properties of the classical Gauss sums, we have

$$\begin{aligned} A(m) &= \sum_{a=1}^{p-1} \chi_2(a^3) e\left(\frac{ma^3}{p}\right) = \sum_{a=1}^{p-1} \chi_2(a)(1 + \lambda(a) + \overline{\lambda}(a)) e\left(\frac{ma}{p}\right) \\ &= \sum_{a=1}^{p-1} \chi_2(a) e\left(\frac{ma}{p}\right) + \sum_{a=1}^{p-1} \chi_2(a)\lambda(a) e\left(\frac{ma}{p}\right) + \sum_{a=1}^{p-1} \chi_2(a)\overline{\lambda}(a) e\left(\frac{ma}{p}\right) \\ &= \chi_2(m)\tau(\chi_2) + \chi_2(m)\overline{\lambda}(m)\tau(\chi_2\lambda) + \chi_2(m)\lambda(m)\tau(\chi_2\overline{\lambda}). \end{aligned} \tag{16}$$

Note that $p \equiv 3 \bmod 4$, $\chi_2(-1) = -1$, $\lambda(-1) = 1$, $\tau(\chi_2) = i \cdot \sqrt{p}$, $\lambda^2 = \overline{\lambda}$, and the identity $\tau(\chi_2\lambda)\tau(\chi_2\overline{\lambda}) = \chi_2(-1)\tau(\chi_2\lambda)\overline{\tau(\chi_2\lambda)} = -p$; from (16), we also have

$$\begin{aligned} A^2(m) &= \left(\chi_2(m)\tau(\chi_2) + \chi_2(m)\overline{\lambda}(m)\tau(\chi_2\lambda) + \chi_2(m)\lambda(m)\tau(\chi_2\overline{\lambda})\right)^2 \\ &= -p + 2\tau(\chi_2)\left(\overline{\lambda}(m)\tau(\chi_2\lambda) + \lambda(m)\tau(\chi_2\overline{\lambda})\right) + \left(\overline{\lambda}(m)\tau(\chi_2\lambda) + \lambda(m)\tau(\chi_2\overline{\lambda})\right)^2 \\ &= -3p + 2\tau(\chi_2)(\chi_2(m)A(m) - \tau(\chi_2)) + \lambda(m)\tau^2(\chi_2\lambda) + \overline{\lambda}(m)\tau^2(\chi_2\overline{\lambda}) \\ &= -p + 2\chi_2(m)\tau(\chi_2)A(m) + \lambda(m)\tau^2(\chi_2\lambda) + \overline{\lambda}(m)\tau^2(\chi_2\overline{\lambda}). \end{aligned} \tag{17}$$

From (16) and (17) and Lemma 1, we have

$$\begin{aligned} A^3(m) &= \left(\chi_2(m)\tau(\chi_2) + \chi_2(m)\overline{\lambda}(m)\tau(\chi_2\lambda) + \chi_2(m)\lambda(m)\tau(\chi_2\overline{\lambda})\right) \\ &\quad \times\left(-p + 2\chi_2(m)\tau(\chi_2)A(m) + \lambda(m)\tau^2(\chi_2\lambda) + \overline{\lambda}(m)\tau^2(\chi_2\overline{\lambda})\right) \\ &= -pA(m) + 2\chi_2(m)\tau(\chi_2)A^2(m) + \chi_2(m)\tau(\chi_2)\left(\lambda(m)\tau^2(\chi_2\lambda) + \overline{\lambda}(m)\tau^2(\chi_2\overline{\lambda})\right) \\ &\quad + \chi_2(m)\left(\tau^3(\chi_2\lambda) + \tau^3(\chi_2\overline{\lambda})\right) - p(A(m) - \chi_2(m)\tau(\chi_2)) \\ &= 2\chi_2(m)\tau(\chi_2)A^2(m) + \chi_2(m)\tau(\chi_2)\left(A^2(m) + p - 2\chi_2(m)\tau(\chi_2)A(m)\right) \\ &\quad - 2pA(m) + \chi_2(m)\tau(\chi_2)p - \chi_2(m)\tau(\chi_2)\left(d^2 - 2p\right) \\ &= 3\chi_2(m)\tau(\chi_2)A^2(m) + \chi_2(m)\tau(\chi_2)\left(4p - d^2\right). \end{aligned} \tag{18}$$

Now Lemma 2 follows from (16)–(18). □

Lemma 3. *Let p be an odd prime with $p \equiv 1 \bmod 12$. Then for any m with $(m, p) = 1$, we have the identities*

$$\begin{aligned} A(m) &= \chi_2(m)\tau(\chi_2) + \chi_2(m) \cdot \left(\overline{\lambda}(m)\tau(\chi_2\lambda) + \lambda(m)\tau(\chi_2\overline{\lambda})\right), \\ A^2(m) &= p + 2\chi_2(m)\tau(\chi_2)A(m) + \lambda(m)\tau^2(\chi_2\lambda) + \overline{\lambda}(m)\tau^2(\chi_2\overline{\lambda}), \\ A^3(m) &= 3\chi_2(m)\sqrt{p}A^2(m) - \chi_2(m)\sqrt{p}\left(4p - d^2\right). \end{aligned} \tag{19}$$

Proof. Note that $p \equiv 1$ modulo 4, $\chi_2(-1) = 1$, $\tau(\chi_2) = \sqrt{p}$, $\lambda^2 = \overline{\lambda}$, and $\tau(\chi_2\lambda)\tau(\chi_2\overline{\lambda}) = p$; from (16) and the methods of proving Lemma 2, we also have

$$A(m) = \chi_2(m)\tau(\chi_2) + \chi_2(m)\overline{\lambda}(m)\tau(\chi_2\lambda) + \chi_2(m)\lambda(m)\tau(\chi_2\overline{\lambda}). \tag{20}$$

$$\begin{aligned} A^2(m) &= \left(\chi_2(m)\tau(\chi_2) + \chi_2(m)\overline{\lambda}(m)\tau(\chi_2\lambda) + \chi_2(m)\lambda(m)\tau(\chi_2\overline{\lambda})\right)^2 \\ &= p + 2\tau(\chi_2)\left(\overline{\lambda}(m)\tau(\chi_2\lambda) + \lambda(m)\tau(\chi_2\overline{\lambda})\right) + \left(\overline{\lambda}(m)\tau(\chi_2\lambda) + \lambda(m)\tau(\chi_2\overline{\lambda})\right)^2 \\ &= 3p + 2\tau(\chi_2)\left(\chi_2(m)A(m) - \tau(\chi_2)\right) + \lambda(m)\tau^2(\chi_2\lambda) + \overline{\lambda}(m)\tau^2(\chi_2\overline{\lambda}) \\ &= p + 2\chi_2(m)\tau(\chi_2)A(m) + \lambda(m)\tau^2(\chi_2\lambda) + \overline{\lambda}(m)\tau^2(\chi_2\overline{\lambda}). \end{aligned} \tag{21}$$

$$\begin{aligned} A^3(m) &= \left(\chi_2(m)\tau(\chi_2) + \chi_2(m)\overline{\lambda}(m)\tau(\chi_2\lambda) + \chi_2(m)\lambda(m)\tau(\chi_2\overline{\lambda})\right) \\ &\quad \times \left(p + 2\chi_2(m)\tau(\chi_2)A(m) + \lambda(m)\tau^2(\chi_2\lambda) + \overline{\lambda}(m)\tau^2(\chi_2\overline{\lambda})\right) \\ &= pA(m) + 2\chi_2(m)\tau(\chi_2)A^2(m) + \chi_2(m)\tau(\chi_2)\left(\lambda(m)\tau^2(\chi_2\lambda) + \overline{\lambda}(m)\tau^2(\chi_2\overline{\lambda})\right) \\ &\quad + \chi_2(m)\left(\tau^3(\chi_2\lambda) + \tau^3(\chi_2\overline{\lambda})\right) + p\left(A(m) - \chi_2(m)\tau(\chi_2)\right) \\ &= 2\chi_2(m)\tau(\chi_2)A^2(m) + \chi_2(m)\tau(\chi_2)\left(A^2(m) - p - 2\chi_2(m)\tau(\chi_2)A(m)\right) \\ &\quad + 2pA(m) + \chi_2(m)\tau(\chi_2)p + \chi_2(m)\tau(\chi_2)\left(d^2 - 2p\right) \\ &= 3\chi_2(m)\sqrt{p}A^2(m) - \chi_2(m)\sqrt{p}\left(4p - d^2\right). \end{aligned} \tag{22}$$

It is clear that Lemma 3 follows from (20)–(22). □

3. Proofs of the Theorems

Now we shall complete the proofs of our all results. First we prove Theorem 1. Let p be an odd prime with $p \equiv 7 \bmod 12$, and then note that $\tau^2(\chi_2) = -p$; from Lemma 2 and the properties of the character sums modulo p, we have

$$\begin{aligned} \sum_{m=1}^{p-1} A^2(m) &= \sum_{m=1}^{p-1} \left(-p + 2\chi_2(m)\tau(\chi_2)A(m) + \lambda(m)\tau^2(\chi_2\lambda) + \overline{\lambda}(m)\tau^2(\chi_2\overline{\lambda})\right) \\ &= -p(p-1) + 2\sum_{m=1}^{p-1} \tau(\chi_2)\left(\tau(\chi_2) + \overline{\lambda}(m)\tau(\chi_2\lambda) + \lambda(m)\tau(\chi_2\overline{\lambda})\right) \\ &= -(p-1)p + 2\sum_{m=1}^{p-1} \tau^2(\chi_2) = -3(p-1)p. \end{aligned} \tag{23}$$

From (23) and Lemmas 1 and 2, we have

$$\begin{aligned}\sum_{m=1}^{p-1} A^4(m) &= \sum_{m=1}^{p-1}\left(-p+2\chi_2(m)\tau(\chi_2)A(m)+\lambda(m)\tau^2(\chi_2\lambda)+\overline{\lambda}(m)\tau^2(\chi_2\overline{\lambda})\right)^2\\ &= (p-1)p^2-4p\sum_{m=1}^{p-1}\chi_2(m)\tau(\chi_2)A(m)-4p\sum_{m=1}^{p-1}A^2(m)+2(p-1)p^2\\ &\quad+4\sum_{m=1}^{p-1}\chi_2(m)\tau(\chi_2)\left(\lambda(m)\tau^2(\chi_2\lambda)+\overline{\lambda}(m)\tau^2(\chi_2\overline{\lambda})\right)A(m)\\ &= 19(p-1)p^2+4(p-1)\tau(\chi_2)\left(\tau^3(\chi_2\lambda)+\tau^3(\chi_2\overline{\lambda})\right)\\ &= 19(p-1)p^2+4(p-1)p\left(d^2-2p\right)=(p-1)\left(11p+4d^2\right)p.\end{aligned} \tag{24}$$

If $n\geq 3$, then $2n\geq 6$, from Lemma 2, we have

$$A^6(m)=-9pA^4(m)-6p\left(4p-d^2\right)A^2(m)-p\left(4p-d^2\right)^2, \tag{25}$$

$$\begin{aligned}A^{2n}(m) &= -9pA^{2n-2}(m)-6p\left(4p-d^2\right)A^{2n-4}(m)\\ &\quad-p\left(4p-d^2\right)^2A^{2n-6}(m).\end{aligned} \tag{26}$$

From (23)–(26) and the definition of $G_n(p)$, we may immediately deduce the three-order linear recursion formula

$$\begin{aligned}G_n(p) &= -9pG_{n-1}(p)-6p\left(4p-d^2\right)G_{n-2}(p)\\ &\quad-p\left(4p-d^2\right)^2G_{n-3}(p),\end{aligned} \tag{27}$$

with the three initial values $G_0(p)=p-1$, $G_1(p)=-3(p-1)p$, and

$$G_2(p)=(p-1)\left(11p+4d^2\right)p. \tag{28}$$

This proves Theorem 1.

Now we prove Theorem 2. If p be an odd prime with $p\equiv 1 \bmod 12$, then note that $\chi_2(-1)=1$ and $\tau(\chi_2)=\sqrt{p}$; from Lemma 3, we have

$$A^6(m)=9pA^4(m)-6p\left(4p-d^2\right)A^2(m)+p\left(4p-d^2\right)^2, \tag{29}$$

$$\begin{aligned}A^{2n}(m) &= 9pA^{2n-2}(m)-6p\left(4p-d^2\right)A^{2n-4}(m)\\ &\quad+p\left(4p-d^2\right)^2A^{2n-6}(m).\end{aligned} \tag{30}$$

It is clear that from Lemmas 1 and 3, we have

$$G_1(p)=\sum_{m=1}^{p-1}A^2(m)=p\cdot\sum_{m=1}^{p-1}1+2\sum_{m=1}^{p-1}\chi_2(m)\tau(\chi_2)A(m)=3(p-1)p. \tag{31}$$

$$\begin{aligned}\sum_{m=1}^{p-1} A^4(m) &= \sum_{m=1}^{p-1}\left(p+2\chi_2(m)\tau(\chi_2)A(m)+\lambda(m)\tau^2(\chi_2\lambda)+\overline{\lambda}(m)\tau^2(\chi_2\overline{\lambda})\right)^2\\ &= (p-1)p^2+4(p-1)p^2+4p\sum_{m=1}^{p-1}A^2(m)+2(p-1)p^2\\ &\quad+4\sqrt{p}\sum_{m=1}^{p-1}\chi_2(m)A(m)\left(\lambda(m)\tau^2(\chi_2\lambda)+\overline{\lambda}(m)\tau^2(\chi_2\overline{\lambda})\right)\\ &= 19(p-1)p^2+4(p-1)\sqrt{p}\left(\tau^3(\chi_2\lambda)+\tau^3(\chi_2\overline{\lambda})\right)\\ &= 19(p-1)p^2+4(p-1)p\left(d^2-2p\right)=(p-1)p\left(4d^2+11p\right).\end{aligned} \tag{32}$$

From (30)–(32) and the definition of $G_n(p)$, we have the three-order linear recursion formula

$$\begin{aligned}G_n(p) &= 9pG_{n-1}(p)-6p\left(4p-d^2\right)G_{n-2}(p)\\ &\quad+p\left(4p-d^2\right)^2G_{n-3}(p),\quad n\geq 3,\end{aligned} \tag{33}$$

where the three initial values $G_0(p) = p-1$, $G_1(p) = 3(p-1)p$, and $G_2(p) = (p-1)p(11p+4d^2)$.

This proves Theorem 2.

Now we prove Theorem 3. First for any integer m with $(m,p)=1$, we have $A(m)\neq 0$. In fact if $A(m)=0$; then from (25), we have

$$\begin{aligned} A^6(m) &= -9pA^4(m) - 6p\left(4p-d^2\right)A^2(m) - p\left(4p-d^2\right)^2 \\ &= -p\cdot\left(27b^2\right)^2 = 0. \end{aligned} \tag{34}$$

This is impossible. So we have $A(m)\neq 0$.

On the contrary, if $p\equiv 7 \bmod 12$, then from (25), we have

$$\begin{aligned} \sum_{m=1}^{p-1} A^4(m) &= -9p\sum_{m=1}^{p-1} A^2(m) - 6p\left(4p-d^2\right)(p-1) \\ &\quad - p\left(4p-d^2\right)^2\sum_{m=1}^{p-1}\frac{1}{A^2(m)}. \end{aligned} \tag{35}$$

Combining (23), (24), and (35), we have

$$\begin{aligned} (p-1)\left(11p+4d^2\right)p &= 27p^2(p-1) - 6p(4p-d)^2(p-1) \\ &\quad - p\left(4p-d^2\right)^2\sum_{m=1}^{p-1}\frac{1}{A^2(m)}, \end{aligned} \tag{36}$$

or

$$\sum_{m=1}^{p-1}\frac{1}{A^2(m)} = -\frac{2(p-1)}{4p-d^2}. \tag{37}$$

From (23), (24), and (37), we also have

$$\begin{aligned} \sum_{m=1}^{p-1} A^2(m) &= -9p(p-1) - 6p\left(4p-d^2\right)\sum_{m=1}^{p-1}\frac{1}{A^2(m)} \\ &\quad - p\left(4p-d^2\right)^2\sum_{m=1}^{p-1}\frac{1}{A^4(m)}, \end{aligned} \tag{38}$$

or

$$\sum_{m=1}^{p-1}\frac{1}{A^4(m)} = \frac{6(p-1)}{\left(4p-d^2\right)^2}. \tag{39}$$

Similarly, if $p\equiv 1 \bmod 12$, then from (31), (35), Lemma 3, and the methods of proving (39), we can also deduce that

$$\sum_{m=1}^{p-1}\frac{1}{A^4(m)} = \frac{6(p-1)}{\left(4p-d^2\right)^2}. \tag{40}$$

If $p\equiv 1 \bmod 12$, then $A(m)$ is a real number, so this time we have

$$A^4(m) = |A(m)|^4. \tag{41}$$

If $p\equiv 7 \bmod 12$, then $A(m)$ is a pure imaginary number; this time we also have

$$|A(m)|^4 = A^2(m)\cdot\overline{A(m)}^2 = A^2(m)\cdot(-A(m))^2 = A^4(m). \tag{42}$$

It is clear that from (24), (32), (41), and (42), we can deduce the identity

$$\sum_{m=1}^{p-1}|A(m)|^4 = (p-1)\cdot p\cdot\left(11p+4d^2\right), \quad p\equiv 1 \bmod 3. \tag{43}$$

From (39)–(42) and noting that $4p = d^2+27b^2$, we can also deduce

$$\sum_{m=1}^{p-1}\frac{1}{|A(m)|^4} = \frac{2}{243}\cdot\frac{p-1}{b^4}, \quad p\equiv 1 \bmod 3. \tag{44}$$

This completes the proofs of our all results.

4. Conclusion

The main result of this paper is to prove a three-order linear recursion formula for one kind new Gauss sums. As an application of this result, we obtained following conclusion: for any prime p with $p\equiv 1 \bmod 3$, we have the identities

$$\begin{aligned} \sum_{m=1}^{p-1}|A(m)|^4 &= (p-1)\cdot p\cdot\left(11p+4d^2\right), \\ \sum_{m=1}^{p-1}\frac{1}{|A(m)|^4} &= \frac{2}{243}\cdot\frac{p-1}{b^4}. \end{aligned} \tag{45}$$

These results not only gave the exact values for the fourth power mean and its inverse fourth power mean of a new Gauss sums, they are also some new contribution to research in related fields.

Data Availability

No data were used to support this study.

Conflicts of Interest

The author declares that there are no conflicts of interest regarding the publication of this paper.

Authors' Contributions

The author contributed to the work and read and approved the final manuscript.

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