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Research Article

Investigation of Pseudo-Ricci Symmetric Spacetimes in Gray's Subspaces

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In the present paper, we focused our attention to study pseudo-Ricci symmetric spacetimes in Gray's decomposition subspaces. It is proved that $(PRS)_n$ spacetimes are Ricci flat in trivial, A, and B subspaces, whereas perfect fluid in subspaces I, $I \oplus A$, and $I \oplus B$, and have zero scalar curvature in subspace $A \oplus B$. Finally, it is proved that pseudo-Ricci symmetric GRW spacetimes are vacuum, and as a consequence of this result, we address several corollaries.

1. Introduction

A pseudo-Ricci symmetric manifold (briefly $(PRS)_n$) is a nonflat pseudo-Riemannian manifold whose Ricci tensor satisfies

$$\nabla_k R_{ij} = 2A_k R_{ij} + A_i R_{kj} + A_j R_{ik}, \tag{1}$$

where A is a nonzero 1-form and ∇ indicates the covariant differentiation with respect to the metric g [1].

The class of pseudo-Ricci symmetric manifolds is a subclass of weakly Ricci symmetric manifolds which were first introduced and studied by Tamássy and Binh [2]. There has been much focus on the concept of $(PRS)_n$ manifolds; for instance, a sufficient condition on $(PRS)_n$ manifolds to be quasi-Einstein manifolds was introduced by De and Gazi [3]. $(PRS)_n$ manifolds whose scalar curvature satisfies $\nabla_k R = 0$ have zero scalar curvature [1]. A concrete example of pseudo-Ricci symmetric manifolds was given in [4]. There are many generalizations of $(PRS)_n$ manifolds, for example, see [5, 6].

An invariant orthogonal decomposition of the covariant derivative of the Ricci tensor was coined and studied by Gray in [7] (see also [8–10]). The manifolds in the trivial subspace have parallel Ricci tensor; that is, $\nabla_k R_{ij} = 0$. The subspace $\mathscr{A}$ contains manifolds whose Ricci tensor is Killing; that is,

$$\nabla_j R_{ki} + \nabla_k R_{ji} + \nabla_i R_{kj} = 0. \tag{2}$$

The next subspace is denoted by $\mathscr{B}$. The Ricci tensors of manifolds in $\mathscr{B}$ are Codazzi; that is,

$$\nabla_k R_{ij} = \nabla_i R_{kj}. \tag{3}$$

The subspace $\mathscr{A} \oplus \mathscr{B}$ is characterized by the equation $\nabla R = 0$. Manifolds with

$$\nabla_k R_{ij} = \frac{n}{(n-1)(n+2)} \nabla_{(k R g_{ij})} \tag{4}$$

lie in $\mathcal{I}$. In $\mathcal{I}\oplus\mathcal{A}$, the tensor $R_{ij} - (2R/(n+2))g_{ij}$ is Killing, whereas in $\mathcal{I}\oplus\mathcal{B}$, the tensor $R_{ji} - (R/2(n-1))g_{ji}$ is a Codazzi tensor. Such manifolds are called Einstein-like manifolds [11]. Recently, there has been growing interest in this decomposition. For example, generalized Robertson–Walker spacetimes are either Einstein or perfect fluid in Gray's orthogonal subspaces except one in which the Ricci tensor is not restricted [12].

An n-dimensional Lorentzian manifold is said to be pseudo-Ricci symmetric spacetime if the Ricci tensor satisfies equation (1). Here, we assume the associated vector A_i is a unit time-like vector ($A_i A^i = -1$).

In standard theory of gravity, the relation between the matter of spacetimes and the geometry of spacetimes is given by Einstein's field equation (EFE):

$$R_{ij} - \frac{R}{2}g_{ij} = kT_{ij}, \tag{5}$$

where R_{ij}, R, k, and T_{ij} are the Ricci tensor, scalar curvature tensor, Newtonian constant, and energy-momentum tensor, respectively. EFE implies that the energy-momentum tensor T_{ij} is divergence-free. This requirement is directly satisfied if $\nabla_l T_{ij} = 0$.

This paper is organized as follows: In Section 2, general properties of $(\mathrm{PRS})_n$ spacetimes are considered. In Section 3, $(\mathrm{PRS})_n$ spacetimes are investigated in all Gray's orthogonal subspaces. It is proved that $(\mathrm{PRS})_n$ spacetimes in trivial, $\mathcal{A}$, and $\mathcal{B}$ subspaces are Ricci flat, in subspaces $\mathcal{I}$, $\mathcal{I}\oplus\mathcal{A}$, and $\mathcal{I}\oplus\mathcal{B}$ are perfect fluid spacetimes, and in $\mathcal{A}\oplus\mathcal{B}$ have a zero scalar curvature. In Section 4, we prove that pseudo-Ricci symmetric GRW spacetimes are vacuum and as a consequence, we address some corollaries.

2. On $(\mathrm{PRS})_n$ Spacetimes

In this section, the main properties of $(\mathrm{PRS})_n$ spacetimes are considered. Equation (1) implies

$$\nabla_k R_i^k = 3A^j R_{ij} + A_i R. \tag{6}$$

The use of $\nabla_k R_i^k = (1/2)\nabla_i R$ yields

$$\nabla_i R = 6A^j R_{ij} + 2A_i R. \tag{7}$$

A different contraction of equation (1) with g^{ij} gives

$$\nabla_k R = 2A_k R + 2A^j R_{kj}. \tag{8}$$

Solving equations (7) and (8) together, one gets

$$A^j R_{kj} = 0, \tag{9}$$

$$\nabla_k R = 2A_k R. \tag{10}$$

Lemma 1. *In $(PRS)_n$ spacetimes, the covariant derivative of the scalar curvature is $\nabla_k R = 2A_k R$. Moreover, A^j is an eigenvector of the Ricci tensor R_{ij} with zero eigenvalue.*

Assume that the scalar curvature is constant. Equation (10) directly leads to $R = 0$.

Lemma 2. *In (PRS) spacetimes, the scalar curvature R is constant if and only if $R = 0$.*

Let us consider $R \neq 0$; then, the use of equation (10) in equation (1) implies that

$$\nabla_k R_{ij} = \frac{\nabla_k R}{R} R_{ij} + \frac{\nabla_i R}{2R} R_{kj} + \frac{\nabla_j R}{2R} R_{ik}. \tag{11}$$

This leads us to the following lemma.

Lemma 3. *In $(PRS)_n$ spacetimes with nonzero scalar curvature, the covariant derivative of the Ricci tensor takes the form*

$$\nabla_k R_{ij} = \frac{\nabla_k R}{R} R_{ij} + \frac{\nabla_i R}{2R} R_{kj} + \frac{\nabla_j R}{2R} R_{ik}, \tag{12}$$

provided $R \neq 0$.

The Weyl tensor of type $(0,4)$ has the form [13]

$$C_{ijkl} = R_{ijkl} - \frac{1}{n-1}\left\{g_{il}R_{jk} + g_{jk}R_{il} - g_{ik}R_{jl} - g_{jl}R_{ik}\right\} + \frac{R}{(n-1)(n-2)}\left\{g_{il}g_{jk} - g_{ik}g_{jl}\right\}, \tag{13}$$

and its divergence is

$$\nabla_h \mathcal{C}^h_{ijk} = \frac{n-3}{n-2}\left[\left(\nabla_k R_{ij} - \nabla_j R_{ik}\right) - \frac{1}{2(n-1)}\left(g_{ij}\nabla_k R - g_{ik}\nabla_j R\right)\right]. \tag{14}$$

In virtue of (1) and (10), we have

$$\nabla_h \mathcal{C}^h_{ijk} = \frac{n-3}{n-2}\left[\left(A_k R_{ij} - A_j R_{ik}\right) - \frac{1}{(n-1)}\left(g_{ij}A_k R - g_{ik}A_j R\right)\right]. \tag{15}$$

Assume that the Weyl conformal curvature tensor is divergence-free, that is, $\nabla_h \mathcal{C}^h_{ijk} = 0$; then,

$$A_k R_{ij} - A_j R_{ik} = \frac{(n-2)}{(n-1)(n-3)}\left(g_{ij}A_k R - g_{ik}A_j R\right). \tag{16}$$

Contracting with A^k and using equation (9), we obtain

$$R_{ij} = \frac{(n-2)R}{(n-1)(n-3)}\left(g_{ij} + A_i A_j\right). \tag{17}$$

A multiplication with g^{ij} gives $R = 0$, and hence,

$$R_{ij} = 0. \tag{18}$$

Thus, we can conclude the following theorem:

Theorem 1. *A $(PRS)_n$ spacetime with divergence-free Weyl curvature tensor is Ricci flat.*

The use of this result ($R_{ij} = 0$) in the defining property of the conformal curvature tensor entails that

$$\mathcal{C}_{ijkh} = R_{ijkh}. \tag{19}$$

Hence, we have the following corollary.

Corollary 1. *Semisymmetric and conformally semisymmetric pseudo-Ricci symmetric spacetimes are equivalent.*

The covariant derivative of equation (1) gives

$$\nabla_r \nabla_k R_{ij} = 2\nabla_r\left(A_k R_{ij}\right) + \nabla_r\left(A_i R_{kj}\right) + \nabla_r\left(A_j R_{ik}\right). \tag{20}$$

Interchanging the indices r and k in the last equation, we have

$$\nabla_k \nabla_r R_{ij} = 2\nabla_k\left(A_r R_{ij}\right) + \nabla_k\left(A_i R_{rj}\right) + \nabla_k\left(A_j R_{ir}\right). \tag{21}$$

Subtracting the last two equations, we obtain

$$\begin{aligned}(\nabla_r\nabla_k - \nabla_k\nabla_r)R_{ij} &= 2\nabla_r\left(A_k R_{ij}\right) + \nabla_r\left(A_i R_{kj}\right) + \nabla_r\left(A_j R_{ik}\right)\\ &\quad - 2\nabla_k\left(A_r R_{ij}\right) - \nabla_k\left(A_i R_{rj}\right) - \nabla_k\left(A_j R_{ir}\right),\\ (\nabla_r\nabla_k - \nabla_k\nabla_r)R_{ij} &= 2R_{ij}\nabla_r\left(A_k\right) + R_{kj}\nabla_r\left(A_i\right) + R_{ik}\nabla_r\left(A_j\right)\\ &\quad + 2A_k\nabla_r\left(R_{ij}\right) + A_i\nabla_r\left(R_{kj}\right) + A_j\nabla_r\left(R_{ik}\right)\\ &\quad - 2A_r\nabla_k\left(R_{ij}\right) - A_i\nabla_k\left(R_{rj}\right) - A_j\nabla_k\left(R_{ir}\right)\\ &\quad - 2R_{ij}\nabla_k\left(A_r\right) - R_{rj}\nabla_k\left(A_i\right) - R_{ir}\nabla_k\left(A_j\right).\end{aligned} \tag{22}$$

Making use of equation (1) and simplifying, we get

$$\begin{aligned}(\nabla_r\nabla_k - \nabla_k\nabla_r)R_{ij} &= 2R_{ij}\left[\nabla_r\left(A_k\right) - \nabla_k\left(A_r\right)\right] + R_{kj}\nabla_r\left(A_i\right)\\ &\quad + R_{ik}\nabla_r\left(A_j\right) - R_{rj}\nabla_k\left(A_i\right) - R_{ir}\nabla_k\left(A_j\right)\\ &\quad + A_iA_kR_{jr} + A_iA_jR_{rk} + A_jA_kR_{ir}\\ &\quad - A_iA_rR_{kj} - A_jA_rR_{ki}.\end{aligned} \tag{23}$$

Now, assume that the $(\mathrm{PRS})_n$ is Ricci semisymmetric, that is, $(\nabla_r\nabla_k - \nabla_k\nabla_r)R_{ij} = 0$; we have

$$\begin{aligned}0 &= 2R_{ij}\left[\nabla_r\left(A_k\right) - \nabla_k\left(A_r\right)\right] + R_{kj}\nabla_r\left(A_i\right)\\ &\quad + R_{ik}\nabla_r\left(A_j\right) - R_{rj}\nabla_k\left(A_i\right) - R_{ir}\nabla_k\left(A_j\right)\\ &\quad + A_iA_kR_{jr} + A_iA_jR_{rk} + A_jA_kR_{ir} - A_iA_rR_{kj} - A_jA_rR_{ki}.\end{aligned} \tag{24}$$

Contracting with A^j and using equation (9), we infer

$$-A_iR_{rk} + A_rR_{ki} = 0. \tag{25}$$

Again, contracting with A^i and utilizing equation (9), we get

$$R_{rk} = 0. \tag{26}$$

Thus, we have the following theorem:

Theorem 2. *Ricci semisymmetric* $(PRS)_n$ *spacetimes are Ricci flat.*

3. ERROR!!PRS)$_n$ Spacetimes in Gray's Decomposition Subspaces

This section is devoted to study $(\mathrm{PRS})_n$ spacetimes in Gray's seven subspaces. Three main results are obtained in this section. A Lorentzian manifold M is said to be perfect fluid if its Ricci tensor satisfies

$$R_{ij} = \alpha g_{ij} + \beta u_i u_j, \tag{27}$$

where α and β are scalar fields and u_i is a time-like vector field [14].

Theorem 3. $(PRS)_n$ *spacetimes in trivial,* $\mathscr{A}$*, and* $\mathscr{B}$ *subspaces are Ricci flat.*

Proof. The trivial subspace of Gray's decomposition contains spacetimes whose Ricci tensors are parallel and the scalar curvatures are constant. Thus, equation (10) easily gives $R = 0$. And hence, equation (1) becomes

$$2A_kR_{ij} = -A_iR_{kj} - A_jR_{ik}. \tag{28}$$

A contraction of equation (28) with g^{ij} yields

$$R_{ij} = RA_iA_j = 0. \tag{29}$$

And consequently,

$$R_{ij} = 0, \tag{30}$$

which means that $(\mathrm{PRS})_n$ spacetimes with parallel Ricci tensor are Ricci flat.

In subspace $\mathscr{A}$ $(\mathrm{PRS})_n$ spacetimes have a Killing Ricci tensor; that is,

$$\nabla_jR_{ki} + \nabla_kR_{ji} + \nabla_iR_{kj} = 0. \tag{31}$$

It is well known that in this subspace, the scalar curvature is covariantly constant. Equation (10) implies $R = 0$. Using equation (1) in equation (31), we have

$$A_kR_{ij} + A_jR_{ik} + A_iR_{jk} = 0. \tag{32}$$

Contracting equation (32) with A^k and using equation (9), we get

$$R_{ij} = RA_jA_i = 0, \tag{33}$$

which means that $(\mathrm{PRS})_n$ spacetimes in subspace $\mathscr{A}$ are Ricci flat.

Next, let us consider the subspace $\mathscr{B}$ in which $(\mathrm{PRS})_n$ has a Codazzi type of Ricci tensor [15]. The Codazzi deviation tensor D_{ijk} of $(\mathrm{PRS})_n$ is given by

$$\begin{aligned}D_{ijk} &= \nabla_kR_{ij} - \nabla_iR_{kj}\\ &= 2A_kR_{ij} + A_iR_{kj} + A_jR_{ik} - \left[2A_iR_{kj} + A_kR_{ij} + A_jR_{ik}\right]\\ &= A_kR_{ij} - A_iR_{kj}.\end{aligned} \tag{34}$$

A contraction with g^{ij} implies

$$g^{ij}D_{ijk} = A_kR - A^jR_{kj}. \tag{35}$$

But, in this subspace, the spacetimes have Codazzi-type Ricci tensor (that is, $D_{ijk} = 0$); then,

$$A_kR_{ij} = A_iR_{kj}. \tag{36}$$

Multiplying with g^{ij} and utilizing equation (9), we get

$$R = 0. \tag{37}$$

A contraction of equation (36) by A^k gives

$$R_{ij} = 0, \tag{38}$$

which means that $(\mathrm{PRS})_n$ spacetimes in Gray's subspace $\mathscr{B}$ are Ricci flat. □

Theorem 4. *$(PRS)_n$ spacetimes in $\mathscr{I}$, $\mathscr{I}\oplus\mathscr{A}$, and $\mathscr{I}\oplus\mathscr{B}$ subspaces are perfect fluid spacetimes.*

Proof. In subspace $\mathscr{I}$, the Ricci tensor of pseudo-Ricci symmetric manifold M satisfies the following property:

$$\nabla_k R_{ij} = \frac{n\nabla_k R}{(n-1)(n+2)} g_{ij} + \frac{(n-2)\nabla_i R}{2(n-1)(n+2)} g_{kj} + \frac{(n-2)\nabla_j R}{2(n-1)(n+2)} g_{ik}. \tag{39}$$

Applying equation (1), we obtain

$$2A_k R_{ij} + A_i R_{kj} + A_j R_{ik} = \frac{n\nabla_k R}{(n-1)(n+2) g_{ij}} + \frac{(n-2)\nabla_i R}{2(n-1)(n+2)} g_{kj} + \frac{(n-2)\nabla_j R}{2(n-1)(n+2)} g_{ik}. \tag{40}$$

It follows that

$$2A_k R_{ij} + A_i R_{kj} + A_j R_{ik} = \frac{2nA_k R}{(n-1)(n+2) g_{ij}} + \frac{(n-2)A_i R}{(n-1)(n+2)} g_{kj} + \frac{(n-2)A_j R}{(n-1)(n+2)} g_{ik}. \tag{41}$$

Contracting with A^k implies

$$R_{ij} = \frac{nR}{(n-1)(n+2)} g_{ij} + \frac{(2-n)R}{(n-1)(n+2)} A_i A_j, \tag{42}$$

which means that $(\mathrm{PRS})_n$ spacetimes in subspace $\mathscr{I}$ are perfect fluid.

In subspace $\mathscr{I}\oplus\mathscr{A}$, the Ricci curvature tensor satisfies

$$\nabla_k R_{ij} + \nabla_i R_{kj} + \nabla_j R_{ik} = \frac{2\nabla_k R}{(n+2)} g_{ij} + \frac{2\nabla_i R}{(n+2)} g_{kj} + \frac{2\nabla_j R}{(n+2)} g_{ik}. \tag{43}$$

Using equation (1), we infer

$$A_k R_{ij} + A_j R_{ik} + A_i R_{jk} = \frac{2}{(n+2)} \left(\nabla_k R g_{ij} + \nabla_i R g_{kj} + \nabla_j R g_{ik}\right). \tag{44}$$

Now, equation (10) implies

$$A_k R_{ij} + A_j R_{ik} + A_i R_{jk} = \frac{2R}{(n+2)} \left(A_k g_{ij} + A_i g_{kj} + A_j g_{ik}\right). \tag{45}$$

A contraction with A^k yields

$$R_{ij} = \frac{2R}{(n+2)} \left(g_{ij} - 2A_j A_i\right), \tag{46}$$

which means that $(\mathrm{PRS})_n$ spacetimes in subspace $\mathscr{I}\oplus\mathscr{A}$ are perfect fluid.

Assume that $(\mathrm{PRS})_n$ are in Gray's subspace $\mathscr{I}\oplus\mathscr{B}$; that is,

$$\nabla_k R_{ji} - \nabla_j R_{ki} = \frac{1}{2(n-1)} \left[g_{ji}\nabla_k R - g_{ki}\nabla_j R\right]. \tag{47}$$

Equation (1) implies

$$2(n-1)\left(A_k R_{ij} - A_j R_{ki}\right) = g_{ji}\nabla_k R - g_{ki}\nabla_j R. \tag{48}$$

The use of equation (10) gives

$$(n-1)\left(A_k R_{ij} - A_j R_{ki}\right) = g_{ji} A_k R - g_{ki} A_j R. \tag{49}$$

Contracting with A^k, we obtain

$$R_{ij} = \frac{R}{n-1} \left(g_{ji} + A_i A_j\right), \tag{50}$$

which means that $(\mathrm{PRS})_n$ spacetimes in Gray's subspace $\mathscr{I}\oplus\mathscr{B}$ are perfect fluid. □

Theorem 5. *$(PRS)_n$ spacetimes in $\mathscr{A}\oplus\mathscr{B}$ subspace have zero scalar curvature.*

Proof. In subspace $\mathscr{A}\oplus\mathscr{B}$, the scalar curvature is covariantly constant and hence equation (10) implies

$$R = 0, \tag{51}$$

which means $(\mathrm{PRS})_n$ spacetimes in Gray's subspace $\mathscr{A}\oplus\mathscr{B}$ have zero scalar curvature. □

4. Pseudo-Ricci Symmetric GRW Spacetimes

A generalized Robertson–Walker spacetime (for simplicity, denoted by GRW spacetimes) is the warped product $M = I \times_f M^*$ of an open connected interval $(I, -dt^2)$ and a Riemannian manifold M^*, where $f\colon I \longrightarrow \mathbb{R}^+$ is a positive smooth function. A Lorentzian manifold M is a generalized Robertson–Walker spacetime if and only if M possesses a unit time-like vector field u_i with [16, 17]

$$\nabla_k u_i = \varphi\left(g_{ki} + u_k u_i\right), \tag{52}$$

$$R_{ij} u^j = \xi u_i, \tag{53}$$

where φ and ξ are scalar functions. Vector fields satisfying equation (52) are called torse-forming.

Now, assume that M is a $(\mathrm{PRS})_n$ generalized Robertson–Walker spacetime; that is,

$$\nabla_k R_{ij} = 2A_k R_{ij} + A_i R_{kj} + A_j R_{ik}. \quad (54)$$

A contraction with u^j yields

$$u^j \nabla_k R_{ij} = 2A_k \left(u^j R_{ij}\right) + A_i \left(u^j R_{kj}\right) + A_j \left(u^j R_{ik}\right). \quad (55)$$

Using equation (1), one gets

$$u^j \nabla_k R_{ij} = 2A_k \xi u_i + \xi A_i u_k + u^j A_j R_{ik}. \quad (56)$$

Therefore,

$$u^j \nabla_k R_{ij} = 2\xi A_k u_i + \xi A_i u_k + \left(u^j A_j\right) R_{ik}. \quad (57)$$

However,

$$\begin{aligned} u^j \nabla_k R_{ij} &= \nabla_k \left(R_{ij} u^j\right) - R_{ij} \nabla_k u^j \\ &= u_i \nabla_k \xi + \xi \nabla_k \left(u_i\right) - R_{ij} \nabla_k u^j \\ &= u_i \nabla_k \xi + \varphi \left(\xi g_{ki} - R_{ik}\right). \end{aligned} \quad (58)$$

Thus,

$$u_i \nabla_k \xi + \varphi \left(\xi g_{ki} - R_{ik}\right) = 2\xi A_k u_i + \xi A_i u_k + \left(u^j A_j\right) R_{ik}. \quad (59)$$

It is well known $\nabla_k \xi = -u_k (u^j \nabla_j \xi) = -\rho u_k$, where $\rho = (u^j \nabla_j \xi)$ (see [12]); thus,

$$\left(\varphi + u^j A_j\right) R_{ik} = \xi \varphi g_{ki} - \rho u_i u_k - 2\xi A_k u_i - \xi A_i u_k. \quad (60)$$

Since M is $(\mathrm{PRS})_n$, equation (9) shows that

$$A^j R_{kj} = 0. \quad (61)$$

Multiplying both the sides by u^k, that is,

$$A^j \left(R_{kj} u^k\right) = 0. \quad (62)$$

Using equation (53), one gets

$$\xi A^j u_j = 0. \quad (63)$$

Now, there are two different possible cases. The first one $A^j u_j = 0$ and consequently ξ does not vanish. Then, equation (59) becomes

$$\varphi R_{ik} = \xi \varphi g_{ki} - \rho u_i u_k - 2\xi A_k u_i - \xi A_i u_k. \quad (64)$$

A contraction by A^i implies that

$$0 = \xi \varphi A_k - \xi \left(A^i A_i\right) u_k, \quad (65)$$

which is a contradiction. The second case is $\xi = 0$. Then, equation (60) leads to

$$\left(\varphi + u^j A_j\right) R_{ik} = 0. \quad (66)$$

Thus, either $R_{ik} = 0$ or $\varphi = -u^j A_j$.

Theorem 6. *A pseudo-Ricci symmetric GRW spacetime is vacuum provided the one form A is not codirectional with the torse-forming vector field u.*

Suppose $A_i \neq \varphi u_i$. Then, the spacetime under consideration is Ricci flat, that is, $R_{ij} = 0$, which implies $R = 0$. It is known that

$$\nabla_h \mathscr{C}^h_{ijk} = \frac{n-3}{n-2} \left[\left(\nabla_k R_{ij} - \nabla_j R_{ik}\right) - \frac{1}{2(n-1)} \left(g_{ij} \nabla_k R - g_{ik} \nabla_j R\right) \right], \quad (67)$$

where $\mathscr{C}$ is the conformal curvature tensor [13].

Therefore, using $R_{ij} = 0$ and $R = 0$, equation (67) yields $\nabla_h \mathscr{C}^h_{ijk} = 0$, that is, $\operatorname{div} \mathscr{C} = 0$. In [18], Mantica et al. proved that an n-dimensional GRW spacetime satisfies $\operatorname{div} \mathscr{C} = 0$ if and only if the spacetime is perfect fluid. Therefore, we conclude the following.

Corollary 2. *A pseudo-Ricci symmetric GRW spacetime is a perfect fluid spacetime provided $A_i \neq \varphi u_i$.*

Since $R_{ij} = 0$ and $R = 0$, from the definition of the conformal curvature tensor, it follows that $\mathscr{C}^h_{ijk} = R^h_{ijk}$. Hence, semisymmetric and conformally semisymmetric manifolds are equivalent. Eriksson and Senovilla [19] considered the semisymmetric spacetime and proved that it is of Petrov types D, N, and O. Thus, we have the following.

Corollary 3. *A conformally semisymmetric pseudo-Ricci symmetric GRW spacetime is of Petrov types D, N, and O.*

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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