Research Article

An Inertial Accelerated Algorithm for Solving Split Feasibility Problem with Multiple Output Sets

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The paper proposes an inertial accelerated algorithm for solving split feasibility problem with multiple output sets. To improve the feasibility, the algorithm involves computing of projections onto relaxed sets (half spaces) instead of computing onto the closed convex sets, and it does not require calculating matrix inverse. To accelerate the convergence, the algorithm adopts self-adaptive rules and incorporates inertial technique. The strong convergence is shown under some suitable conditions. In addition, some newly derived results are presented for solving the split feasibility problem and split feasibility problem with multiple output sets. Finally, numerical experiments illustrate that the algorithm converges more quickly than some existing algorithms. Our results extend and improve some methods in the literature.

1. Introduction

Let H_1 and H_2 be two real Hilbert spaces and C and Q be nonempty, closed, and convex subsets of H_1 and H_2 , respectively. Let $B: H_1 \rightarrow H_2$ be a nonzero bounded linear operator and B^T be its adjoint.

The split feasibility problem (SFP) is formulated to find a point $x^* \in H_1$ satisfying

$$x^* \in C \text{ such that } Bx^* \in Q. \quad (1)$$

The SFP was introduced in [1], which has broad applications in many fields such as image reconstruction problem [2, 3] and intensity-modulated radiation therapy (IMRT) [4–6]. Byrne [2] introduced the CQ-algorithm for solving (1) as follows:

$$x^0 \in H_1; x^{k+1} := P_C(x^k - \tau_k B^T(I - P_Q)Bx^k), \quad (2)$$

where P_C and P_Q are the metric projections onto C and Q , respectively, that is, $P_C(x) = \operatorname{argmin}_{y \in C} \|x - y\|$, over all $x \in H_1$. $\tau_k \in (0, 2/B^2)$, where B^2 is the spectral radius of the matrix B^TB .

In 2004, Yang [7] made the following assumptions.

(1) The sets C and Q are denoted as

$$C := \{x \in H_1: c(x) \leq 0\} \text{ and } Q := \{y \in H_2: q(y) \leq 0\}, \quad (3)$$

where $c: H_1 \rightarrow \mathbb{R}$ and $q: H_2 \rightarrow \mathbb{R}$ are convex and subdifferential functions on H_1 and H_2 , respectively.

(2) The subgradients $\partial c(x)$ and $\partial q(y)$ of c and q , respectively, can be calculated; they are

$$\partial c(x) := \{\xi \in H_1: c(z) \geq c(x) + \xi, z - x, \text{ for each } z \in H_1\}, \quad (4)$$

$$\partial q(y) := \{\eta \in H_2: q(z) \geq q(x) + \eta, u - y, \text{ for each } u \in H_2\}, \quad (5)$$

which are bounded operators (i.e., bounded on bounded sets).

Define two half-spaces C_k and Q_k as

$$C_k := \{x \in H_1: c(x^k) \leq \xi^k, x^k - x\}, \quad (6)$$

where $\xi^k \in \partial c(x^k)$ and

$$Q_k := \{y \in H_2: q(Bx^k) \leq \eta^k, Bx^k - y\}, \quad (7)$$

where $\eta^k \in \partial q(Bx)$. It is easy to see that $C_k \supset C$ and $Q_k \supset Q$, for all $k \geq 1$.

Yang, in [7], introduced the following relaxed CQ algorithm for solving the SFP (1) in a finite-dimensional Hilbert space:

$$x^0 \in H_1; x^{k+1} := P_{C_k}(x^k - \tau_k \nabla f_k(x^k)), \quad (8)$$

where $f_k(x^k) := (1/2)(I - P_{Q_k})^2$, $\nabla f_k(x^k) := B^T(I - P_{Q_k})Bx^k$, and $\tau_k \in (0, 2/B^2)$. Since P_{C_k} and P_{Q_k} are easily calculated, this method appears to be very practical. However, to compute the norm of B turns out to be complicated and costly. To overcome this difficulty, in 2012, L'opez et al. [8] introduced a relaxed CQ algorithm for solving the SFP (1) with a new adaptive way of determining the stepsize sequence τ_k defined as follows:

$$\tau_k := \frac{\rho_k f_k(x^k)}{\nabla f_k(x^k)^2}, \quad (9)$$

where $\{\rho_k\} \in (0, 4)$, $\forall k \geq 1$, such that $\inf_{n \rightarrow \infty} \rho_k(4 - \rho_k) > 0$. It was proved that the sequence $\{x^k\}$ generated by (8) with τ_k defined by (9) converges weakly to a solution of the SFP (1). That is, their algorithm has only weak convergence in the framework of infinite-dimensional Hilbert spaces.

Many authors also proposed algorithms that generate a sequence $\{x^k\}$ which converges strongly to a point in the solution set of the SFP (1), see, e.g., [9–12]. In particular, Deepho and Kumam [11] proposed a modified Halpern's iterative scheme for solving the SFP (1) in the setting of infinite-dimensional Hilbert spaces as follows:

$$\begin{aligned} u, x^0 \in H_1; x^{k+1} &:= \beta_k u + \delta_k x^k \\ &+ \gamma_k P_C(x^k - \tau_k B^*(I - P_Q)Bx^k), \forall k \geq 1, \end{aligned} \quad (10)$$

where $0 < \tau_k < 2/B^2$ and $\{\beta_k\}$, $\{\delta_k\}$, and $\{\gamma_k\}$ are three sequences in $[0, 1]$ such that $\beta_k + \delta_k + \gamma_k = 1$. Under some mild and standard assumptions, it was proved that the sequence $\{x^k\}$ generated by (10) converges strongly to a solution of the SFP (1).

Recently, Reich et al. [13] considered and studied the following split feasibility problem with multiple output sets in Hilbert spaces.

Let H and H_i , $i = 1, \dots, N$, be real Hilbert spaces and let $B_i: H \rightarrow H_i$, $i = 1, \dots, N$, be bounded linear operators. Let C and Q_i , $i = 1, \dots, N$, be nonempty, closed, and convex subsets of H and H_i , $i = 1, \dots, N$, respectively. Given H , H_i , and B_i as above, the split feasibility problem with multiple output sets (in short, SFPwMOS) is to find an element x^* such that

$$x^* \in \Gamma := C \cap \left(\bigcap_{i=1}^N B_i^{-1}(Q_i) \right) \neq \emptyset. \quad (11)$$

Reich et al. defined the function $g: H \rightarrow \mathbb{R}$ by

$$g(x) := 1/2 \sum_{i=1}^N (I - P_{Q_i})B_i x^2, \text{ for all } x \in H. \quad (12)$$

It is not difficult to see that an element p^* is a solution to the SFPwMOS (11) if and only if it is a solution to the problem

$$\min_{x \in C} g(x), \quad (13)$$

This is equivalent to

$$0 \in \nabla g(x^*) + N_C(x^*), \quad (14)$$

where $N_C(x)$ is the normal cone of C at the point x (recall: let $C \subset H$ be a closed convex subset of a real Hilbert space H . The normal cone of C at x denoted by $N_C(x)$ is given by $N_C(x) = \{z \in H: z, y - x \leq 0, \forall y \in C\}$), which implies

$$x^* = P_C \left(x^T - \lambda \sum_{i=1}^N B_i^T (I - P_{Q_i}) B_i x^T \right), \quad (15)$$

where λ is an arbitrary positive real number. Motivated by these characterizations, Reich et al. [13] introduced the following iterative method for solving the SFPwMOS (11). For any given point, $x^0 \in H$, $\{x^k\}$ is generated by the following iteration:

$$x^{k+1} := \alpha_k f(x^k) + (1 - \alpha_k) P_C \left(x^k - \lambda_k \sum_{i=1}^N B_i^T (I - P_{Q_i}) B_i x^k \right), \quad (16)$$

where $f: C \rightarrow C$ is a strict contraction mapping of H into itself with the contraction constant $\theta \in [0, 1)$, $\lambda_k \in (0, \infty)$ and $\{\alpha_k\} \subset (0, 1)$. It was proved that if the sequences $\{\lambda_k\}$ and $\{\alpha_k\}$ satisfy the conditions,

$$(I - f)x^*, x - px^* \geq 0 \forall y \in \Gamma. \quad (17)$$

An important observation here is that the method (Scheme (17)) introduced by Reich et al. [13] requires to compute the metric projections on to the sets C and Q_i . Moreover, it needs to compute the operator norm, and it only employs current iterative point to obtain the next iteration, which leads to slow convergence.

Inertial effect was proposed by Polyak [14] to speed up convergence of smooth convex minimization problem. The

main idea is to make use of two previous iterates in order to update the next iterate. Due to its good acceleration character, inertial type algorithms have been widely studied by authors [15–19].

Motivated by the modified Halpern's iterative scheme proposed by [11] for the SFP (1), the relaxed CQ algorithm is proposed by He et al. [11] to solve the SFP (1) and inertial effect is proposed by Polyak [14]. In this paper, we propose a new inertial self-adaptive relaxed CQ algorithm for solving the SFPwMOS (11) in general Hilbert spaces.

In Section 2, we recall some necessary tools which are used in establishing our main results. In Section 3, we propose an inertial self-adaptive relaxed CQ algorithm for solving the SFPwMOS (11), and we establish and analyze a strong convergence theorem for the proposed algorithm. In Section 4, we present some newly derived results for solving the SFPwMOS (11). Finally, in Section 5, we provide one numerical experiment to illustrate the implementation of our proposed method and compare it with some existing results.

2. Preliminaries

In this section, we recall some preliminaries which are needed in the sequel. Let H be a real Hilbert space with the inner product $\langle \cdot, \cdot \rangle$ and induced norm $\|\cdot\|$. Let I stand for the identity operator on H . We denote the fixed point set of an operator $T: H \rightarrow H$ (if T has fixed point) by $\text{Fix}(T)$, i.e., $\text{Fix}(T) = \{x \in H: Tx = x\}$. Let the symbols “ $\rightharpoonup$ ” and “ $\rightarrow$ ” denote the weak and strong convergence, respectively. For any sequence, $\{x^k\} \subset H$, $\omega_\omega(x^k) = \{x \in H: \exists \{x^{k_j}\} \subset \{x^k\} \text{ such that } \{x^{k_j}\} \rightharpoonup x\}$ denotes the weak ω -limit set of $\{x^k\}$.

Definition 1 (see [3]). Let H be a real Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and induced norm $\|\cdot\|$. Let C be a nonempty closed convex subset of H . Let $T: C \rightarrow H$ be a given operator. Then, T is called

- (1) Lipschitz continuous with constant $\sigma > 0$ on C if

$$\|Tx - Ty\| \leq \sigma \|x - y\|, \forall x, y \in C. \quad (18)$$

- (2) Nonexpansive on C if

$$\|Tx - Ty\| \leq \|x - y\|, \forall x, y \in C. \quad (19)$$

- (3) Firmly nonexpansive on C if

$$\|Tx - Ty\|^2 \leq \|x - y\|^2 - \|(I - T)x - (I - T)y\|^2, \forall x, y \in C, \quad (20)$$

which is equivalent to

$$\|Tx - Ty\| \leq \|x - y\|, \forall x, y \in C. \quad (21)$$

- (4) Averaged if there exists a number $\sigma \in (0, 1)$ and a nonexpansive operator $F: C \rightarrow H$ such that

$$T = F + (1 - \sigma)I, \quad (22)$$

where I is the identity operator.

In this case, we say that T is σ -averaged.

Lemma 1. Let $C \subset H$ be a nonempty, closed, and convex set, and $P_C(x)$ denotes the metric (orthogonal) projection of x onto C . Then, the following assertions hold for any $x, y \in H$ and $z \in C$:

- (1) $\langle x - P_C(x), z - P_C(x) \rangle \leq 0$,
- (2) $\|P_C(x) - P_C(y)\| \leq \|x - y\|$,
- (3) $\|P_C(x) - P_C(y)\|^2 \leq \langle x - P_C(x), x - y \rangle$,
- (4) $\|P_C(x) - z\|^2 \leq \|x - z\|^2 - \|x - P_C(x)\|^2$.

Evidently, $I - P_C$ is firmly nonexpansive and also nonexpansive.

Definition 2. Let $\delta \in (0, 1)$ and $f: H \rightarrow \mathbb{R}$ be a proper function. Then,

- (1) f is convex if

$$f(\delta x + (1 - \delta)y) \leq \delta f(x) + (1 - \delta)f(y), \forall x, y \in H. \quad (24)$$

- (2) f is strongly convex with constant σ , where $\sigma > 0$, if

$$f(\delta x + (1 - \delta)y) + \frac{\sigma}{2} \delta(1 - \delta) \|x - y\|^2 \leq \delta f(x) + (1 - \delta)f(y), \forall x, y \in H. \quad (25)$$

- (3) A vector $\omega \in H$ is a subgradient of f at a point x if

$$f(y) \geq f(x) + \langle \omega, y - x \rangle, \forall y \in H. \quad (26)$$

- (4) The set of all subgradients of a convex function $f: H \rightarrow \mathbb{R}$ at $x \in H$, denoted by $\partial f(x)$, is called the subdifferential of f and is defined by

$$\partial f(x) = \{\omega \in H: f(y) \geq f(x) + \langle \omega, y - x \rangle, \text{ for each } y \in H\}. \quad (27)$$

- (5) If $\partial f(x) \neq \emptyset$, f is said to be subdifferentiable at x .

If the function f is continuously differentiable, then $\partial f(x) = \{\nabla f(x)\}$. The convex function is subdifferentiable everywhere.

Definition 3. Let $f: H \rightarrow [-\infty, +\infty)$ be a proper function.

- (1) f is lower semicontinuous (lsc) at x if $x^k \rightarrow x$ implies

$$f(x) \leq \liminf_{k \rightarrow \infty} f(x^k). \quad (28)$$

- (2) f is weakly lower semicontinuous (ω -lsc) at x if $x^k \rightharpoonup x$ implies

$$f(x) \leq \liminf_{k \rightarrow \infty} f(x^k). \quad (29)$$

- (3) f is lower semicontinuous on H if it is lower semicontinuous at every point $x \in H$ and f is weakly lower semicontinuous on H if it is weakly lower semicontinuous at every point $x \in H$.

Lemma 2 (see [20]). Let C and Q be closed convex subsets of real Hilbert spaces H_1 and H_2 , respectively, and $f: H_1 \rightarrow \mathbb{R}$ is given by $f(x) = (I - P_Q)Ax^2$, where $A: H_1 \rightarrow H_2$ be a bounded linear operator. Then, for $\delta > 0$ and $x^* \in H_1$, the following statements are equivalent.

- (1) The point x^* solves the SFP (1), i.e., $x^* \in \{x \in C: Ax \in Q\}$.
- (2) The point x^* is the fixed point of the mapping $P_C(I - \delta \nabla f)$.
- (3) The point x^* solves the variational inequality problem with respect to the gradient of f , that is, find a point $x^* \in C$ such that

$$\nabla f(x), y - x \geq 0, \forall y \in C. \quad (30)$$

Lemma 3 (see [6]). Let H_1 and H_2 be real Hilbert spaces, and $f: H_1 \rightarrow \mathbb{R}$ is given by $f(x) = 1/2(I - P_Q)Ax^2$, where Q is closed convex subset of H_2 and $A: H_1 \rightarrow H_2$ is a bounded linear operator. Then, the following assertions hold:

- (1) f is convex and differentiable
- (2) f is weakly lower semicontinuous on H_1
- (3) $\nabla f(x) = A^T(I - P_Q)Ax$, for $x \in H_1$
- (4) $\nabla f(x)$ is A^2 -Lipschitz, i.e., $\nabla f(x) - \nabla f(y) \leq A^2x - y, \forall x, y \in H_1$

Definition 4. Let $\{\Lambda_k\}$ be a real sequence. Then, $\{\Lambda_k\}$ decreases at infinity if there exists $k_0 \in \mathbb{N}$ such that $\Lambda_{k+1} \leq \Lambda_k$, for $k \geq k_0$. In other words, the sequence $\{\Lambda_k\}$ does not decrease at infinity if there exists a subsequence $\{\Lambda_{k_i}\}_{i \geq 1}$ of $\{\Lambda_k\}$ such that $\Lambda_{k_i} < \Lambda_{k_{i+1}}$, for all $i \geq 1$.

Lemma 4. Let $\{\Lambda_k\}$ be a sequence of real numbers that does not decrease at infinity. Also, consider the sequence of integers $\{\varphi(k)\}_{k \geq k_0}$ defined by

$$\varphi(k) = \max\{m \in \mathbb{N}: m \leq k, \Lambda_m \leq \Lambda_{m+1}\}. \quad (31)$$

Then, $\{\varphi(k)\}_{k \geq k_0}$ is a nondecreasing sequence verifying $\lim_{k \rightarrow \infty} \varphi(k) = \infty$, and for all $k \geq k_0$, the following two estimates hold:

$$\Lambda_{\varphi(k)} \leq \Lambda_{\varphi(k)+1} \text{ and } \Lambda_k \leq \Lambda_{\varphi(k)+1}. \quad (32)$$

Lemma 5 (see [21]). Let $\{s_k\}$ be a sequence of nonnegative real numbers satisfying the following relation:

$$s_{k+1} \leq (1 - \sigma_k)s_k + \sigma_k\mu_k + \beta_k, \quad k \geq 1, \quad (33)$$

where $\{\sigma_k\}$, $\{\mu_k\}$, and $\{\beta_k\}$ satisfy the conditions:

$$\begin{aligned} (1) \quad & \{\sigma_k\} \subset [0, 1], \sum_{k=1}^{\infty} \sigma_k = \infty, \\ (2) \quad & \limsup_{k \rightarrow \infty} \mu_k \leq 0, \\ (3) \quad & \beta_k \geq 0, \sum_{k=1}^{\infty} \beta_k < \infty \text{ Then, } \limsup_{k \rightarrow \infty} s_k = 0. \end{aligned} \quad (34)$$

3. Main Results

In this section, we consider a general case of the SFPwMOS (11), where the nonempty, closed, and convex sets C and Q_i ($i = 1, \dots, N$) are given by level sets of convex functions as in [7]. Throughout this section, we assume that $c: H \rightarrow \mathbb{R}$ and $q_i: H_i \rightarrow \mathbb{R}$ are lower semicontinuous convex functions and the sets C and Q_i are given by

$$C := \{x \in H: c(x) \leq 0\} \text{ and } Q_i := \{y \in H_i: q_i(y) \leq 0\}. \quad (35)$$

The relaxed set (half spaces) C_k at x^k is defined as

$$C_k := \{x \in H: c(x^k) \leq \xi^k, x^k - x\}, \quad (36)$$

where $\xi^k \in \partial c(x^k)$.

The relaxed set (half spaces) Q_i^k ($i = 1, \dots, N$) at x^k are defined as

$$\begin{aligned} Q_i^k &:= \{x \in H: c(x^k) \leq \xi^k, x^k - x\}, \\ &\{y \in H_i: q_i(B_i x^k) \leq \eta_i^k, B_i x^k - y\}, \end{aligned} \quad (37)$$

where $\eta_i^k \in \partial q_i(B_i x^k)$. It follows that $C \subset C_k$ and $Q_i \subset Q_i^k$ hold, for every $k \geq 0$. Now, we define the following (relaxed) proximity function; for $x \in H$,

$$g_k(x) := \frac{1}{2} \sum_{i=1}^N \left(I - P_{Q_i^k} \right) B_i x^2. \quad (38)$$

We note that $g_k(\cdot)$ is differentiable with its gradient given by

$$\nabla g_k(x) := \sum_{i=1}^N B_i^T \left(I - P_{Q_i^k} \right) B_i x, \quad (39)$$

where each Q_i^k are half spaces given in (37). We note that g_k is weakly lower semicontinuous, convex, and differentiable function and ∇g_k is Lipschitz continuous.

Now, we present an inertial self-adaptive relaxed CQ algorithm.

We can see that Algorithm 1 terminates at some iterate (say k) when $\nabla g_k(x^k) = 0$.

The following lemma is important for the convergence analysis.

Lemma 6. If $z^k := x^k + \theta_k(x^k - x^{k-1})$, where $0 \leq \theta_k < 1$, for all $k \in \mathbb{N}$, then, for all $z \in H$,

$$z^k - z^2 \leq x^k - z^2 + \theta_k(x^k - z^2 - x^{k-1} - z^2) + 2\theta_k x^k - x^{k-12}. \quad (40)$$

Initialization: choose positive sequences $0 \leq \theta_k < 1$, $\{\rho_k\} \subset (0, 4)$, $\{\beta_k\} \subset (0, 1)$, $\{\delta_k\} \subset [0, 1)$ and $\{\varepsilon_k\} \subset (0, 1)$ such that $\beta_k + \delta_k + \varepsilon_k = 1$. Let $u \in H$ be a fixed point. Select arbitrary starting points $x^0, x^1 \in H$, and set $k = 0$

Step 1: given the current iterate $x^k \in H$, if $\nabla g_k(x^k) = 0$, for some $k \in \mathbb{N}$, then stop. Otherwise, continue and calculate $\tau_k := \rho_k g_k(z^k) / \nabla g_k(z^k)^2$.

Step 2: compute the next iterate as

$$\begin{cases} z^k = x^k + \theta_k(x^k - x^{k-1}) \\ x^{k+1} := P_{C_k}(\beta_k u + \delta_k z^k + \varepsilon_k(z^k - \tau_k \nabla g_k(z^k))) \end{cases}$$

where C_k is the half space given as in (36)

ALGORITHM 1: Inertial self-adaptive relaxed CQ algorithm for SFPwMOS.

Proof. Using the identity $2a, b = a^2 + b^2 - a - b^2$, we have

$$\begin{aligned} z^k - z^2 &= x^k - z + \theta_k(x^k - x^{k-1})^2 \\ &= x^k - z^2 + 2\theta_k x^k - z, x^k - x^{k-1} + \theta_k^2 x^k - x^{k-12}, \\ &= x^k - z^2 + \theta_k x^k - z^2 + x^k - x^{k-12} - x^{k-1} \\ &\quad - z^2 + \theta_k^2 x^k - x^{k-12}, \\ &= x^k - z^2 + \theta_k(x^k - z^2 - x^{k-1} - z^2) \\ &\quad + \theta_k(1 + \theta_k)x^k - x^{k-12}, \\ &\leq x^k - z^2 + \theta_k(x^k - z^2 - x^{k-1} - z^2) \\ &\quad + 2\theta_k x^k - x^{k-12}. \end{aligned} \quad (41)$$

Next, we show the following strong convergence theorem for Algorithm 1. $\square$

Theorem 1. Let $H, H_i, i = 1, \dots, N$, be real Hilbert spaces and let $B_i: H \rightarrow H_i, i = 1, \dots, N$, be bounded linear operators. Denote Γ as the solution set of the problem (11). Let C and $Q_i, i = 1, \dots, N$, be nonempty, closed, and convex subsets of H and $H_i, i = 1, \dots, N$, respectively. Assume that the SFPwMOS (11) is consistent. Suppose the sequences $\{\rho_k\}, \{\beta_k\}, \{\delta_k\}$, and $\{\varepsilon_k\}$ in Algorithm 1 satisfy the following conditions:

$$(A1) \lim_{k \rightarrow \infty} \beta_k = 0$$

$$\text{and } \sum_{k=1}^{\infty} \beta_k = +\infty, \frac{\beta_{k+1}}{\beta_k} \leq l, l \text{ is a constant}, \quad (42)$$

$$(A2) 0 < \liminf_{k \rightarrow \infty} \varepsilon_k \leq \limsup_{k \rightarrow \infty} \varepsilon_k < 1, \quad (43)$$

$$(A3) \liminf_{k \rightarrow \infty} \rho_k(4 - \rho_k) > 0, \quad (44)$$

$$(A4) \lim_{k \rightarrow \infty} \frac{\theta_k}{\beta_k} \|x^k - x^{k-1}\| = 0. \quad (45)$$

Then, the sequence $\{x^k\}$ generated by Algorithm 1 converges strongly to the solution $p \in \Gamma$, where $p = P_{\Gamma}(u)$.

Proof. We may assume that Algorithm 1 does not terminate in a finite number of iterations. Thus, $\nabla g_k(x^k) \neq 0$, for all $k \geq 0$. Denote Γ as the solution set of problem (11). In the

consistent case of problem (11), Γ is nonempty, closed, and convex. Thus, the metric projection P_{Γ} is well defined.

Let $p \in \Gamma$ and set $s^k = z^k - \tau_k \nabla g_k(z^k)$. Note that $I - P_{Q_i^k}$, for each $i = 1, \dots, N$, is firmly nonexpansive and $\nabla g_k(p) = 0$. Hence, we have, from Lemma 1, that

$$\begin{aligned} \langle \nabla g_k(z^k), z^k - p \rangle &= \langle \sum_{i=1}^N B_i^T (I - P_{Q_i^k}) B_i z^k, z^k - p \rangle, \\ &= \sum_{i=1}^N B_i^T (I - P_{Q_i^k}) B_i z^k, z^k - p, \\ &= \sum_{i=1}^N (I - P_{Q_i^k}) B_i z^k, B_i z^k - B_i p, \\ &\geq \sum_{i=1}^N (I - P_{Q_i^k}) B_i z^k = 2g_k(z^k), \end{aligned} \quad (46)$$

which implies that

$$\begin{aligned} \|s^k - p\|^2 &= \|z^k - p - \tau_k \nabla g_k(z^k)\|^2, \\ &= z^k - p^2 + \tau_k^2 \nabla g_k(z^k)^2 - 2\tau_k \nabla g_k(z^k), z^k - p \\ &\leq z^k - p^2 + \frac{\rho_k^2 g_k^2(z^k)}{\nabla g_k(z^k)^2} - \frac{2\rho_k g_k(z^k)}{\nabla g_k(z^k)^2} (2g_k(z^k)) \\ &= z^k - p^2 + \frac{\rho_k^2 g_k^2(z^k)}{\nabla g_k(z^k)^2} - \frac{4\rho_k g_k^2(z^k)}{\nabla g_k(z^k)^2} \\ &= z^k - p^2 - \rho_k(4 - \rho_k) \frac{g_k^2(z^k)}{\nabla g_k(z^k)^2}. \end{aligned} \quad (47)$$

Using condition (A3), we have

$$s^k - p^2 \leq z^k - p^2, \forall k \geq 0. \quad (48)$$

Next, we show $\{z^k\}$ is bounded. Since $p \in \Gamma \subset C_k$ and the projection operator P_{C_k} is nonexpansive, we obtain from Algorithm 1 and (48) that

$$\begin{aligned}
 x^{k+1} - p &= P_{C_k}(\beta_k u + \delta_k z^k + \varepsilon_k(z^k - \tau_k \nabla g_k(z^k))) - p, \\
 &\leq \beta_k u - p + \delta_k z^k - p + \varepsilon_k s^k - p, \\
 &\leq \beta_k u - p + \delta_k z^k - p + \varepsilon_k z^k - p, \\
 &= \beta_k u - p + (1 - \beta_k)z^k - p, \\
 &= \beta_k u - p + (1 - \beta_k)x^k + \theta_k(x^k - x^{k-1}) - p, \\
 &\leq \beta_k u - p + (1 - \beta_k)x^k - p + \theta_k \|x^k - x^{k-1}\|, \\
 &= (1 - \beta_k) \|x^k - p\| + \beta_k \left[\|u - p\| + \frac{\theta_k}{\beta_k} \|x^k - x^{k-1}\| \right].
 \end{aligned} \tag{49}$$

Condition (A4) implies the sequence $\theta_k/\beta_k x^k - x^{k-1}$ is bounded. Let an upper bound of $\{u - p + \theta_k/\beta_k x^k - x^{k-1}\}$ be M . Then, we rewrite (49) as

$$x^{k+1} - p \leq \max\{x^k - p, M\}, \tag{50}$$

By induction, we have that

$$x^{k+1} - p \leq \max\{x^1 - p, M\}. \tag{51}$$

Hence, $\{x^k - p\}$ is bounded, so is $\{z^k - p\}$ and $\{s^k - p\}$. And, $\{z^k\}$ is bounded. Consequently, $\{s^k\}$ and $\{B_i z^k\}_{i=1}^N$ are also bounded. The rest of the proof will be divided into two parts. $\square$

Case 1. Suppose that there exists $k_0 \in \mathbb{N}$ such that $\{x^k - p^2\}_{k=k_0}^\infty$ is nonincreasing. Then, $\{x^k - p^2\}_{k=1}^\infty$ converges and $x^k - p^2 - x^{k+1} - p^2 \rightarrow 0$ as $k \rightarrow \infty$. From (47), we obtain

$$\rho_k(4 - \rho_k) \frac{g_k^2(z^k)}{\nabla g_k(z^k)^2} \leq z^k - p^2 - s^k - p^2. \tag{52}$$

Since $p \in \Gamma \subset C_k$ and the projection operator P_{C_k} is nonexpansive, from Algorithm 1, we obtain

$$\begin{aligned}
 x^{k+1} - p^2 &= P_{C_k}(\beta_k u + \delta_k z^k + \varepsilon_k s_k) - p^2, \\
 &\leq (\beta_k u + \delta_k z^k + \varepsilon_k s_k) - p^2, \\
 &\leq \beta_k u - p^2 + \delta_k z^k - p^2 + \varepsilon_k s_k - p^2.
 \end{aligned} \tag{53}$$

Since $\beta_k + \delta_k + \varepsilon_k = 1$, from (53) and using the inequality $x + y^2 \leq x^2 + 2y, x + y$, we have the following estimation:

$$\begin{aligned}
 z^k - p^2 - s^k - p^2 &\leq \frac{\beta_k}{\varepsilon_k} u - p^2 + \frac{1 - \beta_k}{\varepsilon_k} z^k - p^2 - \frac{1}{\varepsilon_k} x^{k+1} - p^2, \\
 &= \frac{\beta_k}{\varepsilon_k} u - p^2 - \frac{\beta_k}{\varepsilon_k} z^k - p^2 + \frac{1}{\varepsilon_k} [z^k - p^2 - x^{k+1} - p^2], \\
 &\leq \frac{\beta_k}{\varepsilon_k} u - p^2 + \frac{1}{\varepsilon_k} [z^k - p^2 - x^{k+1} - p^2], \\
 &= \frac{1}{\varepsilon_k} [\beta_k u - p^2 + [z^k - p^2 - x^{k+1} - p^2]], \\
 &= \frac{1}{\varepsilon_k} \left[\beta_k u - p^2 + [x^k - p + \theta_k(x^k - x^{k-1})^2 - x^{k+1} - p^2] \right], \\
 &\leq \frac{1}{\varepsilon_k} [\beta_k u - p^2 + [x^k - p^2 + 2\theta_k x^k - x^{k-1}, z^k - p - x^{k+1} - p^2]], \\
 &\leq \frac{1}{\varepsilon_k} [\beta_k u - p^2 + [x^k - p^2 + 2\theta_k x^k - x^{k-1} z^k - p - x^{k+1} - p^2]], \\
 &= \frac{1}{\varepsilon_k} [\beta_k u - p^2 + [x^k - p^2 - x^{k+1} - p^2 + 2\theta_k x^k - x^{k-1} z^k - p]].
 \end{aligned} \tag{54}$$

Combining (52) and (54), we obtain

$$\begin{aligned}
\rho_k(4 - \rho_k) \frac{g_k^2(z^k)}{\nabla g_k(z^k)^2} &\leq z^k - p^2 - s^k - p^2, \\
&\leq \frac{1}{\varepsilon_k} [\beta_k u - p^2 + [x^k - p^2 - x^{k+1} - p^2 \\
&\quad + 2\theta_k x^k - x^{k-1} z^k - p]].
\end{aligned} \tag{55}$$

By condition (A4), $\beta_k \in (0, 1)$ imply the sequence $\lim_{k \rightarrow \infty} \theta_k x^k - x^{k-1} \rightarrow 0$, and we have proved $\{z^k - p\}$ is bounded, so $2\theta_k x^k - x^{k-1} z^k - p \rightarrow 0$. By conditions (A2) and (A3) and (55), we have, as $k \rightarrow \infty$,

$$\begin{aligned}
0 &< \rho_k(4 - \rho_k) \frac{g_k^2(z^k)}{\nabla g_k(z^k)^2}, \\
&\leq \frac{1}{\varepsilon_k} [\beta_k u - p^2 + [x^k - p^2 - x^{k+1} - p^2 \\
&\quad + 2\theta_k x^k - x^{k-1} z^k - p]] \rightarrow 0,
\end{aligned} \tag{56}$$

which implies that

$$\lim_{k \rightarrow \infty} \frac{g_k^2(z^k)}{\nabla g_k(z^k)^2} = 0. \tag{57}$$

We note that, for each $i = 1, \dots, N$, $B_i^T(I - P_{Q_i^*})B_i(\cdot)$ is Lipschitz continuous. Since the sequence $\{z^k\}$ is bounded and

$$\begin{aligned}
B_i^T(I - P_{Q_i^*})B_i z^k &= B_i^T(I - P_{Q_i^*})B_i z^k \\
&- B_i^T(I - P_{Q_i^*})B_i p \leq \left(\max_{1 \leq i \leq N} B_i\right) z^k - p,
\end{aligned} \tag{58}$$

for all $i = 1, \dots, N$, we have the sequence $\{B_i^T(I - P_{Q_i^*})B_i z^k\}_{i=1}^\infty$ is bounded. Hence, $\{\nabla g_k(z^k)\}_{k=1}^\infty$ is bounded. Consequently, we have from (57) that

$$\lim_{k \rightarrow \infty} (I - P_{Q_i^*})B_i z^k = 0 \tag{59}$$

For

$$z^k - x^k = x^k + \theta_k(x^k - x^{k-1}) - x^k = \theta_k x^k - x^{k-1}, \tag{60}$$

which implies

$$z^k - x^k \rightarrow 0, \text{ as } k \rightarrow \infty, \tag{61}$$

for each $i = 1, \dots, N$, since $s^k = z^k - \tau_k \nabla g_k(z^k)$, we have from (59) and (61) that

$$\begin{aligned}
s^k - z^k &= \tau_k \nabla g_k(z^k) \rightarrow 0, \text{ as } k \rightarrow \infty, \\
s^k - x^k &\leq s^k - z^k + z^k - x^k \rightarrow 0, \text{ as } k \rightarrow \infty.
\end{aligned} \tag{62}$$

That is,

$$\lim_{k \rightarrow \infty} s^k - x^k = 0, \tag{63}$$

$$\lim_{k \rightarrow \infty} B_i^T(I - P_{Q_i^*})B_i z^k = 0 \tag{64}$$

Let

$$l^k = \beta_k u + \delta_k z^k + \varepsilon_k s^k = \beta_k u + (1 - \beta_k) v^k, \tag{65}$$

where

$$v^k = \frac{\delta_k}{1 - \beta_k} z^k + \frac{\varepsilon_k}{1 - \beta_k} s^k. \tag{66}$$

This gives

$$v^k - z^k \leq \frac{\varepsilon_k}{1 - \beta_k} s^k - z^k \rightarrow 0, \text{ as } k \rightarrow \infty, \tag{67}$$

$$\|l^k - v^k\| \leq \beta_k \|u - v^k\| \rightarrow 0, \text{ as } k \rightarrow \infty,$$

$$\begin{aligned}
l^k - z^k &\leq l^k - v^k + v^k - z^k \rightarrow 0, \text{ as } k \rightarrow \infty, \\
l^k - x^k &\leq l^k - z^k + z^k - x^k \rightarrow 0, \text{ as } k \rightarrow \infty.
\end{aligned} \tag{68}$$

Hence,

$$\lim_{k \rightarrow \infty} l^k - z^k = \lim_{k \rightarrow \infty} l^k - x^k = \lim_{k \rightarrow \infty} l^k - v^k = \lim_{k \rightarrow \infty} v^k - z^k = 0. \tag{69}$$

Since $\{x^k\}$ is bounded, there exists a subsequence $\{x^{k_j}\}$ of $\{x^k\}$ such that $\{x^{k_j}\} \rightarrow p^* \in \omega_\omega(x^k)$. Next, we need to show $p^* \in C$ and $B_i p^* \in Q_i$, for each $i = 1, \dots, N$.

We first show that $x^{k+1} - x^k \rightarrow 0$. For $p \in \Gamma$,

$$\begin{aligned}
\frac{1}{2} x^{k+1} - p^2 &= \frac{1}{2} x^{k+1} - x^k + x^k - p^2, \\
&= \frac{1}{2} x^k - p^2 + x^{k+1} - x^k, x^{k+1} - p - \frac{1}{2} x^{k+1} - x^{k2}, \\
&= \frac{1}{2} x^k - p^2 + x^{k+1} - l^k + l^k - x^k, x^{k+1} \\
&\quad - p - \frac{1}{2} x^{k+1} - x^{k2}, \\
&= \frac{1}{2} x^k - p^2 + x^{k+1} - l^k, x^{k+1} - p + l^k \\
&\quad - x^k, x^{k+1} - p - \frac{1}{2} x^{k+1} - x^{k2},
\end{aligned} \tag{70}$$

then

$$\begin{aligned}
\frac{1}{2} x^{k+1} - x^{k2} &\leq \frac{1}{2} x^k - p^2 - \frac{1}{2} x^{k+1} - p^2 + x^{k+1} - l^k, x^{k+1} \\
&\quad - p + l^k - x^k, x^{k+1} - p,
\end{aligned} \tag{71}$$

from $x^{k+1} - l^k, x^{k+1} - p < 0$,

$$\frac{1}{2}x^{k+1} - x^{k2} \leq \frac{1}{2}x^k - p^2 - \frac{1}{2}x^{k+1} - p^2 + l^k - x^k, x^{k+1} - p. \quad (72)$$

Since $x^k - p^2 - x^{k+1} - p^2 \longrightarrow 0$ as $k \longrightarrow \infty$,
 $\lim_{k \longrightarrow \infty} l^k - x^k = 0$, we obtain

$$x^{k+1} - x^k \longrightarrow 0, \text{ as } k \longrightarrow \infty. \quad (73)$$

Since $x^{k_j+1} \in C_{k_j}$, we obtain

$$c(x^{k_j}) + \xi^{k_j}, x^{k_j+1} - x^{k_j} \leq 0. \quad (74)$$

Thus,

$$c(x^{k_j}) \leq -\xi^{k_j}, x^{k_j+1} - x^{k_j} \leq \xi x^{k_j+1} - x^{k_j}, \quad (75)$$

where ξ satisfies $\xi^k \leq \xi$, for all k . By virtue of the continuity of function c and $x^{k_j+1} - x^{k_j} \longrightarrow 0$, we get that

$$c(p^*) = \lim_{j \longrightarrow \infty} c(x^{k_j}) \leq 0. \quad (76)$$

Therefore, $p^* \in C$.

Now, we show that $B_i p^* \in Q_i$; to do this, let $h_i^k = B_i z^k - P_{Q_i^k} B_i z^k \longrightarrow 0$ and let η_i^k be such that $\eta_i^k \leq \eta$, for all k . Since $B_i z^{k_j} - h_i^{k_j} = P_{Q_i^{k_j}}(B_i z^{k_j}) \in Q_i^{k_j}$, we have

$$q_i(B_i x^{k_j}) + \eta_i^{k_j}, (B_i z^{k_j} - h_i^{k_j}) - B_i x^{k_j} \leq 0. \quad (77)$$

Hence,

$$q_i(B_i x^{k_j}) \leq \eta_i^{k_j}, B_i x^{k_j} - B_i z^{k_j} + \eta_i^{k_j}, h_i^{k_j} \leq \eta B_i x^{k_j} - x^{k_j-1} + \eta h_i^{k_j} \longrightarrow 0. \quad (78)$$

By the continuity of q_i and $B_i x^{k_j} \longrightarrow B_i x^*$, we arrive at the conclusion

$$q_i(B_i p^*) = \lim_{j \longrightarrow \infty} q_i(B_i x^{k_j}) \leq 0, \quad (79)$$

namely, $B_i p^* \in Q_i$, for all $i = 1, \dots, N$. Hence, $p^* \in \Gamma$.

Moreover, for $p = P_\Gamma u$, we can see that

$$\limsup_{k \longrightarrow \infty} l^k - p, u - p = \lim_{k \longrightarrow \infty} l^{k_j} - p, u - p \leq p^* - p, u - p \leq 0. \quad (80)$$

By Lemma 1 and (65), we have

$$\begin{aligned} x^{k+1} - p^2 &= P_C(l^k) - p^2 \leq l^k - p^2 = \beta_k u + (1 - \beta_k)v^k - p, l^k - p \\ &= \beta_k u - p, l^k - p + (1 - \beta_k)v^k - p, l^k - p \\ &\leq \beta_k u - p, l^k - p + (1 - \beta_k)v^k - p, l^k - p \\ &= (1 - \beta_k) \|x^k + \theta_k(x^k - x^{k-1}) - p\|^2 + \beta_k \langle u - p, l^k - p \rangle + \frac{\beta_k}{2} \|u - p\|^2, \\ &\leq (1 - \beta_k) \left[\|x^k - p\|^2 + 2\theta_k \langle x^k - x^{k-1}, z^k - p \rangle \right] + \beta_k \langle u - p, l^k - p \rangle + \frac{\beta_k}{2} \|u - p\|^2, \\ &\leq (1 - \beta_k) \left[\|x^k - p\|^2 + 2\theta_k \|x^k - x^{k-1}\| \|z^k - p\| \right] + \beta_k \langle u - p, l^k - p \rangle + \frac{\beta_k}{2} \|u - p\|^2. \end{aligned} \quad (81)$$

By conditions (A1) and (A4), $\beta_k \in (0, 1)$ imply the sequence $\lim_{k \longrightarrow \infty} \theta_k x^k - x^{k-1} \longrightarrow 0$, $\beta_k/2 \|u - p\| \longrightarrow 0$, and using (80) in (81) and applying Lemma 5, we obtain

$$\lim_{k \longrightarrow \infty} x^k - p^2 = 0. \quad (82)$$

Therefore, as $k \longrightarrow \infty$, $x^k \longrightarrow p = P_\Gamma u$

Case 2. Set $\Lambda_k = x^k - p^2$. Assume that $\{\Lambda_k\}$ is not decreasing at infinity. Let $\phi: \mathbb{N} \longrightarrow \mathbb{N}$ be a mapping for all $k \geq k_0$ (for some k_0 large enough) defined by

$$\phi(k) = \max\{t \in \mathbb{N}: t \leq k, \Lambda_t \leq \Lambda_{t+1}\}. \quad (83)$$

By Lemma 4, $\{\phi(k)\}_{k=k_0}^\infty$ is a nondecreasing sequence such that $\phi(k) \longrightarrow \infty$ as $k \longrightarrow \infty$ and

$$\max\{\Lambda_{\phi(k)}, \Lambda_k\} \leq \Lambda_{\phi(k)+1}, \forall k \geq k_0. \quad (84)$$

After a similar conclusion from (59), it is easy to see that

$$\lim_{k \longrightarrow \infty} (I - P_{Q_i^{\phi(k)}}) B_i z^{\phi(k)} = 0. \quad (85)$$

By the similar argument as the above in Case 1, we conclude immediately that

$$\lim_{k \rightarrow \infty} B_i^* \left(I - P_{Q_i^{\phi(k)}} \right) B_i z^{\phi(k)} = 0, \quad (86)$$

$$\limsup_{k \rightarrow \infty} l^{\phi(k)} - p, u - p \leq 0. \quad (87)$$

Since $\{x^{\phi(k)}\}$ is bounded, there exists a subsequence of $\{x^{\phi(k)}\}$, still denoted by $\{x^{\phi(k)}\}$ which converges weakly to p^* . By similar argument as above in Case 1, we conclude immediately that $p^* \in C$ and $B_i p^* \in Q_i \Rightarrow p^* \in \Gamma$.

From (57), we have that

$$\begin{aligned} x^{\phi(k)+1} - p^2 &\leq (1 - \beta_{\phi(k)})z^{\phi(k)} - p^2 + \beta_{\phi(k)}u - p, l^{\phi(k)} - p \\ &\leq (1 - \beta_{\phi(k)})x^{\phi(k)} + \theta_{\phi(k)}(x^{\phi(k)} - x^{\phi(k)-1}) - p^2 + \beta_{\phi(k)}u - p, l^{\phi(k)} - p, \\ &\leq (1 - \beta_{\phi(k)})[x^{\phi(k)} - p^2 + 2\theta_{\phi(k)}x^{\phi(k)} - x^{\phi(k)-1}, z^{\phi(k)} - p] + \beta_{\phi(k)}u - p, l^{\phi(k)} - p \\ &\leq (1 - \beta_{\phi(k)})[x^{\phi(k)} - p^2 + 2\theta_{\phi(k)}x^{\phi(k)} - x^{\phi(k)-1}z^{\phi(k)} - p] + \beta_{\phi(k)}u - p, l^{\phi(k)} - p, \end{aligned} \quad (88)$$

which implies by Lemma 5

$$\lim_{k \rightarrow \infty} \|x^{\phi(k)} - p\|^2 = 0 \text{ and } \lim_{k \rightarrow \infty} x^{\phi(k)+1} - p^2 = 0. \quad (89)$$

Moreover, for $k \geq k_0$, it is easy to see that $\Lambda_{\phi(k)} - \Lambda_{\phi(k)+1} \leq 0$ if $k \neq \phi(k)$ (that is, $k \geq \phi(k)$) because, for $\phi(k) + 1 \leq m \leq n$, $\Lambda_m > \Lambda_{m+1}$. As a consequence, we obtain

$$0 \leq \Lambda_k \leq \max\{\Lambda_{\phi(k)}, \Lambda_{\phi(k)+1}\} = \Lambda_{\phi(k)+1}, \quad \forall k \geq k_0. \quad (90)$$

Therefore, we obtain $\lim_{k \rightarrow \infty} \Lambda_k = 0$, that is, $\{x^k\}$ converges strongly to p . This completes the proof.

4. Some Extensive Results

For the SFPwMOS (11), when $N = 1$, it becomes the SFP (1). Thus, we have the following corollary for solving the SFP (1), which is an immediate consequence of Theorem 1.

Corollary 1. Let H_1 and H_2 be two real Hilbert spaces, and let $B: H_1 \rightarrow H_2$ be bounded linear operator. Let C and Q be nonempty, closed, and convex subsets of H_1 and H_2 , respectively. Assume that $\Omega = C \cap B^{-1}(Q) \neq \emptyset$. Let $u \in H_1$ be a fixed point. For any starting point $x^0, x^1 \in H_1$, let $\{x^k\}$ be the sequence generated by

$$\begin{cases} z^k = x^k + \theta_k(x^k - x^{k-1}), \\ x^{k+1} := P_{C_k}(\beta_k u + \delta_k z^k + \varepsilon_k(z^k - \tau_k \nabla g_k(z^k))), \end{cases} \quad (91)$$

where θ_k, C_k, τ_k , and ∇f_k are given as in (8). Suppose the sequences $\{\beta_k\}, \{\delta_k\}$, and $\{\varepsilon_k\}$ satisfy the conditions in Theorem 1. Then, the sequence $\{x^k\}$ converges strongly to the solution $p \in \Omega$, where $p = P_{\Omega}(u)$.

When we take $u = x^0$ in Algorithm 1, we note also the following results regarding to the SFPwMOS (11).

Corollary 2. Let $H, H_i, i = 1, \dots, N$, be real Hilbert spaces and let $B_i: H \rightarrow H_i, i = 1, \dots, N$, be bounded linear operators. Let C and $Q_i, i = 1, \dots, N$, be nonempty, closed, and convex subsets of H and $H_i, i = 1, \dots, N$, respectively. Assume that problem (11) is consistent. For any initial guess $x^0, x^1 \in H$, let $\{x^k\}$ be the sequence generated by

$$\begin{cases} z^k = x^k + \theta_k(x^k - x^{k-1}), \\ x^{k+1} := P_{C_k}(\beta_k x^0 + \delta_k z^k + \varepsilon_k(z^k - \tau_k \nabla g_k(z^k))), \end{cases} \quad (92)$$

where θ_k, C_k, τ_k , and ∇g_k are given by (36) and (39) and Algorithm 1, respectively. Suppose the sequences $\{\beta_k\}, \{\delta_k\}$, and $\{\varepsilon_k\}$ satisfy the conditions in Theorem 1. Then, the sequence $\{x^k\}$ generated by (92) strongly converges to the solution $p = P_{\Omega}(x^0) \in \Gamma$.

When we take $\delta_k \equiv 0$ in Algorithm 1, we obtain the following result regarding the SFPwMOS (11).

Corollary 3. Let $H, H_i, i = 1, \dots, N$, be real Hilbert spaces, and let $T_i: H \rightarrow H_i, i = 1, \dots, N$, be bounded linear operators. Let C and $Q_i, i = 1, \dots, N$, be nonempty, closed, and convex subsets of H and $H_i, i = 1, \dots, N$, respectively. Assume that problem (11) is consistent. For a fixed point $u \in H$ and any initial guess $x^0 \in H$, let $\{x^k\}$ be the sequence generated by

$$\begin{cases} z^k = x^k + \theta_k(x^k - x^{k-1}), \\ x^{k+1} := P_{C_k}(\beta_k u + (1 - \beta_k)(z^k - \tau_k \nabla g_k(z^k))), \end{cases} \quad (93)$$

where θ_k, C_k, τ_k , and ∇g_k are given by (36) and (39) and Algorithm 1, respectively. Suppose the sequences satisfy conditions (A1). Then, the sequence $\{x^k\}$ generated by (93) strongly converges to the solution $p = P_{\Omega}(u) \in \Gamma$.

Of course, when we take $u = x^0$, we get the following result regarding the SFPwMOS (11).

Corollary 4. Let $H, H_i, i = 1, \dots, N$, be real Hilbert spaces and let $B_i: H \rightarrow H_i, i = 1, \dots, N$, be bounded linear operators. Let C and $Q_i, i = 1, \dots, N$, be nonempty, closed, and convex subsets of H and $H_i, i = 1, \dots, N$, respectively. Assume that problem (11) is consistent. For any initial guess $x^0 \in H$, let $\{x^k\}$ be the sequence generated by

$$\begin{cases} z^k = x^k + \theta_k(x^k - x^{k-1}), \\ x^{k+1} := P_{C_k}(\beta_k x^0 + (1 - \beta_k)(z^k - \tau_k \nabla g_k(z^k))), \end{cases} \quad (94)$$

TABLE 1: Algorithm 1 for different choices of θ_k , ε , and ρ_k .

ε	θ_k	$\rho_k = 3.98$		$\rho_k = 3.00$	
		Iter	E_k	Iter	E_k
10^{-3}	$\theta_k = 0.8$	5	$7.0906e-05$	6	$5.8048e-05$
	$\theta_k = 0.3$	7	$6.9397e-04$		
	$\theta_k = 0$	9	$7.5095e-04$		
10^{-4}	$\theta_k = 0.8$	5	$7.0906e-05$	6	$5.8048e-05$
	$\theta_k = 0.3$	11	$9.1282e-05$		
	$\theta_k = 0$	15	$9.6150e-05$		
10^{-5}	$\theta_k = 0.8$	12	$4.4752e-06$	14	$2.5296e-06$
	$\theta_k = 0.3$	16	$9.1012e-06$		
	$\theta_k = 0$	24	$8.8064e-06$		

where θ_k , C_k , τ_k , and ∇g_k are given by (36) and (39) and Algorithm 1, respectively. Suppose the sequence $\{\beta_k\}$ satisfies conditions (A1). Then, the sequence $\{x^k\}$ generated by (94) strongly converges to the solution $p = P_\Omega(u) \in \Gamma$.

Remark 1. In Corollary 4, for the particular case, where $N = 1$, the iterative scheme (94) reduced exactly to iterative scheme proposed by He et al. in Theorem 3.2 of [10].

5. Numerical Experiment

In this section, we provide one numerical experiment to illustrate the implementation and efficiency of our proposed method. First, we study the behavior of Algorithm 1 for different choices of θ_k and ρ_k . Next, we compare Algorithm 1 ($\theta_k = 0.8$) with no inertial Algorithm 1 ($\theta_k = 0$) and Scheme (17) for different initial points. The numerical results are completed on a standard TOSHIBA laptop with Intel(R) Core(TM) i5-2450M CPU@2.5GHz 2.5 GHz with memory 4 GB. The code is implemented in MATLAB R2020a.

Let

$$\begin{aligned}
 A_1 &= \begin{bmatrix} -0.4 & -0.2 & -0.2 \\ 0.4 & 0.5 & -0.1 \\ 0.2 & -0.5 & 0.3 \end{bmatrix}, \\
 A_2 &= \begin{bmatrix} 0.3 & -0.1 & 0.3 \\ -0.3 & 0.2 & -0.2 \\ 0 & 0.2 & 0.3 \end{bmatrix}, \\
 C &= \{x \in \mathbb{R}^3 | x_1 - x_2^2 + 2x_3 \leq 0\}, \\
 Q_1 &= \{x \in \mathbb{R}^3 | x_1^2 + x_2 - x_3 \leq 0\}, \\
 Q_2 &= \{x \in \mathbb{R}^3 | x_1 + x_2^2 - x_3 \leq 0\}.
 \end{aligned} \tag{95}$$

Find x^* such that

$$x^* \in \Gamma := C \cap \left(\bigcap_{i=1}^2 A_i^{-1}(Q_i) \right) \neq \emptyset. \tag{96}$$

Define the error function as

FIGURE 1: Error against iterations for Algorithm 1 with difference choices of θ_k , under $\rho_k = 3.98$ and $\varepsilon = 10^{-5}$.

$$\begin{aligned}
 E_k &= \frac{1}{3}x^k - P_{C_k}(x^k)^2 + \frac{1}{3}A_1x^k - P_{Q_{1,k}}(A_1x^k)^2 \\
 &\quad + \frac{1}{3}A_2x^k - P_{Q_{2,k}}(A_2x^k)^2.
 \end{aligned} \tag{97}$$

Note that if $E_k = 0$ at the k th step, then $x^k \in \Gamma$.

For Algorithm 1, for different choices of ρ_k and θ_k , we choose $\beta_k = 1/9k + 1$, $\delta_k = k/9k + 1$, $\varepsilon_k = 8k/9k + 1$, $u = (0.1 \ 0.1 \ 0.1)^T$, and $x^0 = (-0.1 \ -0.1 \ -0.1)^T$ and $x^1 = [0.3 \ 0.5 \ -0.6]^T$. Using $E_k < \varepsilon$ as stopping criteria, where ε is a small enough positive number, the results of the numerical experiment are reported in Table 1. The numerical results can be seen from Table 1 and Figures 1 and 2. In Table 1, “Iter” denotes the number of iterations.

The behavior of the function E_k in Table 1 is described in Figures 1 and 2.

From Table 1 and Figures 1 and 2, we can observe that Algorithm 1 for inertial effect case has less number of iterations than Algorithm 1 for $\theta_k = 0$ (that is, no inertial effect) case; Algorithm 1 with bigger θ_k has less number of iterations than Algorithm 1 for small case; for the same θ_k , big ρ_k case has less number of iterations than Algorithm 1 for small case.

FIGURE 2: Error against iterations for Algorithm 1 with difference choices of ρ_k , under $\theta_k = 0.8$ and $\varepsilon = 10^{-5}$.

TABLE 2: Comparison of Algorithm 1 and Scheme (97) for different choice initial point for $\varepsilon = 10^{-4}$

Initiative point x^1	Algorithm 1 with $\theta_k = 0.8$		Algorithm 1 with $\theta_k = 0.0$		(98)	
	Iter	E_k	Iter	E_k	Iter	E_k
[0.4 0.9–0.9]	11	$4.1830e-06$	25	$9.6999e-06$	51	$8.8226e-06$
[0.9 0.3–0.5]	6	$1.4290e-07$	22	$8.6399e-06$	26	$9.6719e-06$
[−0.7 0.3–0.2]	12	$7.8838e-06$	24	$9.2013e-06$	48	$8.4198e-06$
[0.1 0.1–0.9]'	12	$1.5353e-07$	26	$8.7544e-06$	52	$9.9722e-06$

FIGURE 3: Error against of iterations for the comparison of Algorithm 1 and Scheme (97).

Following, we compare our algorithm with iteration (16). For the example, iteration (16) can be written as

$$x^{k+1} := \alpha_k f(x^k) + (1 - \alpha_k) P_C(x^k - \lambda_k (B_1^*(I - P_{Q_1})) B_1 x^k + (B_2^*(I - P_{Q_2})) B_2 x^k).$$

In the process, we take $\alpha_k = 0.8$ and $f(x^k) = u = (111)^T \lambda_k = 0.00005$ for Scheme (97). We choose $\beta_k = 1/2k + 1$, $\delta_k = \varepsilon_k = k/2k + 1$, and $\rho_k = 3$ for Algorithm 1.

From Table 2 and Figure 3, we can observe that Algorithm 1 has less number of iterations than Scheme (97). Moreover, our iterative method is advantageous over Scheme (98) because Algorithm 1 does not require computing of the operator norm.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

Authors' Contributions

This entire work has been completed by the authors: Huijuan Jia, Shufen Liu, and Yazheng Dang. The authors read and approved the final manuscript.

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