Research Article

On Groups Whose Irreducible Character Degrees of All Proper Subgroups are All Prime Powers

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Isaacs, Passman, and Manz have determined the structure of finite groups whose each degree of the irreducible characters is a prime power. In particular, if G is a nonsolvable group and every character degree of a group G is a prime power, then G is isomorphic to $S \times A$, where $S \in \{A_5, \text{PSL}_2(8)\}$ and A is abelian. In this paper, we change the condition, each character degree of a group G is a prime power, into the condition, each character degree of the proper subgroups of a group is a prime power, and give the structure of almost simple groups whose character degrees of all proper subgroups are all prime powers.

1. Introduction

In this paper, we assume that all groups are finite. Denote by $\text{Irr}(G)$ the set of all complex irreducible characters of a group G , write the set of linear characters of G by $\text{Lin}(G)$, and denote the set of character degrees of a group G by $\text{cd}(G)$.

Let G be a group and let p, q be different primes. If $\text{cd}(G) = \{1, m\}$, then either G has an abelian normal subgroup of index m or $m = p^e$ for a prime p and G is the direct product of a p -group and an abelian group (see Theorem 12.5 in [1]); Isaacs and Passman have identified the structure of groups with $\text{cd}(G) \subseteq \{1, p, q\}$ and $\text{cd}(G) \subseteq \{1, p, p^2\}$ (see [2, 3], respectively).

In this paper, the following problem is considered.

Question: what can we say about the structure of a group if irreducible character degrees of all proper subgroups are all of prime powers.

In order to argue in short, we introduce the following definition.

Definition 1. An irreducible character $\chi \in \text{Irr}(G)$ is a DP-character if $\chi(1)$ is a prime power, and a non-DP-character otherwise. A group G is named a DP-group if for every $\chi \in \text{Irr}(G)$, χ is a DP-character, and a non-

DP-group otherwise. We also assume that an abelian group is a DP-group.

Manz in [4, 5] has determined the structure of DP-groups. In particular, nonsolvable DP-groups are determined.

Theorem 1 (see [5]). *If G is a nonsolvable DP-group, then $G = S \times A$, where A is abelian and S is isomorphic to A_5 or $\text{PSL}_2(8)$.*

In order to talk conveniently, we give a definition.

Definition 2. Let $\sum G$ be the set of all proper subgroups of a group G . A group G is called an SDP-group if, for each $M \in \sum G$, M is a DP-group, and a non-SDP-group otherwise.

We will prove the following results.

Theorem 2. *Let G be an almost simple group such that $S \leq G \leq \text{Aut}(S)$, where S is a nonabelian simple group. Assume that G is an SDP-group, then G is isomorphic to one of the groups: $\text{PSL}_2(q)$, where $q = 2r^m + 1 \geq 5$ is a prime with r an odd prime or $q = 2^m$ and $2^m - 1 \geq 3$ is a prime, $\text{PSL}_2(9)$, $\text{PSL}_2(3^5)$, S_5 , or $\text{PSL}_3(3)$.*

We will introduce the structure of this paper. In Section 2, we give the structure of simple SDP-groups and some

properties of SDP-groups; in Section 3, we give the structure of almost simple SDP-groups. For a group G , let $\sum G$ be the set of all proper subgroups of G and $\max G$ be the set of representatives of maximal subgroups of G . The notation and notions are standard, see [1, 6], for instance.

2. Some Lemmas

In this section, we will give some needed results that are used to prove main theorem. We first give some basic results of SDP-group, then some subgroup of the nonabelian simple groups is given, and finally the structure of the nonabelian simple SDP-groups is determined. Now, we give some results about SDP-groups.

Lemma 1

- (1) G is an SDP-group if and only if, for each $M \in \sum G$, M is a DP-group.
- (2) Let G be an SDP-group and let N be a nontrivial normal subgroup of G . Then, N and G/N are DP-groups.
- (3) A DP-group must be an SDP-group.

Proof

- (1) We can get the result by Definition 2.
- (2) N is a DP-group by Definition 2, and $1 < N < G$. If G/N is not a DP-group, then there exists a subgroup $M \in \max G$ with $N \not\leq M$, so $G = MN$ and $1 < M < G$, M is a DP-group. Since $(G/N) \cong (M/M \cap N)$ is not a DP-group, there is a character $\chi \in \text{Irr}(M/N \cap M)$ which is not a DP-character. It follows that M is not a DP-group, a contradiction.
- (3) Let G be a DP-group. We will show that G is an SDP-group. If G is not an SDP-group, then there is a subgroup $M \in \sum G$ and an irreducible character $\chi \in \text{Irr}(M)$ such that χ is a non-DP-character. Let $I = I_G(\chi)$, the inertia group of χ in G . Let $\eta \in \text{Irr}(I)$ such that $[\eta_M, \chi] \neq 0$. Then, by Theorem 6.5 of [1], $\eta^G \in \text{Irr}(G)$. We know that $\chi(1)|\eta(1)|\eta^G(1)$, so there is a maximal subgroup $K \in \sum G$ and an irreducible character $\xi \in \text{Irr}(K)$ such that ξ is not a DP-character, so K is not a DP-group. It follows that G is not a DP-group, a contradiction.

In generality, a group, which is the direct product of two DP-groups, is not an SDP-group. $\square$

Example 1. Let P, Q be nonabelian p - and q -groups for different primes p, q , respectively. Also, assume that $\text{cd}(P) = \{1, p, p^2\}$ and $\text{cd}(Q) = \{1, q\}$. Then, $P \times Q$ is not an SDP-group. In fact, let $P_1 \in \max P$ such that $\chi \in \text{Irr}(P_1) - \text{Lin}(P_1)$. Obviously $P_1 \times Q$ is a subgroup of $P \times Q$. However, by Theorem 4.21 of [1], there is a nonlinear character $\chi \in \text{Irr}(P_1 \times Q)$ such that χ is not a DP-character.

Example 2. Let P be nonabelian p -group with $\text{cd}(P) = \{1, d_1, d_2, \dots, d_s\}$, $s \geq 2$, and $p > 7$. Then, $A_5 \times P$ is

not an SDP-group. In fact, A_5 and P are DP-groups. Let $G = A_5 \times P$; then, there are a maximal proper subgroup P_1 of P and an irreducible character $\chi \in \text{Irr}(P_1) - \text{Lin}(P_1)$, so $A_5 \times P_1$ has a nonlinear character $\theta\chi$ for some $\theta \in \text{Irr}(A_5) - \text{Lin}(A_5)$ and $\theta\chi$ is a non-DP-character.

There are simple groups whose certain subgroup is a DP-group, but itself not.

Example 3. Let J_1 be the simple sporadic group, and let P be the Sylow 2-subgroup of J_1 . Say $N = N_G(P)$, the normalizer of P in G . Then, $2^3 \cdot 3 \cdot 7 \cdot 3$ is a DP-group since $\text{cd}(N) = \{1, 3, 7\}$ by [7].

Example 4. Let M_{11} be the Mathieu group of degree 11. By pp.18 in [6], M_{11} has a maximal subgroup H isomorphic to $2 : S_4$. By [7], $\text{cd}(2 : S_4) = \{1, 2, 3, 4\}$ and so H is a DP-group.

We need some subgroup structure of finite simple groups to shorten the proof of Theorem 3.

Lemma 2. Let q be a prime power.

- (1) Let $n \geq 8$. Then, A_n has a subgroup A_{n-1} .
- (2) Let $n \geq 4$ and $\varepsilon = \pm$. Then, $\text{PSL}_n^\varepsilon(q)$ has a subgroup isomorphic to $\text{SL}_{n-1}^\pm(q)$ or $\text{PSL}_{n-1}^\pm(q)$, and $\text{SL}_n^\varepsilon(q)$ has a subgroup of the form $\text{SL}_{n-1}^\varepsilon(q)$.
- (3) Let $n \geq 2$. Then, $\text{PSp}_{2n}(q)$ has a subgroup $\text{PSp}_{2(n-1)}(q)$.
- (4) Let $n \geq 3$, and q odd. Then, $\Omega_{2n+1}(q)$ contains a subgroup $\Omega_{2n-1}(q)$.
- (5) Let $n \geq 4$, $\varepsilon = \pm$. Then, $\text{P}\Omega_{2n}^\varepsilon(q)$ has a subgroup $\text{P}\Omega_{2n-2}^\varepsilon(q)$ with q odd or $\text{PSp}_{2n-2}(q)$ with q even.

Let $\text{PSL}_n^+(q) = \text{PSL}_n(q)$ and $\text{PSL}_n^-(q) = \text{PSU}_n(q)$. Let $\text{SL}_n^+(q) = \text{SL}_n(q)$ and $\text{SL}_n^-(q) = \text{SU}_n(q)$. Let $\varepsilon = \pm$.

Proof

- (1) By Appendix in [8], S_n , the symmetric group of degree n , has a subgroup $S_{n-k} \times S_k$ with $1 \leq k < n/2$. Take $k = 1$, $S_{n-1} \leq S_n$, and so A_n has a subgroup A_{n-1} as $|S_n : A_n| = 2$.
- (2) By Proposition 4.1.4 of [9], $\text{PSL}_n^\pm(q)$ has a maximal subgroup H isomorphic to

$$\left[\frac{a_{m,n-m}^\varepsilon}{(q - \varepsilon, n)} \right] \cdot (\text{PSL}_m^\varepsilon(q) \times \text{PSL}_{n-m}^\varepsilon(q)) \cdot [b_{m,n-m}^\varepsilon]. \quad (1)$$

Take $m = 1$, and $\text{PSL}_n^\varepsilon(q)$ has a subgroup isomorphic to $\text{SL}_{n-1}^\varepsilon(q)$ or $\text{PSL}_{n-1}^\varepsilon(q)$.

We know that $\text{GL}_n^\varepsilon(q)$ has a subgroup $\text{GL}_{n-1}^\varepsilon(q)$, and $\text{SL}_n^\varepsilon(q) \trianglelefteq \text{GL}_n^\varepsilon(q)$. Note that $(\text{GL}_n^\varepsilon(q)/\text{SL}_n^\varepsilon(q))$ is isomorphic to a cyclic group of order $q - 1$, so $\text{SL}_n^\varepsilon(q)$ has a subgroup isomorphic to $\text{SL}_{n-1}^\varepsilon(q)$.

- (3) Note that $\text{Sp}_{2n}(q) = 2 \cdot \text{PSp}_{2n}(q)$ for odd q and $\text{Sp}_{2n}(q) = \text{PSp}_{2n}(q)$ for even q . Then, by Table 3.5.C of [9], $\text{PSp}_{2n}(q)$ has a subgroup $\text{PSp}_{2n-2}(q)$.
- (4) By Table 3.5.D of [9], $\Omega_{2n+1}(q)$ has a subgroup with form $\Omega_{2m+1}(q) \perp \Omega_{2(n-m)+1}^\varepsilon(q)$ with $1 \leq m < n$, $\varepsilon = \pm$. Taking $m = 1$, $\Omega_{2n+1}(q)$ has a subgroup $\Omega_{2n-1}(q)$.

- (5) By Table 3.5.E of [9], $\Omega_{2n}^+(q)$ has a subgroup of the form $P\Omega_{2m}^\varepsilon(q) \perp P\Omega_{2(n-m)}^\varepsilon(q)$ or $\Omega_{2m+1} \perp P\Omega_{2(n-m)}^\varepsilon(q)$, with $1 \leq m < n$, $\varepsilon \in \{+, -, \circ\}$, q odd if m odd, or the form $\text{Sp}_{2n-2}(q)$ with q even. $\square$

Taking $m = 1$ and $|\Omega_{2n}^+(q): P\Omega_{2n}^+(q)| = \gcd(4, q-1)$, W_n has a subgroup $P\Omega_{2(n-1)}^+(q)$ or $\text{P}\text{Sp}_{2n-2}(q)$.

By Table 3.5.F of [9], we also have that Ω_{2n}^- has a subgroup of the form either $\Omega_{2(n-1)}^-(q)$ for q odd or $\text{Sp}_{2n-2}(q)$ with q even.

Taking $m = 1$ and $|\Omega_{2n}^-(q): P\Omega_{2n}^-(q)| = \gcd(4, q+1)$, W_n has $P\Omega_{2(n-1)}^-(q)$ for odd q or $\text{P}\text{Sp}_{2(n-1)}(q)$ for even q .

The following three lemmas are important to give the structure of the simple groups as Theorem 3.

Lemma 3 (see [10]). *The only solution of the diophantine equation $p^m - q^n = 1$ with p and q prime and $m, n > 1$ is $3^2 - 2^3 = 1$.*

Lemma 4 (see [10, 11]). *With the exceptions of the relations $239^2 - 2 \cdot 13^4 = -1$ and $3^5 - 2 \cdot 11^2 = 1$, every solution of the equation $p^m - 2q^n = \pm 1$ with p, q prime, $m, n > 1$, has exponents $m = n = 2$, i.e., it comes from a unit $p - q \cdot 2^{(1/2)}$ of the quadratic field $\mathbb{Q}(2^{(1/2)})$ for which the coefficients p and q are primes.*

Lemma 5 (Zsigmondy theorem, see [12]). *Let p be a prime and let n be a positive integer. Then, one of the following holds:*

- (i) *There is a primitive prime p' for $p^n - 1$, that is, $p' \mid (p^n - 1)$, but $p' \nmid (p^m - 1)$, for every $1 \leq m < n$*
- (ii) *$p = 2, n = 1$ or 6*
- (iii) *p is a Mersenne prime and $n = 2$*

We say that a group A acts Frobeniusly on a group B if $[B]A$ is a Frobenius group with kernel B and complement A . A group G is a 2-Frobenius group if there is a normal series $N \triangleleft M \triangleleft G$ such that G/N and M are Frobenius groups with M/N , N as their Frobenius kernels, respectively.

Lemma 6 (see Theorems 13.3 and 13.8 of [13]). *Let $G = K:H$ be a Frobenius group with kernel K and complement H . Then,*

- (i) $|H| \mid (|K| - 1)$.
- (ii) *Any subgroup of H of order p^2 or pq is cyclic, where p and q are primes.*
- (iii) *If $|H|$ is even, K is abelian.*
- (iv) *In any case, K is nilpotent.*
- (v) *Assume that K has $h(K)$ conjugacy classes, H has $h(H)$ conjugacy classes, and G has characters with forms:*
 - (a) $h(H)$ is irreducible characters $\chi_1, \dots, \chi_{h(H)}$ with K in their kernel; if $\mu_1, \dots, \mu_{h(H)}$ are the irreducible characters of H , then these satisfy $\chi_i(hk) = \mu_i(h)$, for all $h \in H$ and $k \in K$.
 - (b) *Whenever $\tau \neq 1_K$ is an irreducible character of K , then τ^G is an irreducible character of G . This gives*

$[h(K) - 1]/|H|$ irreducible characters of G with K not in their kernel. Such τ^G satisfy

$$\tau^G|_H = \tau(1)\rho_H, \quad (2)$$

where ρ_H is the regular character of H .

The following result is due to White [14].

Lemma 7 (see Theorem A of [14]). *Let $S = \text{PSL}_2(q)$, with $q = p^f > 3$, for some prime p , $A = \text{Aut}(S)$, and let $S \leq H \leq A$. Given $G = \text{PGL}_2(q)$ if $\bar{\delta} \in H$, $G = S$ if $\bar{\delta} \notin H$ and set $|H: G| = d = 2^a m$ for m odd. If p is odd, set $\varepsilon = (-1)^{(q-1)/2}$. The set of irreducible character degrees of H is*

$$\text{cd}(H) = \left\{ 1, q, \frac{(q+\varepsilon)}{2} \right\} \cup \{(q-1)2^a i: i|m\} \cup \{(q+1)j: j|d\}, \quad (3)$$

with the following exceptions:

- (i) *If p is odd with $H \neq S\langle\varphi\rangle$ or if $p = 2$, then $(q+\varepsilon)/2$ is not a degree of H*
- (ii) *If f is odd, $p = 3$, and $H = S\langle\varphi\rangle$, then $i \neq 1$*
- (iii) *If f is odd, $p = 3$, and $H = A$, then $j \neq 1$*
- (iv) *If f is odd, $p = 2, 3$ or 5 , and $H = S\langle\varphi\rangle$, then $j \neq 1$*
- (v) *If $p \equiv 2 \pmod{4}$, $p = 2$ or 3 , and $H = S\langle\varphi\rangle$, then $j \neq 2$*

Theorem 3. *Let G be a nonabelian simple SDP-group. Then, G is isomorphic to one of the groups:*

- (1) $\text{PSL}_2(9)$, $\text{PSL}_2(3^5)$, and $\text{PSL}_2(q)$, where $q = 2r^m + 1 \geq 5$ is a prime with r prime or $q = 2^m$ and $2^m - 1 \geq 3$ is a prime
- (2) $\text{PSL}_3(3)$

Proof. It is well-known that a nonabelian simple group G is isomorphic to a sporadic simple group, a simple group of exceptional Lie type, an alternating group, or a simple group of classical Lie type. Thus, in the following, we consider the simple groups case by case.

Claim 1: if G is isomorphic to A_n with $n \geq 5$, then G is isomorphic to A_5 or A_6 .

If $n = 7$, then A_7 has a subgroup $L_2(7)$ by (pp.10 in [6]) and $2.3 \in \text{cd}(\text{PSL}_2(7))$, a contradiction. It follows that A_7 is not an SDP-group and also not a DP-group.

Let $n \geq 8$. We know from Appendix in [8] that the symmetric group S_n of degree n contains a subgroup S_{n-1} , so $A_{n-1} < A_n$. It follows from Lemma 2 that there exists a subgroup series:

$$A_7 < A_8 < \dots < A_{n-1} < A_n, \quad (4)$$

so A_n with $n \geq 8$ is not an SDP-group.

If $n = 6$, then by pp.4 in [6], $\max A_6 = \{A_5, 3^2: 4, S_4\}$. We can see that $\text{cd}(A_5) = \{1, 3, 4, 5\}$, $\text{cd}(3^2: 4) = \{1, 4\}$, and $\text{cd}(S_4) = \{1, 2, 3\}$, so $G \cong A_6$ is an SDP-group by Lemma 1. If $n = 5$, then by Theorem 1 and Lemma 1, A_5 is an SDP-group.

It follows that G is isomorphic to A_5 or A_6 .

Claim 2: a sporadic simple group or a Tits group is not an SDP-group.

Let G be a sporadic simple group or a Tits group and let H be a proper subgroup of G ; then, we can get Table 1 by [6] and Theorem 4.21 of [1]. For each $\chi_i \in \text{Irr}(H)$, χ_i is not a DP-character. It follows that a sporadic simple group or a Tits group is a non-SDP-group.

Claim 3: if G is isomorphic to a simple group of classical Lie type, then G is isomorphic to $\text{PSL}_2(9)$, $\text{PSL}_2(3^5)$, $\text{PSL}_2(q)$, where $q = 2r^m + 1 \geq 5$ is a prime with r a prime or $q = 2^m$ and $2^m - 1 \geq 3$ is a prime, and $\text{PSL}_3(3)$. $\square$

2.1. $\text{PSL}_n(q)$, $n \geq 2$

2.1.1. Let $n = 2$. If $q = 4$ or 8 , then $\text{PSL}_2(4)$ and $\text{PSL}_2(8)$ are DP-groups by Theorem 1, so Lemma 1 (3) implies that $\text{PSL}_2(4)$ and $\text{PSL}_2(8)$ are SDP-groups. It follows that G is isomorphic to $\text{PSL}_2(4)$ or $\text{PSL}_2(8)$. Thus, $q \geq 9$ or $q = 7$. If $q = 7$, then by pp.3 in [6], all maximal subgroups of $\text{PSL}_2(7) \cong \text{PSL}_3(2)$ are of the forms S_4 and $7:3$. Note that $\text{cd}(S_4) = \{1, 2, 3\}$ and $\text{cd}(7:3) = \{1, 3\}$, so $\text{PSL}_2(7)$ is an SDP-group. Now, $q \geq 9$. If $q = 9$, then by pp.5 in [6], $\max L_2(9) = \{A_5, 3^2:4, S_4\}$; hence, $\text{PSL}_2(9)$ is an SDP-group since A_5 is a DP-group, $\text{cd}(3^2:4) = \{1, 4\}$ and $\text{cd}(S_4) = \{1, 2, 3\}$. Therefore, G is isomorphic to $\text{PSL}_2(7)$ or $\text{PSL}_2(9)$.

By Table 2, $\text{PSL}_2(q)$ has a subgroup of the form $E_q: Z_{(q-1)/2}$. Note from Lemma 6 that $\text{cd}(E_q: Z_{(q-1)/2}) = \{1, (q-1)/2\}$, so by hypothesis, $(q-1)/2$ is a prime power or a prime. Two cases now are considered as follows: $q > 9$ odd and $q \geq 16$ even.

Case 1: $q > 9$ odd.

Let $q = p^m$ with $m \geq 2$. We first assume that $(q-1)/2$ is a prime. If $m \geq 2$ is even, then $(p-1)(p+1)$ divides $(p^m - 1)$. Note that $(p-1)_2(p+1)_2 = 2^k$, for some $k \geq 3$, so assumption shows $(q-1)/2 = 2^s$ for some $s \geq 3$. Now, Lemma 3 shows that $((q-1)/2) = 2^s$ has no solution in $\mathbb{N}$. If $m \geq 3$ is odd, then $((q-1)/2) = ((p-1)/2) \cdot (p^{m-1} + p^{m-2} + \dots + 1)$. It is easy to see that $p^{m-1} + p^{m-2} + \dots + 1$ is odd, so $(q-1)/2$ cannot be a prime. Assume now that $(q-1)/2$ is a prime power and say $q-1 = 2r^n$ for some integer $n \geq 0$ and a prime r .

- (i) If $n = 0$, then $q = 3$, but $\text{PSL}_2(3)$ is solvable, a contradiction.
- (ii) If $n \geq 1$ and $r = 2$, then by Lemmas 3–5, $q = 5, 9 > 9$, or $q = 2^{n+1} + 1$ is a prime. It follows that G is isomorphic to $\text{PSL}_2(q)$ with $q = 2^{n+1} + 1 \geq 17$ a prime.
- (iii) If $n \geq 2$ and $r \geq 3$, then by Lemma 4, $q = 3^5$, $r = 11$, and $n = 2$ or $q-1 = p^2 - 1 = (p+1)(p-1) = 2r^2$ (rule out this case since 8 divides $(p+1)_2(p-1)_2$). So, we have $G \cong \text{PSL}_2(3^5)$.

TABLE 1: Simple sporadic groups.

G	H	Character	Degree
M_{11}	$\text{PSL}_2(11)$	χ_4	2.5
J_1	$\text{PSL}_2(11)$	χ_4	2.5
J_2	$A_4 \times A_5$	χ_{20}	3.5
HS	M_{11}	χ_2	2.5
M_{24}	M_{23}	χ_2	2.11
He	$\text{PSL}_3(2)$	χ_4	2.3
Suz	A_7	χ_2	2.3
Co_3	$A_4 \times S_5$	χ_3	2.3
Fi_{22}	S_{10}	χ_5	5.7
Ly	$G_2(5)$	χ_2	$2^2 \cdot 31$
Fi_{23}	S_{12}	χ_7	5.11
J_4	$2^{11}: M_{24}$	χ_3	$3^2 \cdot 5$
B	$S_4 \times^2 F_4(2)$	χ_3	$2^2 \cdot 13$
${}^2F_4(2)'$	M_{11}	χ_2	2.5
M_{12}	M_{11}	χ_2	2.5
M_{22}	$\text{PSL}_2(11)$	χ_4	2.5
M_{23}	A_7	χ_2	2.3
J_3	$\text{PSL}_2(19)$	χ_4	$2 \cdot 3^2$
McL	M_{22}	χ_2	3.7
Ru	A_8	χ_3	2.7
ON	A_7	χ_2	2.3
Co_2	M_{23}	χ_2	2.11
HN	A_{12}	χ_4	5.11
Th	$\text{PSL}_2(19): 2$	χ_3	$2 \cdot 3^2$
Co_1	$A_9 \times S_3$	χ_7	3.7
Fi_{24}'	He: 2	χ_3	2.3.17
M	$2^5: S_6$	χ_{17}	2.5

TABLE 2: $\text{PSL}_2(q)$, $q \geq 5$ (chap II, Theorem 8.27 of [15]).

	$\max L_2(q)$	Condition
$\mathcal{E}_1$	$E_q: Z_{(q-1)/k}$	$k = \gcd(q-1, 2)$
$\mathcal{E}_2$	$D_{2(q-1)/k}$	$q \notin \{5, 7, 9, 11\}$
$\mathcal{C}_3$	$D_{2(q-1)/k}$	$q \notin \{7, 9\}$
$\mathcal{E}_5$	$\text{PSL}_2(q_0) \cdot (k, b)$	$q = q_0^b$, b a prime, $q_0 \neq 2$
$\mathcal{E}_6$	S_4	$q = p \equiv \pm 1 \pmod{8}$
	A_4	$q = p \equiv 3, 5, 13, 27, 37 \pmod{40}$
$\mathcal{S}$	A_5	$q \equiv \pm 1 \pmod{10}$, $F_q = F_p[\sqrt{5}]$

Let $q = p$ be a prime. Then, the fact that $(q-1)/2$ is a prime or a prime power implies that $q = p = 1 + 2r^m$, where $m \geq 1$ is an integer and r is a prime, so $G \cong \text{PSL}_2(q)$ with $q = 1 + 2r^m$, for some integer $m \geq 1$ and r a prime.

Case 2: $q \geq 2^4$ even.

Now, Table 2 implies that $\text{PSL}_2(q)$ has a maximal subgroup $E_q: Z_{q-1}$. By Lemma 6, $\text{cd}(E_q: Z_{q-1}) = \{1, q-1\}$. By assumption, $q-1$ is a prime or a prime power. If $q-1$ is a prime and say $q = 2^m \geq 2^4$, then $G \cong \text{PSL}_2(q)$ with $q = 2^m \geq 2^4$, $q-1$ a prime. If $q-1$ is a prime power, then by Lemma 3, it is impossible.

2.1.2. Let $n = 3$. If $q = 2$, then since $\text{PSL}_3(2) \cong \text{PSL}_2(7)$, one has that $\text{PSL}_3(2)$ is an SDP-group. If $q = 3$, then by [6], $\text{PSL}_3(3)$ has maximal subgroups with the forms: $3^2: 2S_4$,

13: 3, and S_4 , so by GAP, one obtains that $\text{cd}(3^2: 2S_4) = \text{cd}(M_9: S_3) = \{1, 2, 3, 4, 8, 16\}$, so $G \cong \text{PSL}_3(3)$. If $q = 4$, then by pp.23 in [6], $\text{PSL}_3(4)$ has a maximal subgroup $\text{PSL}_2(7)$, but $2.3 \in \text{cd}(\text{PSL}_2(7))$, a contradiction to the hypothesis. Now, we assume that $q \geq 5$. By Table 3, $\text{PSL}_3(q)$ has a subgroup $E_{q^2}: (1/k)\text{GL}_2(q) > \text{PGL}_2(q)$. Note that $\text{cd}(\text{PGL}_2(q)) = \{1, q, q-1, q+1\}$. By assumption, $q-1$ and $q+1$ are prime powers. Note that $q-1, q$, and $q+1$ are consecutive integers, so one of the numbers $q-1, q, q+1$ is divisible by 3.

Let $q = p^m$ be odd with $m \geq 2$. Then, $q = 3^n$ for some integer n . Since $q \geq 5$, one has that $n \geq 2$ and that $q-1$ and $q+1$ are divisible by 2. It follows that $q = 3 \nmid 5$, a contradiction.

Let $q = p^m$ be even with $m \geq 3$. Then, 3 divides $q-1$ or $q+1$. If $q-1 = 3$ or $q-1 = 3^s$, for some $s \geq 2$, then, by Lemma 3, $q = 4 \nmid 5$ a contradiction. If $q+1 = 3$ or $q+1 = 3^s$, for some $s \geq 2$, then $q = 2$ or $q = 8$ by Lemma 3. If $G \cong \text{PSL}_3(8)$, then, by pp.74 in [6], $\text{PSL}_3(8)$ has a subgroup

TABLE 3: $\text{PSL}_3(q)$ (see page 191 of [16]).

	$\max L_3(q)$	Condition
$\mathcal{C}_1$	$E_{q^2}: (1/k)\text{GL}_2(q)$	$k = \gcd(3, q-1)$
$\mathcal{C}_2$	$Z_{(q-1)/k}^2 \cdot S_3$	$q \geq 5$
C_3	$Z_{(q^2+q+1)/k} \cdot 3$	$q \neq 4$
$\mathcal{C}_5$	$\text{PSL}_3(q_0) \cdot (\delta, b)$	$q = q_0^b, b$ prime
$\mathcal{C}_6$	$3^2 \cdot \text{SL}_2(3)$ $3^2 \cdot Q_8$	$q = p \equiv 1 \pmod{9}$ $q = p \equiv 4, 7 \pmod{9}$
$\mathcal{C}_8$	$\text{SO}_3(q)$ $\text{PSU}_3(q_0)$	q odd $q = q_0^2$
$\mathcal{S}$	A_6 $\text{PSL}_2(7)$	$p \equiv 1, 2, 4, 7, 8, 13 \pmod{15}$, $F_q = F_p[\sqrt{5}, \sqrt{-3}]$ $2 < q = p \equiv 1, 2, 4 \pmod{7}$

$\text{PSL}_2(7)$ and $2.3 \in \text{cd}(\text{PSL}_2(7))$, a contradiction to the hypothesis.

2.1.3. Let $n \geq 4$. We know from [17] that

$$\text{cd}(\text{PSL}_3(q)) = \{1, q(q+1), q^2+q+1, (q+1)(q-1)^2, (q-1)(q^2+q+1), q^3, q(q^2+q+1), (q+1)(q^2+q+1)\}, \quad (5)$$

so $\text{PSL}_3(q)$ is not a DP-group.

Now, Lemma 2 shows that there exists a subgroup series:

$$\begin{aligned} \text{PSL}_3(q) &< \text{PSL}_n(q) \\ \text{or } \text{SL}_3(q) &< \text{PSL}_n(q), \end{aligned} \quad (6)$$

so $\text{PSL}_n(q)$ for $n \geq 4$ is a non-SDP-group.

2.2. $\text{PSU}_n(q)$, $n \geq 3$

2.2.1. Let $n = 3$. If $q = 2$, then $\text{PSU}_3(2) \cong 3^2: Q_8$ is solvable, so $q \geq 3$. If $q = 3$, then $\text{PSU}_3(3)$ has a subgroup $\text{PSL}_2(7)$ by pp.14 in [6], so $2.3 \in \text{cd}(\text{PSL}_2(7))$, a contradiction. Hence,

$q \geq 4$, so by pp.200 in [16], $\text{PSU}_3(q)$ has a subgroup $\text{GU}_2(q)$. Notice that $\text{PSU}_2(q) \cong \text{PSL}_2(q)$ implies $\text{GU}_2(q) \cong \text{GL}_2(q)$. Thus, by Table 2 in [18], $\text{cd}(\text{GU}_2(q)) = \{1, q-1, q, q+1\}$. By hypothesis, $|\pi(q-1)| = 1 = |\pi(q+1)|$, so $q = 4$ or $q = 8$ by Lemmas 3 and 5. If $q = 4$, then $\text{PSU}_3(4)$ has a maximal subgroup $2 \wedge (2+4): 15$, so by [7], $\text{cd}(2 \wedge (2+4): 15) = \{1, 2^2 \cdot 3, 3 \cdot 5\}$, a contradiction. If $q = 8$, then, by pp.66 in [6], $\text{PSU}_3(8)$ has a subgroup $2 \wedge (3+6): 21$, whence by [7], $\text{cd}(2 \wedge (3+6): 21) = \{1, 3 \cdot 7, 2^3 \cdot 7\}$, a contradiction to the assumption.

2.2.2. Let $n \geq 4$. We see from [17] that

$$\text{cd}(\text{PSU}_3(q)) = \{1, q(q-1), q^2-q+1, (q-1)(q^2-q+1), q(q^2-q+1), q^3, (q+1)(q^2-q+1), (q-1)(q+1)^2\}, \quad (7)$$

so $\text{PSU}_3(q)$ is not a DP-group.

By Lemma 2, one has

$$\text{PSU}_3(q) < \text{PSU}_n(q) \text{ or } \text{SU}_3(q) < \text{PSU}_n(q), \quad (8)$$

so $\text{PSU}_n(q)$ for $n \geq 4$ is not a SDP-group.

2.3. $\text{PSp}_{2n}(q)$ with $n \geq 2$, $\Omega_{2n+1}(q)$ with $n \geq 3$, q Odd, $\text{P}\Omega_{2n}^\epsilon(q)$ with $n \geq 4$, or a Simple Group of Exceptional Lie Type.

First, we consider the simple groups of exceptional Lie type, where $N(A_i): {}^2G_2(q)$ has abelian Hall subgroups A_2 and A_3 of order $q+1-3^{m+1}$ and $q+1+3^{m+1}$,

respectively; $N(A_i)$ is a Frobenius group with kernel A_i , $i = 2, 3$, and a cyclic noninvariant factor of order 6. $A \circ B$ denotes the central product of A and B .

Consider ${}^2G_2(q)$; then, by Table 4, G has a Frobenius subgroup $N(A_i)$ with cyclic complement of order 6 and abelian kernel A_i of order $q \pm \sqrt{3q} + 1$, i.e., $N(A_i)/A_i$ is isomorphic to a cyclic group of order 6. It follows that $N(A_i)/A_i$ is abelian, so $|N(A_i)/A_i| \in \text{cd}(N(A_i))$. This means that there is a proper subgroup H of G and a character $\chi \in \text{Irr}(H)$ such that $\chi(1)$ is divisible by 6, so χ is not a DP-character. It follows that ${}^2G_2(q)$ is a non-SDP-group.

TABLE 4: A proper maximal subgroup H of G .

G	H	Condition	References
${}^2G_2(q)$	$N(A_1)$	$q = 3^{2m+1} \geq 27$	[19]
$G_2(q)$	$SL_3(q) \cdot 2$ $SL_3(q) \cdot 2$	$3 q$ $3 \nmid q$	(Table 1 in [20])
${}^3D_4(q)$	$G_2(q)$ ${}^3D_4(\sqrt{q})$	q square	(Table 1 in [20])
$F_4(q)$	${}^3D_4(q) \cdot S_3$		(Table 1 in [20])
E_6^ε	$F_4(q)$	$\varepsilon = \pm 1$	(Table 1 in [20])
$E_7(q)$	$(SL_2(q) \circ P\Omega_{12}^+(q)) \cdot d$	$ Z(L \cap M) = d$	(Table 1 in [20])
$E_8(q)$	$(SL_2(q) \circ E_7(q)) \cdot d$	$ Z(L \cap M) = d$	(Table 1 in [20])

Now, we do with other cases. By Table 4, we obtain subgroup series

$$E_8(q) > E_7(q) > P\Omega_{12}(q), \quad (9)$$

and either

$$E_6^\varepsilon(q) > F_4(q) > {}^3D_4(q) > G_2(q) > SL(3, q) \cdot 2 > SL_3(q), \quad \text{for } 3|q, \quad (10)$$

or

$$E_6^\varepsilon(q) > F_4(q) > {}^3D_4(q) > G_2(q) > SU(3, q) \cdot 2 > SU_3(q), \quad \text{for } 3 \nmid q. \quad (11)$$

Note that $P\Omega_{12}^+(q) > P\Omega_4^+(q)$ and that $PSL_3(q) \cong (SL_3(q)/Z(SL_3(q)))$ and $PSU_3(q^2) \cong (SU_3(q^2)/Z(SU_3(q^2)))$ are non-DP-groups, so $E_8(q)$, $E_7(q)$, $E_6^\varepsilon(q)$, $F_4(q)$, ${}^3D_4(q)$, and $G_2(q)$ are non-SDP-groups.

Now, consider ${}^2B_2(q)$ with $q = 2^{2m+1} \geq 8$; then, by Theorem 9 of [21], ${}^2B_2(q)$ has a subgroup $E_q^{1+1}: Z_{q-1}$, and $d(q-1) \in \text{cd}(E_q^{1+1}: Z_{q-1})$, where $q = 2^{2m+1} \geq 8$ and $\chi(1) = d \neq 1$, for $\chi \in \text{Irr}(E_q^{1+1})$. It follows that $E_q^{1+1}: Z_{q-1}$ is not a DP-group, so ${}^2B_2(q)$ is a non-SDP-group.

Second, we consider the groups: $P\text{Sp}_{2n}(q)$ with $n \geq 2$, $\Omega_{2n+1}(q)$ with $n \geq 3$, q odd, $P\Omega_{2n}^\varepsilon(q)$ with $n \geq 4$.

Notice that $P\Omega_4^+(q) = \text{PSL}_2(q) \times \text{PSL}_2(q)$, and $P\Omega_4^-(q) = \text{PSL}_2(q^2)$ with $q > 2$, $G_2(q)$, and $\Omega_7(q)$ are non-DP-groups. It follows from Table 5 and Lemma 2 that the groups $P\text{Sp}_{2n}(q)$ with $n \geq 2$, $\Omega_{2n+1}(q)$ with $n \geq 3$, q is odd, and $P\Omega_{2n}^\varepsilon(q)$ with $n \geq 4$ are non-SDP-group. So, we need to consider $P\Omega_8^\varepsilon(2)$. We know from pp. 89 in [6] that $P\Omega_8^-(2)$ has a $\text{PSL}_2(7)$ as a subgroup. As $2.3 \in \text{cd}(\text{PSL}_2(7))$, $P\Omega_8^-(2)$ contains a non-DP-subgroup, so $P\Omega_8^-(2)$ is a non-SDP-group.

3. Proof of Theorem 2

Now, we will prove Theorem 2.

TABLE 5: $P\text{Sp}_{2n}(q)$, $n \geq 2$, $\Omega_{2n+1}(q)$, $n \geq 3$, q is odd, and $P\Omega_{2n}^\varepsilon(q)$, $n \geq 4$.

G	H	Condition	References
$P\text{Sp}_4(q)$	$P\Omega_4^+(q)$ $P\Omega_4^-(q)$	q even q odd	(pp. 209 in [16]) (pp. 209 in [16])
$\Omega_7(q)$	$G_2(q)$	q odd	(pp. 213 in [16])
$P\Omega_8^+(q)$	$\Omega_7(q)$		(pp. 214 in [16])
$P\Omega_8^-(q)$	$P\Omega_4^-(q)$		(pp. 215 in [16])

Let $q = p^f$ for some prime p . Note that the order of outer-automorphism group of $\text{PSL}_2(q)$ is

$$|\text{Out}(\text{PSL}_2(q))| = \gcd(2, q-1) \cdot f. \quad (12)$$

3.1. Proof of Theorem 2

Proof. If G is simple, then by Theorem 3, G is isomorphic to one of the groups. $\text{PSL}_2(q)$, where $q = 2r^m + 1 \geq 5$, is a prime with r an odd prime or $q = 2^m$ and $2^m - 1 \geq 3$ a prime; $\text{PSL}_2(9)$; $\text{PSL}_2(3^5)$. So, if G is nonsimple, then we consider the following cases: $q = 2^f$ with $2^f - 1$ a prime, and $q = 9$, $q = 3^5$, and $q = 2r^m + 1 \geq 5$ is a prime with r an odd prime and $\text{PSL}_3(3)$.

Case 1: $q = 2^f$ with $2^f - 1$ a prime.

If $f = 2$, then $\text{PSL}_2(4) \cong \text{PSL}_2(5) \cong A_5$ and $|\text{Out}(A_5)| = 2$. As G is a not simple group, G is possible isomorphic to S_5 , where S_n is the symmetric group of degree n . By pp. 2 in [6], $\max S_5 = \{A_5, S_4, 5: 4, 2 \times S_3\}$. Note that $\text{cd}(A_5) = \{1, 3, 4, 5\}$, $\text{cd}(S_4) = \{1, 2, 3\}$, $\text{cd}(5: 4) = \{1, 4\}$, and $\text{cd}(S_3) = \{1, 2\}$, so S_5 is an SDP-group. Thus, $G \cong S_5$.

If $f = 3$, then, by pp. 6 in [6], $7: 6 \in \max \text{PSL}_2(8) \cdot 3$, so $\text{PSL}_2(8) \cdot 3$ is not an SDP-group.

If $f \geq 5$, then we easily get that f does divides the order of $\text{PSL}_2(2^f)$, so $\text{PSL}_2(2^f)$ is a maximal subgroup of $\text{PSL}_2(2^f) \cdot f$. We see that $\text{PSL}_2(2^f)$ for $f \geq 5$ is not a DP-group, so $\text{PSL}_2(2^f) \cdot f$ is a non-SDP-group.

Case 2: $q = 9$.

Let $\mathfrak{S} = \{\text{PSL}_2(9) \cdot 2_1, \text{PSL}_2(9) \cdot 2_2, \text{PSL}_2(9) \cdot 2_3, \text{PSL}_2(9) \cdot 2^2\}$.

Now, by pp. 4 in [6], $|\text{Out}(\text{PSL}_2(9))| = 4$, so G is possibly isomorphic to $A \in \mathfrak{S}$. Note that $\text{PSL}_2(9) \in \sum A$ and that $\text{PSL}_2(9)$ is a non-DP-group, so each $A \in \mathfrak{S}$ is a non-SDP-group.

Case 3: $q = 3^5$.

Now, $|\text{Out}(\text{PSL}_2(3^5))| = 2 \cdot 5$. Thus, G possibly is isomorphic to $\text{PGL}_2(3^5)$, $\text{PGL}_2(3^5) \cdot 5$, $\text{PSL}_2(3^5) \cdot 5$, or $\text{PSL}_2(3^5) \cdot 10$. It is easy to see that $\text{PSL}_2(3^5)$ is a maximal subgroup of $\text{PSL}_2(3^5) \cdot 2$. So, $\text{PSL}_2(3^5) \cdot 5$ is a non-SDP-group as $\text{PSL}_2(3^5)$ is not a DP-group and so are $\text{PSL}_2(3^5) \cdot 10$ and $\text{PGL}_2(3^5) \cdot 5$. We know that $\text{PGL}_2(3^5)$ has a normal subgroup $\text{PSL}_2(3^5)$ of index 2, but $\text{PSL}_2(3^5)$ is a non-DP-group. Thus, $\text{PGL}_2(3^5)$ is a non-SDP-group.

Case 4: $q = 2r^m + 1$ is a prime with r an odd prime.

In this case, $|\text{Out}(\text{PSL}_2(q))| = 2$. We know that $\text{PSL}_2(q) \trianglelefteq \text{PGL}_2(q)$ and that $\text{PSL}_2(q)$ is a non-DP-group. Hence, $\text{PGL}_2(q)$ is not an SDP-group.

Case 5: $\text{PSL}_3(3)$.

Now, $|\text{Out}(\text{PSL}_3(3))| = 2$. From pp. 13 in [6], $13: 6 \in \max \text{PSL}_3(3) \cdot 2$, so $6 \in \text{cd}(13: 6)$ is not a prime power. Thus, $\text{PSL}_3(3) \cdot 2$ is a non-SDP-group. $\square$

Data Availability

No data were used to support the findings of the study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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References

- [1] I. M. Isaacs, *Character Theory of Finite Groups*, Dover Publications, Inc., New York, NY, USA, 1994.
- [2] I. M. Isaacs and D. S. Passman, "A characterization of groups in terms of the degrees of their characters. II," *Pacific Journal of Mathematics*, vol. 24, pp. 467–510, 1968.
- [3] D. S. Passman, "Groups whose irreducible representations have degrees dividing p^2 ," *Pacific Journal of Mathematics*, vol. 17, pp. 475–496, 1966.
- [4] O. Manz, "Endliche auflösbare gruppen, deren sämtliche charaktergrade primzahlpotenzen sind," *Journal of Algebra*, vol. 94, no. 1, pp. 211–255, 1985.
- [5] O. Manz, "Endliche nicht-auflösbare gruppen, deren sämtliche charaktergrade primzahlpotenzen sind," *Journal of Algebra*, vol. 96, no. 1, pp. 114–119, 1985.
- [6] J. H. Conway, R. T. Curtis, S. P. Norton, R. A. Parker, and R. A. Wilson, *Atlas of Finite Groups: Maximal Subgroups and Ordinary Characters For Simple Groups*, Oxford University Press, Oxford, UK, 1985.
- [7] T. Breuer, "The GAP character table library," 2012, <http://www.math.rwth-aachen.de/Thomas.Breuer/ctbllib>.
- [8] M. Aschbacher and L. Scott, "Maximal subgroups of finite groups," *Journal of Algebra*, vol. 92, no. 1, pp. 44–80, 1985.
- [9] P. Kleidman and M. Liebeck, *The Subgroup Structure of the Finite Classical Groups*, Cambridge University Press, Cambridge, CA, USA, volume 129 of London Mathematical Society Lecture Note Series, 1990.
- [10] A. Khosravi and B. Khosravi, "A new characterization of some alternating and symmetric groups. II," *International Journal of Mathematics and Mathematical Sciences*, vol. 30, no. 4, pp. 953–967, 2004.
- [11] P. Crescenzo, "A Diophantine equation which arises in the theory of finite groups," *Advances in Mathematics*, vol. 17, no. 1, pp. 25–29, 1975.
- [12] K. Zsigmondy, "Zur theorie der potenzreste," *Monatshefte für Mathematik und Physik*, vol. 3, no. 1, pp. 265–284, 1892.
- [13] L. Dornhoff, *Group Representation Theory. Part A: Ordinary Representation Theory*, Marcel Dekker, Inc., New York, NY, USA, 1971.
- [14] D. L. White, "Character degrees of extensions of $\text{PSL}_2(q)$ and $\text{SL}_2(q)$," *Journal of Group Theory*, vol. 16, no. 1, pp. 1–33, 2013.
- [15] B. Huppert, *Endliche Gruppen. I. Die Grundlehren der Mathematischen Wissenschaften, Band 134*, Springer-Verlag, Berlin, Germany, 1967.
- [16] P. B. Kleidman, *The subgroup structure of some finite simple groups*, (Ph.D. Thesis), Trinity College, Cambridge, CA, USA, 1987.
- [17] F. Lübeck, "Character degrees and their multiplicities for some groups of lie type of rank < 9 ," 2007, <https://www.math.rwth-aachen.de/Frank.Luebeck/chev/DegMult/index.html?LANG=en>.
- [18] W. A. Simpson and J. S. Frame, "The character tables for $\text{SL}(3, q)$, $\text{SU}(3, q^2)$, $\text{PSL}(3, q)$, $\text{PSU}(3, q^2)$," *Canadian Journal of Mathematics*, vol. 25, pp. 486–494, 1973.
- [19] V. M. Levchuk and Y. N. Nuzhin, "The structure of ree groups," *Algebra I Logika*, vol. 24, no. 1, pp. 26–41, 1985.
- [20] M. W. Liebeck and J. Saxl, "On the orders of maximal subgroups of the finite exceptional groups of lie type," *Proceedings of the London Mathematical Society*, vol. 55, no. 2, pp. 299–330, 1987.
- [21] M. Suzuki, "On a class of doubly transitive groups," *Annals of Mathematics*, vol. 79, no. 2, pp. 105–145, 1962.