Research Article

On Convolution and Convex Combination of Harmonic Mappings

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In this paper, the subclass of harmonic univalent functions by shearing construction is studied and this subclass of harmonic mappings needs a necessary and adequate condition to be convex in the horizontal direction. Furthermore, convolutions of two special subclasses of univalent harmonic mappings are shown to be convex in the horizontal direction. Also, the family of univalent harmonic mappings of the unit disk onto a region convex in the direction of the imaginary axis is introduced. Sufficient conditions for convex combinations of harmonic mappings of this family to be univalently convex in the direction of the imaginary axis are obtained.

1. Motivation and Preliminaries

A complex-valued function $f = u + iv$ defined on the unit disk $U = \{z \in \mathbb{C} : |z| < 1\}$ is called harmonic mapping if u and v are real-valued harmonic functions. In addition, since U is a simply connected domain, f has a unique representation $f = h + \bar{g}$, where h and g are analytic and co-analytic parts of f , respectively. It is known from Lewy [1] that the mapping $f = h + \bar{g}$ defined on U is locally univalent and sense-preserving if and only if

$$|g'(z)| < |h'(z)|, \quad \forall z \in U. \quad (1)$$

The class of all univalent, harmonic sense-preserving mappings $f = h + \bar{g}$ defined in U is denoted by $\mathcal{H}$. Moreover, let S_H be the class of all functions $f \in \mathcal{H}$ which is normalized by $f(0) = f_z(0) - 1 = 0$. Therefore, each member of $f \in S_H$ has the following representation:

$$f(z) = z + \sum_{n=2}^{\infty} b_n z^n + \sum_{n=1}^{\infty} \bar{c}_n \bar{z}^n \quad (|c_1| < 1), \quad (2)$$

for all $z \in U$.

The class of such type of functions given in (2) with $c_1 = 0 = f_{\bar{z}}(0)$ is a subclass of S_H and is denoted by S_H^0 . The dilatation of f belonging to S_H^0 is the function $\omega: U \rightarrow \mathbb{C}$

given by $\omega = g'(z)/h'(z)$ and satisfies $|\omega(z)| < 1$ for all $z \in U$. Further, denote by K_H (or K_H^0) all $f \in S_H$ (or S_H^0) which are mapping U onto convex regions. For comprehensive and fundamental knowledge on planar harmonic mappings, see Duren [2]. The subclass of S_H^0 denoted by S_{CHD}^0 consists of all univalent harmonic functions which maps onto domain convex in the direction of the real axis.

Section 2 demonstrates that the convolution of two special subclasses of univalent harmonic mappings is convex in the horizontal direction. The convolution of any two arbitrary harmonic functions

$$\begin{aligned} f(z) &= h(z) + \bar{g}(z) = z + \sum_{n=2}^{\infty} a_n z^n + \sum_{n=1}^{\infty} \bar{b}_n \bar{z}^n, \\ F(z) &= H(z) + \bar{G}(z) = z + \sum_{n=2}^{\infty} A_n z^n + \sum_{n=1}^{\infty} \bar{B}_n \bar{z}^n \end{aligned} \quad (3)$$

is defined by

$$\begin{aligned} (f \ast F)(z) &= (h \ast H)(z) + \overline{(g \ast G)}(z) = z + \sum_{n=2}^{\infty} a_n A_n z^n \\ &\quad + \sum_{n=1}^{\infty} \bar{b}_n \bar{B}_n \bar{z}^n. \end{aligned} \quad (4)$$

The convolution of harmonic functions is different from the convolution of univalent functions (for more details, one can refer to [3]). Indeed, the convolution of two harmonic functions does not preserve the convexity and the convolution of any two univalent harmonic functions $f \in S_H^0$ need not be univalent. These facts generated significant interest in the analysis of harmonic convolutions of univalent harmonic functions, and several articles on this subject recently appeared in the literature [3–11]. Specifically, the collection of the mappings $F = H + \overline{G} \in S_H^0$ that map U onto the right half-plane $\Psi = \{w: \operatorname{Re}(w) > -(1/2)\}$ and have the form $(H + G)(z) = z/(1 - z)$ with $|G'| < |H'|$ for all $z \in U$. Kumar et al. [12] proposed a class of right half-plane harmonic mappings, $F_a = H_a + \overline{G}_a$, $(-1 < a < 1)$, satisfying

$$\begin{aligned} H_a(z) + G_a(z) &= \frac{z}{1 - z}, \\ G'_a(z) &= \frac{a - z}{1 - az} H'_a(z). \end{aligned} \quad (5)$$

Using the shear construction of Clunie and Sheil-Small [13], it follows that

$$\begin{aligned} H_a(z) &= \frac{z/(1 + a) - 1/2z^2}{(1 - z)^2}, \\ G_a(z) &= \frac{(a/1 + a)z - (1/2)z^2}{(1 - z)^2}. \end{aligned} \quad (6)$$

Letting $a = 0$ in (5), the mapping $F_0 = H_0 + G_0$ with $(F_0 + G_0)(z) = z/(1 - z)$ and $G'_0(z) = -zH'_0(z)$, which is called the standard right half-plane mapping, is obtained. In [9], the following result was obtained.

Theorem 1 (see [9]). *Let $f = h + \overline{g} \in K_H^0$ satisfy the condition $(h + g)(z) = z/(1 - z)$ and dilatation $\omega(z) = g'(z)/h'(z) = e^{i\theta}z^n$ ($n \in \mathbb{Z}^+$; $\theta \in \mathbb{R}$). If $n = 1, 2$, then $F_0 * f \in S_{\text{CHD}}^0$.*

Recently, Liu and Ponnusamy [8] generalized Theorem 1 in the case $n = 1$ as follows.

Theorem 2 (see [8]). *Let $f = h + \overline{g} \in K_H^0$ satisfy the condition $(h + g)(z) = (1 + a)z/(1 - z)$ with dilatation $\omega_1(z) = (z + a)/(1 + az)$, where $a \in (-1, 1)$, and $f_1 = h_1 + \overline{g}_1 \in K_H^0$ with dilatation $\omega_2(z) = e^{i\theta}z$ ($\theta \in \mathbb{R}$). Then, $f * f_1 \in S_{\text{CHD}}^0$.*

Liu and Ponnusamy [8] also proposed the following problem.

Problem 1. Let $f = h + \overline{g} \in K_H$ with $(h + g)(z) = (1 + a)z/(1 - z)$ and dilatation function $\omega_1(z) = (z + a)/(1 + az)$, where $a \in (-1, 1)$, and let $f_n = h_n + \overline{g}_n \in K_H^0$ with dilatation $\omega_2(z) = e^{i\theta}z^n$ ($\theta \in \mathbb{R}, n \in \mathbb{N}^+$). Then, $f * f_n \in S_{\text{CHD}}^0$.

Ali et al. [14] solved Problem 1 for $\omega_2(z) = -z$. In [8], the authors conjectured that $f * f_n$ is not locally univalent for $n \geq 2$. In Section 2, new subclass of harmonic mappings $P_a[\phi]$ is obtained. It is also shown that the necessary and sufficient condition for $P_a[\phi] \in S_{\text{CHD}}^0$, if and only if $\phi(z)$ is convex. The function $f_n = h + \overline{g}$ satisfies the condition $(h -$

$g)(z) = z/(1 - z)$ and dilatation $\omega(z) = z^n$; $n \in \mathbb{N}$ with the mapping $P_a[\ell]$ belonging to subclass S_{CHD}^0 for $a \in (-1, (2 - n)/(2 + n))$ and for all $n \in \mathbb{N}$ are also explored.

The results proposed by Hengartner and Schober [15] and Pommerenke [16] are very useful in checking the convexity of an analytic function in the direction of the imaginary axis as well as the convexity in the direction of the real axis, respectively.

Lemma 1 (see [15]). *Suppose f is a nonconstant analytic function of U . Then,*

$$\operatorname{Re}\{(1 - z^2)f'(z)\} \geq 0, \quad (7)$$

if and only if

- (1) f is univalent in U .
- (2) f is convex in the direction of imaginary axis.
- (3) There exist sequences $\{z_n'\}$ converging to $z = 1$ and $\{z_n''\}$ converging to $z = -1$, such that

$$\begin{aligned} \lim_{n \rightarrow \infty} \operatorname{Re}\{f(z_n')\} &= \sup_{|z| < 1} \operatorname{Re}(f(z)), \\ \lim_{n \rightarrow \infty} \operatorname{Re}\{f(z_n'')\} &= \inf_{|z| < 1} \operatorname{Re}(f(z)). \end{aligned} \quad (8)$$

Lemma 2 (see [16]). *Suppose f is a nonconstant analytic function of U satisfying the condition $f(0) = 0$ and $f'(0) = 0$, and suppose*

$$\varphi(z) = \frac{z}{(1 + ze^{i\theta_1})(1 + ze^{i\theta_2})}, \quad (9)$$

where $\theta_i, i = 1, 2 \in \mathbb{R}$. If

$$\operatorname{Re}\left(\frac{zf'(z)}{\varphi(z)}\right) > 0, \quad (10)$$

then f is convex in the direction of the real axis.

Recall that the region Ω is said to be convex in the direction θ , $0 \leq \theta < \pi$, if every line parallel to the line through 0 and $e^{i\theta}$ has either connected or empty intersection with Ω , and if Ω is convex in every direction, then it is called convex. If $\theta = 0$ (or $\theta = \pi/2$), then Ω is convex in the direction of the real axis “CHD” or (is convex in the direction of the imaginary axis “CID”). Clunie and Sheil [13] proposed an integrated way to construct a univalent harmonic mapping convex in a given direction.

Theorem 3 (see [13]). *Let $0 \leq \theta < \pi$. A locally univalent harmonic function $f = h + \overline{g}$ in U is a univalent mapping of U onto a domain convex in the direction of θ if and only if $h - e^{2i\theta}g$ is a univalent mapping of U onto a domain convex in the direction of θ .*

The study of the geometric properties of the convex combination of the univalent mappings is another important topic in the geometric function theory. Dorff and Rolf [17] developed a necessary condition for convex combination to be

univalent maps onto a domain convex in the direction of the imaginary axis.

Theorem 4 (see [17]). Let $f_1 = h_1 + \bar{g}_1$ and $f_2 = h_2 + \bar{g}_2$ be two univalent harmonic mappings defined on U , with f_1 and f_2 having the same second complex dilatation and satisfying conditions in (4); then,

$$\begin{aligned} f_3 &= h_3 + \bar{g}_3 \\ &= [th_1(z) + (1-t)h_2] + [t\bar{g}_1(z) + (1-t)\bar{g}_2(z)] \quad (11) \\ &= tf_1(z) + (1-t)f_2 \end{aligned}$$

is univalent and is convex in the direction of the imaginary axis.

It is known that even if f and g are convex analytic functions, the convex combination of f and g may not be a univalent function (see [18], and for more recent studies of linear combinations of harmonic mappings, see [17, 19–23]). Moreover, Kumar et al. [20] studied the convexity of linear combination of harmonic mappings, which are shears of the analytic mapping $z/(1-z)$ and $z(1-\beta z)/(1-z^2)$, for $-1 \leq \beta \leq 1$. Beig et al. [23] studied and found necessary conditions for the convex combination of the right half-plane mappings, the vertical strip mapping, their rotations, and some other harmonic mappings to be univalent and convex in a particular direction. In Section 3, the convex combination of mappings of the family of sense-preserving and locally univalent harmonic mappings $f_\alpha = h_\alpha + g_\alpha$, by shearing the function $h_\alpha + g_\alpha = K_\alpha$ where

$$K_\alpha(z) = h_\alpha(z) + g_\alpha(z) = \frac{z(1-\alpha z^2)}{1-z^2}, \quad \alpha \in [0, 1], \quad (12)$$

$$\left| \frac{g'_\alpha}{h'_\alpha} \right| < 1,$$

are studied. Further, the appropriate conditions for the convex combination in the vertical direction between univalent harmonic mappings to be univalent and convex in the vertical direction are also established.

2. Convolutions of Subclasses of Univalent Harmonic Right Half-Plane Mappings

In this section, we consider the harmonic mapping $P_a = h_a + \bar{g}_a$ with $(h_a - g_a)(z) = (1-a)z/(1-z)$ with the second complex dilatation $\omega(z) = (a+z)/(1+az)$, $a \in (-1, 1)$. Hence, we can solve for h_a and g_a as follows:

$$\begin{aligned} h_a(z) &= \left(\frac{z - (1/2)(1-a)z^2}{(1-z)^2} \right) = \frac{1}{2} \left(\frac{(1+a)z}{(1-z)^2} + \frac{(1-a)z}{(1-z)} \right), \\ g_a(z) &= h_a(z) - \frac{(1-a)z}{1-z} = \frac{1}{2} \left(\frac{(1+a)z}{(1-z)^2} - \frac{(1-a)z}{(1-z)} \right), \end{aligned} \quad (13)$$

and thus

$$\begin{aligned} P_a(z) &= h_a(z) + \bar{g}_a(z) = \frac{1}{2} \left(\left(\frac{(1+a)z}{(1-z)^2} + \frac{(1-a)z}{(1-z)} \right) \right. \\ &\quad \left. + \overline{\left(\frac{(1+a)z}{(1-z)^2} - \frac{(1-a)z}{(1-z)} \right)} \right). \end{aligned} \quad (14)$$

Taking $\ell(z) = z/(1-z)$, P_a given in (14) can be rewritten as

$$\begin{aligned} P_a(z) &= \frac{1}{2} \left(((1+a)z\ell'(z) + (1-a)\ell(z)) \right. \\ &\quad \left. + \overline{((1+a)z\ell'(z) - (1-a)\ell(z))} \right). \end{aligned} \quad (15)$$

For an analytic univalent function $\phi: U \rightarrow \mathbb{C}$ normalized by $\phi(0) = \phi'(0) - 1 = 0$, define

$$\begin{aligned} P_a[\phi](z) &= \frac{1}{2} \left(((1+a)z\phi'(z) + (1-a)\phi(z)) \right. \\ &\quad \left. + \overline{((1+a)z\phi'(z) - (1-a)\phi(z))} \right) \\ &= H_a[\phi](z) + \bar{G}_a[\phi](z), \end{aligned} \quad (16)$$

for all $z \in U$. It is clear that $P_a[\ell] = P_a$. Therefore,

$$\begin{aligned} P_a[\ell](z) &= \frac{1}{2} \left(((1+a)z\ell'(z) + (1-a)\ell(z)) \right. \\ &\quad \left. + \overline{((1+a)z\ell'(z) - (1-a)\ell(z))} \right) \\ &= H_a(z) + \bar{G}_a(z). \end{aligned} \quad (17)$$

If F is an analytic function with $F(0) = 0$, then

$$(H_a * F)(z) = \frac{1}{2} \left(((1+a)zF'(z) + (1-a)F(z)) \right), \quad (18)$$

and

$$(G_a * F)(z) = \frac{1}{2} \left(((1+a)zF'(z) - (1-a)F(z)) \right). \quad (19)$$

Lemma 3. Let $f = h + \bar{g}$ satisfy the condition $(h - g)(z) = z/(1-z)$ with second complex dilatation $\omega = g'/h'$ such that $|\omega| < 1$ and $P_a[\ell]$ be family of harmonic mappings given in (17). Then, $\hat{\omega}$, the second complex dilatation of $f * P_a[\ell]$, is given by

$$\hat{\omega}(z) = \frac{2(a+z)\omega(1-\omega) + (1+a)z\omega'(1-z)}{2(1+az)(1-\omega) + (1+a)z\omega'(1-z)}. \quad (20)$$

Proof. Since

$$f * P_a[\ell] = (h + \bar{g}) * (H_a + \bar{G}_a) = (h * H_a) + \overline{(g * G_a)}, \quad (21)$$

then

$$\hat{\omega}(z) = \frac{(g * G_a)'(z)}{(h * H_a)'(z)}. \quad (22)$$

Now, from (18) and (19), it follows that

$$\begin{aligned} \hat{\omega}(z) &= \frac{((1+a)zg'(z) - (1-a)g(z))'}{((1+a)zh'(z) + (1-a)h(z))'} \\ &= \frac{2ag'(z) + (1+a)zg''(z)}{2h'(z) + (1+a)zh''(z)}. \end{aligned} \quad (23)$$

Since $g'' = \omega'h' + \omega h''$, then

$$\hat{\omega}(z) = \frac{(2a\omega + (1+a)z\omega')h'(z) + (1+a)z\omega h''(z)}{2h'(z) + (1+a)zh''(z)}. \quad (24)$$

From $(h-g)(z) = z/(1-z)$, it follows that

$$\begin{aligned} h'(z) &= \frac{1}{(1-\omega)(1-z)^2}, \\ h''(z) &= \frac{2(1-\omega) + \omega'(1-z)}{(1-\omega)^2(1-z)^3}, \end{aligned} \quad (25)$$

and thus

$$\begin{aligned} \hat{\omega}(z) &= \frac{(2a\omega + (1+a)z\omega')(1-\omega)(1-z) + (1+a)z\omega[2(1-\omega) + \omega'(1-z)]}{2(1-\omega)(1-z) + (1+a)z[2(1-\omega) + \omega'(1-z)]} \\ &= \frac{2(a+z)\omega(1-\omega) + (1+a)z\omega'(1-z)}{(2+2az)(1-\omega) + (1+a)z\omega'(1-z)}. \end{aligned} \quad (26)$$

□

Lemma 4. Let $f = h + \bar{g} \in S_H^0$ with $(h-g)(z) = z/(1-z)$. If $P_a[\ell] * f$ is locally univalent, then $P_a[\ell] * f$ is in S_{CHD}^0 .

Proof. Since

$$\begin{aligned} H_a - G_a &= (1+a)\ell(z), \\ h - g &= \ell(z), \end{aligned} \quad (27)$$

then

$$\begin{aligned} (1+a)(h+g) &= (H_a - G_a) * (h+g) \\ &= (H_a * h + H_a * g - G_a * h - G_a * g), \\ (H_a + G_a) &= (H_a + G_a) * (h-g) \\ &= (H_a * h) - (H_a * g) + (G_a * h) - (G_a * g). \end{aligned} \quad (28)$$

Thus,

$$\begin{aligned} (H_a * h) - (G_a * g) &= \frac{1}{2}[(1+a)(h+g) + (H_a + G_a)], \\ ((H_a * h) - (G_a * g))' &= \frac{1}{2}[(1+a)(h' + g') + (H_a' + G_a')]. \end{aligned} \quad (29)$$

Let $\varphi(z) = z/(1-z)^2$. Consequently, we have

$$\begin{aligned} &\operatorname{Re}\left(\frac{z(H_a * h - G_a * g)'}{\varphi}\right) \\ &= \operatorname{Re}\left(\frac{(1+a)z}{2\varphi}(h' + g') + \frac{z}{2\varphi}(H_a' + G_a')\right) \\ &= \frac{1}{2}\left[\operatorname{Re}\left((1+a)\frac{z}{\varphi}(h' - g')\frac{(h' + g')}{(h' - g')}\right) + \operatorname{Re}\left(\frac{z}{\varphi}(H_a' - G_a')\frac{(H_a' + G_a')}{(H_a' - G_a')}\right)\right] \\ &= \frac{(1+a)}{2}\left[\operatorname{Re}\left(\frac{h' + g'}{h' - g'}\right) + \operatorname{Re}\left(\frac{H_a' + G_a'}{H_a' - G_a'}\right)\right] \\ &= \frac{(1+a)}{2}[\operatorname{Re}p_1(z) + \operatorname{Re}p_2(z)] > 0, \end{aligned} \quad (30)$$

where $p_1(z) = (1 + \omega_f)/(1 - \omega_f)$ and $p_2(z) = (1 + \omega_{P_a[\ell]})/(1 - \omega_{P_a[\ell]})$. Thus, from Lemma 3, we deduce that $(H_a * h) - (G_a * g)$ is convex in the direction of the real axis. Theorem 3 implies that $P_a[\ell] * f \in S_{\text{CHD}}^0$. $\square$

Lemma 5. *The mapping $P_a[\phi]$ given in (16) is locally univalent if and only if the function ϕ is convex.*

Proof. The mapping $P_a[\phi] = H_a[\phi] + \overline{G_a}[\phi]$ is given in (16). Since $P_a[\phi]$ is locally univalent if and only if $|G'_a[\phi]/H'_a[\phi]| < 1$,

$$\left| \frac{2a\phi'(z) + (1+a)z\phi''(z)}{2\phi'(z) + (1+a)z\phi''(z)} \right| < 1, \quad (31)$$

and hence

$$\left| \frac{2a/(1+a) + z\phi''(z)/\phi'(z)}{2/(1+a) + z\phi''(z)/\phi'(z)} \right| < 1. \quad (32)$$

Thus,

$$\left| \frac{(a-1)}{(1+a)} + \left(1 + \frac{z\phi''(z)}{\phi'(z)}\right) \right| < \left| \frac{(1-a)}{(1+a)} + \left(1 + \frac{z\phi''(z)}{\phi'(z)}\right) \right|. \quad (33)$$

It can be easily shown that the above inequality is identical to

$$\operatorname{Re} \left(1 + \frac{z\phi''(z)}{\phi'(z)} \right) > 0. \quad (34)$$

Hence, $P_a[\phi]$ is locally univalent if $\phi(z)$ is convex function. $\square$

Theorem 5. *The mapping $P_a[\phi]$ is in the class S_{CHD}^0 , if and only if ϕ is convex.*

Proof. The mapping $P_a[\phi] = H_a[\phi] + \overline{G_a}[\phi]$ is given in (16), and $H_a[\phi] - G_a[\phi] = (1-a)\phi$ where ϕ is convex. This implies that $H_a[\phi] - G_a[\phi]$ is convex in the direction of the real axis. Consequently, from Lemma 5 and Theorem 3, we deduce that $P_a[\phi]$ in S_{CHD}^0 .

In order to prove the next result, we will employ the distribution theory of the polynomial function roots of the disc unit. Given a polynomial function of degree m :

$$p(z) = a_m z^m + a_{m-1} z^{m-1} + \cdots + a_1 z + a_0, \quad (35)$$

with complex coefficients, the parallel algorithm for discovering polynomial zeros for (35) inside U is worth exploring. Let

$$p^*(z) = z^n \overline{p\left(\frac{1}{\bar{z}}\right)} = \bar{a}_m + \bar{a}_{m-1} z + \cdots + \bar{a}_1 z^{m-1} + \bar{a}_0 z^m. \quad (36)$$

The inverse zeros for (35) and (36) can be easily checked with respect to U . We need the following lemma to prove the main theorem. $\square$

Lemma 6 (Cohn's rule) (see [24]). *Let p be a polynomial function as given in (35) of degree m and let p^* be as given in (36). The number of zeros of p in or on the unit circle is denoted by r and s , respectively. If $|a_0| < |a_m|$, then*

$$p_1(z) = \frac{\bar{a}_m p_1(z) - a_0 p^*(z)}{z} \quad (37)$$

is a polynomial function of degree $m-1$ with $r_1 = r-1$ and $s_1 = s$, where r_1 and s_1 are the number of zeros of the polynomial p_1 in or on the unit circle, respectively.

Theorem 6. *Let $P_a[\ell]$ be mapping given in (17) and let $f_n = h + \bar{g}$ with $(h-g)(z) = z/(1-z)$ and dilatation $\omega_1(z) = z^n$, $n \in \mathbb{N}$. Then, $P_a[\ell] * f_n \in S_{\text{CHD}}^0$ for $a \in (-1, (2-n)/(2+n))$ and for all n belonging to $\mathbb{N}$.*

Proof. According Lemma 4, it suffices to show that the mapping $P_a[\ell] * f_n$ has dilatation ω such that $|\omega| < 1$ for all $z \in U$. Setting $\omega_1(z) = z^n$ in (20), we obtain

$$\begin{aligned} \omega(z) &= -z^n \frac{(a(2+n)+n) + (2-n-an)z - 2az^n - 2z^{1+n}}{-2 - 2az + (2-n-an)z^n + (a(2+n)+n)z^{1+n}} \\ &= -z^n \frac{q(z)}{q^*(z)}, \end{aligned} \quad (38)$$

where

$$\begin{aligned} q(z) &= (a(2+n)+n) + (2-n-an)z - 2az^n - 2z^{1+n} \\ &= a_0 + a_1 z + a_n z^n + a_{n+1} z^{n+1}, \\ q^*(z) &= z^{n+1} \overline{q\left(\frac{1}{\bar{z}}\right)} = -2 - 2az + (2-n-an)z^n \\ &\quad + (a(2+n)+n)z^{1+n} \\ &= \bar{a}_{n+1} + \bar{a}_n + \bar{a}_1 z^n + \bar{a}_0 z^{n+1}. \end{aligned} \quad (39)$$

If $z_0 \neq 0$, is zero of the polynomial function $q(z)$, then $1/z_0$ will be zero of the polynomial function q^* . Hence, we can rewrite ω as

$$\omega(z) = -z^n \frac{(z+A_1)(z+A_2)(z+A_3), \dots, (z+A_{n+1})}{(1+\bar{A}_1 z)(1+\bar{A}_2 z)(1+\bar{A}_3 z), \dots, (1+\bar{A}_{n+1} z)}. \quad (40)$$

In order to show that $|\omega| < 1$ in U , it suffices to prove that $|A_i| < 1$ for $i = 1, 2, 3, \dots, n+1$, or equivalently, all the zeros A_i for $i = 1, 2, 3, \dots, n+1$ of the polynomial function $q(z)$ lie in or on the unit circle $|z| = 1$.

Note that

$$|a_0| = |a(2+n)+n| < |2| = |a_{n+1}|, \quad \forall a \in \left(-1, \frac{2-n}{2+n}\right), n \in \mathbb{N}, \quad (41)$$

so Cohn's rule can be applied on q . Consider the polynomial function $q_1(z)$ given by

$$\begin{aligned} q_1(z) &= \frac{\bar{a}_{n+1}q(z) - a_0q^*(z)}{z} \\ &= (1+a)(-2+n+a(2+n))(2+nz^{n-1} - (2+n)z^n). \end{aligned} \quad (42)$$

It is easy to verify that Cohn's rule is applicable to q_1 also. So, let a polynomial function q_2 given by

$$\begin{aligned} q_2(z) &= \frac{-(2+n)q_1(z) - 2q^*(z)}{z} \\ &= (1+a)(-2+n+a(2+n))(4n+n^2)(-1+z)z^{n-2}. \end{aligned} \quad (43)$$

Then, the zeros of polynomial function q_2 lie on ∂U or in U . We deduce that all zeros of q_1 lie inside U or on ∂U , and this implies that all the zeros of q lie inside U or on ∂U . $\square$

Corollary 1. Let $P_a[\ell]$ be mapping given in (17) and let $f_2 = h + \bar{g}$ with $(h-g)(z) = z/(1-z)$ and dilatation $\omega_1(z) =$

$-z^n$, $n \in \mathbb{N}$. Then, $P_a[\ell] * f_n \in S_{CHD}^0$ for $a \in (-1, (2-n)/(2+n))$.

The following examples illustrate Theorems 5 and 6.

Example 1. In Theorem 6, if $n = 2$, $(h-g)(z) = z/(1-z)$, and $\omega_{f_2}(z) = z^2$, then

$$\begin{aligned} f_2(z) &= h(z) + \bar{g}(z) \\ &= \frac{1}{8} \left(\frac{2z(3-2z)}{(1-z)^2} + \log\left(\frac{1+z}{1-z}\right) \right) \\ &\quad + \frac{1}{8} \overline{\left(\frac{2z(2z-1)}{(1-z)^2} + \log\left(\frac{1+z}{1-z}\right) \right)}. \end{aligned} \quad (44)$$

It follows from (17) that

$$\begin{aligned} P_a[\ell](z) &= h_a + \bar{g}_a = \frac{1}{2} \left(\left(\frac{(1+a)z}{(1-z)^2} + \frac{(1-a)z}{(1-z)} \right) \right. \\ &\quad \left. + \overline{\left(\frac{(1+a)z}{(1-z)^2} - \frac{(1-a)z}{(1-z)} \right)} \right), \end{aligned} \quad (45)$$

and $P_a[\ell](z) * f_2(z) = H(z) + \bar{G}(z)$. Further,

$$\begin{aligned} H(z) &= (h_a * h)(z) \\ &= \frac{1}{16} \left[\left(\frac{(1+a)z}{(1-z)^2} + \frac{(1-a)z}{(1-z)} \right) * \left(\frac{2z(3-2z)}{(1-z)^2} + \log\left(\frac{1+z}{1-z}\right) \right) \right] \\ &= \frac{1}{16} \left[\frac{8(1+a)z}{(1-z)^3(1+z)} + (1-a) \left(\frac{2z(3-2z)}{(1-z)^2} + \log\left(\frac{1+z}{1-z}\right) \right) \right], \\ G(z) &= (g_a * g)(z) \\ &= \frac{1}{16} \left[\left(\frac{(1+a)z}{(1-z)^2} - \frac{(1-a)z}{(1-z)} \right) * \left(\frac{2z(2z-1)}{(1-z)^2} + \log\left(\frac{1+z}{1-z}\right) \right) \right] \\ &= \frac{1}{16} \left[\frac{8(1+a)z^3}{(1-z)^3(1+z)} - (1-a) \left(\frac{2z(2z-1)}{(1-z)^2} + \log\left(\frac{1+z}{1-z}\right) \right) \right]. \end{aligned} \quad (46)$$

Figures 1–4, respectively, display representations of concentrated circles within U under the harmonic mapping $P_a[\ell](z)$ and concentrated circles under the convolution map $P_a[\ell](z) * f_2(z)$. We take various values of $a = -0.001, -0.5 \in (-1, 0)$.

Example 2. Consider an analytic univalent function $\phi(z) = z + (z^2/4)$, where $\phi(0) = 0$ and $\phi'(0) = 1$. Since $\phi'(z) = 1 + (z/2)$ and $\phi''(z) = 1/2$,

$$\operatorname{Re} \left(1 + \frac{z\phi''(z)}{\phi'(z)} \right) = \operatorname{Re} \left(1 + \frac{(1/2)z}{(1+(1/2)z)} \right) \geq 0. \quad (47)$$

This implies that the function $\phi(z)$ maps U into convex region. In view of Theorem 1, the mapping $P_a[\phi](z)$ is in S_{CHD}^0 for $-1 < a < 1$. Now

$$\begin{aligned} P_a[\phi](z) &= \frac{1}{2} \left((1-a) \left(\frac{z^2}{4} + z \right) + (a+1) \left(\frac{z^2}{2} + z \right) \right) \\ &\quad + \frac{1}{2} \overline{\left((a+1) \left(\frac{z^2}{2} + z \right) - (1-a) \left(\frac{z^2}{4} + z \right) \right)}, \quad (z \in U). \end{aligned} \quad (48)$$

Figures 5–7 display images of U under $P_a[z + (1/4)z^2]$ for various values of $a \in (-1, 1)$, respectively. The image of

FIGURE 1: Image of $P_{-0.001}[\ell](z)$. FIGURE 4: Image of $P_{-0.5}[\ell](z) * f_2(z)$. FIGURE 2: Image of $P_{-0.5}[\ell](z)$. FIGURE 5: Image of $P_{-0.09}[z + (z^2/4)]$. FIGURE 3: Image of $P_{-0.001}[\ell](z) * f_2(z)$.

U under $P_a[z + (1/4)z^2]$ is convex in the direction of the real axis.

3. Convex Combination of a Family of Univalent Harmonic Mappings

Let $f_\alpha(z) = h_\alpha(z) + \overline{g}_\alpha(z)$, where

$$K_\alpha(z) = h_\alpha(z) + g_\alpha(z) = \frac{z(1 - \alpha z^2)}{1 - z^2}, \quad \alpha \in [0, 1], \quad (49)$$

$$\left| \frac{g'_\alpha}{h'_\alpha} \right| < 1,$$

be normalized, sense-preserving, and locally univalent harmonic mapping defined on U . First of all, we have to

FIGURE 6: Image of $P_0[z + (z^2/4)]$. FIGURE 7: Image of $P_{0.09}[z + (z^2/4)]$.

show that f_α belongs to class S_H and maps onto region convex in the direction of the imaginary axis. Let

$$\Psi(z) = (1 - z^2)K'_\alpha(z) = \frac{1 + \alpha z^4 + (1 - 3\alpha)z^2}{1 - z^2}, \quad (50)$$

for all $z \in U$, $0 \leq \alpha \leq 1$, and we will prove that

$$\operatorname{Re}(\Psi(z)) = \operatorname{Re} \frac{1 + \alpha z^4 + (1 - 3\alpha)z^2}{1 - z^2} > 0. \quad (51)$$

Since $|1 - z^2|^2 \neq 0$, for all z in U , $0 \leq \alpha \leq 1$ and

$$\begin{aligned} & \left((1 - \bar{z}^2)(1 + \alpha z^4 + (1 - 3\alpha)z^2) \right) \\ &= (1 - \bar{z}^2 + z^2 - |z|^4 + \alpha(z^4 - 3z^2 + 3|z|^4 - |z|^4 z^2)), \end{aligned} \quad (52)$$

it follows that

$$\operatorname{Re}(\Psi(z)) = \operatorname{Re} \frac{(1 - \bar{z}^2 + z^2 - |z|^4 + \alpha(z^4 - 3z^2 + 3|z|^4 - |z|^4 z^2))}{|1 - z^2|^2}. \quad (53)$$

Note that $\Psi(0) = 1$ and for each $\gamma \in \mathbb{R}$,

$$\begin{aligned}
\operatorname{Re}(\Psi(e^{i\gamma})) &= \operatorname{Re} \frac{\left(1 - (e^{-i\gamma})^2 + (e^{i\gamma})^2 - |e^{i\gamma}|^4 + \alpha \left((e^{i\gamma})^4 - 3(e^{i\gamma})^2 + 3|e^{i\gamma}|^4 - |e^{i\gamma}|^4 (e^{i\gamma})^2 \right)\right)}{|1 - (e^{i\gamma})^2|^2} \\
&= \operatorname{Re} \frac{\left(1 - (e^{-i\gamma})^2 + (e^{i\gamma})^2 - 1 + \alpha \left((e^{i\gamma})^4 - 3(e^{i\gamma})^2 + 3 - (e^{i\gamma})^2 \right)\right)}{|1 - (e^{i\gamma})^2|^2} \\
&= \operatorname{Re} \frac{\left(-4i \cos(\gamma) \sin(\gamma) + \alpha \left((e^{i\gamma})^4 - 3(e^{i\gamma})^2 + 3 - (e^{i\gamma})^2 \right)\right)}{|1 - (e^{i\gamma})^2|^2} \\
&= \operatorname{Re} \frac{\left(i(\sin(4\gamma) - 4 \sin(2\gamma)) - 4 \cos(\gamma) \sin(\gamma) + 2\alpha((\cos(2\gamma) - 1)^2)\right)}{|1 - (e^{i\gamma})^2|^2} \\
&= \frac{2\alpha}{|1 - (e^{i\gamma})^2|^2} ((\cos(2\gamma) - 1)^2) \geq 0, \quad 0 \leq \alpha \leq 1.
\end{aligned} \tag{54}$$

Hence, by the minimum principle for harmonic functions, we obtain

$$\operatorname{Re}(\Psi(z)) = \operatorname{Re}((1 - z^2)K'_\alpha(z)) > 0, \tag{55}$$

for all $z \in U$, and this yields the result.

In addition, by Lemma 1, we can deduce that the analytic function $K_\alpha = h_\alpha + g_\alpha$ is univalent and maps onto domain convex in the direction of the imaginary axis. Further, Theorem 3 yields that the harmonic mapping $h_\alpha + \bar{g}_\alpha$ belongs to the class S_H and is convex in the direction of

imaginary axis. Figure 8 gives an illustration of images of U under K_α for several values of α .

Let us begin by presenting our own dilatation ω .

Theorem 7. Let $f_{\alpha_j} = h_{\alpha_j} + \bar{g}_{\alpha_j}$, for $j = 1, 2$, be two normalized harmonic mapping satisfying $h_{\alpha_j}(z) + g_{\alpha_j}(z) = z(1 - \alpha_j z^2)/(1 - z^2)$, $0 \leq \alpha_j \leq 1$, and dilatation $\omega_j = g'_j/h'_j$ with $|\omega_j| < 1$ in U . Then, the second complex dilatation ω of the mapping $f = t f_{\alpha_1} + (1 - t) f_{\alpha_2}$, $0 \leq t \leq 1$, is defined to be

$$\omega(z) = \frac{(1 + z^2)(\omega_1 \omega_2 + t \omega_1 + (1 - t) \omega_2) + z^2(z^2 - 3)(\alpha_1 t \omega_1(1 + \omega_2) + \alpha_2(1 - t)(1 + \omega_1) \omega_2)}{(1 + z^2)(1 + t \omega_2 + (1 - t) \omega_1) + z^2(z^2 - 3)(\alpha_1 t(1 + \omega_2) + \alpha_2(1 - t)(1 + \omega_1))}. \tag{56}$$

Proof. Since

$f = t f_{\alpha_1} + (1 - t) f_{\alpha_2} = t h_{\alpha_1} + (1 - t) h_{\alpha_2} + t \bar{g}_{\alpha_1} + (1 - t) \bar{g}_{\alpha_2}$, it follows that the dilatation ω of the mapping f is given by

$$\omega = \frac{t g'_{\alpha_1} + (1 - t) g'_{\alpha_2}}{t h'_{\alpha_1} + (1 - t) h'_{\alpha_2}} = \frac{t \omega_1 h'_{\alpha_1} + (1 - t) \omega_2 h'_{\alpha_2}}{t h'_{\alpha_1} + (1 - t) h'_{\alpha_2}}. \tag{57}$$

Making use of the mapping $h_{\alpha_j}(z) + g_{\alpha_j}(z) = z(1 - \alpha_j z^2)/(1 - z^2)$ and $\omega_j = g'_{\alpha_j}/h'_{\alpha_j}$, for $j = 1, 2$, we obtain

$$\begin{aligned}
h'_{\alpha_1}(z) &= \frac{1 + z^2(1 + \alpha_1(z^2 - 3))}{(1 + \omega_1)(1 - z^2)^2}, \\
h'_{\alpha_2}(z) &= \frac{1 + z^2(1 + \alpha_2(z^2 - 3))}{(1 + \omega_2)(1 - z^2)^2}.
\end{aligned} \tag{58}$$

Replacing the above expressions with h'_{α_1} and h'_{α_2} in (57), it follows that

FIGURE 8: Images of (a) $K_0(U)$, (b) $K_{1/3}(U)$, (c) $K_{1/2}(U)$, and (d) $K_{0.9}(U)$.

$$\omega = \frac{t\omega_1 \left(1 + z^2(1 + \alpha_1(z^2 - 3)) / (1 + \omega_1)(1 - z^2)^2\right) + (1-t)\omega_2 \left(1 + z^2(1 + \alpha_2(z^2 - 3)) / (1 + \omega_2)(1 - z^2)^2\right)}{t \left(1 + z^2(1 + \alpha_1(z^2 - 3)) / (1 + \omega_1)(1 - z^2)^2\right) + (1-t) \left(1 + z^2(1 + \alpha_2(z^2 - 3)) / (1 + \omega_2)(1 - z^2)^2\right)} \quad (59)$$

$$= \frac{t\omega_1(1 + z^2 + \alpha_1 z^2(z^2 - 3))(1 + \omega_2) + (1-t)\omega_2(1 + z^2 + \alpha_2 z^2(z^2 - 3))(1 + \omega_1)}{t(1 + z^2 + \alpha_1 z^2(z^2 - 3))(1 + \omega_2) + (1-t)(1 + z^2 + \alpha_2 z^2(z^2 - 3))(1 + \omega_1)},$$

which yields the results in (56) through some computation.

The next result clarifies that the necessary and sufficient condition for the convex combination of f_{α_1} and f_{α_2} is in the class S_H and convex in the direction of the imaginary axis if it is locally univalent and sense-preserving. $\square$

Theorem 8. Let $f_{\alpha_j} = h_{\alpha_j} + \overline{g_{\alpha_j}}$, for $j = 1, 2$, be two normalized harmonic mappings satisfying $h_{\alpha_j}(z) + g_{\alpha_j}(z) = z(1 - \alpha_j z^2)/(1 - z^2)$, $0 \leq \alpha_j \leq 1$, and dilatation $\omega_j = g'_{\alpha_j}/h'_{\alpha_j}$

with $|\omega_j| < 1$ in U . Then, the mapping $f = tf_{\alpha_1} + (1-t)f_{\alpha_2}$, $0 \leq t \leq 1$ with dilatation ω as given in (56), is in S_H and maps into region convex in the direction of the imaginary axis, provided f is locally univalent and sense-preserving.

Proof. Define

$$f = h + \overline{g} = (th_{\alpha_1} + (1-t)h_{\alpha_2}) + \overline{tg_{\alpha_1} + (1-t)g_{\alpha_2}}. \quad (60)$$

Let

$$\begin{aligned}
h + g &= t(h_{\alpha_1} + g_{\alpha_1}) + (1-t)(h_{\alpha_2} + g_{\alpha_2}) \\
&= t(h_{\alpha_1} + g_{\alpha_1}) + (1-t)(h_{\alpha_2} + g_{\alpha_2}) \quad (61) \\
&= tK_{\alpha_1} + (1-t)K_{\alpha_2},
\end{aligned}$$

where $K_{\alpha_j} = h_{\alpha_j} + g_{\alpha_j}$, $j = 1, 2$. Since the mapping f_{α_j} satisfies (49), it follows from (7) that

$$\begin{aligned}
\operatorname{Re}\{(1-z^2)(h' + g')\} &= t\operatorname{Re}\{(1-z^2)K_{\alpha_1}'\} \\
&\quad + (1-t)\operatorname{Re}\{(1-z^2)K_{\alpha_2}'\} > 0, \quad (62)
\end{aligned}$$

on U for all $0 \leq t \leq 1$. Hence, according to Lemma 1, one can deduce that the mapping $h + g$ is univalently convex in the direction of the imaginary axis. Also, If $f = h + \bar{g}$ is sense-

preserving locally univalent mapping in U , then in view of Theorem 3, $f \in S_H$ is univalent analytic mapping of U onto a region convex in the direction of imaginary axis. $\square$

Theorem 9. For $j = 1, 2$, define $f_{\alpha_j} = h_{\alpha_j} + \bar{g}_{\alpha_j}$ to be normalized sense-preserving harmonic mapping such that $h_{\alpha_j}(z) + g_{\alpha_j}(z) = z(1 - \alpha_j z^2)/(1 - z^2)$, $0 \leq \alpha_j \leq 1$ with $\alpha_j = \alpha$. Let ω_j be the dilatation of f_{α_j} ($|\omega_j| < 1$ in U). If $\omega_1 \neq \omega_2$, then the mapping $f = tf_{\alpha_1} + (1-t)f_{\alpha_2}$, $0 \leq t \leq 1$, is in S_H and is convex in the direction of the imaginary axis.

Proof. From Theorem 8, it will be sufficient if we show that the dilatation of f given in (56) satisfies the condition $|\omega| < 1$ in U . Suppose that $\alpha_1 = \alpha_2 = \alpha$ in (56); then,

$$\begin{aligned}
\omega &= \frac{(1+z^2)[\omega_1\omega_2 + t\omega_1 + (1-t)\omega_2] + \alpha z^2(z^2-3)[\omega_1\omega_2 + t\omega_1 + (1-t)\omega_2]}{(1+z^2)[1+t\omega_2 + (1-t)\omega_1] + \alpha z^2(z^2-3)[1+t\omega_2 + (1-t)\omega_1]} \\
&= \frac{(\omega_1\omega_2 + t\omega_1 + (1-t)\omega_2)(1 + \alpha z^4 + (1-3\alpha)z^2)}{(1+t\omega_2 + (1-t)\omega_1)(1 + z^2 + \alpha z^2(z^2-3))} \quad (63) \\
&= \frac{\omega_1\omega_2 + t\omega_1 + (1-t)\omega_2}{1+t\omega_2 + (1-t)\omega_1}.
\end{aligned}$$

From the proof of Theorem 3 in [21], it is clear that $|\omega| < 1$. Hence, we deduce that the mapping f is locally univalent and sense-preserving.

Next, we will consider one of the harmonic mappings involved in the linear combination induced by shearing analytic mapping $z(1 - \beta z)/(1 - z^2)$ where $\beta \in [-1, 1]$. $\square$

Theorem 10. Define $f_\beta = h_\beta + \bar{g}_\beta$, satisfying $h_\beta(z) + g_\beta(z) = z(1 - \beta z)/(1 - z^2)$, $-1 \leq \beta \leq 1$, with dilatation $\omega_1 = g_\beta'/h_\beta'$ lies in U and let $f_\alpha = h_\alpha + \bar{g}_\alpha$, satisfying $h_\alpha(z) + g_\alpha(z) = z(1 - \alpha z^2)/(1 - z^2)$, $0 \leq \alpha \leq 1$, with dilatation $\omega_2 = g_\alpha'/h_\alpha'$ lies in U , be two normalized harmonic mapping. Then, the mapping $f_{\beta,\alpha} = tf_\beta + (1-t)f_\alpha$, $0 \leq t \leq 1$ is univalent and convex in the direction of the imaginary axis if $f_{\beta,\alpha}$ is locally univalent and sense-preserving.

Proof. Since $h_\alpha(z) + g_\alpha(z) = z(1 - \alpha z^2)/(1 - z^2)$ and $h_\beta(z) + g_\beta(z) = z(1 - \beta z)/(1 - z^2)$, it follows that

$$\begin{aligned}
h + g &= th_\alpha + (1-t)h_\beta + tg_\alpha + (1-t)g_\beta \\
&= t(h_\alpha + g_\alpha) + (1-t)(h_\beta + g_\beta), \quad (64)
\end{aligned}$$

which implies that

$$h' + g' = t(h_\alpha' + g_\alpha') + (1-t)(h_\beta' + g_\beta'). \quad (65)$$

Hence,

$$\begin{aligned}
\operatorname{Re}\{(1-z^2)(h' + g')\} &= t\operatorname{Re}\{(1-z^2)(h_\alpha' + g_\alpha')\} \\
&\quad + (1-t)\operatorname{Re}\{(1-z^2)(h_\beta' + g_\beta')\}, \quad (66)
\end{aligned}$$

from (55). Since

$$\operatorname{Re}\{(1-z^2)(h_\beta' + g_\beta')\} = \frac{(1-|z|^2)(1+|z|^2-2\beta\operatorname{Re}(z))}{|1-z^2|^2} > 0, \quad (67)$$

for all $0 \leq t \leq 1$, we can easily see that

$$\operatorname{Re}\{(1-z^2)(h' + g')\} > 0. \quad (68)$$

In view of Theorem 3, we deduce that the mapping $f = h + \bar{g}$ is univalent and maps onto domain convex in the direction of the imaginary axis provided f is locally univalent and sense-preserving on U .

It is known that for $0 \leq \theta < \pi$, the mapping $f_\theta = h_\theta + \bar{g}_\theta$ is called vertical strip mapping if it satisfies

$$h_\theta(z) + g_\theta(z) = \frac{1}{2i \sin \theta} \log \left(\frac{1 + e^{i\theta} z}{1 - e^{i\theta} z} \right). \quad (69)$$

Thus, Theorem 10 can be expressed in terms of vertical strip mappings. $\square$

Corollary 2. Let $f_\theta = h_\theta + \overline{g}_\theta$, where $h_\theta(z) + g_\theta(z) = 1/2i \sin \theta \log(1 + e^{i\theta}z/1 - e^{i\theta}z)$ with dilatation g'_θ/h'_θ lies in U and $f_\alpha = h_\alpha + \overline{g}_\alpha$, where $h_\alpha(z) + g_\alpha(z) = z(1 - \alpha z^2)/(1 - z^2)$, $0 \leq \alpha \leq 1$ with dilatation g'_α/h'_α lies in U , to be two normalized harmonic mapping. Then, the mapping $f_{\theta,\alpha} = tf_\theta + (1-t)f_\alpha$, $0 \leq t \leq 1$, is univalent and convex in the direction of the imaginary axis if $f_{\theta,\alpha}$ is locally univalent and sense-preserving.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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References

- [1] H. Lewy, "On the non-vanishing of the Jacobian in certain one-to-one mappings," *Bulletin of the American Mathematical Society*, vol. 42, no. 10, pp. 689–693, 1936.
- [2] D. Peter, *Harmonic Mappings in the Plane*, Cambridge University Press, Cambridge, UK, 2004.
- [3] M. Dorff, "Convolutions of planar harmonic convex mappings," *Complex Variables*, vol. 45, pp. 263–271, 2001.
- [4] S. Ruscheweyh and T. Sheil-Small, "Hadamard products of schlicht functions and the Polya-Schoenberg conjecture," *Commentarii Mathematici Helvetici*, vol. 48, pp. 119–135, 1973.
- [5] S. Beig and V. Ravichandran, "Convolution and convex combination of harmonic mappings," *Bulletin of the Iranian Mathematical Society*, vol. 45, no. 5, pp. 1467–1486, 2019.
- [6] R. Kumar, S. Gupta, S. Singh, and M. Dorff, "An application of Cohn's rule to convolutions of univalent harmonic mappings," *Rocky Mountain Journal of Mathematics*, vol. 46, no. 2, pp. 559–570, 2016.
- [7] Z.-H. Liu and Y.-C. Li, "The properties of a new subclass of harmonic univalent mappings," *Abstract and Applied Analysis*, vol. 2013, Article ID 794108, 7 pages, 2013.
- [8] Z. Liu and S. Ponnusamy, "Univalence of convolutions of univalent harmonic right half-plane mappings," *Computational Methods and Function Theory*, vol. 17, no. 2, pp. 289–302, 2017.
- [9] M. Dorff, M. Nowak, and M. Wołoszkiewicz, "Convolutions of harmonic convex mappings," *Complex Variables and Elliptic Equations*, vol. 57, pp. 489–503, 2012.
- [10] M. Goodloe, "Hadamard products of harmonic mappings," *Complex Variable Theory and Applications*, vol. 47, p. 8192, 2002.
- [11] L. Li and S. Ponnusamy, "Solution to an open problem on convolutions of harmonic mappings," *Complex Variables and Elliptic Equations*, vol. 58, pp. 1647–1653, 2013.
- [12] R. Kumar, M. Dorff, S. Gupta, and S. Singh, "Convolution properties of some harmonic mappings in the right half-plane," *The Bulletin of the Malaysian Mathematical Society Series 2*, vol. 39, no. 1, pp. 439–455, 2016.
- [13] J. Clunie and T. Sheil-Small, "Harmonic univalent functions," *Annales Academiae Scientiarum Fennicae Series A I Mathematica*, vol. 9, pp. 3–25, 1984.
- [14] M. F. Ali, V. Allu, and N. Ghosh, "A convolution property of univalent harmonic right half-plane mappings," *Monatshefte für Mathematik*, vol. 193, no. 4, pp. 729–736, 2020.
- [15] W. Hengartner and G. Schober, "On Schlicht mappings to domains convex in one direction," *Commentarii Mathematici Helvetici*, vol. 45, no. 1, pp. 303–314, 1970.
- [16] C. Pommerenke, "On starlike and close-to-convex functions," *Proceedings of the London Mathematical Society*, vol. s3-13, no. 1, pp. 290–304, 1963.
- [17] M. J. Dorff and J. S. Rolf, "Anamorphosis, mapping problems, and harmonic univalent functions," *Explorations in Complex Analysis*, pp. 197–269, Mathematical Association of America, Washington, DC, USA, 2012.
- [18] T. H. MacGregor, "The univalence of a linear combination of convex mappings," *Journal of the London Mathematical Society*, vol. s1-44, no. 1, pp. 210–212, 1969.
- [19] L. Shi, Z.-G. Wang, A. Rasila, and Y. Sun, "Convex combinations of harmonic shears of slit mappings," *Bulletin of the Iranian Mathematical Society*, vol. 43, no. 5, pp. 1495–1510, 2017.
- [20] R. Kumar, S. Gupta, and S. Singh, "Linear combinations of univalent harmonic mappings convex in the direction of the imaginary axis," *Bulletin of the Malaysian Mathematical Sciences Society*, vol. 39, no. 2, pp. 751–763, 2016.
- [21] Z.-G. Wang, Z.-H. Liu, and Y.-C. Li, "On the linear combinations of harmonic univalent mappings," *Journal of Mathematical Analysis and Applications*, vol. 400, no. 2, pp. 452–459, 2013.
- [22] A. Ferrada-Salas, R. Hernández, and M. J. Martín, "On convex combinations of convex harmonic mappings," *Bulletin of the Australian Mathematical Society*, vol. 96, no. 2, pp. 256–262, 2017.
- [23] S. Beig, Y. J. Sim, and N. E. Cho, "On convex combinations of harmonic mappings," *Journal of Inequalities and Applications*, vol. 2020, p. 14, 2020.
- [24] Q. I. Rahman and G. Schmeisser, *Analytic Theory of Polynomials*, Vol. 26, Oxford University Press, Oxford, UK, 2002.