Research Article

On $L(\wp, \wp - 1, \dots, 1)$ Labelings of Circulant Graphs

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An m - $L(\wp, \wp - 1, \dots, 1)$ labeling of a simple graph $\mathcal{G}$ is a mapping τ from the vertex set $V(\mathcal{G})$ to $\{0, 1, \dots, m\}$ such that $|\tau(u) - \tau(v)| \geq \wp + 1 - \mathbf{d}(u, v)$, $\forall u, v \in V(\mathcal{G})$, where length of the shortest route connecting u and v is represented by $\mathbf{d}(u, v)$. The smallest m for which there exists a m - $L(\wp, \wp - 1, \dots, 1)$ labeling of $\mathcal{G}$ is known as the $L(\wp, \wp - 1, \dots, 1)$ labeling number of $\mathcal{G}$, and it is described by $\lambda_{\wp}(\mathcal{G})$. We define m - $L'(\wp, \wp - 1, \dots, 1)$ as the same as the m - $L(\wp, \wp - 1, \dots, 1)$ labeling if τ is one to one. The $L'(\wp, \wp - 1, \dots, 1)$ labeling number of $\mathcal{G}$ represented by $\lambda(\wp, \wp - 1, \dots, 1)(\mathcal{G})$ and is called minimum span. In this paper, we prove that the circulant graphs with specified generating sets admit m - $L(\wp, \wp - 1, \dots, 1)$ and m - $L'(\wp, \wp - 1, \dots, 1)$ labeling and also find $\lambda_{\wp}(\mathcal{G})$ and $\lambda(\wp, \wp - 1, \dots, 1)(\mathcal{G})$. Moreover, we find the $L(\wp, \wp - 1, \dots, 1)$ labeling number of any simple graph with diameter less than $\wp$.

1. Introduction

Let $\mathcal{G}$ be a simple connected graph with finite whose vertex set $V(\mathcal{G})$ and edge set $\mathcal{E}(\mathcal{G})$ with $|V(\mathcal{G})| = \alpha$ and $|\mathcal{E}(\mathcal{G})| = \beta$. The open neighbouring of $v \in V(\mathcal{G})$ is $N(v) = \{u \in V(\mathcal{G}) | \mathbf{d}(v, u) = 1\}$. Assigning integer values to the vertices or edges or both subject to the certain rules is called *labeling* of the graph $\mathcal{G}$. For more about graph labeling, refer the survey [1]. Specially, for graph distance labeling, we refer to [2–5], and we follow [6] for graph theoretic terminology. The frequency assignment problem (FAP) is a process to allocate frequencies to the set of transmitters under the condition that there is no interference between the different transmitters. The aim of the problem is to reduce the span (the variation among lower and higher frequencies) of the allocated frequencies. A good number of graph theoretic methods are used for analyzing FAP and majority of them are NP-hard [7]. Especially, FAP can be modeled as mobile communication, satellite communication, and radio/TV broadcasting. In [8, 9], authors studied the results based on (h, k) -distance labeling of the graph. Also, Nandi et al. [10]

originated and explored the idea of m - $L(\wp, \wp - 1, \dots, 1)$ labeling of graphs.

Throughout this paper, let $n, r \in \mathbb{Z}^+$ such that $1 \leq r \leq \lfloor n - 1/2 \rfloor$, $\Lambda = \{1, 2, \dots, r, n - 1, n - 2, \dots, n - r\}$ is the generating set of $\mathbb{Z}_n$. The circulant graph with vertex set $V(\mathcal{G}) = \{v_0, v_1, \dots, v_{n-1}\}$ and edge set $\mathcal{E}(\mathcal{G}) = \{v_i v_{i+j} : 0 \leq i \leq n - 1, j \in \Lambda\}$, the indices $i + j$ being taken as modulo n . This graph is represented by $\text{Cir}(n, \Lambda)$.

2. m - $L(\wp, \wp - 1, \dots, 1)$ Labeling of Circulant Graphs

We use the following lemma to obtain the minimum span of a m - $L(\wp, \wp - 1, \dots, 1)$ labeling of a graph. Fact: if τ is m - $L(\wp, \wp - 1, \dots, 1)$ labeling function with $\lambda_{\wp}(\mathcal{G}) = m$, then at least one vertex v such that $\tau(v) = 0$.

Lemma 1. Let $\mathcal{G}$ be a simple graph which allows m - $L(\wp, \wp - 1, \dots, 1)$ labeling by the function τ with each vertex $u \in V$; there exists some $v \in V$ such that $\mathbf{d}(u, v) = t$ and $|\tau(u) - \tau(v)| = \wp + 1 - t$; then, $\lambda_{\wp}(\mathcal{G}) = m$.

Proof. Suppose $\lambda_{\wp}(\mathcal{G}) < m$; then, there exists a m' - $\mathbb{L}(\wp, \wp - 1, \dots, 1)$ labeling function τ' such that $m' < m$. Without lack of generalization, $m' = m - 1$, there exist $u, v \in V$ such that $\tau(u) = m$ and $\tau'(v) = m'$. By the assumption, there exists $u', v' \in V$; we obtain $\mathbf{d}(u, u') = t$ and $\mathbf{d}(v, v') = t$ with $|\tau(u) - \tau(u')| = \wp + 1 - t \leq |\tau'(v) - \tau'(v')|$, since, by the definition,

$$\begin{aligned} |\tau(u) - \tau(u')| &\leq |\tau'(v) - \tau'(v')| \\ |m - \tau(u')| &\leq |m' - \tau'(v')| \\ |m - \tau(u')| &\leq |m - (1 + \tau'(v'))|. \end{aligned} \quad (1)$$

Clearly, $\tau'(v') \leq \tau(u') - 1$ if there exist $u'' \neq u$ and $v'' \neq v \in V$ such that $\mathbf{d}(u', u'') = t$ and $\mathbf{d}(v', v'') = t$ with $|\tau(u') - \tau(u'')| = \wp + 1 - t \leq |\tau'(v') - \tau'(v'')| \Rightarrow \tau'(v'') \leq \tau(u'') - 1$. Otherwise, we choose $u'' \in V$, such that $\tau(u'') = \max\{\tau(v)/\tau(u), \tau(u')\}$. Continue this process up to the last vertex; this gives the contradiction to the fact that the distance labeling function is τ' . $\square$

$$|\tau(v_h) - \tau(v_{r((\wp/2)+1)})| = |t - \frac{\wp}{2}| < \frac{\wp}{2} < \frac{\wp}{2} + 1 \leq \wp + 1 - \mathbf{d}(v_h, v_{r((\wp/2)+1)}). \quad (3)$$

Case 3: if $t = \wp/2$, since τ is $\mathbb{L}(\wp, \wp - 1, \dots, 2, 1)$ labeling of $\mathcal{G}$, then the following one of the subcase holds.

Subcase (i): if $\mathbf{d}(v_0, v_h) = \wp/2 + 1$, then the vertices v_h and $v_{r((\wp/2)+1)}$ are adjacent:

$$|\tau(v_h) - \tau(v_{r((\wp/2)+1)})| = 0 < \wp = \wp + 1 - \mathbf{d}(v_h, v_{r((\wp/2)+1)}). \quad (4)$$

Subcase (ii): if $\mathbf{d}(v_{r((\wp/2)+1)}, v_h) = \wp/2 + 1$, then the vertices v_0 and v_h are adjacent:

$$|\tau(v_0) - \tau(v_h)| = \frac{\wp}{2} < \wp = \wp + 1 - \mathbf{d}(v_0, v_h). \quad (5)$$

This gives the contradiction; clearly, no such v_h exists. This argument leads to any other vertex in H which is labeled with multiple $\wp$ under the minimum required condition of this labeling. Then, we are starting with 0; we get the minimum span $[r((\wp/2) + 1) - 1]\wp$. Hence, $\lambda(H) = [(r((\wp/2) + 1) - 1)]\wp$. $\square$

Similarly, we can derive the succeeding lemma.

Lemma 3. Let $\wp, n, r \in \mathbb{Z}^+$ such that $\wp$ is odd and $1 \leq r \leq \lfloor (n-1)/2 \rfloor$. Let $\mathcal{G} = \text{Cir}(n, \Lambda)$, $\Lambda = \{1, 2, \dots, r, n-1, n-2, \dots, n-r\}$, and $V(\mathcal{G}) = \{v_0, v_1, \dots, v_{n-1}\}$. Let H be the induced subgraph of $\mathcal{G}$ whose vertex set is any set of $r\lfloor \wp/2 \rfloor + \lfloor r/2 \rfloor + 1$, consecutive vertices of $V(\mathcal{G})$; if $\mathcal{G}$ admits the m - $\mathbb{L}(\wp, \wp - 1, \dots, 1)$ labeling, then $\lambda(H) = [(r\lfloor \wp/2 \rfloor + \lfloor r/2 \rfloor - 1)]\wp$.

Lemma 2. Let $\wp, n, r \in \mathbb{Z}^+$ such that $\wp$ is even and $1 \leq r \leq \lfloor (n-1)/2 \rfloor$. Let $\mathcal{G} = \text{Cir}(n, \Lambda)$, $\Lambda = \{1, 2, \dots, r, n-1, n-2, \dots, n-r\}$, and $V(\mathcal{G}) = \{v_0, v_1, \dots, v_{n-1}\}$. Let H be the induced subgraph of $\mathcal{G}$ whose vertex set is any set of $r(\wp/2 + 1) + 1$ consecutive vertices of $V(\mathcal{G})$; if $\mathcal{G}$ admits the m - $\mathbb{L}(\wp, \wp - 1, \dots, 1)$ labeling, then $\lambda(H) = [(r((\wp/2) + 1) - 1)]\wp$.

Proof. Let we assume that $V(H) = \{v_0, v_1, \dots, v_{r((\wp/2)+1)}\}$; clearly, at most, the distance between any pair vertices of H is $(\wp/2) + 1$. Let τ be $\mathbb{L}(\wp, \wp - 1, \dots, 2, 1)$ labeling function of $\mathcal{G}$. In general, we assume that $\tau(v_0) = x$ and $\tau(v_{r((\wp/2)+1)}) = x + \wp/2$ because of the distance between $\mathbf{d}(v_0, v_{r((\wp/2)+1)}) = (\wp/2) + 1$. Suppose $v_h \in H$ with $\tau(v_h) = x + t \in [x, x + \wp]$.

Case 1: if $t < \wp/2$,

$$|\tau(v_0) - \tau(v_h)| = |t| < \frac{\wp}{2} < \frac{\wp}{2} + 1 \leq \wp + 1 - \mathbf{d}(v_0, v_h). \quad (2)$$

Case 2: if $t > \wp/2$,

Lemma 4. Let $\mathcal{G}$ be any simple finite graph with n vertices; if $\wp > \text{diam}(\mathcal{G}) = d$, then $\lambda_{\wp}(\mathcal{G}) = (n-1)\wp - \sum_{s=1}^n (\mathbf{d}(v_{s-1}, v_s) - 1)$, where $v_s = y \in V(\mathcal{G})$ such that $\mathbf{d}(v_{s-1}, y) = \max\{\mathbf{d}(v_{s-1}, x) : \forall x \neq v_{i < s} \in V(\mathcal{G})\}$, $s \in \mathbb{N}$, and v_0 is an any arbitrary vertex of $\mathcal{G}$.

Proof. Let us describe the condition:

$$\tau(v_i) = i\wp - \sum_{s=1}^n (\mathbf{d}(v_{s-1}, v_s) - 1). \quad (6)$$

Here, we have to prove that the function is $\mathbb{L}(\wp, \wp - 1, \dots, 2, 1)$ labeling and take any arbitrary vertex v_i and v_j :

$$\begin{aligned} |\tau(v_i) - \tau(v_j)| &\geq \left| i\wp - \sum_{s=1}^i (\mathbf{d}(v_{s-1}, v_s) - 1) \right. \\ &\quad \left. - \left(j\wp - \sum_{s=1}^j (\mathbf{d}(v_{s-1}, v_s) - 1) \right) \right| \\ &\geq \left| (i-j)\wp - \sum_{s=j}^i (\mathbf{d}(v_{s-1}, v_s) - 1) \right| \\ &\geq |(i-j)\wp - (i-j)(d-1)| \\ &\geq |(i-j)(\wp - d) + (i-j)| \\ &\geq \wp - d + 1 \\ |\tau(v_i) - \tau(v_j)| &\geq \wp + 1 - \mathbf{d}(v_i, v_j). \end{aligned} \quad (7)$$

This shows that the function τ is $\mathbb{L}(\wp, \wp - 1, \dots, 1)$ labeling. We have to prove that the minimum span of this

function; let us consider the vertices in the order $v_0, v_1, \dots, v_n$ such that

$$|\tau(v_{s-1}, v_s)| = \wp + 1 - \mathfrak{d}(v_{s-1}, v_s). \quad (8)$$

This sequence of the vertices shows the minimum span of $L(\wp, \wp - 1, \dots, 2, 1)$ labeling of the graph $\mathcal{G}$. Hence, $\lambda_\wp(\mathcal{G}) = (n-1)\wp - \sum_{s=1}^n (\mathfrak{d}(v_{s-1}, v_s) - 1)$. $\square$

Theorem 1. Let $\wp, n \in \mathbb{Z}^+$ with n as even and $\mathcal{G} = \text{Cir}(n, \Lambda)$, where $\Lambda = \{1, 2, \dots, n/2 - 1, n-1, n-2, \dots, n - ((n/2) - 1)\}$. Then,

$$\lambda_\wp(\mathcal{G}) = (n-1)\wp - \frac{n}{2}. \quad (9)$$

Proof. Let $\mathcal{G} = \text{Cir}(n, \Lambda)$, with $\Lambda = \{1, 2, \dots, (n/2) - 1, n-1, n-2, \dots, n - ((n/2) - 1)\}$. Let $\vee(\mathcal{G}) = \{v_0, v_1, v_2, \dots, v_{n-1}\}$ and $\tau: \vee \longrightarrow [0, \infty]$ be given by

$$\tau(v_i) = \begin{cases} 2i\wp - i, & \text{if } i < \frac{n}{2}, \\ (2\wp - 1)\left(i - \frac{n}{2}\right) + \wp - 1, & \text{if } i \geq \frac{n}{2}. \end{cases} \quad (10)$$

$$\tau(v_i) = \begin{cases} \left\lfloor i \bmod r\left(\frac{\wp}{2} + 1\right) \right\rfloor \wp, & \text{if } \lfloor \frac{i}{r((\wp/2) + 1)} \rfloor = 0, 2, \dots, \ell, \\ \left\lfloor i \bmod r\left(\frac{\wp}{2} + 1\right) \right\rfloor \wp + \frac{\wp}{2}, & \text{if } \lfloor \frac{i}{r((\wp/2) + 1)} \rfloor = 1, 3, \dots, \ell - 1. \end{cases} \quad (12)$$

The distance between arbitrary vertices v_i and v_j in $\vee(\mathcal{G})$ is

$$\mathfrak{d}(v_i, v_j) = \begin{cases} \left\lfloor \frac{|i-j|}{r} \right\rfloor, & \text{if } |i-j| \leq \frac{n}{2}, \\ \left\lfloor \frac{n-|i-j|}{r} \right\rfloor, & \text{if } |i-j| > \frac{n}{2}. \end{cases} \quad (13)$$

Case (i): if $\lfloor i/r((\wp/2) + 1) \rfloor = \lfloor j/r((\wp/2) + 1) \rfloor$,

$$|\tau(v_i) - \tau(v_j)| = \left| \left\lfloor (i-j) \bmod r\left(\frac{\wp}{2} + 1\right) \right\rfloor \wp \right|. \quad (14)$$

Here, $\lfloor (i-j) \bmod r((\wp/2) + 1) \rfloor \neq 0$ since i and j are distinct, so $0 < i-j < r((\wp/2) + 1)$ because $\lfloor i/r((\wp/2) + 1) \rfloor = \lfloor j/r((\wp/2) + 1) \rfloor$.

Let $v_i, v_j \in \vee(\mathcal{G})$ with $i, j < n/2$; then, $\mathfrak{d}(v_i, v_j) = 1$ and $|\tau(v_i) - \tau(v_j)| = |(2\wp - 1)i - (2\wp - 1)j| \geq 2\wp - 1 \geq \wp \geq \wp + 1 - \mathfrak{d}(v_i, v_j)$.

Let $v_i, v_j \in \vee(\mathcal{G})$ with $i, j < n/2$; then, $\mathfrak{d}(v_i, v_j) = 1$ and $|\tau(v_i) - \tau(v_j)| = |(2\wp - 1)(i-j)| \geq 2\wp - 1 \geq \wp \geq \wp + 1 - \mathfrak{d}(v_i, v_j)$.

Let $v_i, v_j \in \vee(\mathcal{G})$ with $i < n/2$ and $j \geq n/2$; then, $\mathfrak{d}(v_i, v_j) = 1$ or 2 and $|\tau(v_i) - \tau(v_j)| = |(2\wp - 1)(i - n/2) + \wp - 1 - 2i\wp + i| \geq \wp(1-n) + n/2 - 1 \geq \wp \geq \wp + 1 - \mathfrak{d}(v_i, v_j)$. Thus, τ is $L(\wp, \wp - 1, \dots, 2, 1)$ labeling function, and Lemma 1 holds the theorem. $\square$

Example 1. If $\wp = 4$ and $n = 6$, then the circulant graph $\mathcal{G} = \text{Cir}(6, \{1, 2, 4, 5\})$ has $L(\wp, \wp - 1, \dots, 2, 1)$ labeling number $\lambda_4(\mathcal{G}) = 17$. Figure 1 illustrates that the span is sharp.

Theorem 2. Let $\wp, n, r \in \mathbb{Z}^+$ such that $\wp$ is even and $1 \leq r \leq \lfloor (n-1)/2 \rfloor$. Let $\mathcal{G} = \text{Cir}(n, \Lambda)$ with $\Lambda = \{1, 2, \dots, r, n-1, n-2, \dots, n-r\}$. If $2r((\wp/2) + 1)|n$, then

$$\lambda_\wp(\mathcal{G}) = \left[r\left(\frac{\wp}{2} + 1\right) - 1 \right] \wp + \frac{\wp}{2}. \quad (11)$$

Proof. Let $\mathcal{G} = \text{Cir}(n, \Lambda)$, where $\Lambda = \{1, 2, \dots, r, n-1, n-2, \dots, n-r\}$. Let $\ell = n/r((\wp/2) + 1)$, $\vee(\mathcal{G}) = \{v_0, v_1, v_2, \dots, v_{n-1}\}$, and $\tau: \vee \longrightarrow [0, \infty]$ by

$$|\tau(v_i) - \tau(v_j)| = \left| \left\lfloor (i-j) \bmod r\left(\frac{\wp}{2} + 1\right) \right\rfloor \wp \right| \geq \wp \geq \wp + 1 - \mathfrak{d}(v_i, v_j). \quad (15)$$

This satisfied the $L(\wp, \wp - 1, \dots, 2, 1)$ labeling condition.

Case (ii): if $\lfloor i/r((\wp/2) + 1) \rfloor \neq \lfloor j/r((\wp/2) + 1) \rfloor$.

Subcase (a): if $|\tau(v_i) - \tau(v_j)| = |\lfloor (i-j) \bmod r((\wp/2) + 1) \rfloor \wp - \wp/2|$.

There exist $a, b \in \mathbb{Z}^+$ such that $(i-j) \bmod r((\wp/2) + 1) = a \Rightarrow i-j = b[r((\wp/2) + 1)] + a$ with $1 \leq b < \ell$. Hence,

$$\mathfrak{d}(v_i, v_j) = \left\lfloor \frac{b[r((\wp/2) + 1)] + a}{r} \right\rfloor \geq \frac{\wp}{2} + 1, \text{ or} \quad (16)$$

$$\left\lfloor \frac{\ell[r((\wp/2) + 1)] - \{b[r((\wp/2) + 1)] + a\}}{r} \right\rfloor \geq \frac{\wp}{2} + 1.$$

FIGURE 1: $\lambda_4(\mathcal{G}) = 17$.

Therefore, this implies that $\mathfrak{d}(v_i, v_j) \geq \wp/2 + 1$:

$$|\tau(v_i) - \tau(v_j)| \geq \frac{\wp}{2} = \wp - \frac{\wp}{2} + 1 - 1 \geq \wp + 1 - \mathfrak{d}(v_i, v_j). \quad (17)$$

Subcase (b): if $|\tau(v_i) - \tau(v_j)| = |(i - j) \bmod r ((\wp/2) + 1)|\wp$.

There exist $a, b \in \mathbb{Z}^+$ such that $(i - j) \bmod r ((\wp/2) + 1) = a \Rightarrow i - j = b[(\wp/2) + 1] + a$ with $2 \leq b < \ell - 1$. Hence, by equation (13),

$$\mathfrak{d}(v_i, v_j) \geq 2\left(\frac{\wp}{2} + 1\right) = \wp + 2, \text{ or,} \quad (18)$$

$$\left\lfloor \frac{\ell[r((\wp/2) + 1)] - \{b[r((\wp/2) + 1)] + a\}}{r} \right\rfloor \geq \wp + 2.$$

Therefore, $\mathfrak{d}(v_i, v_j) \geq \wp + 2$ implies that

$$\left| \left[(i - j) \bmod r \left(\frac{\wp}{2} + 1 \right) \right] \wp \right| \geq 0 \geq \wp - \wp + 1 - 1 \geq \wp + 1 - (\wp + 1) \geq \wp + 1 - \mathfrak{d}(v_i, v_j). \quad (19)$$

Now, each vertex has some vertex with distance $(\wp/2) + 1$ and absolute difference between those vertices is $\wp/2$. Hence, by Lemma 1, τ is minimum span function of $\mathcal{G}$; this proves the theorem. $\square$

Example 2. If $\wp = 4$ and $r = 2$, then the circulant graph $\mathcal{G} = \text{Cir}(24, \{1, 2, 22, 23\})$ has $\mathbb{L}(\wp, \wp - 1, \dots, 2, 1)$ labeling

number $\lambda_4(\mathcal{G}) = 22$. Figure 2 illustrates that the span is sharp.

Theorem 3. Let $\wp, n, r \in \mathbb{Z}^+$ such that $\wp$ is odd and $1 \leq r \leq \lfloor (n - 1)/2 \rfloor$. Let $\mathcal{G} = \text{Cir}(n, \Lambda)$, where $\Lambda = \{1, 2, \dots, r, n - 1, n - 2, \dots, n - r\}$. If $2(r\lfloor \wp/2 \rfloor + \lfloor r/2 \rfloor)|n$, then

$$\lambda_\wp(\mathcal{G}) = \left[r\left\lfloor \frac{\wp}{2} \right\rfloor + \left\lfloor \frac{r}{2} \right\rfloor - 1 \right] \wp + \left\lfloor \frac{\wp}{2} \right\rfloor. \quad (20)$$

Proof. Let $\mathcal{G} = \text{Cir}(n, \Lambda)$ with $\Lambda = \{1, 2, \dots, r, n - 1, n - 2, \dots, n - r\}$. Let $\ell = n/r\lfloor \wp/2 \rfloor + \lfloor r/2 \rfloor$, $\mathbb{V}(\mathcal{G}) = \{v_0, v_1, v_2, \dots, v_{n-1}\}$, and $r: \mathbb{V} \rightarrow [0, \infty]$ by

$$\tau(i) = \begin{cases} \left[i \bmod r\left\lfloor \frac{\wp}{2} \right\rfloor + \left\lfloor \frac{r}{2} \right\rfloor \right] \wp, & \text{if } \left\lfloor \frac{i}{r\lfloor \wp/2 \rfloor + \lfloor r/2 \rfloor} \right\rfloor = 0, 2, \dots, \ell, \\ \left[i \bmod r\left\lfloor \frac{\wp}{2} \right\rfloor + \left\lfloor \frac{r}{2} \right\rfloor \right] \wp + \left\lfloor \frac{\wp}{2} \right\rfloor, & \text{if } \left\lfloor \frac{i}{r\lfloor \wp/2 \rfloor + \lfloor r/2 \rfloor} \right\rfloor = 1, 3, \dots, \ell - 1. \end{cases} \quad (21)$$

FIGURE 2: $\lambda_4(\mathcal{G}) = 22$.

The distance between arbitrary vertices v_i and v_j in $\mathcal{V}(\mathcal{G})$ is

$$\mathfrak{d}(v_i, v_j) = \begin{cases} \lfloor \frac{|i-j|}{r} \rfloor, & \text{if } |i-j| \leq \frac{n}{2}, \\ \lfloor \frac{n-|i-j|}{r} \rfloor, & \text{if } |i-j| > \frac{n}{2}. \end{cases} \quad (22)$$

Case (i): if $\lfloor i/r \rfloor \lfloor \wp/2 \rfloor + \lfloor r/2 \rfloor = \lfloor j/r \rfloor \lfloor \wp/2 \rfloor + \lfloor r/2 \rfloor$,

$$|\tau(v_i) - \tau(v_j)| = \left| \left[(i-j) \bmod r \left\lfloor \frac{\wp}{2} \right\rfloor + \left\lfloor \frac{r}{2} \right\rfloor \right] \wp \right|. \quad (23)$$

Here, $(i-j) \bmod (r \lfloor \wp/2 \rfloor + \lfloor r/2 \rfloor) \neq 0$, since i and j are distinct, so $0 < i-j < r \lfloor \wp/2 \rfloor + \lfloor r/2 \rfloor$ because $\lfloor i/r \rfloor \lfloor \wp/2 \rfloor + \lfloor r/2 \rfloor = \lfloor j/r \rfloor \lfloor \wp/2 \rfloor + \lfloor r/2 \rfloor$:

$$|\tau(v_i) - \tau(v_j)| = \left| \left[(i-j) \bmod r \left\lfloor \frac{\wp}{2} \right\rfloor + \left\lfloor \frac{r}{2} \right\rfloor \right] \wp \right| \geq \wp \geq \wp + 1 - \mathfrak{d}(v_i, v_j). \quad (24)$$

This satisfied the $L(\wp, \wp-1, \dots, 2, 1)$ labeling condition.

Case (ii): if $\lfloor i/r \rfloor \lfloor \wp/2 \rfloor + \lfloor r/2 \rfloor = \lfloor j/r \rfloor \lfloor \wp/2 \rfloor + \lfloor r/2 \rfloor$.

Subcase (a): if $|\tau(v_i) - \tau(v_j)| = \left| \left[(i-j) \bmod r \left\lfloor \frac{\wp}{2} \right\rfloor + \left\lfloor \frac{r}{2} \right\rfloor \right] \wp - \left\lfloor \frac{\wp}{2} \right\rfloor \right|$.

There exist $a, b \in \mathbb{Z}^+$ such that $(i-j) \bmod r \lfloor \wp/2 \rfloor + \lfloor r/2 \rfloor = a \Rightarrow i-j = b \lfloor r \lfloor \wp/2 \rfloor + \lfloor r/2 \rfloor \rfloor + a$ with $1 \leq b < \ell$. Hence, by equation (22), we have

$$\mathfrak{d}(v_i, v_j) = \left\lfloor \frac{b \lfloor r \lfloor \wp/2 \rfloor + \lfloor r/2 \rfloor \rfloor + a}{r} \right\rfloor \geq \lfloor \wp/2 \rfloor + \lfloor r/2 \rfloor, \text{ or,} \quad (25)$$

$$\left\lfloor \frac{\ell(r \lfloor \wp/2 \rfloor + \lfloor r/2 \rfloor) - \{b \lfloor r \lfloor \wp/2 \rfloor + \lfloor r/2 \rfloor \rfloor + a\}}{r} \right\rfloor \geq \left\lfloor \frac{\wp}{2} \right\rfloor + \left\lfloor \frac{r}{2} \right\rfloor.$$

Therefore, $\mathfrak{d}(v_i, v_j) \geq \lfloor \wp/2 \rfloor + \lfloor r/2 \rfloor$:

$$|\tau(v_i) - \tau(v_j)| \geq \left\lfloor \frac{\wp}{2} \right\rfloor \geq \wp - \left\lfloor \frac{\wp}{2} \right\rfloor \geq \wp + 1 - \mathfrak{d}(v_i, v_j). \quad (26)$$

Subcase (b): if $|\tau(v_i) - \tau(v_j)| = \left| \left[(i-j) \bmod r \left\lfloor \frac{\wp}{2} \right\rfloor + \left\lfloor \frac{r}{2} \right\rfloor \right] \wp \right|$.

Let $(i-j) \bmod r \lfloor \wp/2 \rfloor + \lfloor r/2 \rfloor = a \Rightarrow i-j = b \lfloor r \lfloor \wp/2 \rfloor + \lfloor r/2 \rfloor \rfloor + a$ with $2 \leq b < \ell - 1$. Hence, by equation (22),

$$\mathfrak{d}(v_i, v_j) = \left\lfloor \frac{b \lfloor r \lfloor \wp/2 \rfloor + \lfloor r/2 \rfloor \rfloor + a}{r} \right\rfloor \geq 2 \left(\left\lfloor \frac{\wp}{2} \right\rfloor + \left\lfloor \frac{r}{2} \right\rfloor \right), \text{ or} \quad (27)$$

$$\left\lfloor \frac{\ell(r \lfloor \wp/2 \rfloor + \lfloor r/2 \rfloor) - \{b \lfloor r \lfloor \wp/2 \rfloor + \lfloor r/2 \rfloor \rfloor + a\}}{r} \right\rfloor \geq 2 \left(\left\lfloor \frac{\wp}{2} \right\rfloor + \left\lfloor \frac{r}{2} \right\rfloor \right).$$

Therefore, $\mathfrak{d}(v_i, v_j) \geq 2(\lfloor \varphi/2 \rfloor + \lfloor r/2 \rfloor) > \varphi$.

Thus, τ is $\mathbb{L}(\varphi, \varphi - 1, \dots, 2, 1)$ labeling and $\lambda_\varphi(\mathcal{G}) = \lfloor r\lfloor \varphi/2 \rfloor + \lfloor r/2 \rfloor - 1 \rfloor \varphi$. Now, each vertex has some vertex with distance $\lfloor \varphi/2 \rfloor + \lfloor r/2 \rfloor$ and absolute difference between those vertices is $\lfloor \varphi/2 \rfloor$. Hence, by Lemma 1, τ is minimum span function of $\mathcal{G}$; this proves the theorem. $\square$

Example 3. If $k = 3$ and $r = 3$, then the circulant graph $\mathcal{G} = \text{Cir}(14, \{1, 2, 3, 11, 12, 13\})$ has $\mathbb{L}(\varphi, \varphi - 1, \dots, 2, 1)$ labeling number $\lambda_3(\mathcal{G}) = 19$. Figure 3 illustrates that the span is sharp.

3. $\mathbb{L}'(\varphi, \varphi - 1, \dots, 2, 1)$ Labeling of Circulant Graphs

We prove the following theorems for the circulant graph $\text{Cir}(n, \Lambda)$, $\Lambda = \{1, 2, \dots, r, n-1, n-2, \dots, n-r\}$, and $1 \leq r < \lfloor n/2 \rfloor$ admits $\mathbb{L}'(\varphi, \varphi - 1, \dots, 2, 1)$ labeling with $\lambda_{(\varphi, \varphi-1, \dots, 1)}'(\mathcal{G}) = n-1$.

Theorem 4. Let $n, r \in \mathbb{Z}^+$ with $1 \leq r < \lfloor n/2 \rfloor$. If $n \geq 2(r+1) + 1$ and $(r+1) \nmid n$, then the circulant graph $\text{Cir}(n, \Lambda)$, $\Lambda = \{1, 2, \dots, r, n-1, n-2, \dots, n-r\}$ admits $\mathbb{L}'(2, 1)$ labeling with $\lambda_{(2,1)}'(\mathcal{G}) = n-1$.

Proof. Let $n, \ell \in \mathbb{Z}^+$, $\mathcal{G} = \text{Cir}(n, \Lambda)$ be the circulant graph with $n \geq 2(r+1) + 1$ where $1 \leq r < \lfloor n/2 \rfloor$ and $\ell = n/(r+1)$. By the division algorithm, for every vertex $v \in \mathcal{V}(\mathcal{G})$, we can write $v = v_{i(r+1)+j}$, where $0 \leq i \leq \ell-1, 0 \leq j \leq r$. By using $\tau: \mathcal{V}(\mathcal{G}) \rightarrow \{0, 1, \dots, n-1\}$,

$$\tau(v_{i(r+1)+j}) = i + \ell j. \quad (28)$$

For $i = 0, 0 \leq j \leq r-1$, and $1 \leq d \leq \ell/2$ we have

$$\begin{aligned} N^d(v_j) = & \{v_{(d-1)(r+1)+j+1}, v_{(d-1)(r+1)+j+2}, \dots, \\ & \cdot v_{(d-1)(r+1)+j+r}, v_{((\ell-1)+(d-1))(r+1)+j+1}, \\ & \cdot v_{((\ell-1)+(d-1))(r+1)+j+2}, \dots, v_{((\ell-1)+(d-1))(r+1)+j+r}\}. \end{aligned} \quad (29)$$

Also, for $1 \leq i \leq \ell-1, 0 \leq j \leq r-1$, and $1 \leq d \leq \ell/2$, we have

$$\begin{aligned} N^d(v_{i(r+1)+j}) = & \{v_{(i+d-1)(r+1)+j+1}, v_{(i+d-1)(r+1)+j+2}, \dots, \\ & \cdot v_{(i+d-1)(r+1)+j+r}, v_{((i-1)+(d-1))(r+1)+j+1}, \\ & \cdot v_{((i-1)+(d-1))(r+1)+j+2}, \dots, \\ & \cdot v_{((i-1)+(d-1))(r+1)+j+r}\}. \end{aligned} \quad (30)$$

The function τ is injective, and the values assigned to $\mathcal{V}(\mathcal{G})$ is from $\{0, 1, 2, \dots, n-1\}$. So, for every $e = uv \in \mathcal{E}(\mathcal{G})$, the difference of the labels $|\tau(u) - \tau(v)|$ gets one of the values in the set $\{1, 2, \dots, n-1\}$. By the function τ , for every $u \in \mathcal{V}(\mathcal{G})$ and $v \in N^d(u)$, we get the difference values $|\tau(u) - \tau(v)|$ in the set $\{\ell + d - 1, 2\ell + d - 1, \dots, r\ell + d - 1, \ell - 1 +$

FIGURE 3: $\lambda_3(\mathcal{G}) = 19$.

$r\ell + d - 1, \ell - 1 + (r-1)\ell + d - 1, \ell - 1 + (r-2)\ell + d - 1, \dots, \ell - 1 + \ell + d - 1\}$. Since $n \geq 2(r+1) + 1$, $n/(r+1) = \ell$ which implies $\ell \geq 2$. For $\varphi = 1$ and $\varphi = 2$, the difference $|\tau(u) - \tau(v)| \geq \varphi + 1 - \mathfrak{d}(u, v)$, for all $u, v \in \mathcal{V}(\mathcal{G})$. Hence, the function τ is a $\mathbb{L}'(2, 1)$ -labeling with $\lambda_{(2,1)}'(\mathcal{G}) = n-1$. $\square$

Theorem 5. Let $n, r \in \mathbb{Z}^+$ with $1 \leq r < \lfloor n/2 \rfloor$. If $n \geq 2(r+1) + 1$ and $(r+1) \mid n$ with $\gcd(n, r+1) = 1$, then the circulant graph $\text{Cir}(n, \Lambda)$, $\Lambda = \{1, 2, \dots, r, n-1, n-2, \dots, n-r\}$, admits $\mathbb{L}'(2, 1)$ labeling with $\lambda_{(2,1)}'(\mathcal{G}) = n-1$.

Proof. Let $n, \ell \in \mathbb{Z}^+$ and $\text{Cir}(n, \Lambda)$ be the circulant graph with $n \geq 2(r+1) + 1$, where $1 \leq r < \lfloor n/2 \rfloor$ and $\ell = \lfloor n/(r+1) \rfloor$. By the division algorithm, for every vertex $v \in \mathcal{V}(\mathcal{G})$, we can write $v = v_{i(r+1) \bmod n}$, where $0 \leq i \leq \ell-1, 0 \leq j \leq r$. Let $\tau: \mathcal{V}(\mathcal{G}) \rightarrow \{0, 1, \dots, n-1\}$ by

$$\tau(v_{i(r+1) \bmod n}) = i \text{ for } i = 0, 1, \dots, n-1. \quad (31)$$

For $0 \leq i \leq n-1$, we have

$$\begin{aligned} N(v_{i(r+1) \bmod n}) = & \{v_{(i(r+1) \bmod n)+1}, v_{(i(r+1) \bmod n)+2}, \dots, \\ & \cdot v_{(i(r+1) \bmod n)+r}, v_{(i(r+1) \bmod n)+n-1}, \\ & \cdot v_{(i(r+1) \bmod n)+n-2}, \dots, v_{(i(r+1) \bmod n)+n-r}\}. \end{aligned} \quad (32)$$

Since $\gcd(n, r+1) = 1$, then the function τ is injective, and the values assigned to $\mathcal{V}(\mathcal{G})$ is from $\{0, 1, 2, \dots, n-1\}$. So, for every $e = uv \in \mathcal{E}(\mathcal{G})$, the difference of the labels $|\tau(u) - \tau(v)|$ gets one of the values in the set $\{1, 2, \dots, n-1\}$. By the function τ , for every $u \in \mathcal{V}(\mathcal{G})$, $v \in N^d(u)$, and $1 \leq d \leq \ell/2$, we get the difference values $|\tau(u) - \tau(v)|$ in the set $\{\ell + d - 1, 2\ell + d - 1, \dots, r\ell + d - 1, \ell - 1 + r\ell + d - 1, \ell - 1 + (r-1)\ell + d - 1, \ell - 1 + (r-2)\ell + d - 1, \dots, \ell - 1 + \ell + d - 1\}$. Since $n \geq 2(r+1) + 1$, $n/(r+1) = \ell$ which implies $\ell \geq 2$. For

$\wp = 1$ and $\wp = 2$, the difference $|\tau(u) - \tau(v)| \geq \wp + 1 - \mathfrak{d}(u, v)$ for all $u, v \in V(\mathcal{G})$. Hence, the function τ is a $L(2, 1)$ -labeling with $\lambda(2, 1)_\ell(\mathcal{G}) = n - 1$. $\square$

Theorem 6. Let $n, \wp, r \in \mathbb{Z}^+$ with $1 \leq r < \lfloor n/2 \rfloor$. If $n \geq \wp((\wp - 1)r + 1) - r(\wp - 3)$, for $\wp \geq 3$ and $((\wp - 1)r + 1) | n$, then the circulant graph $\text{Cir}(n, \Lambda)$, $\Lambda = \{1, 2, \dots, r, n - 1, n - 2, \dots, n - r\}$, admits $L'(\wp, \wp - 1, \dots, 2, 1)$ labeling with $\lambda(\wp, \wp - 1, \dots, 2, 1)_\ell(\mathcal{G}) = n - 1$.

Proof. Let $n, \wp, \ell \in \mathbb{Z}^+$ and $\mathcal{G} = \text{Cir}(n, \Lambda)$ with $n \geq \wp((\wp - 1)r + 1) - r(\wp - 3)$, where $1 \leq r < \lfloor n/2 \rfloor$. $(\wp - 1)r + 1$ divides n .

Take $\ell = n/((\wp - 1)r + 1)$. By the division algorithm, for every vertex $v \in V(\mathcal{G})$, we can write $v = v_{i((\wp - 1)r + 1) + j}$, where $0 \leq i \leq \ell - 1, 0 \leq j \leq r$. Let $\tau: V(\mathcal{G}) \rightarrow \{0, 1, \dots, n - 1\}$ by

$$\tau(v_{i((\wp - 1)r + 1) + j}) = i + \ell j. \quad (33)$$

For $i = 0, 0 \leq j \leq r - 1$, and $1 \leq d \leq \ell/2$, we have

$$\begin{aligned} N^d(v_j) = & \{v_{(d-1)((\wp-1)r+1)+j+1}, v_{(d-1)((\wp-1)r+1)+j+2}, \dots, \\ & \cdot v_{(d-1)((\wp-1)r+1)+j+r}, v_{((\ell-1)+(d-1))((\wp-1)r+1)+j+1}, \\ & \cdot v_{((\ell-1)+(d-1))((\wp-1)r+1)+j+2}, \dots, \\ & \cdot v_{((\ell-1)+(d-1))((\wp-1)r+1)+j+r}\}. \end{aligned} \quad (34)$$

Also, for $1 \leq i \leq \ell - 1, 0 \leq j \leq r - 1$, and $1 \leq d \leq \ell/2$, we have

$$\begin{aligned} N^d(v_{i((\wp-1)r+1)+j}) = & \{v_{(i+d-1)((\wp-1)r+1)+j+1}, \\ & \cdot v_{(i+d-1)((\wp-1)r+1)+j+2}, \dots, \\ & \cdot v_{(i+d-1)((\wp-1)r+1)+j+r} \\ & \cdot v_{((i-1)+(d-1))((\wp-1)r+1)+j+1}, \\ & \cdot v_{((i-1)+(d-1))((\wp-1)r+1)+j+2}, \dots, \\ & \cdot v_{((i-1)+(d-1))((\wp-1)r+1)+j+r}\}. \end{aligned} \quad (35)$$

The function τ is injective, and the values assigned to $v(\mathcal{G})$ is from $\{0, 1, 2, \dots, n - 1\}$. So, for every $e = uv \in \mathcal{E}(\mathcal{G})$, the difference of the labels $|\tau(u) - \tau(v)|$ gets one of the values in the set $\{1, 2, \dots, n - 1\}$. By the function τ , for every $u \in V(\mathcal{G})$ and $v \in N^d(u)$, we get the difference values $|\tau(u) - \tau(v)|$ in the set $\{\ell + d - 1, 2\ell + d - 1, \dots, r\ell + d - 1, \ell - 1 + r\ell + d - 1, \ell - 1 + (r - 1)\ell + d - 1, \dots, \ell - 1 + (r - 2)\ell + d - 1, \dots, \ell - 1 + \ell + d - 1\}$. Since $n \geq \wp((\wp - 1)r + 1) - r(\wp - 3)$, $n/((\wp - 1)r + 1) = \ell$ which implies $\ell \geq \wp$. For $\wp \geq 3$, the difference $|\tau(u) - \tau(v)| \geq \wp + 1 - \mathfrak{d}(u, v)$, for all $u, v \in V(\mathcal{G})$. Hence, the function τ is a $L'(\wp, \wp - 1, \dots, 1)$ -labeling for $\wp \geq 3$ with $\lambda(\wp, \wp - 1, \dots, 2, 1)_\ell(\mathcal{G}) = n - 1$. $\square$

4. Conclusion

We show in this study that the circulant graphs with specific generating sets are allowed m - $L(\wp, \wp - 1, \dots, 2, 1)$ labeling and $L(\wp, \wp - 1, \dots, 2, 1)$ -labeling. Also, we identified the

minimum span of m - $L(\wp, \wp - 1, \dots, 2, 1)$ labeling. However, a major challenge of the problems m - $L(\wp, \wp - 1, \dots, 2, 1)$ labeling and $L'(\wp, \wp - 1, \dots, 1)$ labeling with minimum span for the arbitrary circulant graphs remains under investigation. Another future work is addressed to find m - $L(\wp, \wp - 1, \dots, 1)$ and $L'(\wp, \wp - 1, \dots, 2, 1)$ labeling with minimum span for the Cayley graphs.

Data Availability

No data were used to support the findings of the study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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