Research Article

Developing Reverse Order Law for the Moore–Penrose Inverse with the Product of Three Linear Operators

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In this paper, we study the reverse order law for the Moore–Penrose inverse of the product of three bounded linear operators in Hilbert spaces. We first present some equivalent conditions for the existence of the reverse order law $(ABC)^{\dagger} = C^{\dagger}B^{\dagger}A^{\dagger}$. Moreover, several equivalent statements of $\mathcal{R}(AA^*(ABC)) = \mathcal{R}(ABC)$ and $\mathcal{R}(C^*C(ABC)^*) = \mathcal{R}((ABC)^*)$ are also deduced by the theory of operators.

1. Introduction

Let $\mathcal{L}(X, Y)$ denote the set of all bounded linear operators from Hilbert space X to Hilbert space Y . Especially, it is briefly written as $\mathcal{L}(X)$ when $Y = X$. For $A \in \mathcal{L}(X, Y)$, the symbols $\mathcal{N}(A)$, $\mathcal{R}(A)$, and $A^{\dagger}$ denote null space, range, and Moore–Penrose inverse of A , respectively.

The Moore–Penrose inverse of $A \in \mathcal{L}(X, Y)$ is the operator $A^{\dagger} \in \mathcal{L}(Y, X)$ that satisfies the Penrose equations, i.e.,

$$\begin{aligned} AA^{\dagger}A &= A; \\ A^{\dagger}AA^{\dagger} &= A^{\dagger}; \\ (AA^{\dagger})^* &= AA^{\dagger}; \\ (A^{\dagger}A)^* &= A^{\dagger}A. \end{aligned} \quad (1)$$

If the Moore–Penrose inverse $A^{\dagger}$ of A exists, then it is unique. $A^{\dagger}$ exists if and only if $\mathcal{R}(A)$ is closed in Y (see [1, 18] for details).

The reverse order law for the generalized inverse of a product of matrices or bounded linear operators yields a class of interesting fundamental problems in the theory of

the generalized inverse of matrices or operators. It has been widely studied since the middle 1960s. Greville [7] first discussed the reverse order law for the Moore–Penrose inverse of the product of two matrices $(AB)^{\dagger} = B^{\dagger}A^{\dagger}$ and gave a necessary and sufficient condition for the law. Hartwig et al. [8] studied the reverse order law for the Moore–Penrose inverse of the product of three matrices and derived the equivalent conditions for the existence of the reverse order law. Ji and Wei [10] studied the weighted Moore–Penrose inverse and extended the fundamental theorem of linear algebra for matrix spaces to tensor spaces. Recently, Panigrahy and Mishra presented some properties for the weighted Moore–Penrose inverse for an arbitrary order tensor via the Einstein product [12]; they also proposed the expression for the Moore–Penrose inverse of the product of two tensors via the Einstein product [13, 14]. Until now, some reverse order laws for the known generalized inverses have been widely studied and extended into a complex number field, a Hilbert space, or an associative ring (see [2, 3, 5, 6, 9, 11, 15–17, 19–23] and references therein).

Recently, based on the results in [8], Dinčić and Djordjević [4] extended some finite dimensional results to infinite dimensional settings and obtained new results related to the mixed-type reverse order law for the

Moore–Penrose inverse of various products of three operators in Hilbert space. This motivates us to have a further study concerning the reverse order law of three operators in Hilbert space and to extend some results in [4, 8].

In the remainder of this section, we will give some denotations. In Section 2, we will present several lemmas. In Section 3, we will investigate some equivalent conditions for the existence of $(ABC)^\dagger = C^\dagger B^\dagger A^\dagger$. Moreover, several equivalent statements of $\mathcal{R}(AA^*(ABC)) = \mathcal{R}(ABC)$ and $\mathcal{R}(C^*C(ABC)^*) = \mathcal{R}((ABC)^*)$ are also deduced by the theory of operators.

Now, we describe several denotations. If $P, Q \in \mathcal{L}(X)$, then the symbol $[P, Q]$ stands for the commutator of P and Q , i.e., $[P, Q] \stackrel{\text{def}}{=} PQ - QP$. Also, for $A \in \mathcal{L}(X, Y)$, the symbols $A^{-*} \stackrel{\text{def}}{=} (A^*)^{-1}$ and $A^{*\dagger} \stackrel{\text{def}}{=} (A^*)^\dagger$ are obtained. Throughout this paper, we always assume that $A \in \mathcal{L}(X_2, X_1)$, $B \in \mathcal{L}(X_3, X_2)$, and $C \in \mathcal{L}(X_4, X_3)$, where $X_i, i = 1, \dots, 4$, denote arbitrary Hilbert spaces. And then, ABC will be well-defined in what follows.

2. Lemmas

First of all, we list two lemmas, which come from [4, 6], and the first is a rewritten version of Lemmas 1.1 and 1.2 in [6].

Lemma 2.1 (see [6]). *Let $K \in \mathcal{L}(X, Y)$ have a closed range. Let X_1 and X_2 be closed and mutually orthogonal subspaces of X such that $X = X_1 \oplus X_2$ (here, $\oplus$ represents direct sum). Let Y_1 and Y_2 be closed and mutually orthogonal subspaces of Y such that $Y = Y_1 \oplus Y_2$. Then, the operator K has the following matrix representations with respect to the orthogonal sums of subspaces $X = X_1 \oplus X_2 = \mathcal{R}(K^*) \oplus \mathcal{N}(K)$ and $Y = \mathcal{R}(K) \oplus \mathcal{N}(K^*) = Y_1 \oplus Y_2$:*

(a) *Representations with respect to the orthogonal sums of subspaces $X = X_1 \oplus X_2 = \mathcal{R}(K^*) \oplus \mathcal{N}(K)$ are*

$$K = \begin{bmatrix} K_1 & K_2 \\ 0 & 0 \end{bmatrix} : \begin{bmatrix} X_1 \\ X_2 \end{bmatrix} \longrightarrow \begin{bmatrix} \mathcal{R}(K) \\ \mathcal{N}(K^*) \end{bmatrix}, \quad (2)$$

where $D_K = K_1 K_1^* + K_2 K_2^*$ maps $\mathcal{R}(K)$ into itself and $D_K > 0$ (meaning $D_K > 0$ is invertible). Also,

$$K^\dagger = \begin{bmatrix} K_1^* D_K^{-1} & 0 \\ K_2^* D_K^{-1} & 0 \end{bmatrix}, \quad (3)$$

$$KK^\dagger = \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix}.$$

(b) *Representations with respect to the orthogonal sums of subspaces $Y = \mathcal{R}(K) \oplus \mathcal{N}(K^*) = Y_1 \oplus Y_2$ are*

$$K = \begin{bmatrix} K_1 & 0 \\ K_2 & 0 \end{bmatrix} : \begin{bmatrix} \mathcal{R}(K^*) \\ \mathcal{N}(K) \end{bmatrix} \longrightarrow \begin{bmatrix} Y_1 \\ Y_2 \end{bmatrix}, \quad (4)$$

where $D_K = K_1^* K_1 + K_2^* K_2$ maps $\mathcal{R}(K^*)$ into itself and $D_K > 0$ (meaning $D_K > 0$ is invertible). Also,

$$K^\dagger = \begin{bmatrix} D_K^{-1} K_1^* & D_K^{-1} K_2^* \\ 0 & 0 \end{bmatrix}, \quad (5)$$

$$K^\dagger K = \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix}.$$

Here, $K_i, i = 1, 2$, denote different operators in any of these two cases.

In particular,

$$K = \begin{bmatrix} K_1 & 0 \\ 0 & 0 \end{bmatrix} : \begin{bmatrix} \mathcal{R}(K^*) \\ \mathcal{N}(K) \end{bmatrix} \longrightarrow \begin{bmatrix} \mathcal{R}(K) \\ \mathcal{N}(K^*) \end{bmatrix}, \quad (6)$$

where K_1 is invertible. Also,

$$K^\dagger = \begin{bmatrix} K_1^{-1} & 0 \\ 0 & 0 \end{bmatrix} : \begin{bmatrix} \mathcal{R}(K) \\ \mathcal{N}(K^*) \end{bmatrix} \longrightarrow \begin{bmatrix} \mathcal{R}(K^*) \\ \mathcal{N}(K) \end{bmatrix}. \quad (7)$$

Lemma 2.2 (see [4]). *Let X and Y be Hilbert spaces. Let $B \in \mathcal{L}(X, Y)$ have a closed range, and let $H \in \mathcal{L}(Y)$ be Hermitian and invertible. Then, $\mathcal{R}(HB) = \mathcal{R}(B)$ if and only if $[H, BB^\dagger] = 0$.*

Next, the following lemma can guarantee the existence of the Moore–Penrose inverses of various operators appearing in what follows.

Lemma 2.3. *Let A, C , and ABC have closed ranges. Then $(A^*A)^\dagger, (CC^*)^\dagger, (A^\dagger ABC)^\dagger, (A^* ABC)^\dagger, (ABCC^\dagger)^\dagger, (ABCC^*)^\dagger, (A^\dagger ABCC^*)^\dagger, (A^* ABCC^\dagger)^\dagger, (A^* ABCC^*)^\dagger$, and $(A^\dagger ABCC^\dagger)^\dagger$ exist.*

Proof. Since $\mathcal{R}(C) = \mathcal{R}(CC^\dagger) = \mathcal{R}(CC^*)$, $\mathcal{R}(ABCC^\dagger) = \mathcal{R}(ABCC^*) = \mathcal{R}(ABC)$. Furthermore, we have $(CC^*)^\dagger, (ABCC^\dagger)^\dagger$, and $(ABCC^*)^\dagger$ exist since C and ABC have closed ranges. Then,

$$\begin{aligned} \mathcal{R}(CC^* B^* A^* A^{\dagger*}) &= \mathcal{R}(CC^* B^* A^*) \\ &\Leftrightarrow \mathcal{R}(A^\dagger ABCC^*) \text{ is closed} \\ &\Leftrightarrow \mathcal{R}(ABCC^*) = \mathcal{R}(ABC) \text{ is closed.} \end{aligned} \quad (8)$$

Hence, $(A^\dagger ABCC^*)^\dagger$ exists.

Similarly, we can obtain $(A^*A)^\dagger, (A^\dagger ABC)^\dagger, (A^* ABC)^\dagger, (A^\dagger ABCC^\dagger)^\dagger, (A^* ABCC^\dagger)^\dagger, (A^* ABCC^*)^\dagger$, and $(A^\dagger ABCC^\dagger)^\dagger$ exist.

Finally, the following two lemmas are used for proving some results in the next section. Here, the symbol $[A, B] = AB - BA$, which denotes the commutator of A and B . $\square$

Lemma 2.4. *Let A and C be invertible and B have a closed range and $(AB)^\dagger, (BC)^\dagger$, and $(ABC)^\dagger$ exist. Then,*

(i) $(AB)^\dagger = B^\dagger A^{-1}$ if and only if $[A^*A, BB^\dagger] = 0$ if and only if $[(A^*A)^{-1}, BB^\dagger] = 0$ if and only if $[AA^*, AB(AB)^\dagger] = 0$

- (ii) $(BC)^\dagger = C^{-1}B^\dagger$ if and only if $[CC^*, B^\dagger B] = 0$ if and only if $[(CC^*)^{-1}, B^\dagger B] = 0$ if and only if $[C^*C, BC(BC)^\dagger] = 0$
- (iii) $(ABC)^\dagger = C^{-1}B^\dagger A^{-1}$ if and only if $[A^*A, BB^\dagger] = 0$ and $[CC^*, B^\dagger B] = 0$ if and only if $[(A^*A)^{-1}, BB^\dagger] = 0$ and $[(CC^*)^{-1}, B^\dagger B] = 0$

Proof. Obviously, $\mathcal{R}(BC) = \mathcal{R}(B)$ due to the invertibility of C . Then, BC has a closed range since $\mathcal{R}(B)$ is closed. Hence, $(BC)^\dagger$ exists. So does $(AB)^*$. And then, $(AB)^\dagger$ exists. Similarly, $(ABC)^\dagger$ exists. Hence,

- (i) Clearly $B^\dagger A^{-1}(AB)B^\dagger A^{-1} = B^\dagger A^{-1}$, $(AB)B^\dagger A^{-1}(AB) = AB$, and $B^\dagger A^{-1}(AB) = (B^\dagger A^{-1}(AB))^*$.

Note that $(ABB^\dagger A^{-1})^* = ABB^\dagger A^{-1}$ holds if and only if $BB^\dagger A^*A = A^*ABB^\dagger$, namely, $[A^*A, BB^\dagger] = 0$, which is equivalent to $[(A^*A)^{-1}, BB^\dagger] = 0$. Hence, the proposition is true by the definition of the Moore–Penrose inverse.

The proofs of (ii) and (iii) are similar. $\square$

Lemma 2.5. Let A and C be Hermitian and invertible and B have a closed range. Then, for arbitrary rational numbers n and p , we have the following properties:

- (i) $[A, (A^{n+p}B)(A^{n+p}B)^\dagger] = 0$ if and only if $[A, (A^n B)(A^n B)^\dagger] = 0$
- (ii) $[C, (BC^{n+p})^\dagger (BC^{n+p})] = 0$ if and only if $[C, (BC^n)^\dagger (BC^n)] = 0$
- (iii) $[A, BB^\dagger] = 0$ implies $[A^2, BB^\dagger] = 0$
- (iv) $[C, B^\dagger B] = 0$ implies $[C^2, B^\dagger B] = 0$

Proof. For arbitrary rational numbers k , $(A^k B)^\dagger$ and $(BC^k)^\dagger$ exist by Lemma 2.4, since A^k and C^k are obviously invertible.

- (i) By Lemma 2.2, $[A, (A^{n+p}B)(A^{n+p}B)^\dagger] = 0$ if and only if $\mathcal{R}(A^{n+p}B) = \mathcal{R}(A(A^{n+p}B))$ if and only if $A^{-p}\mathcal{R}(A^{n+p}B) = A^{-p}\mathcal{R}(A(A^{n+p}B))$, i.e., $\mathcal{R}(A^n B) = \mathcal{R}(A(A^n B))$ if and only if $[A, (A^n B)^\dagger (A^n B)] = 0$.
- (iii) By Lemma 2.2, $[A, BB^\dagger] = 0$ leads to $\mathcal{R}(AB) = \mathcal{R}(B)$. Thus, $\mathcal{R}(A^2 B) = A\mathcal{R}(B) = \mathcal{R}(B)$ and therefore $[A^2, BB^\dagger] = 0$.

The proofs of (ii) and (iv) are similar. $\square$

Remark 1. If matrix $A \in R^n$ is Hermitian and invertible, let $\lambda_1, \dots, \lambda_n$ be eigenvalues for A ; then, the matrix A is similar to a diagonal matrix. And, it can be written as follows:

$$A = Q\Lambda Q^T = Q \begin{bmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_n \end{bmatrix} Q^T, \quad (9)$$

where Q is a unitary matrix. Therefore, we have

$$A^p = Q\Lambda^p Q^T = Q \begin{bmatrix} \lambda_1^p & 0 & \cdots & 0 \\ 0 & \lambda_2^p & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_n^p \end{bmatrix} Q^T, \quad (10)$$

where p denotes arbitrary rational numbers.

3. Reverse Order Laws

In this section, we will prove the result concerning the reverse order law for the Moore–Penrose inverse of various products of bounded linear operators in Hilbert spaces.

Firstly, we shall show some equivalent conditions for the existence of the reverse order law $(ABC)^\dagger = C^\dagger B^\dagger A^\dagger$, which extends Theorem 1 in [8].

Theorem 3.1. Let A, B, C , and ABC have closed ranges. ien, the following statements are equivalent:

- (i) $(ABC)^\dagger = C^\dagger B^\dagger A^\dagger$
- (ii) $(A^\dagger ABCC^\dagger)^\dagger = CC^\dagger B^\dagger A^\dagger A$, $A^* ABCC^\dagger B^\dagger A^\dagger A$, and $CC^\dagger B^\dagger A^\dagger ABCC^*$ are Hermitian
- (iii) $(A^* ABCC^\dagger)^\dagger = CC^\dagger B^\dagger A^\dagger A^*$, $A^* ABCC^\dagger B^\dagger A^\dagger A$, and $CC^\dagger B^\dagger A^\dagger ABCC^*$ are Hermitian
- (iv) $(A^\dagger ABCC^*)^\dagger = C^* C^\dagger B^\dagger A^\dagger A$, $A^* ABCC^\dagger B^\dagger A^\dagger A$, and $CC^\dagger B^\dagger A^\dagger ABCC^*$ are Hermitian
- (v) $(A^* ABCC^*)^\dagger = C^* C^\dagger B^\dagger A^\dagger A^*$, $A^* ABCC^\dagger B^\dagger A^\dagger A$, and $CC^\dagger B^\dagger A^\dagger ABCC^*$ are Hermitian

Proof. By Lemma 2.3, the Moore–Penrose inverses $(ABC)^\dagger$, $(A^\dagger ABCC^\dagger)^\dagger$, $(A^* ABCC^\dagger)^\dagger$, and $(A^\dagger ABCC^*)^\dagger$ exist.

Set $K = C$ in Lemma 2.1(b). By Lemma 2.1(b),

$$C = \begin{bmatrix} C_1 & 0 \\ C_2 & 0 \end{bmatrix} : \begin{bmatrix} \mathcal{R}(K^*) \\ \mathcal{N}(K) \end{bmatrix} \longrightarrow \begin{bmatrix} Y_1 \\ Y_2 \end{bmatrix}, \quad (11)$$

where $D_C = C_1^* C_1 + C_2^* C_2$ maps $\mathcal{R}(K^*)$ into itself and $D_C > 0$,

$$\begin{aligned} CC^\dagger &= \begin{bmatrix} C_1 D_C^{-1} C_1^* & C_1 D_C^{-1} C_2^* \\ C_2 D_C^{-1} C_1^* & C_2 D_C^{-1} C_2^* \end{bmatrix}, \\ C^\dagger C &= \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix}. \end{aligned} \quad (12)$$

Set $K = B$ in Lemma 2.1(b). Then,

$$\begin{aligned} B &= \begin{bmatrix} B_1 & 0 \\ B_2 & 0 \end{bmatrix}, \\ ABC &= \begin{bmatrix} A_1 B_1 C_1 & 0 \\ 0 & 0 \end{bmatrix}, \\ (ABC)^\dagger &= \begin{bmatrix} (A_1 B_1 C_1)^\dagger & 0 \\ 0 & 0 \end{bmatrix}, \\ A^\dagger ABCC^* &= \begin{bmatrix} B_1 C_1 C_1^* & B_1 C_1 C_2^* \\ 0 & 0 \end{bmatrix}. \end{aligned} \quad (13)$$

Since $A^\dagger ABCC^*$ has a closed range, by Lemma 2.1, we can obtain that

$$\begin{aligned} D &= B_1 C_1 C_1^* (B_1 C_1 C_1^*)^* + B_1 C_1 C_2^* (B_1 C_1 C_2^*)^* \\ &= B_1 C_1 D_C C_1^* B_1^* \\ &= B_1 C_1 D_C^{1/2} D_C^{1/2} C_1^* B_1^* \\ &= (B_1 C_1 D_C^{1/2}) (B_1 C_1 D_C^{1/2})^*. \end{aligned} \quad (14)$$

Thus, D is invertible. Furthermore, we have

$$(B_1 C_1)^* D^{-1} = (D_C^{-1/2})^* (B_1 C_1 D_C^{1/2})^* D^{-1} = D_C^{-1/2} (B_1 C_1 D_C^{1/2})^\dagger, \quad (15)$$

and then,

$$(A^\dagger ABCC^*)^\dagger = \begin{bmatrix} (B_1 C_1 C_1^*)^* D^{-1} & 0 \\ (B_1 C_1 C_2^*)^* D^{-1} & 0 \end{bmatrix} = \begin{bmatrix} C_1 D_C^{-1/2} (B_1 C_1 D_C^{1/2})^\dagger & 0 \\ C_2 D_C^{-1/2} (B_1 C_1 D_C^{1/2})^\dagger & 0 \end{bmatrix}. \quad (16)$$

Similarly, we can obtain

$$(A^\dagger ABCC^\dagger)^\dagger = \begin{bmatrix} C_1 D_C^{-1/2} (B_1 C_1 D_C^{-1/2})^\dagger & 0 \\ C_2 D_C^{-1/2} (B_1 C_1 D_C^{-1/2})^\dagger & 0 \end{bmatrix}, \quad (17)$$

$$(A^* ABCC^\dagger)^\dagger = \begin{bmatrix} C_1 D_C^{-1/2} (A_1^* A_1 B_1 C_1 D_C^{-1/2})^\dagger & 0 \\ C_2 D_C^{-1/2} (A_1^* A_1 B_1 C_1 D_C^{-1/2})^\dagger & 0 \end{bmatrix}, \quad (18)$$

$$(A^* ABCC^*)^\dagger = \begin{bmatrix} C_1 D_C^{-1/2} (A_1^* A_1 B_1 C_1 D_C^{1/2})^\dagger & 0 \\ C_2 D_C^{-1/2} (A_1^* A_1 B_1 C_1 D_C^{1/2})^\dagger & 0 \end{bmatrix}. \quad (19)$$

In addition, since $A^* ABCC^\dagger B^\dagger A^\dagger A$ is Hermitian, we have

$$\begin{aligned} A^* ABCC^\dagger B^\dagger A^\dagger A &= \begin{bmatrix} A_1^* A_1 B_1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} C_1 D_C^{-1} C_1^* D_B^{-1} B_1^* & 0 \\ C_2 D_C^{-1} C_1^* D_B^{-1} B_1^* & 0 \end{bmatrix} \\ &= \begin{bmatrix} A_1^* A_1 B_1 C_1 D_C^{-1} C_1^* D_B^{-1} B_1^* & 0 \\ 0 & 0 \end{bmatrix}, \end{aligned} \quad (20)$$

$$A_1^* A_1 B_1 C_1 D_C^{-1} C_1^* D_B^{-1} B_1^* = (B_1 C_1 D_C^{-1} C_1^* D_B^{-1} B_1^*)^* A^* A. \quad (21)$$

And, $CC^\dagger B^\dagger A^\dagger ABCC^*$ is Hermitian, so

$$\begin{aligned} &\begin{bmatrix} C_1 D_C^{-1} C_1^* D_B^{-1} B_1^* B_1 C_1 C_1^* & C_1 D_C^{-1} C_1^* D_B^{-1} B_1^* B_1 C_1 C_2^* \\ C_2 D_C^{-1} C_1^* D_B^{-1} B_1^* B_1 C_1 C_1^* & C_2 D_C^{-1} C_1^* D_B^{-1} B_1^* B_1 C_1 C_2^* \end{bmatrix} \\ &= \begin{bmatrix} C_1 C_1^* B_1^* B_1 D_B^{-1} C_1 D_C^{-1} C_1^* & C_1 C_1^* B_1^* B_1 D_B^{-1} C_1 D_C^{-1} C_2^* \\ C_2 C_1^* B_1^* B_1 D_B^{-1} C_1 D_C^{-1} C_1^* & C_2 C_1^* B_1^* B_1 D_B^{-1} C_1 D_C^{-1} C_2^* \end{bmatrix}. \end{aligned} \quad (22)$$

Multiplying by C^* and C , respectively, on the left and right sides of equation (22) yields

$$D_C^{-1} C_1^* D_B^{-1} B_1^* B_1 C_1 = (C_1^* D_B^{-1} B_1^* B_1 C_1)^* D_C^{-1}. \quad (23)$$

(i) $\Leftrightarrow$ (ii): similar to the proof of Theorem 1 in [8].

(ii) $\Rightarrow$ (iii): by (12), we have

$$\begin{aligned} CC^\dagger B^\dagger A^\dagger A &= \begin{bmatrix} C_1 D_C^{-1} C_1^* & C_1 D_C^{-1} C_2^* \\ C_2 D_C^{-1} C_1^* & C_2 D_C^{-1} C_2^* \end{bmatrix} \begin{bmatrix} D_B^{-1} B_1^* & D_B^{-1} B_2^* \\ 0 & 0 \end{bmatrix} \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix} \\ &= \begin{bmatrix} C_1 D_C^{-1} C_1^* D_B^{-1} B_1^* & 0 \\ C_2 D_C^{-1} C_1^* D_B^{-1} B_1^* & 0 \end{bmatrix}. \end{aligned} \quad (24)$$

Then, $(A^\dagger ABCC^\dagger)^\dagger = CC^\dagger B^\dagger A^\dagger A$ implies $C^* (A^\dagger ABCC^\dagger)^\dagger = C^* CC^\dagger B^\dagger A^\dagger A$, and therefore, by (17) and (24),

$$(B_1 C_1 D_C^{-1/2})^\dagger = D_C^{-1/2} C_1^* D_B^{-1} B_1^*. \quad (25)$$

By (21) and (25),

$$\begin{aligned} (A_1^* A_1) B_1 C_1 D_C^{-1/2} (B_1 C_1 D_C^{-1/2})^\dagger &= (A_1^* A_1) B_1 C_1 D_C^{-1} C_1^* D_B^{-1} B_1^* \\ &= (B_1 C_1 D_C^{-1} C_1^* D_B^{-1} B_1^*)^* (A_1^* A_1) \\ &= (B_1 C_1 D_C^{-1/2} (B_1 C_1 D_C^{-1/2})^\dagger)^* (A_1^* A_1), \end{aligned} \quad (26)$$

and then,

$$(A_1^* A_1)^2 B_1 C_1 D_C^{-1/2} (B_1 C_1 D_C^{-1/2})^\dagger = B_1 C_1 D_C^{-1/2} (B_1 C_1 D_C^{-1/2})^\dagger (A_1^* A_1)^2. \quad (27)$$

As a result, by Lemma 2.4(i), we have

$$(A_1^* A_1 B_1 C_1 D_C^{-1/2})^\dagger = (B_1 C_1 D_C^{-1/2})^\dagger (A_1^* A_1)^{-1}. \quad (28)$$

Hence, by (18), (25), and (28), we can obtain

$$\begin{aligned} (A^* ABCC^\dagger)^\dagger &= \begin{bmatrix} C_1 D_C^{-1/2} (B_1 C_1 D_C^{-1/2})^\dagger (A_1^* A_1)^{-1} & 0 \\ C_2 D_C^{-1/2} (B_1 C_1 D_C^{-1/2})^\dagger (A_1^* A_1)^{-1} & 0 \end{bmatrix} \\ &= \begin{bmatrix} C_1 D_C^{-1} C_1^* D_B^{-1} B_1^* (A_1^* A_1)^{-1} & 0 \\ C_2 D_C^{-1} C_1^* D_B^{-1} B_1^* (A_1^* A_1)^{-1} & 0 \end{bmatrix} \\ &= \begin{bmatrix} C_1 D_C^{-1} C_1^* & C_1 D_C^{-1} C_2^* \\ C_2 D_C^{-1} C_1^* & C_2 D_C^{-1} C_2^* \end{bmatrix} \begin{bmatrix} D_B^{-1} B_1^* & D_B^{-1} B_2^* \\ 0 & 0 \end{bmatrix} \begin{bmatrix} (A_1^* A_1)^{-1} & 0 \\ 0 & 0 \end{bmatrix} \\ &= CC^\dagger B^\dagger A^\dagger A^{*\dagger}. \end{aligned} \quad (29)$$

(iii) $\Rightarrow$ (v): since

$$CC^\dagger B^\dagger A^\dagger A^{*\dagger} = \begin{bmatrix} C_1 D_C^{-1} C_1^* D_B^{-1} B_1^* (A_1^* A_1)^{-1} & 0 \\ C_2 D_C^{-1} C_1^* D_B^{-1} B_1^* (A_1^* A_1)^{-1} & 0 \end{bmatrix}, \quad (30)$$

$(A^* ABCC^\dagger)^\dagger = CC^\dagger B^\dagger A^\dagger A^{*\dagger}$ implies

$$(A_1^* A_1 B_1 C_1 D_C^{-1/2})^\dagger = D_C^{-1/2} C_1^* D_B^{-1} B_1^* (A_1^* A_1)^{-1}. \quad (31)$$

By (18), (24), and (31), we have

$$\begin{aligned} D_C^{-1} (A_1^* A_1 B_1 C_1 D_C^{-1/2})^\dagger A_1^* A_1 B_1 C_1 D_C^{-1/2} &= D_C^{-1/2} C_1^* D_B^{-1} B_1^* B_1 C_1 D_C^{-1/2} \\ &= D_C^{-1/2} (C_1^* D_B^{-1} B_1^* B_1 C_1)^* D_C^{-3/2} \\ &= \left((A_1^* A_1 B_1 C_1 D_C^{-1/2})^\dagger A_1^* A_1 B_1 C_1 D_C^{-1/2} \right)^* D_C^{-1} \end{aligned} \quad (32)$$

and $[D_C^{-2}, (A_1^* A_1 B_1 C_1 D_C^{-1/2})^\dagger A_1^* A_1 B_1 C_1 D_C^{-1/2}] = 0$.
Consequently, by Lemma 2.4(ii), we have

$$(A_1^* A_1 B_1 C_1 D_C^{-1/2})^\dagger = D_C^{-1} (A_1^* A_1 B_1 C_1 D_C^{-1/2})^\dagger. \quad (33)$$

Combined with (19), (31), and (33), we can further obtain

$$\begin{aligned} (A^* ABCC^*)^\dagger &= \begin{bmatrix} C_1 D_C^{-1/2} D_C^{-3/2} C_1^* D_B^{-1} B_1^* (A_1^* A_1)^{-1} & 0 \\ C_2 D_C^{-1/2} D_C^{-3/2} C_1^* D_B^{-1} B_1^* (A_1^* A_1)^{-1} & 0 \end{bmatrix} \\ &= \begin{bmatrix} C_1 D_C^{-1} & 0 \\ C_2 D_C^{-1} & 0 \end{bmatrix} \begin{bmatrix} D_C^{-1} C_1^* & D_C^{-1} C_2^* \\ 0 & 0 \end{bmatrix} \begin{bmatrix} D_B^{-1} B_1^* & D_B^{-1} B_2^* \\ 0 & 0 \end{bmatrix} \begin{bmatrix} (A_1^* A_1)^{-1} & 0 \\ 0 & 0 \end{bmatrix} \\ &= C^{*\dagger} C^\dagger B^\dagger A^\dagger A^{*\dagger}. \end{aligned} \quad (34)$$

(v) $\Rightarrow$ (iv): since

$$C^{*\dagger}C^\dagger B^\dagger A^\dagger A^{*\dagger} = \begin{bmatrix} C_1 D_C^{-2} C_1^* D_B^{-1} B_1^* (A_1^* A_1)^{-1} & 0 \\ C_2 D_C^{-2} C_1^* D_B^{-1} B_1^* (A_1^* A_1)^{-1} & 0 \end{bmatrix}, \quad (35)$$

using (19) and $(A^* ABCC^*)^\dagger = C^{*\dagger}C^\dagger B^\dagger A^\dagger A^{*\dagger}$, we can obtain

$$(A_1^* A_1 B_1 C_1 D_C^{1/2})^\dagger = D_C^{-3/2} C_1^* D_B^{-1} B_1^* (A_1^* A_1)^{-1}. \quad (36)$$

Combining (21) and (36), we have

$$\begin{aligned} & (A_1^* A_1)^{-1} A_1^* A_1 B_1 C_1 D_C^{1/2} (A_1^* A_1 B_1 C_1 D_C^{1/2})^\dagger \\ &= B_1 C_1 D_C^{-1} C_1^* D_B^{-1} B_1^* (A_1^* A_1)^{-1} \\ &= (A_1^* A_1)^{-1} (B_1 C_1 D_C^{-1} C_1^* D_B^{-1} B_1^*)^* \\ &= \left(A_1^* A_1 B_1 C_1 D_C^{1/2} (A_1^* A_1 B_1 C_1 D_C^{1/2})^\dagger \right)^* (A_1^* A_1)^{-1}, \end{aligned} \quad (37)$$

and then, $[(A_1^* A_1)^{-2}, A_1^* A_1 B_1 C_1 D_C^{1/2} (A_1^* A_1 B_1 C_1 D_C^{1/2})^\dagger] = 0$. Hence, by Lemma 2.4(i), we have

$$(A_1^* A_1 B_1 C_1 D_C^{1/2})^\dagger = (B_1 C_1 D_C^{1/2})^\dagger (A_1^* A_1)^{-1}, \quad (38)$$

and therefore, by (16), (38), and (36), we have

$$\begin{aligned} (A^\dagger ABCC^*)^\dagger &= \begin{bmatrix} C_1 D_2^{-1/2} (A_1^* A_1 B_1 C_1 D_C^{1/2})^\dagger (A_1^* A_1) & 0 \\ C_2 D_2^{-1/2} (A_1^* A_1 B_1 C_1 D_C^{1/2})^\dagger (A_1^* A_1) & 0 \end{bmatrix} \\ &= \begin{bmatrix} C_1 D_C^{-2} C_1^* D_B^{-1} B_1^* & 0 \\ C_2 D_C^{-2} C_1^* D_B^{-1} B_1^* & 0 \end{bmatrix} \\ &= \begin{bmatrix} C_1 D_C^{-1} & 0 \\ C_2 D_C^{-1} & 0 \end{bmatrix} \begin{bmatrix} D_C^{-1} C_1^* & D_C^{-1} C_2^* \\ 0 & 0 \end{bmatrix} \begin{bmatrix} D_B^{-1} B_1^* & D_B^{-1} B_2^* \\ 0 & 0 \end{bmatrix} \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix} \\ &= C^{*\dagger} C^\dagger B^\dagger A^\dagger A. \end{aligned} \quad (39)$$

(iv) $\Rightarrow$ (ii): since

$$C^{*\dagger}C^\dagger B^\dagger A^\dagger A = \begin{bmatrix} C_1 D_C^{-2} C_1^* D_B^{-1} B_1^* & 0 \\ C_2 D_C^{-2} C_1^* D_B^{-1} B_1^* & 0 \end{bmatrix}, \quad (40)$$

by (16), $(A^\dagger ABCC^*)^\dagger = C^{*\dagger}C^\dagger B^\dagger A^\dagger A$ implies

$$(B_1 C_1 D_C^{1/2})^\dagger = D_C^{-3/2} C_1^* D_B^{-1} B_1^*. \quad (41)$$

Combining (23) and (41), we have

$$\begin{aligned} D_C (B_1 C_1 D_C^{1/2})^\dagger B_1 C_1 D_C^{1/2} &= D_C^{1/2} C_1^* D_B^{-1} B_1^* B_1 C_1 D_C^{1/2} \\ &= D_C^{1/2} (C_1^* D_B^{-1} B_1^* B_1 C_1)^* D_C^{-1/2} \\ &= \left[(B_1 C_1 D_C^{1/2})^\dagger B_1 C_1 D_C^{1/2} \right]^* D_C. \end{aligned} \quad (42)$$

and then, by Lemma 2.4(ii), we can obtain that

$$(B_1 C_1 D_C^{-1/2})^\dagger = D_C (B_1 C_1 D_C^{1/2})^\dagger. \quad (43)$$

Accordingly, by (17), (41), and (43), we have

$$\begin{aligned} (A^\dagger ABCC^*)^\dagger &= \begin{bmatrix} C_1 D_C^{1/2} (B_1 C_1 D_C^{1/2})^\dagger & 0 \\ C_2 D_C^{1/2} (B_1 C_1 D_C^{1/2})^\dagger & 0 \end{bmatrix} \\ &= \begin{bmatrix} C_1 D_C^{-1} C_1^* D_B^{-1} B_1^* & 0 \\ C_2 D_C^{-1} C_1^* D_B^{-1} B_1^* & 0 \end{bmatrix} \\ &= \begin{bmatrix} C_1 D_C^{-1} C_1^* & C_1 D_C^{-1} C_2^* \\ C_2 D_C^{-1} C_1^* & C_2 D_C^{-1} C_2^* \end{bmatrix} \begin{bmatrix} D_B^{-1} B_1^* & D_B^{-1} B_2^* \\ 0 & 0 \end{bmatrix} \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix} \\ &= CC^\dagger B^\dagger A^\dagger A. \end{aligned} \quad (44)$$

In [4], Dinčić and Djordjević showed some equivalent statements of $\mathcal{R}[AA^*(ABC)] = \mathcal{R}(ABC)$ and $\mathcal{R}[C^*C(ABC)^*] = \mathcal{R}((ABC)^*)$. Here, we will present other equivalent statements. $\square$

Theorem 3.2. *Let A, C , and ABC have closed ranges. Then, the following statements are equivalent:*

- (i) $(A^\dagger ABCC^*)^\dagger = C^{*\dagger} (ABC)^\dagger A$
- (ii) $(A^* ABCC^\dagger)^\dagger = C (ABC)^\dagger A^{*\dagger}$
- (iii) $C^\dagger (A^\dagger ABCC^*)^\dagger A^* = (C^* C)^\dagger (ABC)^\dagger AA^*$
- (iv) $C^* (A^* ABCC^\dagger)^\dagger A^\dagger = C^* C (ABC)^\dagger (AA^*)^\dagger$

- (v) $(ABC)^\dagger = C^*(A^\dagger ABCC^*)^\dagger A^\dagger$
 (vi) $(ABC)^\dagger = C^\dagger (A^* ABCC^\dagger)^\dagger A^*$
 (vii) $\mathcal{R}[AA^*(ABC)] = \mathcal{R}(ABC)$ and $\mathcal{R}[C^*C(ABC)^*] = \mathcal{R}((ABC)^*)$

Proof. By Lemma 2.3, we can obtain that the Moore–Penrose inverse $(ABC)^\dagger$, $(A^\dagger ABCC^*)^\dagger$, and $(A^* ABCC^\dagger)^\dagger$ exist.

(i) $\Rightarrow$ (iii): multiplying $C^\dagger$ and A^* on the left and the right side of (i), respectively, we then have

$$C^\dagger (A^\dagger ABCC^*)^\dagger A^* = (C^*C)^\dagger (ABC)^\dagger AA^*, \quad (45)$$

where $(C^*C)^\dagger = C^\dagger C^*$.

(iii) $\Rightarrow$ (v): multiplying C^*C and $(AA^*)^\dagger$ on the left and the right side of (iii), respectively, we thus have

$$C^*CC^\dagger (A^\dagger ABCC^*)^\dagger A^* (AA^*)^\dagger = C^*C(C^*C)^\dagger (ABC)^\dagger AA^* (AA^*)^\dagger, \quad (46)$$

namely,

$$C^* (A^\dagger ABCC^*)^\dagger A^\dagger = C^\dagger C (ABC)^\dagger AA^\dagger. \quad (47)$$

Thus, by (12), (v) obviously holds.

(v) $\Rightarrow$ (i): multiplying $C^{*\dagger}$ and A on the left and the right side of (v), respectively, then we have

$$C^{*\dagger} (ABC)^\dagger A = CC^\dagger (A^\dagger ABCC^*)^\dagger A^\dagger A. \quad (48)$$

By (12) and (16), $CC^\dagger (A^\dagger ABCC^*)^\dagger = (A^\dagger ABCC^*)^\dagger$ and then (i) easily holds.

(v) $\Leftrightarrow$ (vii): by (16), we have

$$C^* (A^\dagger ABCC^*)^\dagger A^\dagger = \begin{bmatrix} D_C^{1/2} (B_1 C_1 D_C^{1/2})^\dagger A_1^{-1} & 0 \\ 0 & 0 \end{bmatrix}. \quad (49)$$

Therefore, (v) is equivalent to

$$(B_1 C_1 D_C^{1/2})^\dagger = [A_1^{-1} (A_1 B_1 C_1) D_C^{1/2}]^\dagger = D_C^{-1/2} (A_1 B_1 C_1)^\dagger A_1. \quad (50)$$

By Lemma 2.4(iii), we can obtain that (50) holds if and only if the following equations are satisfied:

$$\begin{aligned} (A_1 B_1 C_1) (A_1 B_1 C_1)^\dagger A_1 A_1^* &= A_1 A_1^* (A_1 B_1 C_1) (A_1 B_1 C_1)^\dagger, \\ D_C (A_1 B_1 C_1)^\dagger (A_1 B_1 C_1) &= (A_1 B_1 C_1)^\dagger (A_1 B_1 C_1) D_C. \end{aligned} \quad (51)$$

Namely,

$$\begin{aligned} [AA^*, (ABC)(ABC)^\dagger] &= 0, \\ [C^*C, (ABC)^\dagger (ABC)] &= 0. \end{aligned} \quad (52)$$

On the other hand, by Lemma 2.2, (v) is equivalent to (vii).

(vi) $\Leftrightarrow$ (vii): the equation $(A_1^* A_1 B_1 C_1 D_C^{-1/2})^\dagger = D_C^{1/2} (A_1 B_1 C_1)^\dagger A_1^{-*}$ can be deduced and then, the equivalence between (vi) and (vii) is similarly proved.

In addition, we can deduce a key equation, similar to (16), and therefore obtain the equivalence of (ii), (iv), and (vi).

By Theorems 2.1 and 2.2 in [4], we can prove the following theorem, similar to the proof of Theorem 3.2. $\square$

Theorem 3.3. Let A, C , and ABC have closed ranges. ien, the following statements are equivalent:

- (i) $(A^* ABCC^*)^\dagger = C^{*\dagger} (ABC)^\dagger A^\dagger$
- (ii) $(A^\dagger ABCC^\dagger)^\dagger = C^\dagger (ABC)^\dagger A$
- (iii) $C^\dagger (A^* ABCC^\dagger)^\dagger A^\dagger = (C^*C)^\dagger (ABC)^\dagger (AA^*)^\dagger$
- (iv) $C^* (A^\dagger ABCC^\dagger)^\dagger A^* = C^*C (ABC)^\dagger AA^*$
- (v) $(ABC)^\dagger = C^\dagger (A^\dagger ABCC^\dagger)^\dagger A^\dagger$
- (vi) $(ABC)^\dagger = C^* (A^* ABCC^*)^\dagger A^*$
- (vii) $\mathcal{R}[AA^*(ABC)] = \mathcal{R}(ABC)$ and $\mathcal{R}[C^*C(ABC)^*] = \mathcal{R}((ABC)^*)$

Next, we will further deduce some statements, each of which is equivalent to $\mathcal{R}[AA^*(ABC)] = \mathcal{R}(ABC)$ and $\mathcal{R}[C^*C(ABC)^*] = \mathcal{R}((ABC)^*)$.

Theorem 3.4. Let A, C , and ABC have closed ranges. ien, the following statements are equivalent:

- (i) $(A^\dagger ABC)^\dagger A^\dagger = C^\dagger (ABCC^\dagger)^\dagger$
- (ii) $(A^\dagger ABC)^\dagger A^* = C^\dagger (A^\dagger BCC^\dagger)^\dagger$
- (iii) $(A^\dagger ABC)^\dagger A^\dagger = C^* (ABCC^*)^\dagger$
- (iv) $(A^* ABC)^\dagger A^* = C^\dagger (ABCC^\dagger)^\dagger$
- (v) $(A^\dagger ABC^*)^\dagger A^\dagger = C^\dagger (ABCC^\dagger)^\dagger$
- (vi) $(A^* ABC^*)^\dagger A^* = C^\dagger [AB(CC^*)^\dagger]^\dagger$
- (vii) $[(A^* A)^\dagger BC]^\dagger A^\dagger = C^* (A^\dagger BCC^*)^\dagger$
- (viii) $\mathcal{R}[AA^*(ABC)] = \mathcal{R}(ABC)$ and $\mathcal{R}[C^*C(ABC)^*] = \mathcal{R}((ABC)^*)$

Proof. By Lemma 2.3, we can obtain that $(A^* A)^\dagger$, $(CC^*)^\dagger$, $(A^\dagger ABC)^\dagger$, $(A^* ABC)^\dagger$, $(ABCC^\dagger)^\dagger$, $(ABCC^*)^\dagger$, $(A^* ABC^*)^\dagger$, $[(A^* A)^\dagger BC]^\dagger$, and $[AB(CC^*)^\dagger]^\dagger$ exist. By (12), we have

$$ABCC^\dagger = \begin{bmatrix} A_1 B_1 C_1 D_C^{-1} C_1^* & A_1 B_1 C_1 D_C^{-1} C_2^* \\ 0 & 0 \end{bmatrix}. \quad (53)$$

Since $ABCC^\dagger$ has a closed range, by Lemma 2.1, we have

$$\begin{aligned} D &= A_1 B_1 C_1 D_C^{-1} C_1^* (A_1 B_1 C_1 D_C^{-1} C_1^*)^* + A_1 B_1 C_1 D_C^{-1} C_2^* (A_1 B_1 C_1 D_C^{-1} C_2^*)^* \\ &= A_1 B_1 C_1 D_C^{-1} C_1^* A_1^* \\ &= (A_1 B_1 C_1 D_C^{-1/2}) (A_1 B_1 C_1 D_C^{-1/2})^*, \end{aligned} \quad (54)$$

which is invertible, and

$$\begin{aligned} (A_1 B_1 C_1 D_C^{-1})^* D^{-1} &= (D_C^{-1/2})^* (A_1 B_1 C_1 D_C^{-1/2})^* D^{-1} \\ &= D_C^{-1/2} (A_1 B_1 C_1 D_C^{-1/2})^\dagger, \end{aligned} \quad (55)$$

and then,

$$(ABCC^\dagger)^\dagger = \begin{bmatrix} C_1 D_C^{-1/2} (A_1 B_1 C_1 D_C^{-1/2})^\dagger & 0 \\ C_2 D_C^{-1/2} (A_1 B_1 C_1 D_C^{-1/2})^\dagger & 0 \end{bmatrix}. \quad (56)$$

Similarly, we can get that

$$(A^* BCC^\dagger)^\dagger = \begin{bmatrix} C_1 D_C^{1/2} (A_1^{-*} B_1 C_1 D_C^{-1/2})^\dagger & 0 \\ C_2 D_C^{1/2} (A_1^{-*} B_1 C_1 D_C^{-1/2})^\dagger & 0 \end{bmatrix}, \quad (57)$$

$$(ABCC^*)^\dagger = \begin{bmatrix} C_1 D_C^{-1/2} (A_1 B_1 C_1 D_C^{1/2})^\dagger & 0 \\ C_2 D_C^{-1/2} (A_1 B_1 C_1 D_C^{1/2})^\dagger & 0 \end{bmatrix}, \quad (58)$$

$$(AB(CC^*)^\dagger)^\dagger = \begin{bmatrix} C_1 D_C^{-1/2} (A_1 B_1 C_1 D_C^{-3/2})^\dagger & 0 \\ C_2 D_C^{-1/2} (A_1 B_1 C_1 D_C^{-3/2})^\dagger & 0 \end{bmatrix}, \quad (59)$$

$$(A^* BCC^*)^\dagger = \begin{bmatrix} C_1 D_C^{-1/2} (A_1^{-*} B_1 C_1 D_C^{1/2})^\dagger & 0 \\ C_1 D_C^{-1/2} (A_1^{-*} B_1 C_1 D_C^{1/2})^\dagger & 0 \end{bmatrix}, \quad (60)$$

$$(A^\dagger ABC)^\dagger = \begin{bmatrix} (B_1 C_1)^\dagger & 0 \\ 0 & 0 \end{bmatrix}, \quad (61)$$

$$(A^* ABC)^\dagger = \begin{bmatrix} (A_1^* A_1 B_1 C_1)^\dagger & 0 \\ 0 & 0 \end{bmatrix}, \quad (62)$$

$$(A^\dagger ABC^*)^\dagger = \begin{bmatrix} (B_1 C_1 D_C^{-1})^\dagger & 0 \\ 0 & 0 \end{bmatrix}, \quad (63)$$

$$(A^* ABC^*)^\dagger = \begin{bmatrix} (A_1^* A_1 B_1 C_1 D_C^{-1})^\dagger & 0 \\ 0 & 0 \end{bmatrix}, \quad (64)$$

$$((A^* A)^\dagger BC)^\dagger = \begin{bmatrix} [(A_1^* A_1)^{-1} B_1 C_1]^\dagger & 0 \\ 0 & 0 \end{bmatrix}. \quad (65)$$

(i) $\Rightarrow$ (viii): using (56) and (61), we have

$$\begin{aligned} (A^\dagger ABC)^\dagger A^\dagger &= \begin{bmatrix} (B_1 C_1)^\dagger A_1^{-1} & 0 \\ 0 & 0 \end{bmatrix}, \\ C^\dagger (ABCC^\dagger)^\dagger &= \begin{bmatrix} D_C^{-1/2} (A_1 B_1 C_1 D_C^{-1/2})^\dagger & 0 \\ 0 & 0 \end{bmatrix}. \end{aligned} \quad (66)$$

Now, (i) holds; i.e., $(A^\dagger ABC)^\dagger A^\dagger = C^\dagger (ABCC^\dagger)^\dagger$ holds if and only if

$$(A_1 B_1 C_1 D_C^{-1/2})^\dagger = D_C^{1/2} (B_1 C_1)^\dagger A_1^{-1}. \quad (67)$$

By Lemma 2.4, (67) is equivalent to

$$\begin{aligned} [A_1^* A_1, (B_1 C_1) (B_1 C_1)^\dagger] &= 0, \\ [D_C, (B_1 C_1)^\dagger (B_1 C_1)] &= 0. \end{aligned} \quad (68)$$

Since A_1 is invertible, by Lemma 2.4, we have

$$\begin{aligned} (B_1 C_1)^\dagger &= (A_1 B_1 C_1)^\dagger A_1^{-1} \\ &\Leftrightarrow [A_1^* A_1, (B_1 C_1) (B_1 C_1)^\dagger] = 0 \\ &\Leftrightarrow [A_1 A_1^*, (AB_1 C_1) (AB_1 C_1)^\dagger] = 0. \end{aligned} \quad (69)$$

In addition, $[D_C, (B_1 C_1)^\dagger (B_1 C_1)] = 0$ is equivalent to $[D_C, (A_1 B_1 C_1)^\dagger (A_1 B_1 C_1)] = 0$ whenever $(B_1 C_1)^\dagger A_1^{-1} = (A_1 B_1 C_1)^\dagger$. From this, we can get that (68) is equivalent to

$$\begin{aligned} [A_1 A_1^*, (A_1 B_1 C_1) (A_1 B_1 C_1)^\dagger] &= 0, \\ [D_C, (A_1 B_1 C_1)^\dagger (A_1 B_1 C_1)] &= 0, \end{aligned} \quad (70)$$

namely,

$$\begin{aligned} [AA^*, (ABC) (ABC)^\dagger] &= 0, \\ [C^* C, (ABC)^\dagger (ABC)] &= 0. \end{aligned} \quad (71)$$

By Lemma 2.2, (71) holds if and only if

$$\begin{aligned} \mathcal{R}[AA^* (ABC)] &= \mathcal{R}(ABC), \\ \mathcal{R}[C^* C (ABC)^*] &= \mathcal{R}((ABC)^*). \end{aligned} \quad (72)$$

Namely, (i) is equivalent to (viii).

(viii) $\Rightarrow$ (vii): statement (viii) implies that

$$\begin{aligned} [AA^*, ABC(ABC)^\dagger] &= 0, \\ [C^*C, (ABC)^\dagger ABC] &= 0, \end{aligned} \quad (73)$$

which is equivalent to

$$\begin{aligned} [A_1A_1^*, A_1B_1C_1(A_1B_1C_1)^\dagger] &= 0, \\ [D_C, (A_1B_1C_1)^\dagger A_1B_1C_1] &= 0. \end{aligned} \quad (74)$$

Thus,

$$\begin{aligned} [(A_1A_1^*)^{-1}, A_1B_1C_1(A_1B_1C_1)^\dagger] &= 0, \\ [D_C, (A_1B_1C_1)^\dagger A_1B_1C_1] &= 0, \end{aligned} \quad (75)$$

and then, the left equation in (75) implies

$$[(A_1A_1^*)^{-1}, A_1^{-*}B_1C_1(A_1^{-*}B_1C_1)^\dagger] = 0. \quad (76)$$

By Lemma 2.5, we have

$$[(A_1A_1^*)^2, A_1^{-*}B_1C_1(A_1^{-*}B_1C_1)^\dagger] = 0. \quad (77)$$

Hence, by Lemma 2.4, we have $(A_1B_1C_1)^\dagger = (A_1^{-*}B_1C_1)^\dagger(A_1A_1^*)^{-1}$. From this, the right equation in (75) becomes $[D_C, (A_1^{-*}B_1C_1)^\dagger A_1^{-*}B_1C_1] = 0$ and then

$$[D_C^{-1}, (A_1^{-*}B_1C_1D_C^{1/2})^\dagger A_1^{-*}B_1C_1D_C^{1/2}] = 0. \quad (78)$$

Therefore, we have $(A_1^{-*}B_1C_1)^\dagger = D_C^{1/2}(A_1^{-*}B_1C_1D_C^{1/2})^\dagger$. Consequently, (76) becomes

$$[(A_1A_1^*)^2, A_1^{-*}B_1C_1D_C^{1/2}(A_1^{-*}B_1C_1D_C^{1/2})^\dagger] = 0. \quad (79)$$

Thus, combining Lemma 2.4, (78), and (79), we have

$$[(A_1^*A_1)^{-1}B_1C_1]^\dagger = D_C^{1/2}(A_1^{-*}B_1C_1D_C^{1/2})^\dagger A_1. \quad (80)$$

(vii) $\Rightarrow$ (vi): by (60) and (65), (vii) holds if and only if

$$(A_1^*A_1)^{-1}(B_1C_1)^\dagger = D_C^{1/2}(A_1^{-*}B_1C_1D_C^{1/2})^\dagger A_1. \quad (81)$$

Then, (81) is equivalent to

$$\begin{aligned} [(A_1^*A_1)^{-1}, A_1^{-*}B_1C_1D_C^{1/2}(A_1^{-*}B_1C_1D_C^{1/2})^\dagger] &= 0, \\ [D_C^{-1}, (A_1^{-*}B_1C_1D_C^{1/2})^\dagger A_1^{-*}B_1C_1D_C^{1/2}] &= 0. \end{aligned} \quad (82)$$

By Lemmas 2.4 and 2.5, the first equation in (82) is equivalent to

$$\begin{aligned} [(A_1^*A_1)^{-1}, A_1^{-*}B_1C_1D_C^{1/2}(A_1^{-*}B_1C_1D_C^{1/2})^\dagger] &= 0 \\ \Leftrightarrow [A_1A_1^*, A_1^{-*}B_1C_1D_C^{1/2}(A_1^{-*}B_1C_1D_C^{1/2})^\dagger] &= 0 \\ \Leftrightarrow [A_1A_1^*, A_1B_1C_1D_C^{1/2}(A_1B_1C_1D_C^{1/2})^\dagger] &= 0. \end{aligned} \quad (83)$$

Thus,

$$[(A_1A_1^*)^2, A_1B_1C_1D_C^{1/2}(A_1B_1C_1D_C^{1/2})^\dagger] = 0, \quad (84)$$

and therefore, $(A_1^{-*}B_1C_1D_C^{1/2})^\dagger = (A_1B_1C_1D_C^{1/2})^\dagger A_1A_1^*$.

The second equation in (82) can be further rewritten as follows:

$$\begin{aligned} [D_C^{-1}, (A_1B_1C_1D_C^{1/2})^\dagger A_1B_1C_1D_C^{1/2}] &= 0 \\ \Leftrightarrow [D_C^2, (A_1B_1C_1D_C^{-3/2})^\dagger A_1B_1C_1D_C^{-3/2}] &= 0 \\ \Leftrightarrow (A_1B_1C_1D_C^{1/2})^\dagger &= D_C^{-2}(A_1B_1C_1D_C^{-3/2})^\dagger. \end{aligned} \quad (85)$$

Consequently, (82) becomes

$$\begin{aligned} [A_1A_1^*, A_1B_1C_1D_C^{-3/2}(A_1B_1C_1D_C^{-3/2})^\dagger] &= 0, \\ [D_C, (A_1B_1C_1D_C^{-3/2})^\dagger A_1B_1C_1D_C^{-3/2}] &= 0, \end{aligned} \quad (86)$$

and then, $(A_1^*A_1B_1C_1D_C^{-1})^\dagger = D_C^{-1/2}(A_1B_1C_1D_C^{-3/2})^\dagger A_1^{-*}$, namely, (vi) holds.

(vi) $\Leftrightarrow$ (v): by (59) and (64), we can obtain that (vi) is equivalent to

$$(A_1^*A_1B_1C_1D_C^{-1})^\dagger = D_C^{-1/2}(A_1B_1C_1D_C^{-3/2})^\dagger A_1^{-*}, \quad (87)$$

or, equivalently,

$$\begin{aligned} [A_1A_1^*, A_1B_1C_1D_C^{-3/2}(A_1B_1C_1D_C^{-3/2})^\dagger] &= 0, \\ [D_C, (A_1B_1C_1D_C^{-3/2})^\dagger A_1B_1C_1D_C^{-3/2}] &= 0. \end{aligned} \quad (88)$$

By Lemma 2.4 and 2.5, the right equation in (88) is equivalent to

$$\begin{aligned} [D_C, (A_1B_1C_1D_C^{-1})^\dagger A_1B_1C_1D_C^{-1}] &= 0 \\ \Leftrightarrow (A_1B_1C_1D_C^{-3/2})^\dagger &= D_C^{1/2}(A_1B_1C_1D_C^{-1})^\dagger. \end{aligned} \quad (89)$$

By Lemma 2.5, the left equation in (88) is equivalent to

$$\begin{aligned}
& \left[A_1 A_1^*, A_1 B_1 C_1 D_C^{-1} (A_1 B_1 C_1 D_C^{-1})^\dagger \right] = 0 \\
& \Leftrightarrow \left[A_1^* A_1, B_1 C_1 D_C^{-1} (B_1 C_1 D_C^{-1})^\dagger \right] = 0 \quad (90) \\
& \Leftrightarrow (A_1 B_1 C_1 D_C^{-1})^\dagger = (B_1 C_1 D_C^{-1})^\dagger A_1^{-1}.
\end{aligned}$$

$$\left[A_1 A_1^*, B_1 C_1 D_C^{-1} (B_1 C_1 D_C^{-1})^\dagger \right] = 0 \text{ and } \left[D_C, (B_1 C_1 D_C^{-1})^\dagger B_1 C_1 D_C^{-1} \right] = 0, \quad (91)$$

if and only if $(A_1 B_1 C_1 D_C^{-1/2})^\dagger = D_C^{-1/2} (B_1 C_1 D_C^{-1})^\dagger A_1^{-1}$, namely, (v) holds.

(v) $\Rightarrow$ (iv): by (56) and (63), (v) holds if and only if

$$(A_1 B_1 C_1 D_C^{-1/2})^\dagger = D_C^{-1/2} (B_1 C_1 D_C^{-1})^\dagger A_1^{-1}. \quad (92)$$

Using Lemma 2.4, we have

$$\begin{aligned}
& \left[A_1^* A_1, (B_1 C_1 D_C^{-1}) (B_1 C_1 D_C^{-1})^\dagger \right] = 0, \\
& \left[D_C, (B_1 C_1 D_C^{-1})^\dagger (B_1 C_1 D_C^{-1}) \right] = 0. \quad (93)
\end{aligned}$$

By Lemma 2.5, the right equation in (93) is equivalent to $[D_C, (B_1 C_1)^\dagger (B_1 C_1)] = 0$ and implies $[D_C^2, (B_1 C_1)^\dagger (B_1 C_1)] = 0$. Thus, by Lemma 2.4, we have $(B_1 C_1 D_C^{-1})^\dagger = D_C (B_1 C_1)^\dagger$. Accordingly, (93) becomes

$$\begin{aligned}
& \left[A_1^* A_1, (B_1 C_1) (B_1 C_1)^\dagger \right] = 0, \\
& \left[D_C^{-1}, (B_1 C_1)^\dagger (B_1 C_1) \right] = 0. \quad (94)
\end{aligned}$$

Similarly, combining Lemmas 2.5 and 2.4, from the left equation in (94), it follows that $(B_1 C_1)^\dagger = (A_1^* A_1 B_1 C_1)^\dagger A_1^* A_1$.

Consequently, (94) becomes

$$\begin{aligned}
& \left[(A_1^* A_1)^{-1}, A_1^* A_1 B_1 C_1 (A_1^* A_1 B_1 C_1)^\dagger \right] = 0, \\
& \left[D_C^{-1}, (A_1^* A_1 B_1 C_1)^\dagger A_1^* A_1 B_1 C_1 \right] = 0, \quad (95)
\end{aligned}$$

if and only if $(A_1 B_1 C_1 D_C^{-1/2})^\dagger = D_C^{1/2} (A_1^* A_1 B_1 C_1)^\dagger A_1^*$, namely, (iv) holds.

(iv) $\Rightarrow$ (iii): by (56) and (62), (iv) holds if and only if

$$\begin{aligned}
& (A_1 B_1 C_1 D_C^{-1/2})^\dagger = D_C^{1/2} (A_1^* A_1 B_1 C_1)^\dagger A_1^*, \\
& \left[A_1^* A_1, A_1^* A_1 B_1 C_1 (A_1^* A_1 B_1 C_1)^\dagger \right] = 0, \quad (96) \\
& \left[D_C, (A_1^* A_1 B_1 C_1)^\dagger A_1^* A_1 B_1 C_1 \right] = 0.
\end{aligned}$$

Notice that the left equation in (96) holds if and only if $[A_1^* A_1, B_1 C_1 (B_1 C_1)^\dagger] = 0$ with Lemma 2.5. Hence, $[(A_1^* A_1)^2, B_1 C_1 (B_1 C_1)^\dagger] = 0$ and then, by Lemma 2.4, we have $(A_1^* A_1 B_1 C_1)^\dagger = (B_1 C_1)^\dagger (A_1^* A_1)^{-1}$. Thus, (96) becomes

As a result, (88) is equivalent to

$$\begin{aligned}
& \left[A_1^* A_1, (B_1 C_1) (B_1 C_1)^\dagger \right] = 0, \\
& \left[D_C, (B_1 C_1)^\dagger (B_1 C_1) \right] = 0. \quad (97)
\end{aligned}$$

Hence, $(A_1 B_1 C_1 D_C^{1/2})^\dagger = D_C^{-1/2} (B_1 C_1)^\dagger A_1^{-1}$ and therefore, (iii) holds.

(iii) $\Rightarrow$ (ii): by (58) and (61), (iii) holds if and only if

$$(A_1 B_1 C_1 D_C^{1/2})^\dagger = D_C^{-1/2} (B_1 C_1)^\dagger A_1^{-1}. \quad (98)$$

Similarly, $[A_1^* A_1, (B_1 C_1) (B_1 C_1)^\dagger] = 0$ and $[D_C, (B_1 C_1)^\dagger (B_1 C_1)] = 0$ if and only if $[(A_1^* A_1)^{-1}, (B_1 C_1) (B_1 C_1)^\dagger] = 0$ and $[D_C^{-1}, (B_1 C_1)^\dagger (B_1 C_1)] = 0$. Therefore,

$$(A_1^{-1} B_1 C_1 D_C^{-1/2})^\dagger = D_C^{1/2} (B_1 C_1)^\dagger A_1^* \quad (99)$$

if and only if (ii) holds.

(ii) $\Rightarrow$ (i): by (57) and (61), (ii) is equivalent to

$$(A_1^{-1} B_1 C_1 D_C^{-1/2})^\dagger = D_C^{1/2} (B_1 C_1)^\dagger A_1^*. \quad (100)$$

Therefore, by Lemma 2.4, we have

$$\begin{aligned}
& \left[A_1^* A_1, (B_1 C_1) (B_1 C_1)^\dagger \right] = 0, \\
& \left[D_C, (B_1 C_1)^\dagger (B_1 C_1) \right] = 0, \quad (101)
\end{aligned}$$

if and only if $(A_1 B_1 C_1 D_C^{-1/2})^\dagger = D_C^{1/2} (B_1 C_1)^\dagger A_1^{-1}$, i.e., (i) holds.

By Theorems 3.1–3.3, we can obtain the following theorem. $\square$

Theorem 3.5. Let A, C , and ABC have closed ranges. ie following statements are equivalent:

- (i) $(ABC)^\dagger = C^\dagger B^\dagger A^\dagger$
- (ii) $(A^* \dagger BC)^\dagger = C^\dagger B^\dagger A^*$
- (iii) $(ABC^*)^\dagger = C^* B^\dagger A^\dagger$
- (iv) $(ABC)^\dagger = C^\dagger (A^\dagger ABCC^\dagger)^\dagger A^\dagger$ and $(A^\dagger ABCC^\dagger)^\dagger = CC^\dagger B^\dagger A^\dagger A$
- (v) $(ABC)^\dagger = C^\dagger (A^* ABCC^\dagger)^\dagger A^*$ and $(A^* ABCC^\dagger)^\dagger = CC^\dagger B^\dagger A^\dagger A^*$
- (vi) $(ABC)^\dagger = C^* (A^\dagger ABCC^*)^\dagger A^\dagger$ and $(A^\dagger ABCC^*)^\dagger = C^* C^\dagger B^\dagger A^\dagger A$

$$\begin{aligned}
(vii) \quad & (ABC)^\dagger = C^*(A^*ABCC^*)^\dagger A^* \text{ and } (A^*ABCC^*)^\dagger \\
& = C^{*\dagger}C^\dagger B^\dagger A^\dagger A^{*\dagger} \\
(viii) \quad & (ABC)^\dagger = (A^\dagger ABC)^\dagger A^\dagger \text{ and } (A^\dagger ABC)^\dagger = \\
& C^\dagger B^\dagger A^\dagger A \\
(ix) \quad & (ABC)^\dagger = C^\dagger (ABCC^\dagger)^\dagger \text{ and } (ABCC^\dagger)^\dagger = CC^\dagger B^\dagger \\
& A^\dagger \\
(x) \quad & (ABC)^\dagger = (A^*ABC)^\dagger A^* \text{ and } (A^*ABC)^\dagger = C^\dagger B^\dagger \\
& A^\dagger A^{*\dagger} \\
(xi) \quad & (ABC)^\dagger = C^*(ABCC^*)^\dagger \text{ and } (ABCC^*)^\dagger = C^{*\dagger}C^\dagger \\
& B^\dagger A^\dagger
\end{aligned}$$

$$\begin{aligned}
(xii) \quad & (ABCC^\dagger)^\dagger = CC^\dagger B^\dagger A^\dagger \text{ and } (A^\dagger ABC)^\dagger = C^\dagger B^\dagger A^\dagger A \\
(xiii) \quad & (ABCC^*)^\dagger = C^{*\dagger}C^\dagger B^\dagger A^\dagger \text{ and } (A^*ABC)^\dagger = C^\dagger B^\dagger \\
& A^\dagger A^{*\dagger}
\end{aligned}$$

Proof. By Lemma 2.3, those Moore–Penrose inverses exist.

$$(i) \Leftrightarrow (ii): \text{ if } (i) \text{ holds, then } (C^\dagger B^\dagger A^*)(A^{*\dagger}BC) = C^\dagger B^\dagger A^\dagger ABC = (ABC)^\dagger ABC \text{ and then}$$

$$(C^\dagger B^\dagger A^*)(A^{*\dagger}BC)(C^\dagger B^\dagger A^*) = (ABC)^\dagger ABC (ABC)^\dagger AA^* = C^\dagger B^\dagger A^\dagger AA^* = C^\dagger B^\dagger A^*. \quad (102)$$

The other Penrose equations are similarly verified. Hence, (ii) holds.

Conversely, the proof is same.

(i) $\Leftrightarrow$ (iii): this is similar to the proof above.

(i) $\Rightarrow$ (iv): by Theorem 3.1, we have $(A^\dagger ABCC^\dagger)^\dagger = CC^\dagger B^\dagger A^\dagger A$ and

$$C^\dagger (A^\dagger ABCC^\dagger)^\dagger A^\dagger = C^\dagger CC^\dagger B^\dagger A^\dagger AA^\dagger = C^\dagger B^\dagger A^\dagger = (ABC)^\dagger. \quad (103)$$

(iv) $\Rightarrow$ (v): by Theorems 3.2 and 3.3, we have $(ABC)^\dagger = C^\dagger (A^*ABCC^\dagger)^\dagger A^*$ and

$$\begin{aligned}
(A^*ABCC^\dagger)^\dagger &= C(ABC)^\dagger A^{*\dagger} = C(ABC)^\dagger AA^\dagger A^{*\dagger} \\
&= (A^\dagger ABCC^\dagger)^\dagger A^\dagger A^{*\dagger} = CC^\dagger B^\dagger A^\dagger AA^\dagger A^{*\dagger} \\
&= CC^\dagger B^\dagger A^\dagger A^{*\dagger}.
\end{aligned} \quad (104)$$

(v) $\Rightarrow$ (vi) and (vi) $\Rightarrow$ (vii): This is similar to the proof of “(iv) $\Rightarrow$ (v).”

(vii) $\Rightarrow$ (i): The proof is obvious.

(i) $\Rightarrow$ (viii): if (i) holds, then we can obtain that

$A^\dagger ABCC^\dagger B^\dagger A^\dagger A = A^\dagger ABC (ABC)^\dagger A = \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix}$ is Hermitian by (13). Thus, it is easily verified that $(A^\dagger ABC)^\dagger = C^\dagger B^\dagger A^\dagger A$ holds by the definition of the Moore–Penrose inverse. Therefore,

$$(A^\dagger ABC)^\dagger A^\dagger = C^\dagger B^\dagger A^\dagger AA^\dagger = (ABC)^\dagger. \quad (105)$$

Conversely, the following is apparent:

(i) $\Leftrightarrow$ (ix), (i) $\Leftrightarrow$ (x), and (i) $\Leftrightarrow$ (xi): these are analogous to the proof above.

(i) $\Leftrightarrow$ (xii): since (i), (viii), and (ix) are equivalent, (i) implies (xii).

Conversely, $(ABC)(C^\dagger B^\dagger A^\dagger) = ABCC^\dagger (ABCC^\dagger)^\dagger$ and then,

$$(ABC)(C^\dagger B^\dagger A^\dagger)(ABC) = ABCC^\dagger (ABCC^\dagger)^\dagger ABCC^\dagger C = ABCC^\dagger C = ABC. \quad (106)$$

Thus, the first and third Penrose equations are satisfied. Similarly, the other Penrose equations are satisfied:

(i) $\Leftrightarrow$ (xiii): the proof is the same as the above. $\square$

Data Availability

Data are available upon request to the corresponding author.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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