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Research Article

Generalized *m*-Polar Fuzzy Positive Implicative Ideals of BCK-Algebras

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This study focuses on combining the theories of m-polar fuzzy sets over BCK-algebras and establishing a new framework of m-polar fuzzy BCK-algebras. In this paper, we define the idea of m-polar fuzzy positive implicative ideals in BCK-algebras and investigate some related properties. Then, we introduce the concepts of m-polar $(\in, \in\vee q)$-fuzzy positive implicative ideals and m-polar $(\overline{\in}, \overline{\in}\vee\overline{q})$-fuzzy positive implicative ideals in BCK-algebras as a generalization of m-polar fuzzy positive implicative ideals. Several properties, examples, and characterization theorems of these concepts are considered.

1. Introduction

The inception of the idea of BCK-algebras, presented by Imai and Iséki [1], laid the frameworks and foundations as well as gave birth to great research studies. Such algebras generalize Boolean D-poset (MV-algebras) as well as Boolean rings. BCK-algebras have many applications in several fields, such as groups, semigroups, graphs, topology and functional analysis, and so on. The study of ideals forms an essential aspect of the theory of BCK-algebra. Since Imai and Iséki [1] introduced the notion of ideals in BCK-algebra, several kinds of ideals in BCK-algebras have occurred, for example, H-ideals, positive implicative ideals, implicative ideals, and so on.

The essential idea of a fuzzy set, proposed by Zadeh [2] in 1965, provides a natural framework for generalizing many fundamental concepts of algebras. Moreover, the idea of fuzzy sets in BCK/BCI-algebras was proposed by Xi [3]. The theory of fuzzy algebraic structures plays a prominent role in different domains of mathematics and other sciences such as theoretical physics, topological spaces, real analysis, coding theory, set theory, logic, and information sciences. In 1994, bipolar fuzzy (BF) set theory was proposed by Zhang [4] as a new platform that extends crisp (classical) and fuzzy sets. BF sets' membership grades (positive and negative) belong to the interval $[-1, 1]$. Hybrid models of fuzzy sets have been applied in various algebraic structures, for instance, hemirings [5], UP-algebras [6], and BCK/BCI-algebras [7]. In many real-world problems, multipolar information play a fundamental role in distinct areas of sciences, such as neurobiology and technology. Data sometimes come from m components $(m \geq 2)$; for example, consider the following statement "Harvard University is a Good University." In this statement, the degree of membership may not be a real number in the standard interval $[0, 1]$. In fact, Harvard University is a good university in several components: good in ranking, location, facilities and education, etc. Any component may be a real number in $[0, 1]$. If we have m components under consideration, then the degree of the membership of the fuzzy statement is an element of $[0, 1]^m$, that is, an m-tuple of real number in $[0, 1]$. Based on the above discussion, Chen et al. [8] generalized the theory of BF set theory to get a new notion, called m-polar fuzzy (m-pF) set theory in 2014. In m-pF sets, the grade of membership function is extended from [0, 1] to m-power of $[0,1]$ or $[0,1]^m$.

The framework of the fuzzy subgroup, initially presented by Rosenfeld [9] in 1971, is a fundamental concept of fuzzy algebras. Pu and Liu [10] and Murali [11] defined "quasi-coincidence" and "belongingness" of a fuzzy point with a fuzzy set, respectively. These notions played a fundamental role to establish distinct kinds of fuzzy subgroups. In the literature, Bhakat and Das [12] first generalized fuzzy subgroups to (α,β)-fuzzy subgroups and they proposed and discussed the idea of $(\in,\in\vee q)$-fuzzy subgroups. In this aspect, Dudek et al. [13], Ibrara et al. [14], Jun and Song [15], and Narayanan and Manikantan [16] extended these results to semigroups, nearrings, and hemirings. In BCK/BCI-algebras, Jun [17] presented (α,β)-fuzzy ideals as an extension of fuzzy ideals. In [18], Zulfiqar introduced (α,β)-fuzzy positive implicative ideals in BCK-algebras. Zhan and Jun [19] defined $(\overline{\in},\overline{\in}\vee\overline{q})$-fuzzy ideals in BCI-algebras.

The notion of m-pF set theory was applied to many practical problems, particularly in the field of graph theory (see, for e.g., [20–22]). In [23], Sarwar and Akram applied m-pF set theory to matroid theory. In addition, various applications of m-pF sets and other hybrid models of fuzzy sets in pure and applied mathematics are studied in [24–32]. Recently, m-pF set theory has been applied to various algebraic structures on different aspects, namely, Farooq et al. applied m-pF set theory to groups [33], Akram and Farooq applied m-pF set theory to Lie algebras [34, 35], and the authors applied m-pF set theory to BCK/BCI-algebras (see [36–40]. Motivated by a lot of work on m-pF sets, we present m-polar fuzzy positive implicative (m-pFPI) ideals in BCK-algebras and discuss some related results. Using the concept of quasi-coincidence of an m-pF point within an m-pF set and as a generalization of m-pFPI ideals, we introduce the concepts of m-polar $(\in,\in\vee q)$-fuzzy positive implicative ($m-p_{(\in,\in\vee q)}-\mathrm{FPI}$) ideals and m-polar $(\overline{\in},\overline{\in}\vee\overline{q})$-fuzzy positive implicative ($m-p_{(\overline{\in},\overline{\in}\vee\overline{q})-\mathrm{FPI}}$) ideals in BCK-algebras. Several properties, examples, and characterization theorems of these concepts are considered.

2. Preliminaries

Here are some of the important concepts of BCK-algebras, m-pF sets, and m-pF ideals that are useful for further discussions. Throughout this paper, for the convenience, Ω stands for a BCK-algebra.

An algebra $(\Omega;\ *,0)$ of type $(2,0)$ is called a BCK-algebra if the axioms below are satisfied for all $\varpi,\varrho,\eta\in\Omega$:

(i) $((\varpi*\varrho)*(\varpi*\eta))\le(\eta*\varrho)$

(ii) $(\varpi*(\varpi*\varrho))\le\varrho$

(iii) $\varpi*\varpi=0$

(iv) $0*\varpi=0$

(v) $\varpi\le\varrho$ and $\varrho\le\varpi$ imply $\varpi=\varrho$

where $\le$ can be presented by $\varpi\le\varrho$ if and only if $\varpi*\varrho=0$. Every BCK-algebra Ω satisfies the following axioms for all $\varpi,\varrho,\eta\in\Omega$:

(1) $\varpi*0=\varpi$.

(2) $(\varpi*\varrho)*\eta=(\varpi*\eta)*\varrho$.

Definition 1 (see [8]). An m-pF set $\widehat{\mathscr{M}^{\mathscr{P}}}$ on $\Omega(\neq\phi)$ is a mapping $\widehat{\mathscr{M}^{\mathscr{P}}}\colon\Omega\longrightarrow[0,1]^m$. The membership grade of any element $\varpi\in\Omega$ is given as

$$\widehat{\mathscr{M}^{\mathscr{P}}}(\varpi)=\left(p_1\circ\widehat{\mathscr{M}^{\mathscr{P}}}(\varpi),p_2\circ\widehat{\mathscr{M}^{\mathscr{P}}}(\varpi),\ldots,p_m\circ\widehat{\mathscr{M}^{\mathscr{P}}}(\varpi)\right),\tag{1}$$

where $p_i\circ\widehat{\mathscr{M}^{\mathscr{P}}}\colon[0,1]^m\longrightarrow[0,1]$ is the i-th projection mapping. The grades $\widehat{1}=(1,1,\ldots,1)$ and $\widehat{0}=(0,0,\ldots,0)$ are the largest and the smallest grades in $[0,1]^m$, respectively.

Definition 2 (see [36]). An m-pF set $\widehat{\mathscr{M}^{\mathscr{P}}}$ of Ω is called an m-pF ideal of Ω if for any $\varpi,\varrho\in\Omega$,

(1) $\widehat{\mathscr{M}^{\mathscr{P}}}(0)\ge\widehat{\mathscr{M}^{\mathscr{P}}}(\varpi)$.

(2) $\widehat{\mathscr{M}^{\mathscr{P}}}(\varpi)\ge\inf\left\{\widehat{\mathscr{M}^{\mathscr{P}}}(\varpi*\varrho),\widehat{\mathscr{M}^{\mathscr{P}}}(\varrho)\right\}$.

That is,

(1) $p_i\circ\widehat{\mathscr{M}^{\mathscr{P}}}(0)\ge p_i\circ\widehat{\mathscr{M}^{\mathscr{P}}}(\varpi)$

(2) $p_i\circ\widehat{\mathscr{M}^{\mathscr{P}}}(\varpi)\ge\inf\left\{p_i\circ\widehat{\mathscr{M}^{\mathscr{P}}}(\varpi*\varrho),p_i\circ\widehat{\mathscr{M}^{\mathscr{P}}}(\varrho)\right\}$

for all $i=1,2,\ldots,m$.

Definition 3 (see [36]). The set $\widehat{\mathscr{M}^{\mathscr{P}}}_{\widehat{\xi}}=\left\{\varpi\in\Omega|\widehat{\mathscr{M}^{\mathscr{P}}}(\varpi)\ge\widehat{\xi}\right\}$, where $\widehat{\mathscr{M}^{\mathscr{P}}}$ is an m-pF set of Ω, is called be the level cut subset of $\widehat{\mathscr{M}^{\mathscr{P}}}$ for all $\widehat{\xi}\in(0,1]^m$.

Lemma 1 (see [36]). *Every m-pF ideal $\widehat{\mathscr{M}^{\mathscr{P}}}$ of Ω satisfies the following assertion for all $\varpi,\varrho\in\Omega$:*

$$\varpi\le\varrho\Rightarrow\widehat{\mathscr{M}^{\mathscr{P}}}(\varpi)\ge\widehat{\mathscr{M}^{\mathscr{P}}}(\varrho).\tag{2}$$

3. m-Polar Fuzzy Positive Implicative Ideals

Definition 4 (see [41]). A nonempty subset $\mathscr{J}$ of Ω is called a positive implicative ideal of Ω if for all $\varpi,\varrho,\eta\in\Omega$,

(1) $0\in\mathscr{J}$.

(2) $(\varpi*\varrho)*\eta\in\mathscr{J}$ and $\varrho*\eta\in\mathscr{J}$ imply $\varpi*\eta\in\mathscr{J}$.

Definition 5. An m-pF set $\widehat{\mathscr{M}^{\mathscr{P}}}$ of Ω is called an m-pFPI ideal of Ω if for any $\varpi,\varrho,\eta\in\Omega$,

(1) $\widehat{\mathscr{M}^{\mathscr{P}}}(0)\ge\widehat{\mathscr{M}^{\mathscr{P}}}(\varpi)$.

(2) $\widehat{\mathscr{M}^{\mathscr{P}}}(\varpi*\eta)\ge\inf\left\{\widehat{\mathscr{M}^{\mathscr{P}}}((\varpi*\varrho)*\eta),\widehat{\mathscr{M}^{\mathscr{P}}}(\varrho*\eta)\right\}$.

That is,

(1) $p_i\circ\widehat{\mathscr{M}^{\mathscr{P}}}(0)\ge p_i\circ\widehat{\mathscr{M}^{\mathscr{P}}}(\varpi)$

(2) $p_i\circ\widehat{\mathscr{M}^{\mathscr{P}}}(\varpi*\eta)\ge\inf\left\{p_i\circ\widehat{\mathscr{M}^{\mathscr{P}}}((\varpi*\varrho)*\eta),p_i\circ\widehat{\mathscr{M}^{\mathscr{P}}}(\varrho*\eta)\right\}$

for all $i=1,2,\ldots,m$.

Example 1. Let $\Omega = \{0,1,2,3,4\} \subseteq \mathbb{N} \cup \{0\}$ and a binary operation " $*$ " be given as follows:

$$0 * x = 0, \quad \forall x \in \Omega,$$
$$1 * y = \begin{cases} 1, & \text{if } y = 0, \\ 0, & \text{if } y \in \{1,2,3,4\}, \end{cases}$$
$$2 * z = \begin{cases} 2, & \text{if } z \in \{0,1,4\}, \\ 0, & \text{if } z \in \{2,3\}, \end{cases} \tag{3}$$
$$3 * w = \begin{cases} 3, & \text{if } w \in \{0,1,2,4\}, \\ 0, & \text{if } w = 3, \end{cases}$$
$$4 * v = \begin{cases} 4, & \text{if } v \in \{0,1,2,3\}, \\ 0, & \text{if } v = 4. \end{cases}$$

Then, $(\Omega, *, 0)$ is a BCK-algebra. Let $\widehat{\mathscr{M}^{\mathscr{P}}}$ be a 4-pF set defined as

$$\widehat{\mathscr{M}^{\mathscr{P}}}(\varpi) = \begin{cases} (0.70, 0.40, 0.90, 0.71), & \text{if } \varpi = 0, \\ (0.60, 0.30, 0.80, 0.61), & \text{if } \varpi = 1, \\ (0.50, 0.20, 0.60, 0.51), & \text{if } \varpi = 2, \\ (0.40, 0.10, 0.30, 0.41), & \text{if } \varpi = 3, \\ (0.30, 0.20, 0.50, 0.31), & \text{if } \varpi = 4. \end{cases} \tag{4}$$

Since conditions (1) and (2) of Definition 5 are satisfied. Then, $\widehat{\mathscr{M}^{\mathscr{P}}}$ is a 4-pFPI ideal of Ω.

Theorem 1. *Any m-pFPI ideal of Ω is an m-pF ideal of Ω, but the converse does not hold.*

Proof. Let $\widehat{\mathscr{M}^{\mathscr{P}}}$ be an m-pFPI ideal of Ω. Then, $\widehat{\mathscr{M}^{\mathscr{P}}}(0) \geq \widehat{\mathscr{M}^{\mathscr{P}}}(\varpi)$. By taking $\eta = 0$ in Definition 5 (2) and since $\varpi * 0 = \varpi \, \forall \varpi \in \Omega$, we have $\widehat{\mathscr{M}^{\mathscr{P}}}(\varpi) \geq \inf \{\widehat{\mathscr{M}^{\mathscr{P}}}(\varpi * \varrho), \widehat{\mathscr{M}^{\mathscr{P}}}(\varrho)\}$. Hence, $\widehat{\mathscr{M}^{\mathscr{P}}}$ is an m-pF ideal of Ω.

The last part is shown by the following example. □

Example 2. Let $\Omega = \{e, \imath, \jmath, \ell\}$ and a binary operation " $*$ " be given as follows:

$$e * x = e, \quad \forall x \in \Omega,$$
$$\imath * y = \begin{cases} \imath, & \text{if } y \in \{e, \ell\} \\ e, & \text{if } y \in \{\imath, \jmath\}, \end{cases}$$
$$\jmath * z = \begin{cases} \jmath, & \text{if } z \in \{e, \ell\}, \\ \imath, & \text{if } z = \imath, \\ e, & \text{if } z = \jmath, \end{cases} \tag{5}$$
$$\ell * w = \begin{cases} \ell, & \text{if } w \in \{e, \imath, \jmath\}, \\ e, & \text{if } w = \ell. \end{cases}$$

Then, $(\Omega, *, e)$ is a BCK-algebra. Let $\widehat{\mathscr{M}^{\mathscr{P}}}$ be a 3-pF set defined as

$$\widehat{\mathscr{M}^{\mathscr{P}}}(\varpi) = \begin{cases} (0.6, 0.6, 0.09), & \text{if } \varpi = 0, \\ (0.5, 0.5, 0.08), & \text{if } \varpi = \imath, \jmath, \\ (0.3, 0.3, 0.3), & \text{if } \varpi = \ell. \end{cases} \tag{6}$$

Since conditions (1) and (2) of Definition 2 are satisfied. Then, $\widehat{\mathscr{M}^{\mathscr{P}}}$ is a 3-pF ideal of Ω, but it is not a 3-pFPI ideal of Ω since $\widehat{\mathscr{M}^{\mathscr{P}}}(\jmath * \imath) = \widehat{\mathscr{M}^{\mathscr{P}}}(\imath) = (0.5, 0.5, 0.08) < \inf\{\widehat{\mathscr{M}^{\mathscr{P}}}((\jmath * \imath) * \imath), \widehat{\mathscr{M}^{\mathscr{P}}}(\imath * \imath)\} = \inf\{\widehat{\mathscr{M}^{\mathscr{P}}}(0), \widehat{\mathscr{M}^{\mathscr{P}}}(0)\} = \widehat{\mathscr{M}^{\mathscr{P}}}(0) = (0.6, 0.6, 0.09)$.

We now give the conditions for an m-pF ideal to be an m-pFPI ideal of Ω.

Theorem 2. *An m-pF set of Ω is an m-pFPI ideal of Ω if and only if it is an m-pF ideal of Ω and $\widehat{\mathscr{M}^{\mathscr{P}}}(\varpi * \varrho) \geq \widehat{\mathscr{M}^{\mathscr{P}}}((\varpi * \varrho) * \varrho)$ for all $\varpi, \varrho \in \Omega$.*

Proof. Suppose $\widehat{\mathscr{M}^{\mathscr{P}}}$ is an m-pFPI ideal of Ω. By Theorem 1, $\widehat{\mathscr{M}^{\mathscr{P}}}$ is an m-pF ideal of Ω. If η is replaced by ϱ in Definition 5 (2), then

$$\begin{aligned} \widehat{\mathscr{M}^{\mathscr{P}}}(\varpi * \varrho) &\geq \inf\left\{\widehat{\mathscr{M}^{\mathscr{P}}}((\varpi * \varrho) * \varrho), \widehat{\mathscr{M}^{\mathscr{P}}}(\varrho * \varrho)\right\} \\ &= \inf\left\{\widehat{\mathscr{M}^{\mathscr{P}}}((\varpi * \varrho) * \varrho), \widehat{\mathscr{M}^{\mathscr{P}}}(0)\right\} \\ &= \widehat{\mathscr{M}^{\mathscr{P}}}((\varpi * \varrho) * \varrho), \end{aligned} \tag{7}$$

for all $\varpi, \varrho \in \Omega$.

Conversely, let $\widehat{\mathscr{M}^{\mathscr{P}}}$ be an m-pF ideal of Ω. Then, $\widehat{\mathscr{M}^{\mathscr{P}}}(0) \geq \widehat{\mathscr{M}^{\mathscr{P}}}(\varpi)$ for all $\varpi \in \Omega$. Also, since $((\varpi * \eta) * \eta) * (\varrho * \eta) \leq (\varpi * \eta) * \varrho = (\varpi * \varrho) * \eta$ for all $\varpi, \varrho \in \Omega$, it follows by Lemma 1 that

$$\widehat{\mathscr{M}^{\mathscr{P}}}(((\varpi * \eta) * \eta) * (\varrho * \eta)) \geq \widehat{\mathscr{M}^{\mathscr{P}}}((\varpi * \varrho) * \eta). \tag{8}$$

Now, by assumption

$$\begin{aligned} \widehat{\mathscr{M}^{\mathscr{P}}}(\varpi * \eta) &\geq \widehat{\mathscr{M}^{\mathscr{P}}}((\varpi * \eta) * \eta) \\ &\geq \inf\left\{\widehat{\mathscr{M}^{\mathscr{P}}}(((\varpi * \eta) * \eta) * (\varrho * \eta)), \widehat{\mathscr{M}^{\mathscr{P}}}(\varrho * \eta)\right\} \\ &\geq \inf\left\{\widehat{\mathscr{M}^{\mathscr{P}}}((\varpi * \varrho) * \eta), \widehat{\mathscr{M}^{\mathscr{P}}}(\varrho * \eta)\right\}. \end{aligned} \tag{9}$$

Hence, $\widehat{\mathscr{M}^{\mathscr{P}}}$ is an m-pFPI ideal of Ω. □

Theorem 3. *An m-pF set $\widehat{\mathscr{M}^{\mathscr{P}}}$ of Ω is an m-pFPI ideal of Ω if and only if $\widehat{\mathscr{M}^{\mathscr{P}}}_{\widehat{\xi}} \neq \phi$ is a positive implicative ideal of Ω for all $\widehat{\xi} \in (0,1]^m$.*

Proof. Suppose that $\widehat{\mathscr{M}^{\mathscr{P}}}$ is an m-pFPI ideal of Ω. Let $\widehat{\xi} \in (0,1]^m$ be such that $\varpi \in \widehat{\mathscr{M}^{\mathscr{P}}}_{\widehat{\xi}}$. Then, $\widehat{\mathscr{M}^{\mathscr{P}}}(0) \geq \widehat{\mathscr{M}^{\mathscr{P}}}(\varpi) \geq \widehat{\xi}$, and we have $0 \in \widehat{\mathscr{M}^{\mathscr{P}}}_{\widehat{\xi}}$. Let $\varpi, \varrho, \eta \in \Omega$ be such that

$(\varpi * \varrho) * \eta, \varrho * \eta \in \widehat{H}_{\widehat{\xi}}$. Then, $\widehat{\mathscr{M}^{\mathscr{P}}}((\varpi * \varrho) * \eta) \geq \widehat{\xi}$ and $\widehat{\mathscr{M}^{\mathscr{P}}}(\varrho * \eta) \geq \widehat{\xi}$. It follows from Definition 5 (2) that

$$\widehat{\mathscr{M}^{\mathscr{P}}}(\varpi * \eta) \geq \inf\left\{\widehat{\mathscr{M}^{\mathscr{P}}}((\varpi * \varrho) * \eta), \widehat{\mathscr{M}^{\mathscr{P}}}(\varrho * \eta)\right\} \geq \widehat{\xi}. \quad (10)$$

Thus, $\varpi * \eta \in \widehat{\mathscr{M}^{\mathscr{P}}}_{\widehat{\xi}}$. Hence, $\widehat{\mathscr{M}^{\mathscr{P}}}_{\widehat{\xi}}$ is a positive implicative ideal of Ω.

Conversely, assume that $\widehat{\mathscr{M}^{\mathscr{P}}}_{\widehat{\xi}} \neq \phi$ is a positive implicative ideal of Ω for all $\widehat{\xi} \in (0,1]^m$. If there exists $t \in \Omega$ such that $\widehat{\mathscr{M}^{\mathscr{P}}}(0) < \widehat{\mathscr{M}^{\mathscr{P}}}(t)$, then $\widehat{\mathscr{M}^{\mathscr{P}}}(0) < \widehat{\psi} \leq \widehat{\mathscr{M}^{\mathscr{P}}}(t)$ for some $\widehat{\psi} \in (0,1]^m$. Then, $0 \notin \widehat{\mathscr{M}^{\mathscr{P}}}_{\widehat{\psi}}$, a contradiction. Thus, $\widehat{\mathscr{M}^{\mathscr{P}}}(0) \geq \widehat{\mathscr{M}^{\mathscr{P}}}(\varpi)$ for all $\varpi \in \Omega$. If there exist $t, k, q \in \Omega$ such that $\widehat{\mathscr{M}^{\mathscr{P}}}(t * q) < \inf\left\{\widehat{\mathscr{M}^{\mathscr{P}}}((t * k) * q), \widehat{\mathscr{M}^{\mathscr{P}}}(k * q)\right\}$, then

$$\widehat{\mathscr{M}^{\mathscr{P}}}(t * q) < \widehat{\zeta} \leq \inf\left\{\widehat{\mathscr{M}^{\mathscr{P}}}((t * k) * q), \widehat{\mathscr{M}^{\mathscr{P}}}(k * q)\right\}, \quad (11)$$

for some $\widehat{\xi} \in (0,1]^m$. It follows that $(t * k) * q \in \widehat{\mathscr{M}^{\mathscr{P}}}_{\widehat{\zeta}}$ and $k * q \in \widehat{\mathscr{M}^{\mathscr{P}}}_{\widehat{\zeta}}$, but $t * q \notin \widehat{\mathscr{M}^{\mathscr{P}}}_{\widehat{\zeta}}$. This is a contradiction. Thus, $\widehat{\mathscr{M}^{\mathscr{P}}}(\varpi * \eta) \geq \inf\left\{\widehat{\mathscr{M}^{\mathscr{P}}}((\varpi * \varrho) * \eta), \widehat{\mathscr{M}^{\mathscr{P}}}(\varrho * \eta)\right\}$ for all $\varpi, \varrho, \eta \in \Omega$. Hence, $\widehat{\mathscr{M}^{\mathscr{P}}}$ is an m-pFPI ideal of Ω. □

4. m-Polar $(\in, \in\vee q)$-Fuzzy Positive Implicative Ideals

Al-Masarwah and Ahmad [40] extended the concepts of "belongingness" and "quasi-coincidence" of a fuzzy point with a fuzzy set and proposed the concepts of "belongingness" and "quasi-coincidence" of an m-pF point with an m-pF set as follows.

An m-pF set $\widehat{\mathscr{M}^{\mathscr{P}}}$ of Ω of the form

$$\widehat{\mathscr{M}^{\mathscr{P}}}(\varrho) = \begin{cases} \widehat{\xi} = (\xi_1, \xi_2, \ldots, \xi_m) \in (0,1]^m, & \text{if } \varrho = \varpi, \\ \widehat{0} = (0, 0, \ldots, 0), & \text{if } \varrho \neq \varpi \end{cases} \quad (12)$$

is said to be an m-pF point, written as $\varpi_{\widehat{\xi}}$, with support ϖ and value $(\xi_1, \xi_2, \ldots, \xi_m) = \widehat{\xi}$.

An m-pF point $\varpi_{\widehat{\xi}}$

(1) Belongs to $\widehat{\mathscr{M}^{\mathscr{P}}}$, written as $\varpi_{\widehat{\xi}} \in \widehat{\mathscr{M}^{\mathscr{P}}}$, if $\widehat{\mathscr{M}^{\mathscr{P}}}(\varpi) \geq \widehat{\xi}$, i.e., $\forall i = 1, 2, \ldots, m,\ p_i \circ \widehat{\mathscr{M}^{\mathscr{P}}}(\varpi) \geq \xi_i$.

(2) Is quasi-coincident with $\widehat{\mathscr{M}^{\mathscr{P}}}$, written as $\varpi_{\widehat{\xi}} q \widehat{\mathscr{M}^{\mathscr{P}}}$, if $\widehat{\mathscr{M}^{\mathscr{P}}}(\varpi) + \widehat{\xi} > \widehat{1}$, i.e., $\forall i = 1, 2, \ldots, m,\ p_i \circ \widehat{\mathscr{M}^{\mathscr{P}}}(\varpi) + \xi_i > 1$.

We say that

(1) $\varpi_{\widehat{\xi}} \overline{\alpha} \widehat{\mathscr{M}^{\mathscr{P}}}$ if $\varpi_{\widehat{\xi}} \alpha \widehat{\mathscr{M}^{\mathscr{P}}}$ does not hold.

(2) $\varpi_{\widehat{\xi}} \in\vee q \widehat{\mathscr{M}^{\mathscr{P}}}$ (resp., $\varpi_{\widehat{\xi}} \in\wedge q \widehat{\mathscr{M}^{\mathscr{P}}}$) if $\varpi_{\widehat{\xi}} \in \widehat{\mathscr{M}^{\mathscr{P}}}$ or $\varpi_{\widehat{\xi}} q \widehat{\mathscr{M}^{\mathscr{P}}}$ (resp., $\varpi_{\widehat{\xi}} \in \widehat{\mathscr{M}^{\mathscr{P}}}$ and $\varpi_{\widehat{\xi}} q \widehat{\mathscr{M}^{\mathscr{P}}}$).

Next, we introduce $m - p_{(\in, \in\vee q)}$ – FPI ideals of Ω and discuss several results.

Definition 6. An m-pF set $\widehat{\mathscr{M}^{\mathscr{P}}}$ of a BCK-algebra Ω is called an $m - p_{(\in, \in\vee q)}$ – FPI ideal of Ω if for all $\varpi, \varrho, \eta \in \Omega$ and $\widehat{\xi}, \widehat{\vartheta} \in (0,1]^m$,

(1) $\varpi_{\widehat{\xi}} \in \widehat{\mathscr{M}^{\mathscr{P}}}$ implies $0_{\widehat{\xi}} \in\vee q \widehat{\mathscr{M}^{\mathscr{P}}}$.

(2) $((\varpi * \varrho) * \eta)_{\widehat{\xi}} \in \widehat{\mathscr{M}^{\mathscr{P}}}$ and $(\varrho * \eta)_{\widehat{\vartheta}} \in \widehat{\mathscr{M}^{\mathscr{P}}}$ imply $(\varpi * \eta)_{\inf\{\widehat{\xi}, \widehat{\vartheta}\}} \in\vee q \widehat{\mathscr{M}^{\mathscr{P}}}$.

Example 3. Consider a BCK-algebra $\Omega = \{0, \imath, \jmath, \ell\}$ which is given in Example 2. Let $\widehat{\mathscr{M}^{\mathscr{P}}}$ be an m-pF set defined as

$$\widehat{\mathscr{M}^{\mathscr{P}}}(\varpi) = \begin{cases} (0.7, \ldots, 0.7), & \text{if } \varpi = 0. \\ (0.6, \ldots, 0.6), & \text{if } \varpi = \imath, \jmath, \ell. \end{cases} \quad (13)$$

Then, $\widehat{\mathscr{M}^{\mathscr{P}}}$ is an $m - p_{(\in, \in\vee q)}$ – FPI ideal of Ω.

Theorem 4. *An m-pF set $\widehat{\mathscr{M}^{\mathscr{P}}}$ of Ω is an $m - p_{(\in, \in\vee q)}$ – FPI ideal of Ω if and only if for all $\varpi, \varrho, \eta \in \Omega$,*

(i) $\widehat{\mathscr{M}^{\mathscr{P}}}(0) \geq \inf\left\{\widehat{\mathscr{M}^{\mathscr{P}}}(\varpi), \widehat{0.5}\right\}$.

(ii) $\widehat{\mathscr{M}^{\mathscr{P}}}(\varpi * \eta) \geq \inf\left\{\widehat{\mathscr{M}^{\mathscr{P}}}((\varpi * \varrho) * \eta), \widehat{\mathscr{M}^{\mathscr{P}}}(\varrho * \eta), \widehat{0.5}\right\}$.

Proof. Suppose that $\widehat{\mathscr{M}^{\mathscr{P}}}$ is an $m - p_{(\in, \in\vee q)}$ – FPI ideal of Ω. Let $\varpi \in \Omega$ and assume $\widehat{\mathscr{M}^{\mathscr{P}}}(\varpi) < \widehat{0.5}$. If $\widehat{\mathscr{M}^{\mathscr{P}}}(0) < \widehat{\mathscr{M}^{\mathscr{P}}}(\varpi)$, then

$$\widehat{\mathscr{M}^{\mathscr{P}}}(0) < \widehat{\xi} \leq \widehat{\mathscr{M}^{\mathscr{P}}}(\varpi), \quad (14)$$

for some $\widehat{0} < \widehat{\xi} < \widehat{0.5}$. This implies that $\varpi_{\widehat{\xi}} \in \widehat{\mathscr{M}^{\mathscr{P}}}$, but $0_{\widehat{\xi}} \overline{\in} \widehat{\mathscr{M}^{\mathscr{P}}}$. Since $\widehat{\mathscr{M}^{\mathscr{P}}}(0) + \widehat{\xi} < \widehat{1}$, we get $0_{\widehat{\xi}} \overline{q} \widehat{\mathscr{M}^{\mathscr{P}}}$. Therefore, $0_{\widehat{\xi}} \overline{\in\vee q} \widehat{\mathscr{M}^{\mathscr{P}}}$, a contradiction. Thus, $\widehat{\mathscr{M}^{\mathscr{P}}}(0) \geq \widehat{\mathscr{M}^{\mathscr{P}}}(\varpi)$ for all $\varpi \in \Omega$. If $\widehat{\mathscr{M}^{\mathscr{P}}}(\varpi) \geq \widehat{0.5}$, then $\varpi_{\widehat{0.5}} \in \widehat{\mathscr{M}^{\mathscr{P}}}$, and so $0_{\widehat{0.5}} \in\vee q \widehat{\mathscr{M}^{\mathscr{P}}}$. Hence, $\widehat{\mathscr{M}^{\mathscr{P}}}(0) \geq \widehat{0.5}$. Otherwise, $\widehat{\mathscr{M}^{\mathscr{P}}}(0) + \widehat{0.5} < \widehat{1}$, a contradiction. Hence, $\widehat{\mathscr{M}^{\mathscr{P}}}(0) \geq \inf\left\{\widehat{\mathscr{M}^{\mathscr{P}}}(\varpi), \widehat{0.5}\right\}$ for all $\varpi \in \Omega$. Let $\varpi, \varrho, \eta \in \Omega$. Assume that

$$\inf\left\{\widehat{\mathscr{M}^{\mathscr{P}}}((\varpi * \varrho) * \eta), \widehat{\mathscr{M}^{\mathscr{P}}}(\varrho * \eta)\right\} < \widehat{0.5}. \quad (15)$$

Then,

$$\widehat{\mathscr{M}^{\mathscr{P}}}(\varpi * \eta) \geq \inf\left\{\widehat{\mathscr{M}^{\mathscr{P}}}((\varpi * \varrho) * \eta), \widehat{\mathscr{M}^{\mathscr{P}}}(\varrho * \eta)\right\}. \quad (16)$$

If not, then

$$\widehat{\mathscr{M}^{\mathscr{P}}}(\varpi * \eta) < \widehat{\xi} \leq \inf\left\{\widehat{\mathscr{M}^{\mathscr{P}}}((\varpi * \varrho) * \eta), \widehat{\mathscr{M}^{\mathscr{P}}}(\varrho * \eta)\right\}, \quad (17)$$

for some $\widehat{0} < \widehat{\xi} < \widehat{0.5}$. This implies that $((\varpi * \varrho) * \eta)_{\widehat{\xi}} \in \widehat{\mathscr{M}^{\mathscr{P}}}$ and $(\varrho * \eta)_{\widehat{\xi}} \in \widehat{\mathscr{M}^{\mathscr{P}}}$, but $(\varpi * \eta)_{\widehat{\xi}} \overline{\in\vee q} \widehat{\mathscr{M}^{\mathscr{P}}}$, a contradiction.

Hence, $\widehat{\mathscr{M}^{\mathscr{P}}}(\varpi * \eta) \geq \inf\left\{\widehat{\mathscr{M}^{\mathscr{P}}}((\varpi * \varrho) * \eta), \widehat{\mathscr{M}^{\mathscr{P}}}(\varrho * \eta)\right\}$ whenever $\inf\left\{\widehat{\mathscr{M}^{\mathscr{P}}}((\varpi * \varrho) * \eta), \widehat{\mathscr{M}^{\mathscr{P}}}(\varrho * \eta)\right\} < \widehat{0.5}$. If $\inf\left\{\widehat{\mathscr{M}^{\mathscr{P}}}((\varpi * \varrho) * \eta), \widehat{\mathscr{M}^{\mathscr{P}}}(\varrho * \eta)\right\} \geq \widehat{0.5}$, then

$$\begin{aligned} ((\varpi * \varrho) * \eta)_{\widehat{0.5}} &\in \widehat{\mathscr{M}^{\mathscr{P}}}, \\ (\varrho * \eta)_{\widehat{0.5}} &\in \widehat{\mathscr{M}^{\mathscr{P}}}. \end{aligned} \tag{18}$$

It follows that $(\varpi * \eta)_{\widehat{0.5}} = (\varpi * \eta)_{\inf\{\widehat{0.5}, \widehat{0.5}\}} \in \vee q \widehat{\mathscr{M}^{\mathscr{P}}}$. Therefore, $\widehat{\mathscr{M}^{\mathscr{P}}}(\varpi * \eta) \geq \widehat{0.5}$ or $\widehat{\mathscr{M}^{\mathscr{P}}}(\varpi * \eta) + \widehat{0.5} > \widehat{1}$. If $\widehat{\mathscr{M}^{\mathscr{P}}}(\varpi * \eta) < \widehat{0.5}$, then

$$\widehat{\mathscr{M}^{\mathscr{P}}}(\varpi * \eta) + \widehat{0.5} < \widehat{1}, \tag{19}$$

a contradiction. Therefore, $\widehat{\mathscr{M}^{\mathscr{P}}}(\varpi * \eta) \geq \inf\left\{\widehat{\mathscr{M}^{\mathscr{P}}}((\varpi * \varrho) * \eta), \widehat{\mathscr{M}^{\mathscr{P}}}(\varrho * \eta), \widehat{0.5}\right\}$ for all $\varpi, \varrho, \eta \in \Omega$.

Conversely, suppose that (i) and (ii) hold. Let $\varpi \in \Omega$ and $\widehat{\xi} \in (0,1]^m$ be such that $\varpi_{\widehat{\xi}} \in \widehat{\mathscr{M}^{\mathscr{P}}}$. Then, $\widehat{\mathscr{M}^{\mathscr{P}}}(\varpi) \geq \widehat{\xi}$. Assume $\widehat{\mathscr{M}^{\mathscr{P}}}(0) < \widehat{\xi}$. If $\widehat{\mathscr{M}^{\mathscr{P}}}(\varpi) < \widehat{0.5}$, then

$$\begin{aligned} \widehat{\mathscr{M}^{\mathscr{P}}}(0) &\geq \inf\left\{\widehat{\mathscr{M}^{\mathscr{P}}}(\varpi), \widehat{0.5}\right\} \\ &= \widehat{\mathscr{M}^{\mathscr{P}}}(\varpi) \geq \widehat{\xi}, \end{aligned} \tag{20}$$

a contradiction. Therefore, $\widehat{\mathscr{M}^{\mathscr{P}}}(\varpi) \geq \widehat{0.5}$, which implies that

$$\begin{aligned} \widehat{\mathscr{M}^{\mathscr{P}}}(0) + \widehat{\xi} &> 2\widehat{\mathscr{M}^{\mathscr{P}}}(0) \\ &\geq 2\inf\left\{\widehat{\mathscr{M}^{\mathscr{P}}}(\varpi), \widehat{0.5}\right\} \\ &= \widehat{1}. \end{aligned} \tag{21}$$

Thus, $0_{\widehat{\xi}} \in \vee q \widehat{\mathscr{M}^{\mathscr{P}}}$. Let $\varpi, \varrho, \eta \in \Omega$ and $\widehat{\xi}, \widehat{\vartheta} \in (0,1]^m$ be such that $((\varpi * \varrho) * \eta)_{\widehat{\xi}} \in \widehat{\mathscr{M}^{\mathscr{P}}}$ and $(\varrho * \eta)_{\widehat{\vartheta}} \in \widehat{\mathscr{M}^{\mathscr{P}}}$. Then,

$$\begin{aligned} \widehat{\mathscr{M}^{\mathscr{P}}}((\varpi * \varrho) * \eta) &\geq \widehat{\xi}, \\ \widehat{\mathscr{M}^{\mathscr{P}}}(\varrho * \eta) &\geq \widehat{\vartheta}. \end{aligned} \tag{22}$$

Suppose that $\widehat{\mathscr{M}^{\mathscr{P}}}(\varpi * \eta) < \inf\{\widehat{\xi}, \widehat{\vartheta}\}$. If $\inf\left\{\widehat{\mathscr{M}^{\mathscr{P}}}((\varpi * \varrho) * \eta), \widehat{\mathscr{M}^{\mathscr{P}}}(\varrho * \eta)\right\} < \widehat{0.5}$, then

$$\begin{aligned} \widehat{\mathscr{M}^{\mathscr{P}}}(\varpi * \eta) &\geq \inf\left\{\widehat{\mathscr{M}^{\mathscr{P}}}((\varpi * \varrho) * \eta), \widehat{\mathscr{M}^{\mathscr{P}}}(\varrho * \eta), \widehat{0.5}\right\} \\ &= \inf\left\{\widehat{\mathscr{M}^{\mathscr{P}}}((\varpi * \varrho) * \eta), \widehat{\mathscr{M}^{\mathscr{P}}}(\varrho * \eta)\right\} \\ &\geq \inf\{\widehat{\xi}, \widehat{\vartheta}\}, \end{aligned} \tag{23}$$

a contradiction. Hence, $\inf\left\{\widehat{\mathscr{M}^{\mathscr{P}}}((\varpi * \varrho) * \eta), \widehat{\mathscr{M}^{\mathscr{P}}}(\varrho * \eta)\right\} \geq \widehat{0.5}$. This implies that

$$\begin{aligned} \widehat{\mathscr{M}^{\mathscr{P}}}(\varpi * \eta) + \inf\{\widehat{\xi}, \widehat{\vartheta}\} &> 2\widehat{\mathscr{M}^{\mathscr{P}}}(\varpi * \eta) \\ &\geq 2\inf\left\{\widehat{\mathscr{M}^{\mathscr{P}}}((\varpi * \varrho) * \eta), \widehat{\mathscr{M}^{\mathscr{P}}}(\varrho * \eta), \widehat{0.5}\right\} \\ &= \widehat{1}. \end{aligned} \tag{24}$$

So, $(\varpi * \eta)_{\inf\{\widehat{\xi}, \widehat{\vartheta}\}} \in \vee q \widehat{\mathscr{M}^{\mathscr{P}}}$. Hence, $\widehat{\mathscr{M}^{\mathscr{P}}}$ is an $m-p_{(\in, \in\vee q)}-$FPI ideal of X. □

Theorem 5. *Every $m-p_{(\in, \in\vee q)}-$FPI ideal of Ω is an $m-p_{(\in, \in\vee q)}-F$ ideal of X, but the converse does not hold.*

Proof. Let $\widehat{\mathscr{M}^{\mathscr{P}}}$ be an $m-p_{(\in, \in\vee q)}-$FPI ideal of Ω. Then, (1) of Definition 6 holds. Put $\eta = 0$ in Definition 6 (2), and we get

$$\begin{aligned} &((\varpi * \varrho) * 0)_{\widehat{\xi}} \in \widehat{\mathscr{M}^{\mathscr{P}}}, \\ &(\varrho * 0)_{\widehat{\vartheta}} \in \widehat{\mathscr{M}^{\mathscr{P}}} \text{ imply } (\varpi * 0)_{\inf\{\widehat{\xi}, \widehat{\vartheta}\}} \in \vee q \widehat{\mathscr{M}^{\mathscr{P}}}. \end{aligned} \tag{25}$$

So,

$$\begin{aligned} &(\varpi * \varrho)_{\widehat{\xi}} \in \widehat{\mathscr{M}^{\mathscr{P}}}, \\ &\varrho_{\widehat{\vartheta}} \in \widehat{\mathscr{M}^{\mathscr{P}}} \text{ imply } \varpi_{\inf\{\widehat{\xi}, \widehat{\vartheta}\}} \in \vee q \widehat{\mathscr{M}^{\mathscr{P}}}. \end{aligned} \tag{26}$$

Hence, $\widehat{\mathscr{M}^{\mathscr{P}}}$ is an $m-p_{(\in, \in\vee q)}-F$ ideal of Ω.

The last part is shown by the following example. □

Example 4. Reconsider the BCK-algebras Ω given in Example 2. Let $\widehat{\mathscr{M}^{\mathscr{P}}}$ be a 3-pF set defined as

$$\widehat{\mathscr{M}^{\mathscr{P}}}(\varpi) = \begin{cases} (0.7, 0.6, 0.5), & \text{if } \varpi = 0, \\ (0.4, 0.3, 0.2), & \text{if } \varpi = \imath, \jmath, \\ (0.3, 0.2, 0.1), & \text{if } \varpi = \ell. \end{cases} \tag{27}$$

Then, $\widehat{\mathscr{M}^{\mathscr{P}}}$ is a 3-polar $(\in, \in\vee q)$-fuzzy ideal of Ω, but it is not a $3-p_{(\in, \in\vee q)}-$FPI ideal of Ω since $\widehat{\mathscr{M}^{\mathscr{P}}}(j * \imath) = \widehat{\mathscr{M}^{\mathscr{P}}}(\imath) = (0.4, 0.3, 0.2) < \inf\left\{\widehat{\mathscr{M}^{\mathscr{P}}}((\mathrm{j} * \imath) * \imath), \widehat{\mathscr{M}^{\mathscr{P}}}(\imath * \imath), \widehat{0.5}\right\} = \inf\left\{\widehat{\mathscr{M}^{\mathscr{P}}}(0), \widehat{\mathscr{M}^{\mathscr{P}}}(0), \widehat{0.5}\right\} = \widehat{0.5}$.

Lemma 2 (see [40]). *Every $m-p_{(\in, \in\vee q)}-F$ ideal $\widehat{\mathscr{M}^{\mathscr{P}}}$ of Ω satisfies the following assertion: for all $\varpi, \varrho \in \Omega$,*

$$\varpi \leq \varrho \Rightarrow \widehat{\mathscr{M}^{\mathscr{P}}}(\varpi) \geq \inf\left\{\widehat{\mathscr{M}^{\mathscr{P}}}(\varrho), \widehat{0.5}\right\}. \tag{28}$$

Theorem 6. *Let $\widehat{\mathscr{M}^{\mathscr{P}}}$ be an $m-p_{(\in,\in\vee q)}-F$ ideal of Ω. Then, $\widehat{\mathscr{M}^{\mathscr{P}}}$ is an $m-p_{(\in,\in\vee q)}-FPI$ ideal of Ω if and only if $\widehat{\mathscr{M}^{\mathscr{P}}}(\varpi*\varrho)\geq\inf\left\{\widehat{\mathscr{M}^{\mathscr{P}}}((\varpi*\varrho)*\varrho),\widehat{0.5}\right\}$ for all $\varpi,\varrho\in\Omega$.*

Proof. Assume $\widehat{\mathscr{M}^{\mathscr{P}}}$ is an $m-p_{(\in,\in\vee q)}-$ FPI ideal of Ω. If η is replaced by ϱ in Theorem 4 (2), then

$$\begin{aligned}\widehat{\mathscr{M}^{\mathscr{P}}}(\varpi*\varrho)&\geq\inf\left\{\widehat{\mathscr{M}^{\mathscr{P}}}((\varpi*\varrho)*\varrho),\widehat{\mathscr{M}^{\mathscr{P}}}(\varrho*\varrho),\widehat{0.5}\right\}\\&=\inf\left\{\widehat{\mathscr{M}^{\mathscr{P}}}((\varpi*\varrho)*\varrho),\widehat{\mathscr{M}^{\mathscr{P}}}(0),\widehat{0.5}\right\}\\&=\inf\left\{\widehat{\mathscr{M}^{\mathscr{P}}}((\varpi*\varrho)*\varrho),\widehat{0.5}\right\},\end{aligned}\tag{29}$$

for all $\varpi,\varrho\in\Omega$.

Conversely, let $\widehat{\mathscr{M}^{\mathscr{P}}}$ be an $m-p_{(\in,\in\vee q)}-F$ ideal of Ω. Then, $\widehat{\mathscr{M}^{\mathscr{P}}}(0)\geq\inf\left\{\widehat{\mathscr{M}^{\mathscr{P}}}(\varpi),\widehat{0.5}\right\}$ for all $\varpi\in\Omega$. Also, since $((\varpi*\eta)*\eta)*(\varrho*\eta)\leq(\varpi*\eta)*\varrho=(\varpi*\varrho)*\eta$ for all $\varpi,\varrho\in\Omega$, it follows by Lemma 2 that

$$\widehat{\mathscr{M}^{\mathscr{P}}}(((\varpi*\eta)*\eta)*(\varrho*\eta))\geq\inf\left\{\widehat{\mathscr{M}^{\mathscr{P}}}((\varpi*\varrho)*\eta),\widehat{0.5}\right\}.\tag{30}$$

Now, by assumption

$$\begin{aligned}\widehat{\mathscr{M}^{\mathscr{P}}}(\varpi*\eta)&\geq\inf\left\{\widehat{\mathscr{M}^{\mathscr{P}}}((\varpi*\eta)*\eta),\widehat{0.5}\right\}\\&\geq\inf\left\{\widehat{\mathscr{M}^{\mathscr{P}}}(((\varpi*\eta)*\eta)*(\varrho*\eta)),\right.\\&\qquad\left.\widehat{\mathscr{M}^{\mathscr{P}}}(\varrho*\eta),\widehat{0.5},\widehat{0.5}\right\}\\&\geq\inf\left\{\widehat{\mathscr{M}^{\mathscr{P}}}((\varpi*\varrho)*\eta),\widehat{\mathscr{M}^{\mathscr{P}}}(\varrho*\eta),\widehat{0.5}\right\}.\end{aligned}\tag{31}$$

Hence, $\widehat{\mathscr{M}^{\mathscr{P}}}$ is an $m-p_{(\in,\in\vee q)}-$ FPI ideal of Ω. □

Theorem 7. *An m-pF set $\widehat{\mathscr{M}^{\mathscr{P}}}$ of Ω is an $m-p_{(\in,\in\vee q)}-FPI$ ideal of a BCK-algebra Ω if and only if $\widehat{\mathscr{M}^{\mathscr{P}}}_{\widehat{\xi}}\neq\phi$ is a positive implicative ideal of Ω for all $\widehat{\xi}\in(0,0.5]^m$.*

Proof. Assume that $\widehat{\mathscr{M}^{\mathscr{P}}}$ is an $m-p_{(\in,\in\vee q)}-$ FPI ideal of a BCK-algebra X. Let $\widehat{\xi}\in(0,0.5]^m$ and $\varpi\in\widehat{\mathscr{M}^{\mathscr{P}}}_{\widehat{\xi}}$. Then, $\widehat{\mathscr{M}^{\mathscr{P}}}(\varpi)\geq\widehat{\xi}$. Theorem 4 (i) implies that

$$\widehat{\mathscr{M}^{\mathscr{P}}}(0)\geq\inf\left\{\widehat{\mathscr{M}^{\mathscr{P}}}(\varpi),\widehat{0.5}\right\}=\widehat{\xi}.\tag{32}$$

Thus, $0\in\widehat{\mathscr{M}^{\mathscr{P}}}_{\widehat{\xi}}$. Again, let $(\varpi*\rangle)*z,y*z\in\widehat{\mathscr{M}^{\mathscr{P}}}_{\widehat{\xi}}$. Then, $\widehat{\mathscr{M}^{\mathscr{P}}}((\varpi*\varrho)*\eta)\geq\widehat{\xi}$ and $\widehat{\mathscr{M}^{\mathscr{P}}}(\varrho*\eta)\geq\widehat{\xi}$. Theorem 4 (ii) implies that

$$\begin{aligned}\widehat{\mathscr{M}^{\mathscr{P}}}(\varpi*\eta)&\geq\inf\left\{\widehat{\mathscr{M}^{\mathscr{P}}}((\varpi*\varrho)*\eta),\widehat{\mathscr{M}^{\mathscr{P}}}(\varrho*\eta),\widehat{0.5}\right\}\\&\geq\inf\left\{\widehat{\xi},\widehat{\xi},t\widehat{0.5}\right\}=\widehat{\xi}.\end{aligned}\tag{33}$$

Therefore, $\varpi*\eta\in\widehat{\mathscr{M}^{\mathscr{P}}}_{\widehat{\xi}}$. Thus, $\widehat{\mathscr{M}^{\mathscr{P}}}_{\widehat{\xi}}$ is a positive implicative ideal of Ω.

Conversely, let $\widehat{\mathscr{M}^{\mathscr{P}}}$ be an m-pF set of Ω be such that $\widehat{\mathscr{M}^{\mathscr{P}}}_{\widehat{\xi}}\neq\phi$ is a positive implicative ideal of Ω for all $\widehat{\xi}\in(0,0.5]^m$. If there exists $k\in\Omega$ such that $\widehat{\mathscr{M}^{\mathscr{P}}}(0)<\inf\left\{\widehat{\mathscr{M}^{\mathscr{P}}}(k),\widehat{0.5}\right\}$, then

$$\widehat{\mathscr{M}^{\mathscr{P}}}(0)<\widehat{\xi}_k\leq\inf\left\{\widehat{\mathscr{M}^{\mathscr{P}}}(k),\widehat{0.5}\right\},\tag{34}$$

for some $\widehat{\xi}\in(0,0.5]^m$. It follows that $k\in\widehat{\mathscr{M}^{\mathscr{P}}}_{\widehat{\xi}_k}$, but $0\overline{\in}\widehat{\mathscr{M}^{\mathscr{P}}}_{\widehat{t}_k}$, a contradiction. Therefore, $\widehat{\mathscr{M}^{\mathscr{P}}}(0)\geq\inf\left\{\widehat{\mathscr{M}^{\mathscr{P}}}(\varpi),\widehat{0.5}\right\}$ for all $\varpi\in\Omega$. Suppose there exist $u,v,w\in\Omega$ such that

$$\widehat{\mathscr{M}^{\mathscr{P}}}(u*w)<\inf\left\{\widehat{\mathscr{M}^{\mathscr{P}}}((u*v)*w),\widehat{\mathscr{M}^{\mathscr{P}}}(v*w),\widehat{0.5}\right\}.\tag{35}$$

Then, $\widehat{\mathscr{M}^{\mathscr{P}}}(u*w)<\widehat{\xi}_u\leq\inf\left\{\widehat{\mathscr{M}^{\mathscr{P}}}((u*v)*w),\widehat{\mathscr{M}^{\mathscr{P}}}(v*w),\widehat{0.5}\right\}$ for some $\widehat{\xi}_u\in(0,0.5]^m$. This implies that $(u*v)*w\in\widehat{\mathscr{M}^{\mathscr{P}}}_{\widehat{\xi}_u}$ and $v*w\in\widehat{\mathscr{M}^{\mathscr{P}}}_{\widehat{\xi}_u}$, but $u*w\overline{\in}\widehat{\mathscr{M}^{\mathscr{P}}}_{\widehat{\xi}_u}$. This is impossible. Thus,

$$\widehat{\mathscr{M}^{\mathscr{P}}}(\varpi*\eta)\geq\inf\left\{\widehat{\mathscr{M}^{\mathscr{P}}}((\varpi*\varrho)*\eta),\widehat{\mathscr{M}^{\mathscr{P}}}(\varrho*\eta),\widehat{0.5}\right\},\tag{36}$$

for all $\varpi,\varrho,\eta\in\Omega$. Hence, $\widehat{\mathscr{M}^{\mathscr{P}}}$ is an $m-p_{(\in,\in\vee q)}-$ FPI ideal of Ω by Theorem 4. □

5. m-Polar $(\overline{\in},\overline{\in}\vee\overline{q})$-Fuzzy Positive Implicative Ideals

In the current section, we introduce $m-p_{(\overline{\in},\overline{\in}\vee\overline{q})-\mathrm{FPI}}$ ideals of Ω and discuss some relevant results and properties.

Definition 7 (see [38]). An m-pF set $\widehat{\mathscr{M}^{\mathscr{P}}}$ of Ω is called an $m-p_{(\overline{\in},\overline{\in}\vee\overline{q})-\mathrm{FPI}}$ ideal of Ω if for all $\varpi,\varrho\in\Omega$ and $\widehat{\xi},\widehat{\vartheta}\in(0,1]^m$,

(1) $0_{\widehat{\xi}}\overline{\in}\widehat{\mathscr{M}^{\mathscr{P}}}$ implies $\varpi_{\widehat{\xi}}\overline{\in}\vee\overline{q}\widehat{\mathscr{M}^{\mathscr{P}}}$.

(2) $\varpi_{\inf\{\widehat{\xi},\widehat{\vartheta}\}}\overline{\in}\widehat{\mathscr{M}^{\mathscr{P}}}$ implies $(\varpi*\varrho)_{\widehat{\xi}}\overline{\in}\vee\overline{q}\widehat{\mathscr{M}^{\mathscr{P}}}$ or $\varrho_{\widehat{\vartheta}}\in\vee\overline{q}\widehat{\mathscr{M}^{\mathscr{P}}}$.

Theorem 8 (see [38]). *An m-pF set $\widehat{\mathscr{M}^{\mathscr{P}}}$ of Ω is an $m-p_{(\overline{\in},\overline{\in}\vee\overline{q})-F}$ ideal of Ω if and only if for all $\varpi,\varrho\in\Omega$,*

(i) $\sup\left\{\widehat{\mathscr{M}^{\mathscr{P}}}(0),\widehat{0.5}\right\}\geq\widehat{\mathscr{M}^{\mathscr{P}}}(\varpi)$.

(ii) $\sup\left\{\widehat{\mathscr{M}^{\mathscr{P}}}(\varpi),\widehat{0.5}\right\}\geq\inf\left\{\widehat{\mathscr{M}^{\mathscr{P}}}(\varpi*\varrho),\widehat{\mathscr{M}^{\mathscr{P}}}(\varrho)\right\}$.

Lemma 3 (see [38]). *Any $m-p_{(\overline{\in},\overline{\in}\vee\overline{q})-F}$ ideal $\widehat{\mathscr{M}^{\mathscr{P}}}$ of Ω satisfies the following assertion for all $\varpi,\varrho\in\Omega$:*

$$\varpi\leq\varrho\Rightarrow\sup\left\{\widehat{\mathscr{M}^{\mathscr{P}}}(\varpi),\widehat{0.5}\right\}\geq\left\{\widehat{\mathscr{M}^{\mathscr{P}}}(\varrho)\right\}.\tag{37}$$

Definition 8. An m-pF set $\widehat{\mathscr{M}^{\mathscr{P}}}$ of Ω is called an $m-p_{(\overline{\in},\overline{\in}\vee\overline{q})-\mathrm{FPI}}$ ideal of Ω if for all $\varpi,\varrho,\eta\in\Omega$ and $\hat{\xi},\hat{\vartheta}\in(0,1]^m$,

(1) $0_{\hat{\xi}}\overline{\in}\widehat{\mathscr{M}^{\mathscr{P}}}$ implies $\varpi_{\hat{\xi}}\overline{\in}\vee\overline{q}\widehat{\mathscr{M}^{\mathscr{P}}}$.

(2) $(\varpi*\eta)_{\inf\{\hat{\xi},\hat{\vartheta}\}}\overline{\in}\widehat{\mathscr{M}^{\mathscr{P}}}$ implies $((\varpi*\varrho)*\eta)_{\hat{\xi}}\overline{\in}\vee\overline{q}\widehat{\mathscr{M}^{\mathscr{P}}}$ or $(\varrho*\eta)_{\hat{\vartheta}}\overline{\in}\vee\overline{q}\widehat{\mathscr{M}^{\mathscr{P}}}$.

Example 5. Consider a BCK-algebra $X=\{0,\imath,\jmath,\ell\}$ which is given in Example 2. Let $\widehat{\mathscr{M}^{\mathscr{P}}}$ be a 3-pF set defined as

$$\widehat{\mathscr{M}^{\mathscr{P}}}(\varpi)=\begin{cases}(0.5,0.5,0.5), & \text{if } \varpi=0,\\ (0.3,0.3,0.4), & \text{if } \varpi=\imath,\jmath,\\ (0.2,0.2,0.2), & \text{if } \varpi=\ell.\end{cases} \tag{38}$$

Then, $\widehat{\mathscr{M}^{\mathscr{P}}}$ is a $3-p_{(\overline{\in},\overline{\in}\vee\overline{q})-\mathrm{FPI}}$ ideal of Ω.

Theorem 9. *An m-pF set $\widehat{\mathscr{M}^{\mathscr{P}}}$ of Ω is an $m-p_{(\overline{\in},\overline{\in}\vee\overline{q})-FPI}$ ideal of Ω if and only if for all $\varpi,\varrho,\eta\in\Omega$,*

(i) $\sup\{\widehat{\mathscr{M}^{\mathscr{P}}}(0),\widehat{0.5}\}\geq\widehat{\mathscr{M}^{\mathscr{P}}}(\varpi)$.

(ii) $\sup\{\widehat{\mathscr{M}^{\mathscr{P}}}(\varpi*\eta),\widehat{0.5}\}\geq\inf\{\widehat{\mathscr{M}^{\mathscr{P}}}((\varpi*\varrho)*\eta),\widehat{\mathscr{M}^{\mathscr{P}}}(\varrho*\eta)\}$.

Proof. Let $\widehat{\mathscr{M}^{\mathscr{P}}}$ be an $m-p_{(\overline{\in},\overline{\in}\vee\overline{q})-\mathrm{FPI}}$ ideal of Ω. Assume there exists $\varpi\in\Omega$ such that $\sup\{\widehat{\mathscr{M}^{\mathscr{P}}}(0),\widehat{0.5}\}<\hat{\xi}=\widehat{\mathscr{M}^{\mathscr{P}}}(\varpi)$. Then,

$$\begin{aligned}&\hat{\xi}\in(0.5,1]^m,\\ &0_{\hat{\xi}}\overline{\in}\widehat{\mathscr{M}^{\mathscr{P}}},\varpi_{\hat{\xi}}\in\widehat{\mathscr{M}^{\mathscr{P}}}.\end{aligned} \tag{39}$$

By Definition 8 (1), we have $\varpi_{\hat{\xi}}\overline{\in}\vee\overline{q}\widehat{\mathscr{M}^{\mathscr{P}}}$, i.e., $\widehat{\mathscr{M}^{\mathscr{P}}}(\varpi)<\hat{\xi}$ or $\widehat{\mathscr{M}^{\mathscr{P}}}(\varpi)+\hat{\xi}\leq\hat{1}$. Since $\widehat{\mathscr{M}^{\mathscr{P}}}(\varpi)=\hat{\xi}$, $\hat{\xi}\leq\widehat{0.5}$. This is a contradiction. Hence, $\sup\{\widehat{\mathscr{M}^{\mathscr{P}}}(0),\widehat{0.5}\}\geq\widehat{\mathscr{M}^{\mathscr{P}}}(\varpi)$ for all $\varpi\in\Omega$. Suppose there exist $\varpi,\varrho,\eta\in\Omega$ such that $\sup\{\widehat{\mathscr{M}^{\mathscr{P}}}(\varpi*\eta),\widehat{0.5}\}<\hat{\xi}=\inf\{\widehat{\mathscr{M}^{\mathscr{P}}}((\varpi*\varrho)*\eta),\widehat{\mathscr{M}^{\mathscr{P}}}(\varrho*\eta)\}$. Then,

$$\hat{\xi}\in(0.5,1]^m,(\varpi*\eta)_{\hat{\xi}}\overline{\in}\widehat{\mathscr{M}^{\mathscr{P}}},((\varpi*\varrho)*\eta)_{\hat{\xi}},(\varrho*\eta)_{\hat{\xi}}\in\widehat{\mathscr{M}^{\mathscr{P}}}. \tag{40}$$

It follows that $((\varpi*\varrho)*\eta)_{\hat{\xi}}\overline{q}\widehat{\mathscr{M}^{\mathscr{P}}}$ or $(\varrho*\eta)_{\hat{\xi}}\overline{q}\widehat{\mathscr{M}^{\mathscr{P}}}$. Then, $\widehat{\mathscr{M}^{\mathscr{P}}}((\varpi*\varrho)*\eta)+\hat{\xi}\leq\hat{1}$ or $\widehat{\mathscr{M}^{\mathscr{P}}}(\varrho*\eta)+\hat{\xi}\leq\hat{1}$. Since $\widehat{\mathscr{M}^{\mathscr{P}}}((\varpi*\varrho)*\eta)\geq\hat{\xi}$ and $\widehat{\mathscr{M}^{\mathscr{P}}}(\varrho*\eta)\geq\hat{\xi}$, it follows that $\hat{\xi}\leq\widehat{0.5}$, a contradiction. Hence, $\sup\{\widehat{\mathscr{M}^{\mathscr{P}}}(\varpi*\eta),\widehat{0.5}\}\geq\inf\{\widehat{\mathscr{M}^{\mathscr{P}}}((\varpi*\varrho)*\eta),\widehat{\mathscr{M}^{\mathscr{P}}}(\varrho*\eta)\}$ for all $\varpi,\varrho,\eta\in\Omega$.

Conversely, let $0_{\hat{\xi}}\overline{\in}\widehat{\mathscr{M}^{\mathscr{P}}}$. Then, $\widehat{\mathscr{M}^{\mathscr{P}}}(0)<\hat{\xi}$, either $\widehat{\mathscr{M}^{\mathscr{P}}}(0)\geq\widehat{\mathscr{M}^{\mathscr{P}}}(\varpi)$ or $\widehat{\mathscr{M}^{\mathscr{P}}}(0)<\widehat{\mathscr{M}^{\mathscr{P}}}(\varpi)$. If $\widehat{\mathscr{M}^{\mathscr{P}}}(0)\geq\widehat{\mathscr{M}^{\mathscr{P}}}(\varpi)$, then $\widehat{\mathscr{M}^{\mathscr{P}}}(\varpi)<\hat{\xi}$, and so $\varpi_{\hat{\xi}}\overline{\in}\widehat{\mathscr{M}^{\mathscr{P}}}$. That is, $\varpi_{\hat{\xi}}\overline{\in}\vee\overline{q}\widehat{\mathscr{M}^{\mathscr{P}}}$. If $\widehat{\mathscr{M}^{\mathscr{P}}}(0)<\widehat{\mathscr{M}^{\mathscr{P}}}(\varpi)$, then by (i), $\widehat{\mathscr{M}^{\mathscr{P}}}(\varpi)\leq\widehat{0.5}$. We consider two cases:

Case (1). If $\widehat{\mathscr{M}^{\mathscr{P}}}(\varpi)<\hat{\xi}$, then $\varpi_{\hat{\xi}}\overline{\in}\widehat{\mathscr{M}^{\mathscr{P}}}$, and so $\varpi_{\hat{\xi}}\overline{\in}\vee\overline{q}\widehat{\mathscr{M}^{\mathscr{P}}}$.

Case (2). If $\widehat{\mathscr{M}^{\mathscr{P}}}(\varpi)\geq\hat{\xi}$, then $\hat{\xi}\leq\widehat{\mathscr{M}^{\mathscr{P}}}(\varpi)\leq\widehat{0.5}$; it follows that $\varpi_{\hat{\xi}}\overline{q}\widehat{\mathscr{M}^{\mathscr{P}}}$, and so $\varpi_{\hat{\xi}}\overline{\in}\vee\overline{q}\widehat{\mathscr{M}^{\mathscr{P}}}$.

Again, let $(\varpi*\eta)_{\inf\{\hat{\xi},\hat{\vartheta}\}}\overline{\in}\widehat{\mathscr{M}^{\mathscr{P}}}$ for $\hat{\xi},\hat{\vartheta}\in(0,1]^m$. Then, $\widehat{\mathscr{M}^{\mathscr{P}}}(\varpi*\eta)<\inf\{\hat{\xi},\hat{\vartheta}\}$. We consider two cases:

Case (1). If $\widehat{\mathscr{M}^{\mathscr{P}}}(\varpi*\eta)\geq\widehat{0.5}$, then

$$\begin{aligned}&\inf\{\widehat{\mathscr{M}^{\mathscr{P}}}((\varpi*\varrho)*\eta),\widehat{\mathscr{M}^{\mathscr{P}}}(\varrho*\eta)\}\\ &\leq\widehat{\mathscr{M}^{\mathscr{P}}}(\varpi*\eta)<\inf\{\hat{\xi},\hat{\vartheta}\}.\end{aligned} \tag{41}$$

Consequently, $\widehat{\mathscr{M}^{\mathscr{P}}}((\varpi*\varrho)*\eta)<\hat{\xi}$ or $\widehat{\mathscr{M}^{\mathscr{P}}}(\varrho*\eta)<\hat{\vartheta}$. That is, $((\varpi*\varrho)*\eta)_{\hat{\xi}}\overline{\in}\widehat{\mathscr{M}^{\mathscr{P}}}$ or $(\varrho*\eta)_{\hat{\vartheta}}\overline{\in}\widehat{\mathscr{M}^{\mathscr{P}}}$. Hence, $((\varpi*\varrho)*\eta)_{\hat{\xi}}\overline{\in}\vee\overline{q}\widehat{\mathscr{M}^{\mathscr{P}}}$ or $(\varrho*\eta)_{\hat{\vartheta}}\overline{\in}\vee\overline{q}\widehat{\mathscr{M}^{\mathscr{P}}}$.

Case (2). If $\widehat{\mathscr{M}^{\mathscr{P}}}(\varpi*\eta)<\widehat{0.5}$, then

$$\inf\{\widehat{\mathscr{M}^{\mathscr{P}}}((\varpi*\varrho)*\eta),\widehat{\mathscr{M}^{\mathscr{P}}}(\varrho*\eta)\}\leq\widehat{0.5}. \tag{42}$$

Assume $((\varpi*\varrho)*\eta)_{\hat{\xi}}\in\widehat{\mathscr{M}^{\mathscr{P}}}$ or $(\varrho*\eta)_{\hat{\vartheta}}\in\widehat{\mathscr{M}^{\mathscr{P}}}$. Then, $\hat{\xi}\leq\widehat{\mathscr{M}^{\mathscr{P}}}((\varpi*\varrho)*\eta)\leq\widehat{0.5}$ or $\hat{\vartheta}\leq\widehat{\mathscr{M}^{\mathscr{P}}}(\varrho*\eta)\leq\widehat{0.5}$. Thus, $\widehat{\mathscr{M}^{\mathscr{P}}}((\varpi*\varrho)*\eta)+\hat{\xi}\leq\widehat{0.5}+\widehat{0.5}=\hat{1}$ or $\widehat{\mathscr{M}^{\mathscr{P}}}(\varrho*\eta)+\hat{\vartheta}\leq\widehat{0.5}+\widehat{0.5}=\hat{1}$. It follows that $((\varpi*\varrho)*\eta)\overline{q}\widehat{\mathscr{M}^{\mathscr{P}}}$ or $(\varrho*\eta)_s\overline{q}\widehat{\mathscr{M}^{\mathscr{P}}}$, and so $((\varpi*\varrho)*\eta)_{\hat{\xi}}\overline{\in}\vee\overline{q}\widehat{\mathscr{M}^{\mathscr{P}}}$ or $(\varrho*\eta)_{\hat{\vartheta}}\overline{\in}\vee\overline{q}\widehat{\mathscr{M}^{\mathscr{P}}}$. Hence, $(\varpi*\eta)_{\inf\{\hat{\xi},\hat{\vartheta}\}}\overline{\in}\widehat{\mathscr{M}^{\mathscr{P}}}$ implies $((\varpi*\varrho)*\eta)_{\hat{\xi}}\overline{\in}\vee\overline{q}\widehat{\mathscr{M}^{\mathscr{P}}}$ or $(\varrho*\eta)_{\hat{\vartheta}}\overline{\in}\vee\overline{q}\widehat{\mathscr{M}^{\mathscr{P}}}$. □

Theorem 10. *Every $m-p_{(\overline{\in},\overline{\in}\vee\overline{q})-FPI}$ ideal of Ω is an m-polar $(\overline{\in},\overline{\in}\vee\overline{q})$-fuzzy ideal of Ω, but the converse does not hold.*

Proof. Let $\widehat{\mathscr{M}^{\mathscr{P}}}$ be an $m-p_{(\overline{\in},\overline{\in}\vee\overline{q})-\mathrm{FPI}}$ ideal of Ω. Then, for all $\varpi,\varrho,\eta\in\Omega$ and $\hat{\xi},\hat{\vartheta}\in(0,1]^m$, we get

$$0_{\hat{\xi}}\overline{\in}\widehat{\mathscr{M}^{\mathscr{P}}}\text{ implies }\varpi_{\hat{\xi}}\overline{\in}\vee\overline{q}\widehat{\mathscr{M}^{\mathscr{P}}}, \tag{43}$$

$$\begin{aligned}&(\varpi*\eta)_{\inf\{\hat{\xi},\hat{\vartheta}\}}\overline{\in}\widehat{\mathscr{M}^{\mathscr{P}}}\text{ implies }((\varpi*\varrho)*\eta)_{\hat{\xi}}\overline{\in}\vee\overline{q}\widehat{\mathscr{M}^{\mathscr{P}}}\\ &\text{or }(\varrho*\eta)_{\hat{\vartheta}}\overline{\in}\vee\overline{q}\widehat{\mathscr{M}^{\mathscr{P}}}.\end{aligned} \tag{44}$$

Put $\eta = 0$ in (44), and we get

$$(\varpi * 0)_{\inf\{\widehat{\xi},\vartheta\}}\overline{\in}\widehat{\mathscr{M}^{\mathscr{P}}} \text{ implies } ((\varpi * \varrho) * 0)_{\widehat{\xi}}\overline{\in}\vee\overline{q}\widehat{\mathscr{M}^{\mathscr{P}}} \text{ or } (\varrho * 0)_{\widehat{\vartheta\vartheta}}\overline{\in}\vee\overline{q}\widehat{\mathscr{M}^{\mathscr{P}}}. \tag{45}$$

This implies

$$\varpi_{\inf\{\widehat{\xi},\vartheta\}}\overline{\in}\widehat{\mathscr{M}^{\mathscr{P}}} \text{ implies } (\varpi * \varrho)_{\widehat{\xi}}\overline{\in}\vee\overline{q}\widehat{\mathscr{M}^{\mathscr{P}}} \text{ or } \varrho_{\widehat{\vartheta}}\overline{\in}\vee\overline{q}\widehat{\mathscr{M}^{\mathscr{P}}}. \tag{46}$$

Hence, $\widehat{\mathscr{M}^{\mathscr{P}}}$ is an m-polar $(\overline{\in},\overline{\in}\vee\overline{q})$-fuzzy ideal of Ω.

The last part is shown by the following example. □

Example 6. Reconsider the BCK-algebras $X = \{0, \imath, \jmath, \ell\}$ given in Example 2. Let $\widehat{\mathscr{M}^{\mathscr{P}}}$ be a 3-pF set defined as

$$\widehat{\mathscr{M}^{\mathscr{P}}}(\varpi) = \begin{cases} (0.6, 0.6, 0.6), & \text{if } \varpi = 0, \\ (0.3, 0.4, 0.3), & \text{if } \varpi = \imath, \jmath, \\ (0.2, 0.3, 0.2), & \text{if } \varpi = \ell. \end{cases} \tag{47}$$

Then, $\widehat{\mathscr{M}^{\mathscr{P}}}$ is an $m - p_{(\overline{\in},\overline{\in}\vee\overline{q})-F}$ ideal of Ω. But it is not an $m - p_{(\overline{\in},\overline{\in}\vee\overline{q})-\text{FPI}}$ ideal of Ω since $\sup\{\widehat{\mathscr{M}^{\mathscr{P}}}(\jmath * \imath), \widehat{0.5}\} = (0.5, 0.5, 0.5) < \inf\{\widehat{\mathscr{M}^{\mathscr{P}}}((\jmath * \imath) * \imath), \widehat{\mathscr{M}^{\mathscr{P}}}(\imath * \imath)\} = \inf\{\widehat{\mathscr{M}^{\mathscr{P}}}(0), \widehat{\mathscr{M}^{\mathscr{P}}}(0)\} = (0.6, 0.6, 0.6)$.

Theorem 11. *Let $\widehat{\mathscr{M}^{\mathscr{P}}}$ be an m-polar $(\overline{\in},\overline{\in}\vee\overline{q})$-fuzzy ideal of Ω. Then, $\widehat{\mathscr{M}^{\mathscr{P}}}$ is an $m - p_{(\overline{\in},\overline{\in}\vee\overline{q})-FPI}$ ideal of Ω if and only if $\sup\{\widehat{\mathscr{M}^{\mathscr{P}}}(\varpi * \varrho), \widehat{0.5}\} \geq \widehat{\mathscr{M}^{\mathscr{P}}}((\varpi * \varrho) * \varrho)$ for all $\varpi, \varrho \in \Omega$.*

Proof. Let $\widehat{\mathscr{M}^{\mathscr{P}}}$ be an $m - p_{(\overline{\in},\overline{\in}\vee\overline{q})-\text{FPI}}$ ideal of Ω. If η is replaced by ϱ in Theorem 9 (ii), we get

$$\begin{aligned} \sup\{\widehat{\mathscr{M}^{\mathscr{P}}}(\varpi * \varrho), \widehat{0.5}\} &\geq \inf\{\widehat{\mathscr{M}^{\mathscr{P}}}((\varpi * \varrho) * \varrho), \widehat{\mathscr{M}^{\mathscr{P}}}(\varrho * \varrho)\} \\ &= \inf\{\widehat{\mathscr{M}^{\mathscr{P}}}((\varpi * \varrho) * \varrho), \widehat{\mathscr{M}^{\mathscr{P}}}(0)\} \\ &\geq \widehat{\mathscr{M}^{\mathscr{P}}}((\varpi * \varrho) * \varrho), \end{aligned} \tag{48}$$

for all $\varpi, \varrho \in \Omega$.

Conversely, Assume that $\widehat{\mathscr{M}^{\mathscr{P}}}$ is an m-polar $(\overline{\in},\overline{\in}\vee\overline{q})$-fuzzy ideal of Ω and $\sup\{\widehat{\mathscr{M}^{\mathscr{P}}}(\varpi * \varrho), \widehat{0.5}\} \geq \widehat{\mathscr{M}^{\mathscr{P}}}((\varpi * \varrho) * \varrho)$ for all $\varpi, \varrho \in \Omega$. Then, $\sup\{\widehat{\mathscr{M}^{\mathscr{P}}}(0), \widehat{0.5}\} \geq \widehat{\mathscr{M}^{\mathscr{P}}}(\varpi)$ for all $\varpi \in \Omega$. Also, since $((\varpi * \eta) * \eta) * (\varrho * \eta) \leq (\varpi * \eta) * \varrho = (\varpi * \varrho) * \eta$ for all $\varpi, \varrho \in \Omega$, it follows by Lemma 3 that

$$\sup\{\widehat{\mathscr{M}^{\mathscr{P}}}(((\varpi * \eta) * \eta) * (\varrho * \eta)), \widehat{0.5}\} \geq \widehat{\mathscr{M}^{\mathscr{P}}}((\varpi * \varrho) * \eta). \tag{49}$$

Now, by assumption,

$$\begin{aligned} \sup\{\widehat{\mathscr{M}^{\mathscr{P}}}(\varpi * \eta), \widehat{0.5}\} &\geq \widehat{\mathscr{M}^{\mathscr{P}}}((\varpi * \eta) * \eta) \\ &\geq \inf\{\widehat{\mathscr{M}^{\mathscr{P}}}(((\varpi * \eta) * \eta) * (\varrho * \eta)), \widehat{\mathscr{M}^{\mathscr{P}}}(\varrho * \eta), \widehat{0.5}\} \\ &\geq \inf\{\widehat{\mathscr{M}^{\mathscr{P}}}((\varpi * \varrho) * \eta), \widehat{\mathscr{M}^{\mathscr{P}}}(\varrho * \eta)\}. \end{aligned} \tag{50}$$

Hence, $\widehat{\mathscr{M}^{\mathscr{P}}}$ is an $m - p_{(\overline{\in},\overline{\in}\vee\overline{q})-\text{FPI}}$ ideal of Ω. □

Theorem 12. *The intersection of any family of an $m - p_{(\overline{\in},\overline{\in}\vee\overline{q})-FPI}$ ideal of Ω is an $m - p_{(\overline{\in},\overline{\in}\vee\overline{q})-FPI}$ ideal of Ω.*

Proof. Let $\{\widehat{\mathscr{M}^{\mathscr{P}}}_j\}_{j\in I}$ be a family of $m - p_{(\overline{\in},\overline{\in}\vee\overline{q})-\text{FPI}}$ ideals of Ω and $\varpi \in \Omega$. Then,

$$\sup\{\widehat{\mathscr{M}^{\mathscr{P}}}_j(0), \widehat{0.5}\} \geq \widehat{\mathscr{M}^{\mathscr{P}}}_j(\varpi), \tag{51}$$

for all $j \in I$. Thus,

$$\begin{aligned} \sup\{(\wedge_{j\in I}\widehat{\mathscr{M}^{\mathscr{P}}}_j)(0), \widehat{0.5}\} &= \sup\{\wedge_{j\in I}\widehat{\mathscr{M}^{\mathscr{P}}}_i(0), \widehat{0.5}\} \\ &\geq \wedge_{j\in I}(\widehat{\mathscr{M}^{\mathscr{P}}}_j(\varpi)) \\ &= (\wedge_{j\in I}\widehat{\mathscr{M}^{\mathscr{P}}}_j)(\varpi). \end{aligned} \tag{52}$$

Therefore, $\sup\{(\wedge_{j\in I}\widehat{\mathscr{M}^{\mathscr{P}}}_j)(0), \widehat{0.5}\} \geq (\wedge_{j\in I}\widehat{\mathscr{M}^{\mathscr{P}}}_j)(\varpi)$. Let $\varpi, \varrho, \eta \in \Omega$. Since every $\widehat{\mathscr{M}^{\mathscr{P}}}_j$ is an $m - p_{(\overline{\in},\overline{\in}\vee\overline{q})-\text{FPI}}$ ideal of Ω,

$$\sup\{\widehat{\mathscr{M}^{\mathscr{P}}}_j(\varpi * \eta), \widehat{0.5}\} \geq \inf\{\widehat{\mathscr{M}^{\mathscr{P}}}_j((\varpi * \varrho) * \eta), \widehat{\mathscr{M}^{\mathscr{P}}}_j(\varrho * \eta)\}, \tag{53}$$

for all $j \in I$. Thus,

$$\begin{aligned} &\sup\{(\wedge_{j\in I}\widehat{\mathscr{M}^{\mathscr{P}}}_i)(\varpi * \eta), \widehat{0.5}\} \\ &= \sup\{\wedge_{j\in I}\widehat{\mathscr{M}^{\mathscr{P}}}_i(\varpi * \eta), \widehat{0.5}\} \\ &\geq \inf\{\wedge_{j\in I}\widehat{\mathscr{M}^{\mathscr{P}}}_j((\varpi * \varrho) * \eta), \wedge_{j\in I}\widehat{\mathscr{M}^{\mathscr{P}}}_j(\varrho * \eta)\} \\ &= \inf\{(\wedge_{j\in I}\widehat{\mathscr{M}^{\mathscr{P}}}_j)((\varpi * \varrho) * \eta), (\wedge_{j\in I}\widehat{\mathscr{M}^{\mathscr{P}}}_j)(\varrho * \eta)\}. \end{aligned} \tag{54}$$

Therefore, $\sup\{(\wedge_{j\in I}\widehat{\mathscr{M}^{\mathscr{P}}}_i)(\varpi * \eta), \widehat{0.5}\} \geq \inf\{(\wedge_{j\in I}\widehat{\mathscr{M}^{\mathscr{P}}}_j)((\varpi * \varrho) * \eta), (\wedge_{j\in I}\widehat{\mathscr{M}^{\mathscr{P}}}_j)(\varrho * \eta)\}$. Hence, $\wedge_{j\in I}\widehat{\mathscr{M}^{\mathscr{P}}}_j$ is an $m - p_{(\overline{\in},\overline{\in}\vee\overline{q})-\text{FPI}}$ ideal of Ω. □

Theorem 13. *An m-pF set $\widehat{\mathscr{M}^{\mathscr{P}}}$ of Ω is an $m - p_{(\overline{\in},\overline{\in}\vee\overline{q})-FPI}$ ideal of Ω if and only if $\widehat{\mathscr{M}^{\mathscr{P}}}_{\widehat{\xi}} \neq \phi$ is a positive implicative ideal of Ω for all $\widehat{\xi} \in (0.5, 1]^m$.*

Proof. Assume that $\widehat{\mathscr{M}^{\mathscr{P}}}$ is an $m - p_{(\overline{\in},\overline{\in}\vee\overline{q})-\text{FPI}}$ ideal of Ω and $\widehat{\xi} \in (0.5, 1]^m$. Suppose $\varpi \in \widehat{\mathscr{M}^{\mathscr{P}}}_{\widehat{\xi}}$. Then, $\widehat{\mathscr{M}^{\mathscr{P}}}(\varpi) \geq \widehat{\xi}$. Now,

$$\sup\left\{\widehat{\mathscr{M}^{\mathscr{P}}}(0), \widehat{0.5}\right\} \geq \widehat{\mathscr{M}^{\mathscr{P}}}(\varpi) \geq \widehat{\xi}. \quad (55)$$

Thus, $\widehat{\mathscr{M}^{\mathscr{P}}}(0) \geq \widehat{\xi}$. Hence, $0 \in \widehat{\mathscr{M}^{\mathscr{P}}}_{\widehat{\xi}}$. Let $(\varpi * \varrho) * \eta$, $\varrho * \eta \in \widehat{\mathscr{M}^{\mathscr{P}}}_{\widehat{\xi}}$. Then, $\widehat{\mathscr{M}^{\mathscr{P}}}((\varpi * \varrho) * \eta) \geq \widehat{\xi}$ and $\widehat{\mathscr{M}^{\mathscr{P}}}(\varrho * \eta) \geq \widehat{\xi}$. Now,

$$\sup\left\{\widehat{\mathscr{M}^{\mathscr{P}}}(\varpi * \eta), \widehat{0.5}\right\} \geq \inf\left\{\widehat{\mathscr{M}^{\mathscr{P}}}((\varpi * \varrho) * \eta), \mathscr{H}(\varrho * \eta)\right\} \geq \widehat{\xi}. \quad (56)$$

Thus, $\widehat{\mathscr{M}^{\mathscr{P}}}(\varpi * \eta) \geq \widehat{\xi}$, that is, $\varpi * \eta \in \widehat{\mathscr{M}^{\mathscr{P}}}_{\widehat{\xi}}$. Therefore, $\widehat{\mathscr{M}^{\mathscr{P}}}_{\widehat{\xi}}$ is a positive implicative ideal of Ω.

Conversely, assume $\widehat{\mathscr{M}^{\mathscr{P}}}_{\widehat{\xi}} \neq \phi$ is a positive implicative ideal of Ω. Let $\varpi \in \Omega$ be such that $\sup\left\{\widehat{\mathscr{M}^{\mathscr{P}}}(0), \widehat{0.5}\right\} < \widehat{\mathscr{M}^{\mathscr{P}}}(\varpi)$. Choose $\widehat{\xi} \in (0.5, 1]^m$ such that

$$\sup\left\{\widehat{\mathscr{M}^{\mathscr{P}}}(0), \widehat{0.5}\right\} < \widehat{\xi} \leq \widehat{\mathscr{M}^{\mathscr{P}}}(\varpi). \quad (57)$$

Then, $\widehat{\mathscr{M}^{\mathscr{P}}}(0) < \widehat{\xi}$ and $\varpi \in \widehat{\mathscr{M}^{\mathscr{P}}}_{\widehat{\xi}}$. Since $\widehat{\mathscr{M}^{\mathscr{P}}}_{\widehat{\xi}}$ is a positive implicative ideal of Ω, we have $0 \in \widehat{\mathscr{M}^{\mathscr{P}}}_{\widehat{\xi}}$, and so $\widehat{\mathscr{M}^{\mathscr{P}}}(0) \geq \widehat{\xi}$, a contradiction. Hence, $\sup\left\{\widehat{\mathscr{M}^{\mathscr{P}}}(0), \widehat{0.5}\right\} \geq \widehat{\mathscr{M}^{\mathscr{P}}}(\varpi)$ for all $\varpi \in \Omega$. Assume $\varpi, \varrho, \eta \in \Omega$ such that $\sup\left\{\widehat{\mathscr{M}^{\mathscr{P}}}(\varpi * \eta), \widehat{0.5}\right\} < \inf\left\{\widehat{\mathscr{M}^{\mathscr{P}}}((\varpi * \varrho) * \eta), \widehat{\mathscr{M}^{\mathscr{P}}}(\varrho * \eta)\right\}$. Choose $\widehat{\xi} \in (0.5, 1]^m$ such that

$$\sup\left\{\widehat{\mathscr{M}^{\mathscr{P}}}(\varpi * \eta), \widehat{0.5}\right\} < \widehat{\xi} \leq \inf\left\{\widehat{\mathscr{M}^{\mathscr{P}}}((\varpi * \varrho) * \eta), \widehat{\mathscr{M}^{\mathscr{P}}}(\varrho * \eta)\right\}. \quad (58)$$

Then, $\widehat{\mathscr{M}^{\mathscr{P}}}(\varpi * \eta) < \widehat{\xi}$. Since $(\varpi * \varrho) * \eta, \varrho * \eta \in \widehat{\mathscr{M}^{\mathscr{P}}}_{\widehat{\xi}}$ and $\widehat{\mathscr{M}^{\mathscr{P}}}_{\widehat{\xi}}$ is a positive implicative ideal of Ω, $x * z \in \widehat{\mathscr{M}^{\mathscr{P}}}_{\widehat{\xi}}$. That is, $\widehat{\mathscr{M}^{\mathscr{P}}}(\varpi * \eta) \geq \widehat{\xi}$. This is a contradiction. Thus, $\sup\left\{\widehat{\mathscr{M}^{\mathscr{P}}}(\varpi * \eta), \widehat{0.5}\right\} \geq \inf\left\{\widehat{\mathscr{M}^{\mathscr{P}}}((\varpi * \varrho) * \eta), \widehat{\mathscr{M}^{\mathscr{P}}}(\varrho * \eta)\right\}$ for all $\varpi, \varrho, \eta \in \Omega$. Hence, $\widehat{\mathscr{M}^{\mathscr{P}}}$ is an $m - p_{(\overline{\in}, \overline{\in} \vee \overline{q})-\text{FPI}}$ ideal of Ω. □

Corollary 1. *Every m-pFPI ideal of Ω is an $m - p_{(\overline{\in}, \overline{\in} \vee \overline{q})-FPI}$ ideal of Ω.*

6. Conclusions

m-pF algebraic structures play a significant role in several fields of pure and applied mathematics. In order to broaden m-pF algebraic structures, in this study, we have introduced m-pFPI ideals, $m - p_{(\in, \in \vee q)}$ − FPI ideals, and $m - p_{(\overline{\in}, \overline{\in} \vee \overline{q})-\text{FPI}}$ ideals in BCK-algebras and discussed several relevant properties. We have considered some characterizations of these concepts in BCK-algebras. In our further research, we will focus on adopting this approach to some more algebraic structures, such as KU-algebras, UP-algebras, semigroups, KU-semigroups, and hemirings, and to some more complicated applications from the domains of computer sciences and information systems.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that there are no conflicts of interest.

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