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Retraction

Retracted: Interval Information Content of Fuzzy Relation and the Application in the Fuzzy Implication Operators

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This article has been retracted by Hindawi following an investigation undertaken by the publisher [1]. This investigation has uncovered evidence of one or more of the following indicators of systematic manipulation of the publication process:

(1) Discrepancies in scope
(2) Discrepancies in the description of the research reported
(3) Discrepancies between the availability of data and the research described
(4) Inappropriate citations
(5) Incoherent, meaningless and/or irrelevant content included in the article
(6) Peer-review manipulation

The presence of these indicators undermines our confidence in the integrity of the article's content and we cannot, therefore, vouch for its reliability. Please note that this notice is intended solely to alert readers that the content of this article is unreliable. We have not investigated whether authors were aware of or involved in the systematic manipulation of the publication process.

Wiley and Hindawi regrets that the usual quality checks did not identify these issues before publication and have since put additional measures in place to safeguard research integrity.

We wish to credit our own Research Integrity and Research Publishing teams and anonymous and named external researchers and research integrity experts for contributing to this investigation.

The corresponding author, as the representative of all authors, has been given the opportunity to register their agreement or disagreement to this retraction. We have kept a record of any response received.

References

[1] Y. Shi, "Interval Information Content of Fuzzy Relation and the Application in the Fuzzy Implication Operators," *Journal of Mathematics*, vol. 2021, Article ID 6611367, 11 pages, 2021.

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Research Article

Interval Information Content of Fuzzy Relation and the Application in the Fuzzy Implication Operators

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In rule optimization, some rule characteristics were extracted to describe the uncertainty correlations of fuzzy relations, but the concrete numbers cannot express correlations with uncertainty, such as "at least 0.1 and up to 0.5." To solve this problem, a novel definition concerning interval information content of fuzzy relation has been proposed in this manuscript to realize the fuzziness measurement of the fuzzy relation. Also, its definition and expressions have also been constructed. Meanwhile based on the interval information content, the issues of fuzzy implication ranking and clustering were analyzed. Finally, utilizing the combination of possibility's interval comparison equations and interval value's similarity measure, the classifications of implication operators were proved to be realizable. The achievements in the presented work will provide a reasonable index to measure the fuzzy implication operators and lay a solid foundation for further research.

1. Introduction

Nowadays, we are in the midst of an information revolution, which is driving the development and deployment of new kinds of science and technology with ever-increasing depth and breadth. Information is related to data and knowledge, as data represents the values attributed to parameters, and knowledge signifies the understanding of real things or abstract concepts [1]. With the development of computer science, the amount of information generated by people has grown from a trickle to a torrent. In 1948, the definition of information theory was first proposed by Shannon, in which the statistics method was used to measure the information content quantitatively.

The rapid progress of information theory makes people realize its significance [2], and its conception has been applied in many regions such as communication, decision making, and pattern recognition [3–5]. But unfortunately, the application research studies of information theory in semantic and pragmatic information science have not been conducted widely until now. As the era of big data has been opened, useful information must be mined from more and more data. In this process, the information needs to be expressed by various rules. From a practical standpoint, it is difficult to describe decision makers' experience with precise mathematical models. So, how to select and evaluate rules is the key issue to realize the control of fuzzy system, which can be summarized as rule optimization [6, 7].

To solve this issue, many researches have been conducted to develop several methods, which can be divided into two categories: (1) by means of extracting some rule characteristics [8–13], such as uncertainties of operators by Yu et al. [8], information entropy by Sendi and Ayoubi [7], and fuzzy reliance by Hu et al. [12], the optimizations of fuzzy systems have been realized. (2) First, the structure of fuzzy rules was established; then, some algorithms [14–19], such as the gradient descent method [14] and neural networks [16], have been used to optimize the variable parameters in fuzzy systems. In the classical compositional rule of inference methods, the fuzzy rules were often converted into implication operators. So, many fuzzy implication operators can be constructed [20–24], and for them, the research on how to realize better control of fuzzy systems is still lacking. To address these issues, a novel method has been proposed in the paper, utilizing which the interval information contents of fuzzy relations have been extracted to realize the ranking, clustering, and classification.

Information content is used to describe the correlations between the sets in fuzzy relation. But, owing to the complexities of the things and the uncertainties of human cognition, the concrete numbers cannot be used to express the correlations between two sets. For example, when the correlation is “at least 0.1 and up to 0.5,” how to measure it is still an unsolved problem. In order to solve this problem, a new definition of uncertainty measurement is constructed, which is named as the interval information content of the fuzzy relation. First, the fuzzy relation of the interval information content was proposed, and five different expressions were developed, with which the ranking, clustering, and classification of fuzzy implication operators have been realized.

2. Preliminaries

In this section, some definitions and theories involved in this paper are introduced.

Definition 1 (see [8]). Let X and Y be two sets, a fuzzy relation R from X to Y be a fuzzy subset of $X \times Y$, and $R(x, y)$ be the membership degree of x and y to fuzzy relation R, and the class of all fuzzy relations from X to Y can be denoted by $\mathcal{F}(X \times Y)$.

Let $X = \{x_1, x_2, \ldots, x_m\}, Y = \{y_1, y_2, \ldots, y_n\}$ be the finite sets and $r_{ij} = R(x_i, y_i)$; then, the fuzzy relation R can be denoted by fuzzy relation matrix $R = (r_{ij})_{m\times n}$.

Remark 1

(1) For fuzzy relation matrix $R = (r_{ij})_{m\times n}, S = (S_{ij})_{m\times n}$, the operations of the fuzzy relation matrix are defined as follows:

$$\begin{aligned} R \cap S &= \left(r_{ij} \wedge S_{ij}\right)_{m\times n},\\ R \cup S &= \left(r_{ij} \vee S_{ij}\right)_{m\times n},\\ R^c &= \left(1 - r_{ij}\right)_{m\times n}, \end{aligned} \tag{1}$$

where $r_{ij} \wedge s_{ij} \triangleq \min(r_{ij}, s_{ij})$ and $r_{ij} \vee s_{ij} \triangleq \max(r_{ij}, s_{ij})$.

(2) $R_\lambda = \{ (x, y) | R(x, y) \geq \lambda \}$ is defined as the λ-cut relations of R. Furthermore, the $((r_{ij})_\lambda)_{m\times n}$ is defined as λ-cut matrix of R with the expression as follows:

$$\left(r_{ij}\right)_\lambda = \begin{cases} 1, & r_{ij} \geq \lambda, \\ 0, & r_{ij} < \lambda. \end{cases} \tag{2}$$

Definition 2 (see [10]). Let $X = \{x_1, x_2, \ldots, x_m\}$, $Y = \{y_1, y_2, \ldots, y_n\}$, and R be a fuzzy relation from X to Y, and the information content of R is measured as follows:

$$\mathrm{IC}(R) = \frac{m}{m+n}\mathrm{IC}(R|X) + \frac{n}{m+n}\mathrm{IC}(R|Y), \tag{3}$$

where $\mathrm{IC}(R|X), \mathrm{IC}(R|Y)$ are the information contents of R restricted on X and Y, respectively with the expression as follows:

$$\begin{aligned} \mathrm{IC}(R \mid X) &= -\sum_{i=1}^{m} \frac{\sum_{j=1}^{n} R\left(x_i, y_j\right)}{\sum_{i=1}^{m}\sum_{j=1}^{n} R\left(x_i, y_j\right)} \log_2 \frac{\sum_{j=1}^{n} R\left(x_i, y_j\right)}{n},\\ \mathrm{IC}(R|Y) &= -\sum_{j=1}^{n} \frac{\sum_{i=1}^{m} R^{-1}\left(y_j, x_i\right)}{\sum_{j=1}^{n}\sum_{i=1}^{m} R^{-1}\left(y_j, x_i\right)} \log_2 \frac{\sum_{i=1}^{m} R^{-1}\left(y_j, x_i\right)}{m}. \end{aligned} \tag{4}$$

The U-uncertainty of A is also used to measure the information content of fuzzy sets.

Definition 3 (see [25]). A is a fuzzy set defined on $X = \{x_1, x_2, \ldots, x_m\}$, and all $A(x_i)$ $(i = 1, 2, \ldots, m)$ can be designed to an ordered possibility distribution $\{\lambda_1, \lambda_2, \ldots, \lambda_m\}$. It is always the case that $\lambda_{i+1} \leq \lambda_i$; then,

$$\begin{aligned} U(A) &= -\sum_{i=1}^{m} (\lambda_i - \lambda_{i+1}) \log_2 \left|A_{\lambda_i}\right| \\ &= -\sum_{i=1}^{m} \lambda_i \left(\log_2 \left|A_{\lambda_i}\right| - \log_2 \left|A_{\lambda_i - 1}\right|\right) \end{aligned} \tag{5}$$

is defined as the U-uncertainty of A, $|\cdot|$ is the cardinality of a set, and

$$A_{\lambda_i} = \{x \in X | A(x) \geq \lambda_i\}. \tag{6}$$

Definition 4 (see [20]). A fuzzy implication operator is any mapping I: $[0,1] \times [0,1] \longrightarrow [0,1]$ satisfying the border conditions:

(P1) $\exists\ a \in [0,1],\ b \in [0,1],\ I(a, b) = 1$

(P2) $\exists\ c \in [0,1],\ d \in [0,1],\ I(c, d) = 0$

Furthermore,

(P3) If $I(1,0) = 0$, $I(0,1) = I(1,1) = I(0,0) = 1$, then I is a normal implication operator. Otherwise, it is called an abnormal implication operator.

For instance,

(1) Zadeh operator: $I_1(a, b) = (1 - a) \vee (a \wedge b)$
(2) Kleene–Dienes operator: $I_2(a, b) = (1 - a) \vee b$
(3) Lukasiewicz operator: $I_3(a, b) = (1 - a + b) \wedge 1$
(4) Reichenbach operator: $I_4(a, b) = 1 - a + ab$
(5) Mamdani operator: $I_5(a, b) = a \wedge b$
(6) Probability product operator: $I_6(a, b) = ab$
(7) R_0 operator:

$$I_7(a, b) = \begin{cases} 1, & a \leq b, \\ (1 - a) \vee b, & a > b \end{cases} \tag{7}$$

(8) Goguen operator:

$$I_8(a, b) = \begin{cases} 1, & a = 0, \\ \left(\dfrac{b}{a}\right) \wedge 1, & a > 0 \end{cases} \tag{8}$$

(9) Gaines–Reseher operator:

$$I_9(a,b)=\begin{cases}1, & a\le b,\\ 0, & a>b\end{cases} \tag{9}$$

(10) Yager operator:

$$I_{10}(a,b)=b^a \tag{10}$$

(11) Bounded product operator: $I_{11}(a,b)=(a+b-1)\vee 0$

(12) Gödel operator:

$$I_{12}(a,b)=\begin{cases}1, & a\le b,\\ b, & a>b\end{cases} \tag{11}$$

(13) $I_{13}(a,b)=\begin{cases}1, & a\le b,\\ 1-a, & a>b.\end{cases}$

To indicate the degree of similarity of two fuzzy sets, the concept of similarity measure is proposed as follows.

Definition 5 (see [26, 27]). A real function S_I: $D\times D\longrightarrow[0,1]$ is called similarity measure, where $D=\{[a^-,a^+]|0\le a^-\le a^+\le 1\}$, if S_I satisfies the following properties:

(S_I1) $S_I(A,A^c)=0$ if A is a crisp set

(S_I2) $S_I(A,B)=1\Longleftrightarrow A=B$

(S_I 3) $S_I(A,B)=S_I(B,A)$

(S_I 4) $\forall A,B,C\in D$, if $A\subseteq B\subseteq C$, then $S_I(A,C)\le S_I(A,B)$, $S_I(A,C)\le S_I(B,C)$

For instance, let $A=[a^-,a^+]$, $B=[b^-,b^+]\in D$, and

$$S_I(A,B)=S_I([a^-,a^+],[b^-,b^+])=\frac{1}{2}\left(\frac{a^-\wedge b^-}{a^-\vee b^-}+\frac{a^+\wedge b^+}{a^+\vee b^+}\right). \tag{12}$$

3. The Construction of Interval Information Content of the Fuzzy Relation

In fact, IC(R) can be used to measure information content transferred by two fuzzy sets by means of an exact value. But, with uncertainty, the value of the information content between two fuzzy sets cannot be measured precisely. For instance, when it is measured as a maximum of 0.7 and a minimum of 0.1, how about it? It is necessary to extend the value from the exact number to interval value, and then, the definition of interval information content is proposed as follows:

Definition 6. Let $X=\{x_1,x_2,\ldots,x_m\}$, $Y=\{y_1,y_2,\ldots,y_n\}$, the interval information content of fuzzy relation R be the mapping IIC(R): $X\times Y\longrightarrow D$, and

$$\mathrm{IIC}_1(R)=[\mathrm{IC}(R|X)\wedge\mathrm{IC}(R|Y),\mathrm{IC}(R|X)\vee\mathrm{IC}(R|Y)]. \tag{13}$$

Based on U-uncertainty, interval information content of the fuzzy relation can also be expressed as follows.

Definition 7. Let $X=\{x_1,x_2,\ldots,x_m\}$, $Y=\{y_1,y_2,\ldots,y_n\}$, R be the fuzzy relation from X to Y, and $\{R(x_1,y_1),R(x_1,y_2),\ldots,R(x_i,y_j)\}$ be ranked in descending order $\{\lambda_1,\lambda_2,\ldots,\lambda_{mn}\}$, where $\lambda_{i+1}\le\lambda_i$, $\lambda_{mn+1}=0$, and then,

$$\mathrm{IIC}_2(R)=\left[\sum_{i=1}^{mn}(\lambda_i-\lambda_{i+1})\log_2\frac{mn}{|R_{\lambda_{i+1}}|},\sum_{i=1}^{mn}(\lambda_i-\lambda_{i+1})\log_2\frac{mn}{|R_{\lambda_i}|}\right] \tag{14}$$

is the interval information content of R from X to Y.

Similarly, by Definition 3, the interval information content of R can also be constructed as

$$\begin{aligned}
\mathrm{IIC}_3(R)&=\left[\sum_{i=1}^{mn}\frac{\lambda_i}{\sum_{i=1}^{mn}\lambda_i}\log_2\left(\frac{mn}{|R_{\lambda_{i+1}}|}\right),\sum_{i=1}^{mn}\frac{\lambda_i}{\sum_{i=1}^{mn}\lambda_i}\log_2\left(\frac{mn}{|R_{\lambda_i}|}\right)\right],\\
\mathrm{IIC}_4(R)&=\left[\sum_{i=1}^{mn}(\lambda_i-\lambda_{i+1})\left(\mathrm{IC}\left(R_{\lambda_i}|X\right)\wedge\mathrm{IC}\left(R_{\lambda_i}|Y\right)\right),\sum_{i=1}^{mn}(\lambda_i-\lambda_{i+1})\left(\mathrm{IC}\left(R_{\lambda_i}|X\right)\vee\mathrm{IC}\left(R_{\lambda_i}|Y\right)\right)\right],\\
\mathrm{IIC}_5(R)&=\left[\sum_{i=1}^{mn}\frac{\lambda_i}{\sum_{i=1}^{mn}\lambda_i}\left(\mathrm{IC}\left(R_{\lambda_i}|X\right)\wedge\mathrm{IC}\left(R_{\lambda_i}|Y\right)\right),\sum_{i=1}^{mn}\frac{\lambda_i}{\sum_{i=1}^{mn}\lambda_i}\left(\mathrm{IC}\left(R_{\lambda_i}|X\right)\vee\mathrm{IC}\left(R_{\lambda_i}|Y\right)\right)\right].
\end{aligned} \tag{15}$$

Then, we have $\mathrm{IIC}_i(R) \in D, \quad i = 1, 2, \ldots, 5$.

Example 1. Let $X = \{x_1, x_2, \ldots, x_9\}$, $Y = \{y_1, y_2, \ldots, y_9\}$, and R be the fuzzy relation from X to Y; the results of $R(x_i, y_j)$ are listed in Table 1.

Taking $\mathrm{IIC}_1(R)$ for example, we have

$$\begin{aligned}
\mathrm{IC}(R|X) &= -\sum_{i=1}^{9} \frac{\sum_{j=1}^{9} R(x_i, y_j)}{\sum_{i=1}^{9}\sum_{j=1}^{9} R(x_i, y_j)} \log_2 \frac{\sum_{j=1}^{9} R(x_i, y_j)}{9},\\
&= \frac{2\times 3.1}{63.4}\log_2\frac{9}{3.1} + \frac{5.2}{63.4}\log_2\frac{9}{5.2} + \frac{7}{63.4}\log_2\frac{9}{7} + \frac{5\times 9}{63.4}\log_2\frac{9}{9},\\
&= 0.2553,\\
\mathrm{IC}(R|Y) &= -\sum_{j=1}^{9} \frac{\sum_{i=1}^{9} R^{-1}(y_j, x_i)}{\sum_{j=1}^{9}\sum_{i=1}^{9} R^{-1}(y_j, x_i)} \log_2 \frac{\sum_{i=1}^{9} R^{-1}(y_j, x_i)}{9},\\
&= \frac{5\times 5.9}{63.4}\log_2\frac{9}{5.9} + \frac{7.5}{63.4}\log_2\frac{9}{7.5} + \frac{8.4}{63.4}\log_2\frac{9}{8.4} + \frac{2\times 9}{63.4}\log_2\frac{9}{9},\\
&= 0.3278,
\end{aligned} \tag{16}$$

and then, $\mathrm{IIC}_1(R) = [0.\ 2553,\ 0.\ 3278]$.

$\{R(x_i, y_j)\}$ is ranked in descending order {1, 0.7, 0.6, 0.4, 0.3, 0}; then, $\lambda_1 = 1$, $\lambda_2 = 0.7$, $\lambda_3 = 0.6$, $\lambda_4 = 0.4$, $\lambda_5 = 0.3$, and $\lambda_i = 0$ $(i = 6, 7, \ldots, 81)$. Taking the case of $\lambda_1 = 1$, the values of $R_1(x_i, y_j)$ are listed in Table 2.

So,

$$\begin{aligned}
\mathrm{IC}(R_1|X) &= \frac{2\times 2}{56}\log_2\frac{9}{2} + \frac{3}{56}\log_2\frac{9}{3} + \frac{4}{56}\log_2\frac{9}{4} + \frac{5\times 9}{56}\log_2\frac{9}{9} = 0.3235,\\
\mathrm{IC}(R_1|Y) &= \frac{5\times 5}{56}\log_2\frac{9}{5} + \frac{6}{56}\log_2\frac{9}{6} + \frac{7}{56}\log_2\frac{9}{7} + \frac{2\times 9}{56}\log_2\frac{9}{9} = 0.4866.
\end{aligned} \tag{17}$$

Then,

$$\mathrm{IIC}_2(R) = \left[\sum_{i=1}^{mn}(\lambda_i - \lambda_{i+1})\log_2\frac{mn}{|R_{\lambda_{i+1}}|}, \sum_{i=1}^{mn}(\lambda_i - \lambda_{i+1})\log_2\frac{mn}{|R_{\lambda_i}|}\right] = [0.2492, 0.36]. \tag{18}$$

Similarly, we have

$$\begin{aligned}
\mathrm{IIC}_3(R) &= \left[\sum_{i=1}^{mn}\frac{\lambda_i}{\sum_{i=1}^{mn}\lambda_i}\log_2\left(\frac{mn}{|R_{\lambda_{i+1}}|}\right), \sum_{i=1}^{mn}\frac{\lambda_i}{\sum_{i=1}^{mn}\lambda_i}\log_2\left(\frac{mn}{|R_{\lambda_i}|}\right)\right] = [0.3161, 0.4106],\\
\mathrm{IIC}_4(R) &= \left[\sum_{i=1}^{mn}(\lambda_i - \lambda_{i+1})\left(\mathrm{IC}(R_{\lambda_i}|X)\wedge \mathrm{IC}(R_{\lambda_i}|Y)\right), \sum_{i=1}^{mn}(\lambda_i - \lambda_{i+1})\left(\mathrm{IC}(R_{\lambda_i}|X)\vee \mathrm{IC}(R_{\lambda_i}|Y)\right)\right] = [0.2342, 0.3292],\\
\mathrm{IIC}_5(R) &= \left[\sum_{i=1}^{mn}\frac{\lambda_i}{\sum_{i=1}^{mn}\lambda_i}\left(\mathrm{IC}(R_{\lambda_i}|X)\wedge \mathrm{IC}(R_{\lambda_i}|Y)\right), \sum_{i=1}^{mn}\frac{\lambda_i}{\sum_{i=1}^{mn}\lambda_i}\left(\mathrm{IC}(R_{\lambda_i}|X)\vee \mathrm{IC}(R_{\lambda_i}|Y)\right)\right] = [0.2683, 0.3722].
\end{aligned} \tag{19}$$

4. The Ranking for Fuzzy Implication Operators Based on Interval Information Content

In data mining, it is necessary to extract rules from large databases, which means that a large number of rules will be generated during the process. So, how to evaluate these rules and get valid and useful information by determining the ranking of rules has become a new hotspot of data mining area. Here, the ranking of fuzzy implication operators can be realized by the interval information content of fuzzy relation. Let $I = \{I_1, I_2, \ldots, I_n\}$be the set of fuzzy implication operators, and the ranking method is defined as follows:

Step 1: to calculate the interval information content $\mathrm{IIC}(I_i)$ of fuzzy implication operator I_i $(i = 1, 2, \ldots, n)$.

Step 2: to calculate the possibility-based comparison value of interval information content p_{ij}, where $p_{ij} = P(\mathrm{IIC}(I_i) > \mathrm{IIC}(I_j))$ [28].

Step 3: to construct interval information content possibility-based comparison matrix P, where $P = (p_{ij})$.

Step 4: let $P_i = \sum_{j=1}^{n} p_{ij}$, and the ranking of implication operator is determined by the value of P_i. That is to say, if $P_i > P_j$, then $I_i > I_j$.

For implication operators, the ranking can be confirmed by extracting the interval information content of the corresponding fuzzy relation, but as the fuzzy relation matrix is only aimed for the discrete domain, it is necessary discretize the interval [0,1] by dividing into n parts, that is to say, let $X = \{m_0 = 0, m_1, m_2, \ldots, m_{n-1}, m_n = 1\}$, and the implication operators can be expressed as

$$I\colon X\times X \longrightarrow [0,1]\ (m_i, m_j) \longmapsto I(m_i, m_j), \quad i, j \in \{0, 1, 2, \ldots, n\}. \tag{20}$$

Here, four insertions can be adopted for the discretization: the average insertion of 9 points (a scale of zero

Table 1: The value of $R(x_i, y_j)$ in Example 1.

R	y_1	y_2	y_3	y_4	y_5	y_6	y_7	y_8	y_9	$\sum_{j=1}^{9} R(x_i, y_j)$
x_1	0	0	0	0	0	0.4	0.7	1	1	3.1
x_2	0	0	0	0	0	0.4	0.7	1	1	3.1
x_3	0.3	0.3	0.3	0.3	0.3	0.7	1	1	1	5.2
x_4	0.6	0.6	0.6	0.6	0.6	1	1	1	1	7
x_5	1	1	1	1	1	1	1	1	1	9
x_6	1	1	1	1	1	1	1	1	1	9
x_7	1	1	1	1	1	1	1	1	1	9
x_8	1	1	1	1	1	1	1	1	1	9
x_9	1	1	1	1	1	1	1	1	1	9
$\sum_{i=1}^{9} R(x_i, y_j)$	5.9	5.9	5.9	5.9	5.9	7.5	8.4	9	9	63.4

Table 2: The value of $R_1(x_i, y_j)$.

R_1	y_1	y_2	y_3	y_4	y_5	y_6	y_7	y_8	y_9	$\sum_{j=1}^{9} R_1(x_i, y_j)$
x_1	0	0	0	0	0	0	0	1	1	2
x_2	0	0	0	0	0	0	0	1	1	2
x_3	0	0	0	0	0	0	1	1	1	3
x_4	0	0	0	0	0	1	1	1	1	4
x_5	1	1	1	1	1	1	1	1	1	9
x_6	1	1	1	1	1	1	1	1	1	9
x_7	1	1	1	1	1	1	1	1	1	9
x_8	1	1	1	1	1	1	1	1	1	9
x_9	1	1	1	1	1	1	1	1	1	9
$\sum_{i=1}^{9} R_1(x_i, y_j)$	5	5	5	5	5	6	7	9	9	56

to ten), the average insertion of 19 points, the average insertion of 99 points, and the random insertion of 9 points. By equation (13), the interval information content of implication operators $I_1, \ldots, I_{13}$ is listed in Table 3.

Taking the average insertion of 9 points among the interval [0,1] as an example, the interval information content comparison matrix can be constructed as follows:

$$P = \left(p_{ij}\right) = \begin{pmatrix} 0 & 1 & 1 & 1 & 0 & 0 & 1 & 1 & 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 & 0 & 0.8199 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0.6501 & 0 & 0 & 0 & 0.0061 & 0.0061 \\ 1 & 1 & 1 & 1 & 0 & 0 & 1 & 1 & 1 & 1 & 0 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 0 & 1 & 1 & 1 & 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0.3591 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0.3499 & 0 & 0 & 0.6409 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 0 & 0 & 1 & 1 & 0 & 1 & 0 & 1 & 1 \\ 0 & 0.1801 & 1 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0.9939 & 0 & 0 & 1 & 0.8067 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0.9939 & 0 & 0 & 1 & 0.8067 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}. \tag{21}$$

Table 3: Interval information content of I_i.

I_i	The average insertion of 9 points	The average insertion of 19 points	The average insertion of 99 points	The random insertion of 9 points
I_1	[0.5995, 0.6340]	[0.6155, 0.6477]	[0.6292, 0.6595]	[0.4978, 0.5060]
I_2	[0.5126, 0.5126]	[0.5304, 0.5304]	[0.5460, 0.5460]	[0.3506, 0.3944]
I_3	[0.2592, 0.2592]	[0.2495, 0.2495]	[0.2410, 0.2410]	[0.2400, 0.2679]
I_4	[0.3825, 0.3825]	[0.3853, 0.3853]	[0.3875, 0.3875]	[0.2972, 0.3305]
I_5	[1.0813, 1.0813]	[1.2214, 1.2214]	[1.3760, 1.3760]	[0.7492, 0.7905]
I_6	[1.2058, 1.2058]	[1.4142, 1.4142]	[1.6487, 1.6487]	[0.8869, 0.9217]
I_7	[0.3509, 0.3509]	[0.3603, 0.3603]	[0.3692, 0.3692]	[0.3301, 0.3301]
I_8	[0.3119, 0.4205]	[0.3077, 0.4075]	[0.3067, 0.3925]	[0.2545, 0.3775]
I_9	[0.6514, 0.6514]	[0.6854, 0.6854]	[0.7141, 0.7141]	[0.5656, 0.6152]
I_{10}	[0.4320, 0.5303]	[0.4394, 0.5192]	[0.4531, 0.5043]	[0.3051, 0.4310]
I_{11}	[1.2174, 1.2174]	[1.4865, 1.4865]	[1.8393, 1.8393]	[0.9064, 0.9215]
I_{12}	[0.3817, 0.5126]	[0.3916, 0.5304]	[0.4016, 0.5460]	[0.2917, 0.4293]
I_{13}	[0.3817, 0.5126]	[0.3916, 0.5304]	[0.4016, 0.5460]	[0.4228, 0.5118]

Then,

$$P_1 = \sum_{j=1}^{n} p_{1j} = 0 + 1 + 1 + 1 + 0 + 0 + 1 + 1 + 0 + 1 + 0 + 1 + 1 = 8. \tag{22}$$

Similarly, we have $P_2 = 6.8199$, $P_3 = 0$, $P_4 = 2.6562$, $P_5 = 10$, $P_6 = 11$, $P_7 = 1.3591$, $P_8 = 1.9908$, $P_9 = 9$, $P_{10} = 4.1801$, $P_{11} = 12$, $P_{12} = 3.8006$, and $P_{13} = 3.8006$; then,

$$I_{11} \succ I_6 \succ I_5 \succ I_9 \succ I_1 \succ I_2 \succ I_{10} \succ \{I_{12}, I_{13}\} \succ I_4 \succ I_8 \succ I_7 \succ I_3. \tag{23}$$

The ranking results indicated that I_{11}, I_6, and I_5 transfer large amounts of information content, whereas I_3 (Luckasiewz operators) transmits the least amount of information content. Also, when the average insertion is concerned, the ranking results of I_{12} and I_{13} cannot be sure, but could be improved with the help of the random ways. All results with different insertions are listed in Table 4. From Table 4, it can be concluded that even though the insertion is different, the ranking results are different, but I_{11}, I_6, and I_5 always transfer large amounts of information content, whereas I_3 (Luckasiewz operators) transmits the least. In fact, the former three operators are used in the construction of fuzzy systems with higher frequency. However, there is almost no research on the advantages of these operators in the construction of fuzzy control systems, and the ranking results based on interval information content provide theoretical basis for the study of the abovementioned problems.

5. The Clustering and Classification of Fuzzy Implication Operators

The clustering analysis is focused on cluster the things with similar attributes into a category by means of extracting the things' attribute. Also, whether the classification is reasonable is a question worth considering. In this section, clustering analysis is carried out for 13 fuzzy operators according to the attributes of interval information content, which are commonly used to construct fuzzy control systems. After confirming the best classification, the similarity measure is used to classify the category of the implication operator.

5.1. Clustering of Fuzzy Implication Operators Based on Interval Information Content. Based on similarity measure of the interval value, fuzzy implication operators can be clustered utilizing interval information content. Let $I = \{I_1, I_2, \ldots, I_n\}$be the set containing finite implication operators; the cluster analysis can be undertaken as follows:

Step 1: to complete the interval information content $\mathrm{IIC}(I_i)$ $(i = 1, 2, \ldots, n)$ of I_i

Step 2: to complete the similarity measure $s_{ij} = S_I(\mathrm{IIC}(I_i), \mathrm{IIC}(I_j))$ by equation (12)

Step 3: to construct similarity matrix $S = (s_{ij})$ based on interval information content

Step 4: to compute transitive closure matrix $t(S)$

Step 5: to cluster the implication operators according to the value of λ

In the same way, four methods are used to disperse the interval [0,1]: the average insertion of 9 points, the average insertion of 19 points, the average insertion of 99 points, and the random insertion of 9 points. By equation (12), the interval information content of $I_1, \ldots, I_{13}$ is listed in Table 3. Next, taking the average insertion of 9 points for example, the 13×13 similarity matrix S_{10} based on interval information content is expressed as

Table 4: Ranking results of implication operators.

Interval segmentation	Ranking results
The average insertion of 9 points	$I_{11} \succ I_6 \succ I_5 \succ I_9 \succ I_1 \succ I_2 \succ I_{10} \succ \{I_{12}, I_{13}\} \succ I_4 \succ I_8 \succ I_7 \succ I_3$
The average insertion of 19 points	$I_{11} \succ I_6 \succ I_5 \succ I_9 \succ I_1 \succ I_2 \succ I_{10} \succ \{I_{12}, I_{13}\} \succ I_4 \succ I_8 \succ I_7 \succ I_3$
The average insertion of 99 points	$I_{11} \succ I_6 \succ I_5 \succ I_9 \succ I_1 \succ I_2 \succ I_{10} \succ \{I_{12}, I_{13}\} \succ I_4 \succ I_8 \succ I_7 \succ I_3$
The random insertion of 9 points	$I_{11} \succ I_6 \succ I_5 \succ I_9 \succ I_1 \succ I_{13} \succ I_2 \succ I_7 \succ I_{10} \succ I_{12} \succ I_4 \succ I_8 \succ I_3$

$$S_{10} = \begin{pmatrix} 1 & 0.8318 & 0.4206 & 0.6207 & 0.5704 & 0.5115 & 0.5694 & 0.5918 & \cdots & 0.7226 \\ 0.8318 & 1 & 0.5057 & 0.7462 & 0.4741 & 0.4251 & 0.6845 & 0.7144 & \cdots & 0.8723 \\ 0.4206 & 0.5057 & 1 & 0.6776 & 0.2397 & 0.2150 & 0.7387 & 0.7237 & \cdots & 0.5924 \\ 0.6207 & 0.7462 & 0.6776 & 1 & 0.3537 & 0.3172 & 0.9174 & 0.8625 & \cdots & 0.8721 \\ 0.5704 & 0.4741 & 0.2397 & 0.3537 & 1 & 0.8967 & 0.3245 & 0.3387 & \cdots & 0.4135 \\ 0.5115 & 0.4251 & 0.2150 & 0.3172 & 0.8967 & 1 & 0.2910 & 0.3037 & \cdots & 0.3708 \\ 0.5694 & 0.6845 & 0.7387 & 0.9174 & 0.3245 & 0.2910 & 1 & 0.8617 & \cdots & 0.8019 \\ 0.5918 & 0.7144 & 0.7237 & 0.8625 & 0.3387 & 0.3037 & 0.8617 & 1 & \cdots & 0.8187 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0.7226 & 0.8723 & 0.5924 & 0.8721 & 0.4135 & 0.3708 & 0.8019 & 0.8187 & \cdots & 1 \end{pmatrix}. \tag{24}$$

Furthermore, the transitive closure matrix is constructed as follows:

$$t_{10}(S) = \begin{pmatrix} 1 & 0.8318 & 0.7387 & 0.8318 & 0.6024 & 0.6024 & 0.8318 & 0.8318 & \cdots & 0.8318 \\ 0.8318 & 1 & 0.7387 & 0.8721 & 0.6024 & 0.6024 & 0.8721 & 0.8625 & \cdots & 0.9047 \\ 0.7387 & 0.7387 & 1 & 0.7387 & 0.6024 & 0.6024 & 0.7387 & 0.7387 & \cdots & 0.7381 \\ 0.8318 & 0.8721 & 0.7387 & 1 & 0.6024 & 0.6024 & 0.9174 & 0.8625 & \cdots & 0.8721 \\ 0.6024 & 0.6024 & 0.6024 & 0.6024 & 1 & 0.8967 & 0.6024 & 0.6024 & \cdots & 0.6024 \\ 0.6024 & 0.6024 & 0.6024 & 0.6024 & 0.8967 & 1 & 0.6024 & 0.6024 & \cdots & 0.6024 \\ 0.8318 & 0.8721 & 0.7387 & 0.9174 & 0.6024 & 0.6024 & 1 & 0.8625 & \cdots & 0.8721 \\ 0.8318 & 0.8625 & 0.7387 & 0.8625 & 0.6024 & 0.6024 & 0.8625 & 1 & \cdots & 0.8625 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0.8318 & 0.9047 & 0.7387 & 0.8721 & 0.6024 & 0.6024 & 0.8721 & 0.8625 & \cdots & 1 \end{pmatrix}. \tag{25}$$

The elements in the abovementioned matrix are arranged in ascending order {0.6024, 0.7387, 0.8318, 0.8625, 0.8721, 0.8967, 0.9047, 0.9174, 0.9251, 0.9468, 0.9905, 1}, then the cluster can be conducted by the abovementioned value, and all cluster results are listed in Figure 1.

Similarly, cluster analyses of average insertion of 19 and 99 points, as well as random insertion of 9 points, are listed in Figures 2–4.

Judging from the abovementioned four figures, the uniformity clustering results are divided into two categories of 13 fuzzy implication operators: $\{I_5, I_6, I_{11}\}$ and $\{I_1, I_2, I_3, I_4, I_7, I_8, I_9, I_{10}, I_{12}, I_{13}\}$. Therefore, it can be granted as the optimum category. According to Definition 4, all fuzzy implication operators can be strictly divided into two categories. Evenly, I_5, I_6, and I_{11} are abnormal implications, and others are normal implications. That is to say, the optimum cluster of the fuzzy implication operators based on interval information content is divided into two categories: normal and abnormal. Therefore, the classification method is reasonable.

5.2. Classification of Implication operators. In the problem of pattern recognition, as soon as the best classifications are selected, it is necessary to determine which category of classification features is the closest to the sample. For any fuzzy implication operator, after determining the best classification by extracting the interval information content characteristics, the final categories are confirmed by the similarity measure between the sample implication operators and the clustering centers of each category. Concretely,

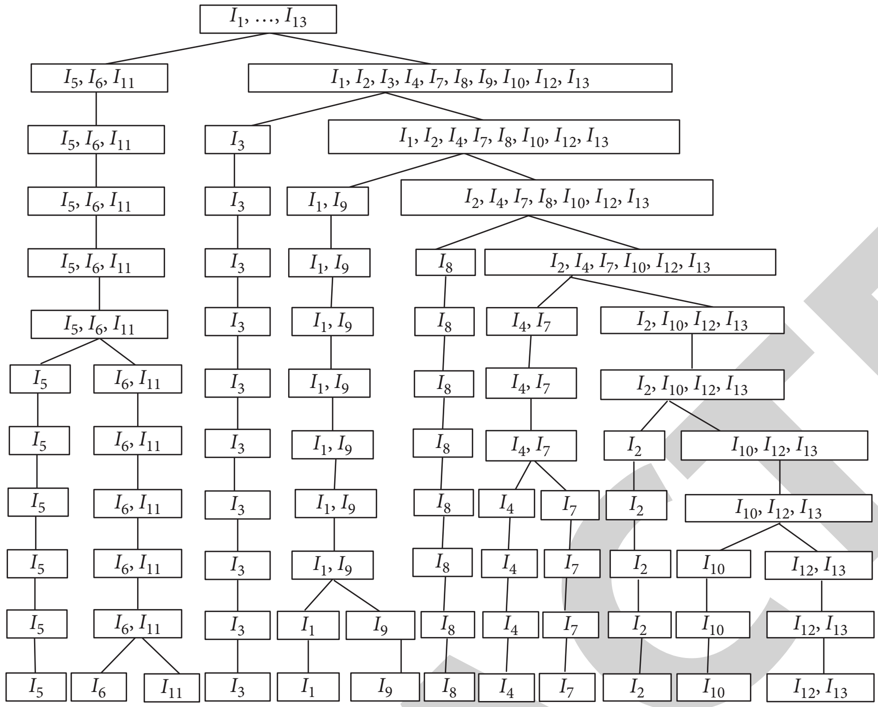

Figure 1: Cluster analysis of average insertion of 9 points.

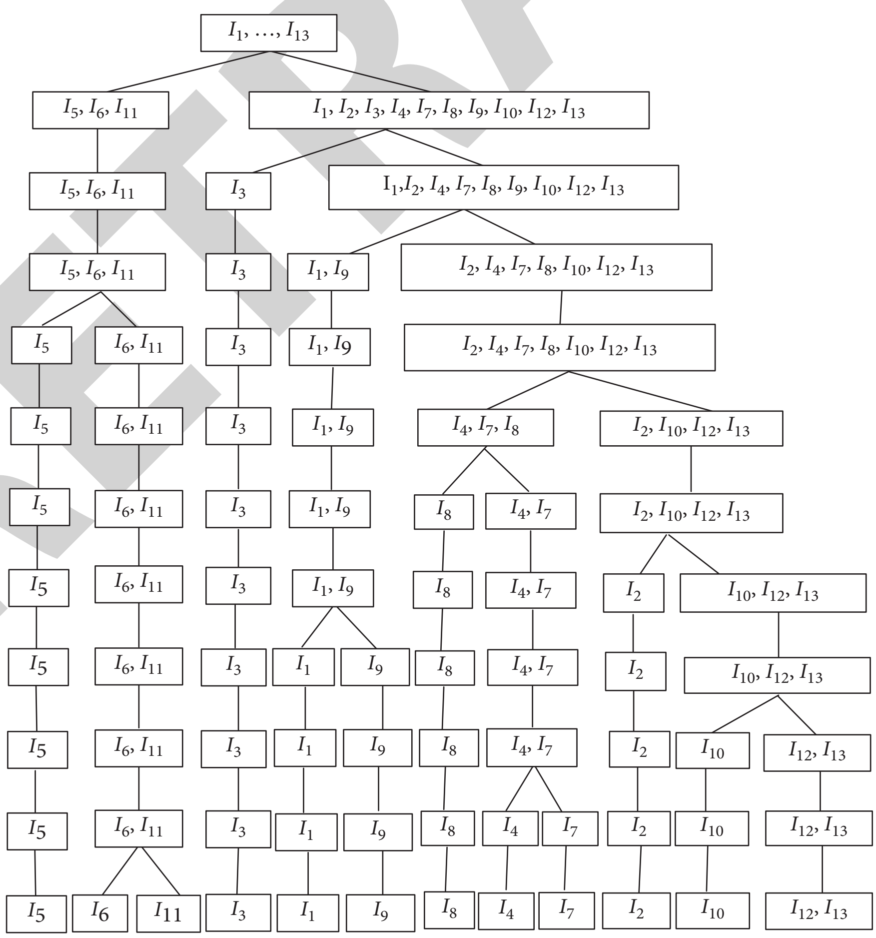

Figure 2: Cluster analysis of average insertion of 19 points.

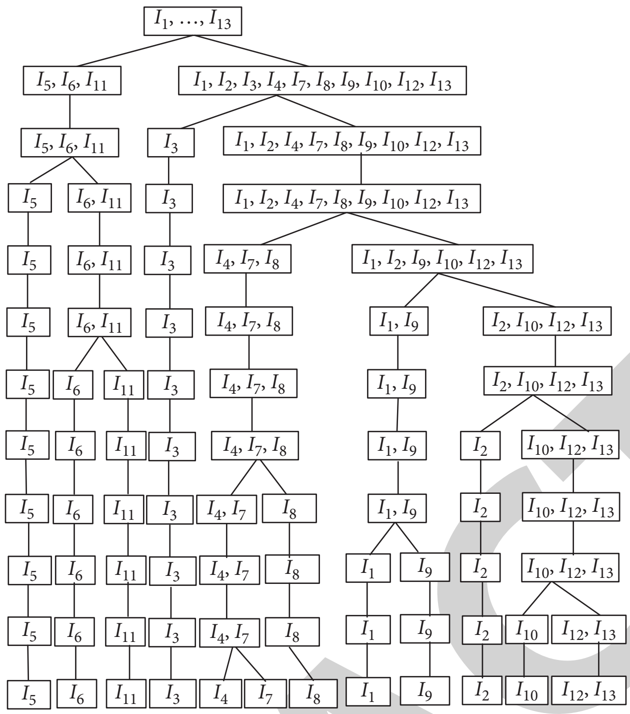

Figure 3: Cluster analysis of average insertion of 99 points.

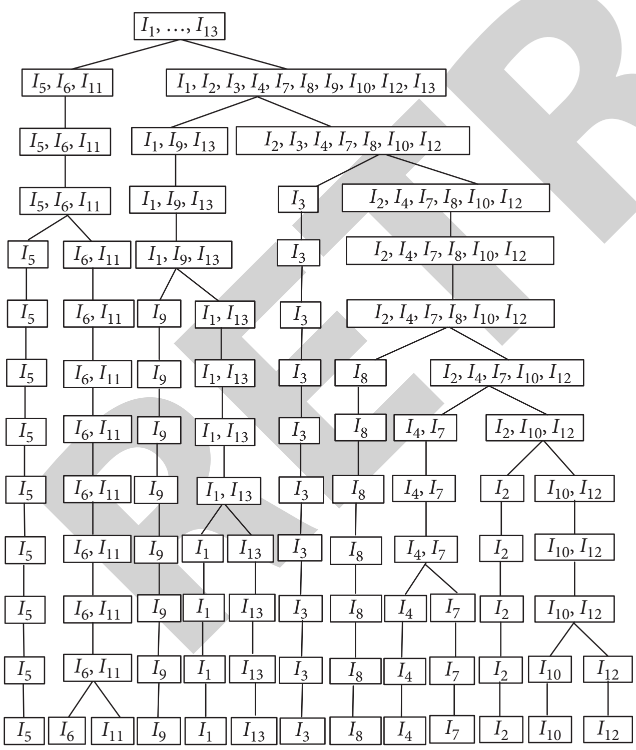

Figure 4: Cluster analysis of random insertion of 9 points.

Step 1: to compute IIC(I) of the sample operator I

Step 2: by equation (12), to compute the similarity measure between the sample operators and the clustering centers of each category, where the center of the i^{th} category $\overline{x}^i = [(\overline{x}^i)^-, (\overline{x}^i)^+]$ and

$$\left(\overline{x}^i\right)^- = \frac{1}{n_i}\sum_{j=1}^{n_i}\left(x_j^i\right)^-, \left(\overline{x}^i\right)^+ = \frac{1}{n_i}\sum_{j=1}^{n_i}\left(x_j^i\right)^+, \tag{26}$$

n_i is the sample number of the i^{th} category

Step 3: to determine the categories according to the maximum similarity principle

For instance, I_{14} and I_{15} are selected as the sample operators for classification:

$$\begin{aligned} I_{14}(a,b) &= \begin{cases} (1-a)\vee b, & (1-a)\wedge b = 0, \\ 1, & \text{else}, \end{cases} \\ I_{15}(a,b) &= \begin{cases} a\wedge b, & a\vee b = 1, \\ 0, & a\vee b < 1. \end{cases} \end{aligned} \tag{27}$$

I_{14} and I_{15} are the normal and abnormal fuzzy implication operators, respectively. Under the abovementioned optimum category, can these two implication operators be classified into correct categories? Firstly, to compute the interval information content with different insertions by equation (13), the results are listed in Table 5.

Secondly, to compute the similarity measure between them and cluster centers, the results are shown in Table 6.

From Table 6, it can be seen that I_{14} is always divided into the category of normal implication operators $\{I_1, I_2, I_3, I_4, I_7, I_8, I_9, I_{10}, I_{12}, I_{13}\}$ and I_{15} is always divided into the contrary, which is consistent with the nature of I_{14} and I_{15} as normal and abnormal implication.

Table 5: Interval information content of I_{14} and I_{15}.

I_i	The average insertion of 9 points	The average of 19 points	The average of 99 points	The random of 9 points
I_{14}	[0.1063, 0.1063]	[0.0570, 0.0570]	[0.0120, 0.0120]	[0.0916, 0.0916]
I_{15}	[1.1719, 1.1719]	[1.2999, 1.2999]	[2.0703, 2.0703]	[1.1093, 1.1093]

Table 6: Similarity measure between the implication operator and cluster center.

Interval segmentation	Cluster Result	Cluster center	Similarity measure between I_{14} and clustering centers	Similarity measure between I_{15} and clustering centers
The average insertion of 9 points	$\{I_5, I_6, I_{11}\}$	[1.1682, 1.1682]	0.0910	**1.0032**
	$\{I_1, I_2, I_3, I_4, I_7, I_8, I_9, I_{10}, I_{12}, I_{13}\}$	[0.4263, 0.4263]	**0.2494**	0.3628
The average insertion of 19 points	$\{I_5, I_6, I_{11}\}$	[1.3790, 1.3790]	0.0413	**0.9426**
	$\{I_1, I_2, I_3, I_4, I_7, I_8, I_9, I_{10}, I_{12}, I_{13}\}$	[0.4357, 0.4357]	**0.1308**	0.3352
The average insertion of 99 points	$\{I_5, I_6, I_{11}\}$	[1.6213, 1.6213]	0.0074	**0.7831**
	$\{I_1, I_2, I_3, I_4, I_7, I_8, I_9, I_{10}, I_{12}, I_{13}\}$	[0.4450, 0.4450]	**0.027**	0.2149
The random insertion of 9 points	$\{I_5, I_6, I_{11}\}$	[0.8475, 0.8779]	0.1156	**0.7495**
	$\{I_1, I_2, I_3, I_4, I_7, I_8, I_9, I_{10}, I_{12}, I_{13}\}$	[0.3555, 0.4194]	**0.2576**	0.3358

6. Conclusions

Facing the era of big data, it is essential to process a large amount of data. So, it is a key issue to extract the attribute of data. The novel attribute in the presented work can be used to realize the rules' ranking and clustering effectively. Utilizing the interval information content, the Mamdani, probability product, and Yager operators show better ranking results than others, which provides a solid theoretical base for the operator selection in constructing fuzzy system. For clustering issues, by means of extracting the interval information content, the operators can be divided into two categories: normal and abnormal. Then, the correct clustering result of the operator with known attribute proves valid.

In the future, the following works will be carried out:

(1) If the axiomatic representation of interval information quantity of fuzzy relation can be established, the research of information quantity will be of great theoretical significance

(2) For the defined interval information content, if it can be applied to data mining to optimize and ranking the inference rules, it will be of practical significance to improve the accuracy of the fuzzy system

Data Availability

All the data have been included in the manuscript, and no additional information can be provided.

Conflicts of Interest

The author declares that there are no conflicts of interest regarding the publication of this paper.

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