Research Article

A Novel Approach of Complex Dual Hesitant Fuzzy Sets and Their Applications in Pattern Recognition and Medical Diagnosis

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Received 14 October 2020; Revised 10 November 2020; Accepted 28 November 2020; Published 28 April 2021

Academic Editor: Ahmed Mostafa Khalil

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Complex dual hesitant fuzzy set (CDHFS) is an assortment of complex fuzzy set (CFS) and dual hesitant fuzzy set (DHFS). In this manuscript, the notion of the CDHFS is explored and its operational laws are discussed. The new methodology of the complex interval-valued dual hesitant fuzzy set (CIVDHFS) and its necessary laws are introduced and are also defensible with the help of examples. Further, the antilogarithmic and with-out exponential-based similarity measures, generalized similarity measures, and their important characteristics are also developed. These similarity measures are applied in the environment of pattern recognition and medical diagnosis to evaluate the proficiency and feasibility of the established measures. We also solved some numerical examples using the established measures to examine the reliability and validity of the proposed measures by comparing these with existing measures. To strengthen the proposed study, the comparative analysis is made and it is conferred that the proposed study is much more superior to the existing studies.

1. Introduction

Zadeh [1] presented the theory of fuzzy sets (FSs), which contain the grade of truth belonging to unit interval. But, in some cases, the theory of FSs has been failed. For instance, when a decision-maker faced information in the form of truth and falsity grades, then the FSs are not able to cope with it. In reality, these sets have given different approaches to allot the participation degree or the nonmembership level of a component to a given set portrayed by various properties. IFSs [2], otherwise called IVFSs from a numerical perspective, can be demonstrated with two capacities that characterize a stretch to mirror some vulnerability on the enrollment work of the components. IVFSs are the speculation of FSs and can demonstrate vulnerability for the need of data, in which a closed subinterval of $[0, 1]$ is relegated to the participation degree. Atanassov and Gargov [3] demonstrated that IFSs and IVFSs are equipollent speculations of FSs and proposed the thought of IIVFS, which has been examined and utilized broadly [4–8].

T2FSs, depicted by enrollment work that is portrayed by more boundaries, grant the fuzzy enrollment as a fuzzy set improving the demonstrating ability compared to the original one. Scientifically, IFSs can be viewed as a specific instance of T2FSs, where the participant's work restores a lot of fresh stretches. In spite of the wide uses of T2FSs [9–12], they experience issues in building up the optional enrollment capacities and troubles in control [13–15]. FMSs are also the extension of the FSs, which contains the grade of truth in the form of the finite subset of the unit interval. Note that in spite of the fact that the highlights of FMSs permit the application to data recovery on the internet, where a web index recovers different events of the same subjects with conceivable various degrees of significance [16], they have issues with the fundamental tasks, for example, the definitions for association and crossing point, which do not sum up the ones for FSs. Rickard [17] gave an elective definition that underlines the value of a commutative property between a set activity and an α -cut, settling this issue. HFSs were initially presented by Torra [4]. The theory of hesitant fuzzy

information is also reliable and more efficient to cope with complicated and vague information in realistic issues. Torra [4] inspected IFSs and FMSs and drew examinations and made characteristic associations among them. The theories of hesitant fuzzy information and fuzzy multisets are both the same concepts because the resultant values of both sets are in the form of the finite subset of the unit interval. HFSs were esteemed IFSs when the HFS is a nonempty shut span. In light of the connections between IFSs and HFSs, Torra [4] gave a definition comparing the envelope of HFS. Xu and Xia [18] explored the conglomeration administrators, separation, what's more, comparability measures for HFSs, and applied them to decision-making problems [19–21].

Recently, Zhu et al. [22] explored the DHFS which encompass fuzzy sets, IFSs, HFSs, and fuzzy multisets as special cases. DHFSs have described in terms of the following two functions: the membership hesitancy function and the nonmembership hesitancy function. Wang et al. [23] defined distance and similarity measures of DHFSs with their applications to multiple attribute decision making. Taking into account such functions provides us with more exemplary and flexible access to assign values for each element in the domain. Apparently, DHFSs can reflect the human's hesitance more objectively than the other existing extensions of the fuzzy set (AIFSs, IVAIFSs, HFSs, etc.). However, in the spirit of what has been done for IVFSs, Farhadinia [24] introduced a dual interval-valued hesitant fuzzy set (DIVHFS) where its fundamental characteristic is that the values of the membership function and nonmembership function are set of intervals rather than a set of exact numbers. Certain scholars have asked a question, when we changed the range of the fuzzy set into unit disc in complex plane, what will be the result. Ramot et al. [25] introduced the idea of complex FS (CFS), which contains the truth grade in the form of a complex number by a member of a unit disc in the complex plane. CFS deals with two dimensions in a single set. CFS is a powerful procedure to illustrate the belief of a human being in the formation of grades. The complex fuzzy set considers only the membership degree, but does not weight on the nonmembership portion of the data entities, which likewise assume an equal part in assessing the object in the decision-making process. However, in the real world, it is regularly hard to express the estimation of the membership degree by an exact value in a fuzzy set. In such cases, it might be easier to depict vagueness and uncertainty in the real world using a 2-dimensional information instead of a single one. Consequently, an extension of the existing theories might be extremely valuable to depict the uncertainties because of his/her reluctant judgment in complex decision-making problems. Many researchers have utilized it in the environment of different fields.

In real-life issues, we run over numerous circumstances where we have to measure the vulnerability existing in the information to settle on ideal choices. Exponential-based similitude measures and with-out exponential-based comparability measures are significant apparatuses for taking care of dubious data present in our day-to-day life issues. Various measures, for example, similitude, exponential, separation, entropy, and incorporation, process the

questionable data and empower us to arrive at some resolution. As of late, these measures have increased a lot of considerations from numerous creators because of their wide applications in different fields, for example, design acknowledgment, clinical determination, grouping examination, and picture portion. All the current methodologies of chiefs, in light of exponential-based closeness measures and with-out exponential-based likeness measures, in FS, CFS, and HFS speculations, manage participation capacities having a place with a unit span as a subset in the idea of HFS. In the CDHFS hypothesis, enrollment and falsity degrees are perplexing esteemed and are spoken to in polar directions. When a decision-maker provides such sorts of information, whose membership and non-membership grades in the form of a finite subset of complex numbers belonging to complex plane with a rule that is the sum of the maximum of the real parts (also for imaginary parts) of the membership and non-membership grades is restricted to the unit interval. For instance, $\{0.9e^{i2\pi(0.3)}, 0.7e^{i2\pi(0.6)}, 0.3e^{i2\pi(0.2)}, 0.1e^{i2\pi(0.2)}\}$ and $\{0.09e^{i2\pi(0.03)}, 0.07e^{i2\pi(0.06)}, 0.03e^{i2\pi(0.02)}, 0.01e^{i2\pi(0.02)}\}$, then all the existing notions are failed. For coping with such kinds of problems, the CDHFS is a proficient technique to resolve realistic decision problems in the environment of the FS theory. CDHFS is more powerful and more general than existing notions such as HFS, CFS, and FS to cope with awkward and complicated information in real-life decisions. Because all these notions are the special cases of the explored CDHFS, the advantages of the presented CDHFS are discussed below:

- (1) When we choose the imaginary parts of the CDHFS as zero, then the CDHFS is reduced into DHFS which is in the form of $\{0.9, 0.7, 0.3, 0.1\}$ and $\{0.09, 0.07, 0.03, 0.01\}$.
- (2) When we choose the CDHFS as a singleton set, then the CDHFS is reduced into CIFS which is in the form of $\{0.9e^{i2\pi(0.3)}\}$ and $\{0.09e^{i2\pi(0.03)}\}$.
- (3) When we choose the CDHFS as a singleton set and the imaginary parts as zero, then the CDHFS is reduced into IFS which is in the form of $\{0.9\}$ and $\{0.09\}$.

In real-life problems, we come across many situations where we need to quantify the uncertainty existing in the data to make optimal decisions. Information measures are important tools for handling uncertain information present in our day-to-day life problems. Different measures of information, such as similarity, distance, entropy, and inclusion, process the uncertain information and enable us to attain some conclusions. Recently, these measures have gained much attention from many authors due to their wide applications in various fields, such as pattern recognition, medical diagnosis, clustering analysis, and image segment. All the existing approaches of decision making, based on information measures, in FS and DHFS theories, deal with membership and nonmembership functions, which are real valued. In CDHFS theory, membership and nonmembership degrees are complex-valued and are represented in polar coordinates. The amplitude term corresponding to

membership (nonmembership) degree gives the extent of belongingness (not belongingness) of an object in a CDHFS, and the phase term associated with membership (nonmembership) degree gives the additional information, generally related with periodicity. The phase terms are novel parameters of the membership and nonmembership degrees and these are the parameters which distinguish the CDHFS and traditional DHFS theories. DHFS theory deals with only one dimension at a time, which results in information loss in some instances. However, in real life, we come across complex natural phenomena where it becomes essential to add the second dimension to the expression of membership and nonmembership grades. By introducing this second dimension, the complete information can be projected in one set, and hence, loss of information can be avoided. To illustrate the significance of phase term, we give an example. Suppose XYZ company decides to set up biometric-based attendance devices (BBADs) in all of its offices spread all over the country. For this, the company consults an expert who gives the information regarding (i) models of BBADs and (ii) production dates of BBADs. The company wants to select the most optimal model of BBADs with its production date simultaneously. Here, the problem is two-dimensional, namely, the model of BBADs and production date of BBADs. This type of problem cannot be modeled accurately using traditional DHFS theory as DHFS theory cannot tackle with both the dimensions simultaneously. The best way to represent all of the information provided by the expert is by using CDHFS theory. The amplitude terms in CDHFS may be used to give company's decision regarding the model of BBADs and the phase terms may be used to represent company's judgment in respect of production date of BBADs. Motivated by the abovementioned challenges and keeping the advantages of the CDHFS, in this manuscript, some key contributions are made:

- (1) Complex dual hesitant fuzzy set (CDHFS) is a combination of two modifications, called complex fuzzy set (CFS) and dual hesitant fuzzy set (DHFS). CDHFS composes two degrees, called truth valued and falsity valued in the form of a finite subset of a unit disc in a complex plane and is a proficient technique to cope with uncertain and unpredictable information in real-life decisions. The aim of this manuscript is to explore the notion of a CDHFS and its operational laws.
- (2) The novel approach of CInvDHFS and its fundamental laws are explored and also justified with the help of examples.
- (3) Further, the antilogarithmic and with-out exponential-based similarity measures and generalized similarity measures and their important characteristics are also explored.
- (4) These similarity measures are applied in the environment of pattern recognition and medical diagnosis to evaluate the proficiency and feasibility of the

established measures. We also solved some numerical examples using the established measures.

- (5) Examining the reliability and validity of the proposed measures by comparing it with existing measures.
- (6) The advantages, comparative analysis, and graphical representation of the explored measures and existing measures are also discussed in detail.

The remainder of this manuscript is as follows. In Section 2, we review some fundamental definitions such as FS, CFS, HFS, DHFS, and IvDHFS. In Section 3, we explore the notion of a CDHFS and its operational laws. The novel approach of CInvDHFS and its fundamental laws are also explored and also justified with the help of examples. In Section 4, the antilogarithmic and with-out exponential-based similarity measures and generalized similarity measures and their important characteristics are also explored. In Section 5, these similarity measures are applied in the environment of pattern recognition and medical diagnosis to evaluate the proficiency and feasibility of the established measures. We also solved some numerical examples using the established measures to examine the reliability and validity of the proposed measures by comparing it with existing measures given in Section 6. The advantages, comparative analysis, and graphical representation of the explored measures and existing measures are also discussed in detail. The conclusion of this manuscript is discussed in Section 7.

2. Preliminaries

In this section, we review fundamental definitions such as FS, CFS, HFS, DHFS, and IvDHFS. Through this article, X speaks to a fix set.

Definition 1 (see [1]). A FS $\mathcal{A}$ is of the following form:

$$\mathcal{A} = \{(x, \mu_{\mathcal{A}}(x)) | x \in nqX\}, \quad (1)$$

with a condition $0 \leq \mu_{\mathcal{A}}(x) \leq 1$, where $\mu_{\mathcal{A}}(x)$ represents the grade of truth. Through this article, the collection of all FSs on X are represented by $FS(X)$. The pair $\mathcal{A} = (x, \mu_{\mathcal{A}}(x))$ is known as fuzzy number (FN).

Definition 2 (see [25]). A CFS $\mathcal{A}$ is of the following form:

$$\mathcal{A} = \{(x, \mu_{\mathcal{A}}(x)) | x \in X\}, \quad (2)$$

where $\mu_{\mathcal{A}}(x) = \gamma_{\mathcal{A}}(x) \cdot e^{i2\pi(\omega_{\gamma_{\mathcal{A}}}(x))}$ represents the complex-valued truth grade in the form of polar coordinate, where $\gamma_{\mathcal{A}}(x), \omega_{\gamma_{\mathcal{A}}}(x) \in [0, 1]$. Additionally, the pair $\mathcal{A} = (x, \gamma_{\mathcal{A}}(x) \cdot e^{i2\pi(\omega_{\gamma_{\mathcal{A}}}(x))})$ is known as complex fuzzy number (CFN).

Definition 3 (see [4]). A HFS $\mathcal{A}$ is of the following form:

$$\mathcal{A} = \{(x, \mu_{\mathcal{A}}(x)) | txn \in qX\}, \quad (3)$$

where $\mu_{\mathcal{A}}(x)$ is the set of different finite values in $[0, 1]$, representing the grade of truth for each element $x \in X$.

Further, the pair $\mathcal{A} = (x, \mu_{\mathcal{A}}(x))$ is known as hesitant fuzzy number (HFN).

Definition 4 (see [23]). A DHFS $\mathcal{A}$ is of the following form:

$$\mathcal{A} = \{ (x, (\mu_{\mathcal{A}}(x), \nu_{\mathcal{A}}(x))) | x \in nX \}, \quad (4)$$

where $\mu_{\mathcal{A}}(x)$ and $\nu_{\mathcal{A}}(x)$ are the two finite subsets in $[0, 1]$, representing the membership grade and nonmembership grade of the component $x \in X$, respectively, with the conditions $0 \leq \gamma_{\mathcal{A}}(x), \delta_{\mathcal{A}}(x) \leq 1$ and $0 \leq \gamma_{\mathcal{A}}^+(x) + \delta_{\mathcal{A}}^+(x) \leq 1$, where $\gamma_{\mathcal{A}}(x) \in \mu_{\mathcal{A}}(x)$, $\delta_{\mathcal{A}}(x) \in \nu_{\mathcal{A}}(x)$, $\gamma_{\mathcal{A}}^+(x) \in \mu_{\mathcal{A}}^+(x) = \cup_{\gamma_{\mathcal{A}}(x) \in \mu_{\mathcal{A}}(x)} \max\{\gamma_{\mathcal{A}}(x)\}$, and $\delta_{\mathcal{A}}^+(x) \in \nu_{\mathcal{A}}^+(x) = \cup_{\delta_{\mathcal{A}}(x) \in \nu_{\mathcal{A}}(x)} \max\{\delta_{\mathcal{A}}(x)\}$.

Definition 5 (see [24]). For any two DHFSs $\mathcal{A}$ and $\mathcal{B}$, the similarity measure $s(\mathcal{A}, \mathcal{B})$ satisfies the following axioms:

- (1) $0 \leq s(\mathcal{A}, \mathcal{B}) \leq 1$
- (2) $s(\mathcal{A}, \mathcal{B}) = 1 \iff \mathcal{A} = \mathcal{B}$
- (3) $s(\mathcal{A}, \mathcal{B}) = s(\mathcal{B}, \mathcal{A})$

Definition 6 (see [24]). For any two DHFSs $\mathcal{A}$ and $\mathcal{B}$, the distance measure $d(\mathcal{A}, \mathcal{B})$ satisfies the following axioms:

- (1) $0 \leq d(\mathcal{A}, \mathcal{B}) \leq 1$
- (2) $d(\mathcal{A}, \mathcal{B}) = 1 \iff \mathcal{A} = \mathcal{B}$
- (3) $d(\mathcal{A}, \mathcal{B}) = d(\mathcal{B}, \mathcal{A})$

From the abovementioned analysis, we get that the $s(\mathcal{A}, \mathcal{B}) = 1 - d(\mathcal{A}, \mathcal{B})$.

Definition 7 (see [26]). A IvDHFS $\mathcal{A}$ is of the following form:

$$\mathcal{A} = \{ (x, (\mu_{\mathcal{A}}(x), \nu_{\mathcal{A}}(x))) | x \in nXq \}, \quad (5)$$

where $\mu_{\mathcal{A}}(x) = \cup_{[\gamma_{\mathcal{A}}^L(x), \gamma_{\mathcal{A}}^U(x)] \in \mu_{\mathcal{A}}(x)} \max\{[\gamma_{\mathcal{A}}^L(x), \gamma_{\mathcal{A}}^U(x)]\}$ and $\nu_{\mathcal{A}}(x) = \cup_{[\delta_{\mathcal{A}}^L(x), \delta_{\mathcal{A}}^U(x)] \in \nu_{\mathcal{A}}(x)} \max\{[\delta_{\mathcal{A}}^L(x), \delta_{\mathcal{A}}^U(x)]\}$ are two finite subsets of some interval-values in $[0, 1]$, representing the membership grade and nonmembership grade of the component $x \in X$, respectively, with the conditions $[\gamma_{\mathcal{A}}^L(x), \gamma_{\mathcal{A}}^U(x)] \subseteq [0, 1]$, and $0 \leq (\gamma_{\mathcal{A}}^U(x))^+ + (\delta_{\mathcal{A}}^U(x))^+ \leq 1$, where $[\gamma_{\mathcal{A}}^L(x), \gamma_{\mathcal{A}}^U(x)] \in \mu_{\mathcal{A}}(x)$, $[\delta_{\mathcal{A}}^L(x), \delta_{\mathcal{A}}^U(x)] \in \nu_{\mathcal{A}}(x)$, $(\gamma_{\mathcal{A}}^U(x))^+ \in \mu_{\mathcal{A}}^+(x) = \cup_{[\gamma_{\mathcal{A}}^L(x), \gamma_{\mathcal{A}}^U(x)] \in \mu_{\mathcal{A}}(x)} \max\{\gamma_{\mathcal{A}}^U(x)\}$, and $(\delta_{\mathcal{A}}^U(x))^+ \in \nu_{\mathcal{A}}^+(x) = \cup_{[\delta_{\mathcal{A}}^L(x), \delta_{\mathcal{A}}^U(x)] \in \nu_{\mathcal{A}}(x)} \max\{\delta_{\mathcal{A}}^U(x)\}$, for all $x \in X$.

3. Complex Dual Hesitant Fuzzy Sets and Complex Interval-Valued Dual Hesitant Fuzzy Sets

The aim of this section is to propose the novel of CDHFS and its operational laws. We also proposed the novel of

CivDHFS and its operational laws. We verified these operation laws with the help of numerical examples.

3.1. Complex Dual Hesitant Fuzzy Sets. We defined the notion of CDHFS which is the combination of CFS and DHFS and discussed its operational laws. Additionally, we verified its operational laws with the help of example.

Definition 8. A CDHFS $\mathcal{A}$ is of the following form:

$$\mathcal{A} = \{ (x, (\mu_{\mathcal{A}}(x), \nu_{\mathcal{A}}(x))) | x \in X \}, \quad (6)$$

where

$$\begin{aligned} \mu_{\mathcal{A}}(x) &= \left\{ \left(x, \gamma_{\mathcal{A}_{\mathcal{F}}}(x) \cdot e^{i2\pi(\omega_{\gamma_{\mathcal{A}_{\mathcal{F}}}(x)})} \right), \mathcal{F} = 1, 2, 3, \dots, g \right\}, \\ \nu_{\mathcal{A}}(x) &= \left\{ \left(x, \delta_{\mathcal{A}_{\mathcal{F}}}(x) \cdot e^{i2\pi(\omega_{\delta_{\mathcal{A}_{\mathcal{F}}}(x)})} \right), \mathcal{F} = 1, 2, 3, \dots, h \right\}, \end{aligned} \quad (7)$$

represented the complex-valued membership grade and nonmembership grade, which are subsets of a unit disc in complex plane with a condition $\gamma_{\mathcal{A}_{\mathcal{F}}}(x), \omega_{\gamma_{\mathcal{A}_{\mathcal{F}}}(x)}, \delta_{\mathcal{A}_{\mathcal{F}}}(x), \omega_{\delta_{\mathcal{A}_{\mathcal{F}}}(x)} \in [0, 1]$, $0 \leq \gamma_{\mathcal{A}_{\mathcal{F}}}^+(x) + \delta_{\mathcal{A}_{\mathcal{F}}}^+(x) \leq 1$, and $0 \leq \omega_{\gamma_{\mathcal{A}_{\mathcal{F}}}}^+(x) + \omega_{\delta_{\mathcal{A}_{\mathcal{F}}}}^+(x) \leq 1$, where $\gamma_{\mathcal{A}_{\mathcal{F}}}^+(x) = \cup_{\gamma_{\mathcal{A}_{\mathcal{F}}}(x) \in \mu_{\mathcal{A}}(x)} \max\{\gamma_{\mathcal{A}_{\mathcal{F}}}(x)\}$, $\omega_{\gamma_{\mathcal{A}_{\mathcal{F}}}}^+(x) = \cup_{\omega_{\gamma_{\mathcal{A}_{\mathcal{F}}}(x)} \in \mu_{\mathcal{A}}(x)} \max\{\omega_{\gamma_{\mathcal{A}_{\mathcal{F}}}(x)}\}$, and $\delta_{\mathcal{A}_{\mathcal{F}}}^+(x) = \cup_{\delta_{\mathcal{A}_{\mathcal{F}}}(x) \in \nu_{\mathcal{A}}(x)} \max\{\delta_{\mathcal{A}_{\mathcal{F}}}(x)\}$, $\omega_{\delta_{\mathcal{A}_{\mathcal{F}}}}^+(x) = \cup_{\omega_{\delta_{\mathcal{A}_{\mathcal{F}}}(x)} \in \nu_{\mathcal{A}}(x)} \max\{\omega_{\delta_{\mathcal{A}_{\mathcal{F}}}(x)}\}$, for $\mathcal{F} = 1, 2, \dots, g$ and $\mathcal{F} = 1, 2, \dots, h$. Further, $\mathcal{A} = (x, (\gamma_{\mathcal{A}_{\mathcal{F}}}(x) \cdot e^{i2\pi(\omega_{\gamma_{\mathcal{A}_{\mathcal{F}}}(x)})}, \delta_{\mathcal{A}_{\mathcal{F}}}(x) \cdot e^{i2\pi(\omega_{\delta_{\mathcal{A}_{\mathcal{F}}}(x)})}))$ is called complex dual hesitant fuzzy number (CDHFN).

Definition 9. Let $\mathcal{A} = (x, (\gamma_{\mathcal{A}_{\mathcal{F}}}(x) \cdot e^{i2\pi(\omega_{\gamma_{\mathcal{A}_{\mathcal{F}}}(x)})}, \delta_{\mathcal{A}_{\mathcal{F}}}(x) \cdot e^{i2\pi(\omega_{\delta_{\mathcal{A}_{\mathcal{F}}}(x)})}))$ and $\mathcal{B} = (x, (\gamma_{\mathcal{B}_{\mathcal{F}}}(x) \cdot e^{i2\pi(\omega_{\gamma_{\mathcal{B}_{\mathcal{F}}}(x)})}, \delta_{\mathcal{B}_{\mathcal{F}}}(x) \cdot e^{i2\pi(\omega_{\delta_{\mathcal{B}_{\mathcal{F}}}(x)})}))$ be two CDHFNs. Then, their complement, union, and intersection are defined as follows:

- (1) $\mathcal{A}^c = \{(x, (\nu_{\mathcal{A}}(x), \mu_{\mathcal{A}}(x)))\}$
- (2) $\mathcal{A} \cup \mathcal{B} = \left\{ \left(x, \left(\max(\gamma_{\mathcal{A}_{\mathcal{F}}}(x), \gamma_{\mathcal{B}_{\mathcal{F}}}(x)) \cdot e^{i2\pi(\max(\omega_{\gamma_{\mathcal{A}_{\mathcal{F}}}(x), \omega_{\gamma_{\mathcal{B}_{\mathcal{F}}}(x)})})}, \min(\delta_{\mathcal{A}_{\mathcal{F}}}(x), \delta_{\mathcal{B}_{\mathcal{F}}}(x)) \cdot e^{i2\pi(\min(\omega_{\delta_{\mathcal{A}_{\mathcal{F}}}(x), \omega_{\delta_{\mathcal{B}_{\mathcal{F}}}(x)})})} \right) \right) \right\}$
- (3) $\mathcal{A} \cap \mathcal{B} = \left\{ \left(x, \left(\min(\gamma_{\mathcal{A}_{\mathcal{F}}}(x), \gamma_{\mathcal{B}_{\mathcal{F}}}(x)) \cdot e^{i2\pi(\min(\omega_{\gamma_{\mathcal{A}_{\mathcal{F}}}(x), \omega_{\gamma_{\mathcal{B}_{\mathcal{F}}}(x)})})}, \max(\delta_{\mathcal{A}_{\mathcal{F}}}(x), \delta_{\mathcal{B}_{\mathcal{F}}}(x)) \cdot e^{i2\pi(\max(\omega_{\delta_{\mathcal{A}_{\mathcal{F}}}(x), \omega_{\delta_{\mathcal{B}_{\mathcal{F}}}(x)})})} \right) \right) \right\}$

Example 1. Let

$$\mathcal{A} = \left\{ \left(x_1, \left\{ \left\{ \frac{0.8e^{i2\pi(0.6)}}{0.5e^{i2\pi(0.4)}} \right\}, \{0.1e^{i2\pi(0.1)}\} \right\} \right), \left(x_2, \left\{ \{0.6e^{i2\pi(0.7)}\}, \{0.3e^{i2\pi(0.2)}\} \right\} \right) \right\},$$

$$\mathcal{B} = \left\{ \left(x_1, \left\{ \left\{ \frac{0.5e^{i2\pi(0.6)}}{0.4e^{i2\pi(0.5)}} \right\}, \{0.1e^{i2\pi(0.3)}\} \right\} \right), \left(x_2, \left\{ \left\{ \frac{0.6e^{i2\pi(0.5)}}{0.3e^{i2\pi(0.4)}} \right\}, \{0.3e^{i2\pi(0.2)}\} \right\} \right) \right\},$$

$$\mathcal{B} = \left\{ \left(x_1, \left\{ \left\{ \frac{0.6e^{i2\pi(0.8)}}{0.5e^{i2\pi(0.55)}} \right\}, \{0.15e^{i2\pi(0.1)}\} \right\} \right), \left(x_2, \left\{ \{0.2e^{i2\pi(0.3)}\}, \left\{ \frac{0.7e^{i2\pi(0.65)}}{0.3e^{i2\pi(0.5)}} \right\} \right\} \right) \right\},$$

$$\mathcal{B} = \left\{ \left(x_3, \left\{ \left\{ \frac{0.65e^{i2\pi(0.35)}}{0.5e^{i2\pi(0.65)}} \right\}, \{0.25e^{i2\pi(0.15)}\} \right\} \right), \left(x_4, \left\{ \{0.55e^{i2\pi(0.45)}\}, \{0.45e^{i2\pi(0.25)}\} \right\} \right) \right\}$$

be two CDHFSs. Then,

$$(1) \mathcal{A}^c = \left\{ \left(x_1, \left\{ \left\{ \frac{0.1e^{i2\pi(0.1)}}{0.8e^{i2\pi(0.6)}} \right\}, \{0.9e^{i2\pi(0.9)}\} \right\} \right), \left(x_2, \left\{ \{0.4e^{i2\pi(0.3)}\}, \{0.7e^{i2\pi(0.8)}\} \right\} \right) \right\},$$

$$(2) \mathcal{A} \cup \mathcal{B} = \left\{ \left(x_1, \left\{ \left\{ \frac{0.8e^{i2\pi(0.8)}}{0.5e^{i2\pi(0.55)}} \right\}, \{0.15e^{i2\pi(0.1)}\} \right\} \right), \left(x_2, \left\{ \{0.2e^{i2\pi(0.3)}\}, \left\{ \frac{0.7e^{i2\pi(0.65)}}{0.3e^{i2\pi(0.5)}} \right\} \right\} \right) \right\},$$

$$(3) \mathcal{A} \cap \mathcal{B} = \left\{ \left(x_1, \left\{ \left\{ \frac{0.6e^{i2\pi(0.6)}}{0.5e^{i2\pi(0.55)}} \right\}, \{0.15e^{i2\pi(0.1)}\} \right\} \right), \left(x_2, \left\{ \left\{ \frac{0.6e^{i2\pi(0.5)}}{0.3e^{i2\pi(0.4)}} \right\}, \{0.3e^{i2\pi(0.2)}\} \right\} \right) \right\}.$$

3.2. Complex Interval-Valued Dual Hesitant Fuzzy Sets.

In this subsection, we explored the novel of CInvDHFS and discussed its operational laws. Additionally, We verified its operational laws with the help of example.

Definition 10. A CInvDHFS $\mathcal{A}$ is of the following form:

$$\mathcal{A} = \{ (x, (\mu_{\mathcal{A}}(x), \nu_{\mathcal{A}}(x))) | x \in X \}, \quad (9)$$

where

$$\mu_{\mathcal{A}}(x) = [\mu_{\mathcal{A}}^L(x), \mu_{\mathcal{A}}^U(x)] = \left\{ \left(x, [\gamma_{\mathcal{A}_j}^L(x), \gamma_{\mathcal{A}_j}^U(x)] \cdot e^{i2\pi([\omega_{\gamma_{\mathcal{A}_j}^L}(x), \omega_{\gamma_{\mathcal{A}_j}^U}(x)])} \right), j = 1, 2, 3, \dots, g \right\},$$

$$\nu_{\mathcal{A}}(x) = [\nu_{\mathcal{A}}^L(x), \nu_{\mathcal{A}}^U(x)] = \left\{ \left(x, [\delta_{\mathcal{A}_j}^L(x), \delta_{\mathcal{A}_j}^U(x)] \cdot e^{i2\pi([\omega_{\delta_{\mathcal{A}_j}^L}(x), \omega_{\delta_{\mathcal{A}_j}^U}(x)])} \right), j = 1, 2, 3, \dots, h \right\}, \quad (10)$$

represented the complex-valued membership grade and nonmembership grade, which are finite subsets of different interval-values of a unit disc in complex plane with a condition

$$\gamma_{\mathcal{A}_j}^L(x), \gamma_{\mathcal{A}_j}^U(x), \omega_{\gamma_{\mathcal{A}_j}^L}(x), \omega_{\gamma_{\mathcal{A}_j}^U}(x), \delta_{\mathcal{A}_j}^L(x), \delta_{\mathcal{A}_j}^U(x), \omega_{\delta_{\mathcal{A}_j}^L}(x), \omega_{\delta_{\mathcal{A}_j}^U}(x) \in [0, 1], \quad 0 \leq (\gamma_{\mathcal{A}_j}^U(x))^+ + (\delta_{\mathcal{A}_j}^U(x))^+ \leq 1, \quad \text{and} \quad 0 \leq (\omega_{\gamma_{\mathcal{A}_j}^U}(x))^+ + (\omega_{\delta_{\mathcal{A}_j}^U}(x))^+ \leq 1, \quad \text{where} \quad (\gamma_{\mathcal{A}_j}^U(x))^+ = \cup_{[\gamma_{\mathcal{A}_j}^L(x), \gamma_{\mathcal{A}_j}^U(x)] \in \mu_{\mathcal{A}}(x)} \max \{ \gamma_{\mathcal{A}_j}^U(x) \}, \quad (\omega_{\gamma_{\mathcal{A}_j}^U}(x))^+ = \cup_{[\omega_{\gamma_{\mathcal{A}_j}^L}(x), \omega_{\gamma_{\mathcal{A}_j}^U}(x)] \in \mu_{\mathcal{A}}(x)} \max \{ \omega_{\gamma_{\mathcal{A}_j}^U}(x) \}, \quad (\delta_{\mathcal{A}_j}^U(x))^U = \cup_{[\delta_{\mathcal{A}_j}^L(x), \delta_{\mathcal{A}_j}^U(x)] \in \nu_{\mathcal{A}}(x)} \max \{ \delta_{\mathcal{A}_j}^U(x) \}, \quad \text{and} \quad (\omega_{\delta_{\mathcal{A}_j}^U}(x))^+ = \cup_{[\omega_{\delta_{\mathcal{A}_j}^L}(x), \omega_{\delta_{\mathcal{A}_j}^U}(x)] \in \nu_{\mathcal{A}}(x)} \max \{ \omega_{\delta_{\mathcal{A}_j}^U}(x) \}.$$

$$\cup_{[\omega_{\delta_{\mathcal{A}_j}^L}(x), \omega_{\delta_{\mathcal{A}_j}^U}(x)] \in \nu_{\mathcal{A}}(x)} \max \{ \omega_{\delta_{\mathcal{A}_j}^U}(x) \}, \quad \text{for } j = 1, 2, \dots, g$$

and $j = 1, 2, \dots, h$. Further, $\mathcal{A} = (x, ([\gamma_{\mathcal{A}_j}^L(x), \gamma_{\mathcal{A}_j}^U(x)] \cdot e^{i2\pi([\omega_{\gamma_{\mathcal{A}_j}^L}(x), \omega_{\gamma_{\mathcal{A}_j}^U}(x)])}, [\delta_{\mathcal{A}_j}^L(x), \delta_{\mathcal{A}_j}^U(x)] \cdot e^{i2\pi([\omega_{\delta_{\mathcal{A}_j}^L}(x), \omega_{\delta_{\mathcal{A}_j}^U}(x)])})$ is called complex interval-valued dual hesitant fuzzy number (CInvDHFN).

Definition 11. Let

$$\mathcal{A} = \left(x, \left([\gamma_{\mathcal{A}_j}^L(x), \gamma_{\mathcal{A}_j}^U(x)] \cdot e^{i2\pi([\omega_{\gamma_{\mathcal{A}_j}^L}(x), \omega_{\gamma_{\mathcal{A}_j}^U}(x)])}, [\delta_{\mathcal{A}_j}^L(x), \delta_{\mathcal{A}_j}^U(x)] \cdot e^{i2\pi([\omega_{\delta_{\mathcal{A}_j}^L}(x), \omega_{\delta_{\mathcal{A}_j}^U}(x)])} \right) \right),$$

$$\mathcal{B} = \left(x, \left([\gamma_{\mathcal{B}_j}^L(x), \gamma_{\mathcal{B}_j}^U(x)] \cdot e^{i2\pi([\omega_{\gamma_{\mathcal{B}_j}^L}(x), \omega_{\gamma_{\mathcal{B}_j}^U}(x)])}, [\delta_{\mathcal{B}_j}^L(x), \delta_{\mathcal{B}_j}^U(x)] \cdot e^{i2\pi([\omega_{\delta_{\mathcal{B}_j}^L}(x), \omega_{\delta_{\mathcal{B}_j}^U}(x)])} \right) \right), \quad (11)$$

be two ClvDHFNS. Then, their complement, union, and intersection are defined as follows:

$$(1) \mathcal{A}^c = \{(x, (\nu_{\mathcal{A}}(x), \mu_{\mathcal{A}}(x)))\}$$

$$(2) \mathcal{A} \cup \mathcal{B} = \left\{ \left(x, \left(\begin{array}{l} [\max(\gamma_{\mathcal{A}}^L(x), \gamma_{\mathcal{B}}^L(x)), \max(\gamma_{\mathcal{A}}^U(x), \gamma_{\mathcal{B}}^U(x))] \cdot e^{i2\pi([\min(\mu_{\mathcal{A}}^L(x), \mu_{\mathcal{B}}^L(x)), \max(\mu_{\mathcal{A}}^L(x), \mu_{\mathcal{B}}^L(x)))}]} \\ [\min(\delta_{\mathcal{A}}^L(x), \delta_{\mathcal{B}}^L(x)), \min(\delta_{\mathcal{A}}^U(x), \delta_{\mathcal{B}}^U(x))] \cdot e^{i2\pi([\min(\mu_{\mathcal{A}}^L(x), \mu_{\mathcal{B}}^L(x)), \max(\mu_{\mathcal{A}}^L(x), \mu_{\mathcal{B}}^L(x)))}]} \end{array} \right) \right\}$$

$$(3) \mathcal{A} \cap \mathcal{B} =$$

$$\left\{ \left(x, \left(\begin{array}{l} [\min(\gamma_{\mathcal{A}}^L(x), \gamma_{\mathcal{B}}^L(x)), \min(\gamma_{\mathcal{A}}^U(x), \gamma_{\mathcal{B}}^U(x))] \cdot e^{i2\pi([\min(\mu_{\mathcal{A}}^L(x), \mu_{\mathcal{B}}^L(x)), \min(\mu_{\mathcal{A}}^U(x), \mu_{\mathcal{B}}^U(x))]} \\ [\max(\delta_{\mathcal{A}}^L(x), \delta_{\mathcal{B}}^L(x)), \max(\delta_{\mathcal{A}}^U(x), \delta_{\mathcal{B}}^U(x))] \cdot e^{i2\pi([\max(\mu_{\mathcal{A}}^L(x), \mu_{\mathcal{B}}^L(x)), \max(\mu_{\mathcal{A}}^U(x), \mu_{\mathcal{B}}^U(x))]} \end{array} \right) \right\}$$

Example 2. Let

$$\mathcal{A} = \left\{ \left(x_1, \left\{ \left\{ [0.4, 0.8]e^{i2\pi([0.2, 0.6])}, [0.2, 0.5]e^{i2\pi([0.1, 0.4])} \right\}, \left\{ [0.05, 0.1]e^{i2\pi([0.1, 0.2])} \right\} \right), \left(x_2, \left\{ \left\{ [0.6, 0.7]e^{i2\pi([0.4, 0.7])} \right\}, \left\{ [0.15, 0.2]e^{i2\pi([0.1, 0.25])} \right\} \right) \right\}, \right. \\ \left. \left(x_3, \left\{ \left\{ [0.3, 0.5]e^{i2\pi([0.5, 0.6])}, [0.4, 0.6]e^{i2\pi([0.5, 0.55])}, [0.2, 0.7]e^{i2\pi([0.1, 0.4])} \right\}, \left\{ [0.1, 0.2]e^{i2\pi([0.15, 0.3])}, [0.2, 0.25]e^{i2\pi([0.1, 0.2])} \right\} \right), \left(x_4, \left\{ \left\{ [0.4, 0.6]e^{i2\pi([0.4, 0.5])}, [0.3, 0.5]e^{i2\pi([0.2, 0.4])} \right\}, \left\{ [0.1, 0.3]e^{i2\pi([0.2, 0.3])}, [0.2, 0.25]e^{i2\pi([0.1, 0.4])} \right\} \right) \right\}, \right. \\ \mathcal{B} = \left\{ \left(x_1, \left\{ \left\{ [0.2, 0.6]e^{i2\pi([0.3, 0.6])}, [0.5, 0.7]e^{i2\pi([0.3, 0.55])}, [0.3, 0.8]e^{i2\pi([0.45, 0.6])} \right\}, \left\{ [0.05, 0.1]e^{i2\pi([0.1, 0.2])}, [0.1, 0.15]e^{i2\pi([0.15, 0.3])} \right\} \right), \left(x_2, \left\{ \left\{ [0.1, 0.2]e^{i2\pi([0.2, 0.3])} \right\}, \left\{ [0.5, 0.7]e^{i2\pi([0.3, 0.6])}, [0.3, 0.6]e^{i2\pi([0.2, 0.5])} \right\} \right) \right\}, \right. \\ \left. \left(x_3, \left\{ \left\{ [0.4, 0.65]e^{i2\pi([0.35, 0.5])}, [0.5, 0.6]e^{i2\pi([0.5, 0.65])} \right\}, \left\{ [0.2, 0.25]e^{i2\pi([0.1, 0.15])}, [0.15, 0.3]e^{i2\pi([0.2, 0.3])} \right\} \right), \left(x_4, \left\{ \left\{ [0.3, 0.55]e^{i2\pi([0.4, 0.45])}, [0.2, 0.45]e^{i2\pi([0.1, 0.25])} \right\} \right) \right\} \right) \quad (12)$$

be two ClvDHFNS. Then,

$$(1) \mathcal{A}^c = \left\{ \left(x_1, \left\{ \left\{ [0.05, 0.1]e^{i2\pi([0.1, 0.2])}, [0.4, 0.8]e^{i2\pi([0.2, 0.6])}, [0.2, 0.5]e^{i2\pi([0.1, 0.4])} \right\}, \left\{ [0.15, 0.2]e^{i2\pi([0.1, 0.25])} \right\} \right), \left(x_2, \left\{ \left\{ [0.6, 0.7]e^{i2\pi([0.4, 0.7])} \right\}, \left\{ [0.15, 0.2]e^{i2\pi([0.1, 0.25])} \right\} \right) \right\}, \right. \\ (2) \mathcal{A} \cup \mathcal{B} = \left\{ \left(x_1, \left\{ \left\{ [0.4, 0.8]e^{i2\pi([0.2, 0.6])}, [0.5, 0.7]e^{i2\pi([0.3, 0.55])}, [0.3, 0.8]e^{i2\pi([0.45, 0.6])} \right\}, \left\{ [0.05, 0.1]e^{i2\pi([0.1, 0.2])}, [0.1, 0.15]e^{i2\pi([0.15, 0.3])} \right\} \right), \left(x_2, \left\{ \left\{ [0.6, 0.7]e^{i2\pi([0.4, 0.7])} \right\}, \left\{ [0.15, 0.2]e^{i2\pi([0.1, 0.25])} \right\} \right) \right\}, \right. \\ (3) \mathcal{A} \cap \mathcal{B} = \left\{ \left(x_1, \left\{ \left\{ [0.2, 0.6]e^{i2\pi([0.3, 0.6])}, [0.5, 0.7]e^{i2\pi([0.3, 0.55])}, [0.3, 0.8]e^{i2\pi([0.45, 0.6])} \right\}, \left\{ [0.05, 0.1]e^{i2\pi([0.1, 0.2])}, [0.1, 0.15]e^{i2\pi([0.15, 0.3])} \right\} \right), \left(x_2, \left\{ \left\{ [0.1, 0.2]e^{i2\pi([0.2, 0.3])} \right\}, \left\{ [0.5, 0.7]e^{i2\pi([0.3, 0.6])}, [0.3, 0.6]e^{i2\pi([0.2, 0.5])} \right\} \right) \right\}, \right. \\ \left. \left(x_3, \left\{ \left\{ [0.4, 0.65]e^{i2\pi([0.35, 0.5])}, [0.5, 0.6]e^{i2\pi([0.5, 0.65])} \right\}, \left\{ [0.2, 0.25]e^{i2\pi([0.1, 0.15])}, [0.15, 0.3]e^{i2\pi([0.2, 0.3])} \right\} \right), \left(x_4, \left\{ \left\{ [0.3, 0.55]e^{i2\pi([0.4, 0.45])}, [0.2, 0.45]e^{i2\pi([0.1, 0.25])} \right\} \right) \right\} \right)$$

4. The Antilogarithmic and Nonexponential-Based Generalized Distance and Similarity Measures for CDHFS

In this section, we have two subsections in which we interpreted some antilogarithmic and nonexponential-based generalized distance and SMs for CDHFSs.

Definition 12. Let $\mathcal{A}$ and $\mathcal{B}$ be two CDHFSs on set X . Then, similarity measure (SM) between $\mathcal{A}$ and $\mathcal{B}$ is indicated by $S_{CD}(\mathcal{A}, \mathcal{B})$, which holds the following axioms:

- (1) $0 \leq S_{CD}(\mathcal{A}, \mathcal{B}) \leq 1$
- (2) $S_{CD}(\mathcal{A}, \mathcal{B}) = 1$ if and only if $\mathcal{A} = \mathcal{B}$
- (3) $S_{CD}(\mathcal{A}, \mathcal{B}) = S_{CD}(\mathcal{B}, \mathcal{A})$

Definition 13. Let $\mathcal{A}$ and $\mathcal{B}$ be two CDHFSs on set X . Then, distance measure between $\mathcal{A}$ and $\mathcal{B}$ is indicated by $d_{CD}(\mathcal{A}, \mathcal{B})$, which holds the following axioms:

- (1) $0 \leq d_{CD}(\mathcal{A}, \mathcal{B}) \leq 1$
- (2) $d_{CD}(\mathcal{A}, \mathcal{B}) = 0$ if and only if $\mathcal{A} = \mathcal{B}$
- (3) $d_{CD}(\mathcal{A}, \mathcal{B}) = d_{CD}(\mathcal{B}, \mathcal{A})$

By investigating Definitions 12 and 13, we can observe that the higher the similarity, the littler the distance between the two CDHFSs will be. It is noted that $d_{CD}(\mathcal{A}, \mathcal{B}) = 1 - S_{CD}(\mathcal{A}, \mathcal{B})$. In like manner, we predominantly discuss the SMs for CDHFSs in this paper, and the relating distance measures can be acquired without any problem.

4.1. The Antilogarithmic-Based Generalized SMs for CDHFS. In this subsection, we defined some antilogarithmic-based generalized SMs for CDHFS.

Definition 14. Let $\mathcal{A}$ and $\mathcal{B}$ be two CDHFS on set X . Then, the antilogarithmic-based generalized SM between $\mathcal{A}$ and $\mathcal{B}$ is given as

$$S_{CD}^1(\mathcal{A}, \mathcal{B}) = \left[\frac{1}{n} \sum_{k=1}^n \left[2^{1 - \left(\frac{(1/g) \sum_{j=1}^g |\gamma_{\mathcal{A}_j}(x_k) - \gamma_{\mathcal{B}_j}(x_k)|^\lambda \vee (1/g) \sum_{j=1}^g |\omega_{\gamma_{\mathcal{A}_j}}(x_k) - \omega_{\gamma_{\mathcal{B}_j}}(x_k)|^\lambda}{\vee (1/h) \sum_{j=1}^h |\delta_{\mathcal{A}_j}(x_k) - \delta_{\mathcal{B}_j}(x_k)|^\lambda \vee (1/h) \sum_{j=1}^h |\omega_{\delta_{\mathcal{A}_j}}(x_k) - \omega_{\delta_{\mathcal{B}_j}}(x_k)|^\lambda} \right)} - 1 \right] \right]^{(1/\lambda)}, \quad (13)$$

where $\lambda > 0$ and $\vee$ represents the maximum operation.

Theorem 1. Let A and B be two CDHES on set X . Then, the SM $S_{CD}^1(\mathcal{A}, \mathcal{B})$ hold the following axioms:

- (1) $0 \leq S_{CD}^1(\mathcal{A}, \mathcal{B}) \leq 1$
- (2) $S_{CD}^1(\mathcal{A}, \mathcal{B}) = 1$ if and only if $\mathcal{A} = \mathcal{B}$
- (3) $S_{CD}^1(\mathcal{A}, \mathcal{B}) = S_{CD}^1(\mathcal{B}, \mathcal{A})$

- (1) Since $(1/g) \sum_{j=1}^g |\gamma_{\mathcal{A}_j}(x_k) - \gamma_{\mathcal{B}_j}(x_k)|^\lambda \in [0, 1]$, $(1/g) \sum_{j=1}^g |\omega_{\gamma_{\mathcal{A}_j}}(x_k) - \omega_{\gamma_{\mathcal{B}_j}}(x_k)|^\lambda \in [0, 1]$, $(1/h) \sum_{j=1}^h |\delta_{\mathcal{A}_j}(x_k) - \delta_{\mathcal{B}_j}(x_k)|^\lambda \in [0, 1]$, and $(1/h) \sum_{j=1}^h |\omega_{\delta_{\mathcal{A}_j}}(x_k) - \omega_{\delta_{\mathcal{B}_j}}(x_k)|^\lambda \in [0, 1]$, then $(1/g) \sum_{j=1}^g |\gamma_{\mathcal{A}_j}(x_k) - \gamma_{\mathcal{B}_j}(x_k)|^\lambda \vee (1/g) \sum_{j=1}^g |\omega_{\gamma_{\mathcal{A}_j}}(x_k) - \omega_{\gamma_{\mathcal{B}_j}}(x_k)|^\lambda \vee (1/h) \sum_{j=1}^h |\delta_{\mathcal{A}_j}(x_k) - \delta_{\mathcal{B}_j}(x_k)|^\lambda \vee (1/h) \sum_{j=1}^h |\omega_{\delta_{\mathcal{A}_j}}(x_k) - \omega_{\delta_{\mathcal{B}_j}}(x_k)|^\lambda \in [0, 1]$. This implies that, for $k = 1$, we obtain

Proof.

$$\frac{1}{2} \left(\frac{(1/g) \sum_{j=1}^g |\gamma_{\mathcal{A}_j}(x_1) - \gamma_{\mathcal{B}_j}(x_1)|^\lambda \vee (1/g) \sum_{j=1}^g |\omega_{\gamma_{\mathcal{A}_j}}(x_1) - \omega_{\gamma_{\mathcal{B}_j}}(x_1)|^\lambda}{\vee (1/h) \sum_{j=1}^h |\delta_{\mathcal{A}_j}(x_1) - \delta_{\mathcal{B}_j}(x_1)|^\lambda \vee (1/h) \sum_{j=1}^h |\omega_{\delta_{\mathcal{A}_j}}(x_1) - \omega_{\delta_{\mathcal{B}_j}}(x_1)|^\lambda} \right) - 1 \in [0, 1]. \quad (14)$$

For $k = 2$,

$$\frac{1}{2} \left(\frac{(1/g) \sum_{j=1}^g |\gamma_{\mathcal{A}_j}(x_2) - \gamma_{\mathcal{B}_j}(x_2)|^\lambda \vee (1/g) \sum_{j=1}^g |\omega_{\gamma_{\mathcal{A}_j}}(x_2) - \omega_{\gamma_{\mathcal{B}_j}}(x_2)|^\lambda}{\vee (1/h) \sum_{j=1}^h |\delta_{\mathcal{A}_j}(x_2) - \delta_{\mathcal{B}_j}(x_2)|^\lambda \vee (1/h) \sum_{j=1}^h |\omega_{\delta_{\mathcal{A}_j}}(x_2) - \omega_{\delta_{\mathcal{B}_j}}(x_2)|^\lambda} \right) - 1 \in [0, 1]. \quad (15)$$

By continuing this process, we get

$$\begin{aligned}
 & \left[\sum_{k=1}^n \left[2 \left(\begin{array}{c} 1- \\ \vee (1/g) \sum_{\mathcal{J}=1}^g |\gamma_{\mathcal{A}_{\mathcal{J}}}(x_k) - \gamma_{\mathcal{B}_{\mathcal{J}}}(x_k)|^\lambda \vee (1/g) \sum_{\mathcal{J}=1}^g |\omega_{\gamma_{\mathcal{A}_{\mathcal{J}}}}(x_k) - \omega_{\gamma_{\mathcal{B}_{\mathcal{J}}}}(x_k)|^\lambda \\ \vee (1/h) \sum_{\mathcal{J}=1}^h |\delta_{\mathcal{A}_{\mathcal{J}}}(x_k) - \delta_{\mathcal{B}_{\mathcal{J}}}(x_k)|^\lambda \vee (1/h) \sum_{\mathcal{J}=1}^h |\bar{\omega}_{\delta_{\mathcal{A}_{\mathcal{J}}}}(x_k) - \bar{\omega}_{\delta_{\mathcal{B}_{\mathcal{J}}}}(x_k)|^\lambda \end{array} \right) - 1 \right] \in n[0, 1] \right. \\
 & \Rightarrow 0 \leq \sum_{k=1}^n \left[2 \left(\begin{array}{c} 1- \\ \vee (1/g) \sum_{\mathcal{J}=1}^g |\gamma_{\mathcal{A}_{\mathcal{J}}}(x_k) - \gamma_{\mathcal{B}_{\mathcal{J}}}(x_k)|^\lambda \vee (1/g) \sum_{\mathcal{J}=1}^g |\omega_{\gamma_{\mathcal{A}_{\mathcal{J}}}}(x_k) - \omega_{\gamma_{\mathcal{B}_{\mathcal{J}}}}(x_k)|^\lambda \\ \vee (1/h) \sum_{\mathcal{J}=1}^h |\delta_{\mathcal{A}_{\mathcal{J}}}(x_k) - \delta_{\mathcal{B}_{\mathcal{J}}}(x_k)|^\lambda \vee (1/h) \sum_{\mathcal{J}=1}^h |\bar{\omega}_{\delta_{\mathcal{A}_{\mathcal{J}}}}(x_k) - \bar{\omega}_{\delta_{\mathcal{B}_{\mathcal{J}}}}(x_k)|^\lambda \end{array} \right) - 1 \right] \leq n \\
 & \Rightarrow 0 \leq \frac{1}{n} \sum_{k=1}^n \left[2 \left(\begin{array}{c} 1- \\ \vee (1/g) \sum_{\mathcal{J}=1}^g |\gamma_{\mathcal{A}_{\mathcal{J}}}(x_k) - \gamma_{\mathcal{B}_{\mathcal{J}}}(x_k)|^\lambda \vee (1/g) \sum_{\mathcal{J}=1}^g |\omega_{\gamma_{\mathcal{A}_{\mathcal{J}}}}(x_k) - \omega_{\gamma_{\mathcal{B}_{\mathcal{J}}}}(x_k)|^\lambda \\ \vee (1/h) \sum_{\mathcal{J}=1}^h |\delta_{\mathcal{A}_{\mathcal{J}}}(x_k) - \delta_{\mathcal{B}_{\mathcal{J}}}(x_k)|^\lambda \vee (1/h) \sum_{\mathcal{J}=1}^h |\bar{\omega}_{\delta_{\mathcal{A}_{\mathcal{J}}}}(x_k) - \bar{\omega}_{\delta_{\mathcal{B}_{\mathcal{J}}}}(x_k)|^\lambda \end{array} \right) - 1 \right] \leq 1 \quad (16) \\
 & \Rightarrow 0 \leq \left[\frac{1}{n} \sum_{k=1}^n \left[2 \left(\begin{array}{c} 1- \\ \vee (1/g) \sum_{\mathcal{J}=1}^g |\gamma_{\mathcal{A}_{\mathcal{J}}}(x_k) - \gamma_{\mathcal{B}_{\mathcal{J}}}(x_k)|^\lambda \vee (1/g) \sum_{\mathcal{J}=1}^g |\omega_{\gamma_{\mathcal{A}_{\mathcal{J}}}}(x_k) - \omega_{\gamma_{\mathcal{B}_{\mathcal{J}}}}(x_k)|^\lambda \\ \vee (1/h) \sum_{\mathcal{J}=1}^h |\delta_{\mathcal{A}_{\mathcal{J}}}(x_k) - \delta_{\mathcal{B}_{\mathcal{J}}}(x_k)|^\lambda \vee (1/h) \sum_{\mathcal{J}=1}^h |\bar{\omega}_{\delta_{\mathcal{A}_{\mathcal{J}}}}(x_k) - \bar{\omega}_{\delta_{\mathcal{B}_{\mathcal{J}}}}(x_k)|^\lambda \end{array} \right) - 1 \right] \right]^{(1/\lambda)} \leq 1 \\
 & \Rightarrow 0 \leq S_{\text{CD}}^1(\mathcal{A}, \mathcal{B}) \leq 1.
 \end{aligned}$$

(2) By Definition 14,

$$\begin{aligned}
S_{\text{CD}}^1(\mathcal{A}, \mathcal{B}) &= \left[\frac{1}{n} \sum_{k=1}^n \left[2 \left(\frac{(1/g) \sum_{j=1}^g |\gamma_{\mathcal{A}_j}(x_k) - \gamma_{\mathcal{B}_j}(x_k)|^\lambda \vee (1/g) \sum_{j=1}^g |\omega_{\gamma_{\mathcal{A}_j}}(x_k) - \omega_{\gamma_{\mathcal{B}_j}}(x_k)|^\lambda}{\vee (1/h) \sum_{j=1}^h |\delta_{\mathcal{A}_j}(x_k) - \delta_{\mathcal{B}_j}(x_k)|^\lambda \vee (1/h) \sum_{j=1}^h |\omega_{\delta_{\mathcal{A}_j}}(x_k) - \omega_{\delta_{\mathcal{B}_j}}(x_k)|^\lambda} \right) - 1 \right] \right]^{(1/\lambda)} \\
&\Leftrightarrow S_{\text{CD}}^2(\mathcal{A}, \mathcal{B}) \left[\frac{1}{n} \sum_{k=1}^n 2 \left[\frac{(1/g) \left(|\gamma_{\mathcal{A}_1}(x_k) - \gamma_{\mathcal{B}_1}(x_k)|^\lambda + |\gamma_{\mathcal{A}_2}(x_k) - \gamma_{\mathcal{B}_2}(x_k)|^\lambda + \dots + |\gamma_{\mathcal{A}_g}(x_k) - \gamma_{\mathcal{B}_g}(x_k)|^\lambda \right) + (1/g) \left(|\omega_{\gamma_{\mathcal{A}_1}}(x_k) - \omega_{\gamma_{\mathcal{B}_1}}(x_k)|^\lambda + |\omega_{\gamma_{\mathcal{A}_2}}(x_k) - \omega_{\gamma_{\mathcal{B}_2}}(x_k)|^\lambda + \dots + |\omega_{\gamma_{\mathcal{A}_g}}(x_k) - \omega_{\gamma_{\mathcal{B}_g}}(x_k)|^\lambda \right) + (1/h) \left(|\delta_{\mathcal{A}_1}(x_k) - \delta_{\mathcal{B}_1}(x_k)|^\lambda + |\delta_{\mathcal{A}_2}(x_k) - \delta_{\mathcal{B}_2}(x_k)|^\lambda + \dots + |\delta_{\mathcal{A}_h}(x_k) - \delta_{\mathcal{B}_h}(x_k)|^\lambda \right) + (1/h) \left(|\omega_{\delta_{\mathcal{A}_1}}(x_k) - \omega_{\delta_{\mathcal{B}_1}}(x_k)|^\lambda + |\omega_{\delta_{\mathcal{A}_2}}(x_k) - \omega_{\delta_{\mathcal{B}_2}}(x_k)|^\lambda + \dots + |\omega_{\delta_{\mathcal{A}_h}}(x_k) - \omega_{\delta_{\mathcal{B}_h}}(x_k)|^\lambda \right)}{1 - (1/4)} \right] - 1 \right]^{(1/\lambda)} \\
&\Leftrightarrow S_{\text{CD}}^1(\mathcal{A}, \mathcal{B}) \left[\frac{1}{n} \sum_{k=1}^n 2 \left[\frac{(1/g) \left(|\gamma_{\mathcal{A}_1}(x_1) - \gamma_{\mathcal{B}_1}(x_1)|^\lambda + |\gamma_{\mathcal{A}_2}(x_1) - \gamma_{\mathcal{B}_2}(x_1)|^\lambda + \dots + |\gamma_{\mathcal{A}_g}(x_1) - \gamma_{\mathcal{B}_g}(x_1)|^\lambda \right) + (1/g) \left(|\omega_{\gamma_{\mathcal{A}_1}}(x_1) - \omega_{\gamma_{\mathcal{B}_1}}(x_1)|^\lambda + |\omega_{\gamma_{\mathcal{A}_2}}(x_1) - \omega_{\gamma_{\mathcal{B}_2}}(x_1)|^\lambda + \dots + |\omega_{\gamma_{\mathcal{A}_g}}(x_1) - \omega_{\gamma_{\mathcal{B}_g}}(x_1)|^\lambda \right) + (1/h) \left(|\delta_{\mathcal{A}_1}(x_1) - \delta_{\mathcal{B}_1}(x_1)|^\lambda + |\delta_{\mathcal{A}_2}(x_1) - \delta_{\mathcal{B}_2}(x_1)|^\lambda + \dots + |\delta_{\mathcal{A}_h}(x_1) - \delta_{\mathcal{B}_h}(x_1)|^\lambda \right) + (1/h) \left(|\omega_{\delta_{\mathcal{A}_1}}(x_1) - \omega_{\delta_{\mathcal{B}_1}}(x_1)|^\lambda + |\omega_{\delta_{\mathcal{A}_2}}(x_1) - \omega_{\delta_{\mathcal{B}_2}}(x_1)|^\lambda + \dots + |\omega_{\delta_{\mathcal{A}_h}}(x_1) - \omega_{\delta_{\mathcal{B}_h}}(x_1)|^\lambda \right)}{1 - (1/4)} \right] - 1 \right]^{(1/\lambda)} \\
&\quad + 2 \left[\frac{(1/g) \left(|\gamma_{\mathcal{A}_1}(x_2) - \gamma_{\mathcal{B}_1}(x_2)|^\lambda + |\gamma_{\mathcal{A}_2}(x_2) - \gamma_{\mathcal{B}_2}(x_2)|^\lambda + \dots + |\gamma_{\mathcal{A}_g}(x_2) - \gamma_{\mathcal{B}_g}(x_2)|^\lambda \right) + (1/g) \left(|\omega_{\gamma_{\mathcal{A}_1}}(x_2) - \omega_{\gamma_{\mathcal{B}_1}}(x_2)|^\lambda + |\omega_{\gamma_{\mathcal{A}_2}}(x_2) - \omega_{\gamma_{\mathcal{B}_2}}(x_2)|^\lambda + \dots + |\omega_{\gamma_{\mathcal{A}_g}}(x_2) - \omega_{\gamma_{\mathcal{B}_g}}(x_2)|^\lambda \right) + (1/h) \left(|\delta_{\mathcal{A}_1}(x_2) - \delta_{\mathcal{B}_1}(x_2)|^\lambda + |\delta_{\mathcal{A}_2}(x_2) - \delta_{\mathcal{B}_2}(x_2)|^\lambda + \dots + |\delta_{\mathcal{A}_h}(x_2) - \delta_{\mathcal{B}_h}(x_2)|^\lambda \right) + (1/h) \left(|\omega_{\delta_{\mathcal{A}_1}}(x_2) - \omega_{\delta_{\mathcal{B}_1}}(x_2)|^\lambda + |\omega_{\delta_{\mathcal{A}_2}}(x_2) - \omega_{\delta_{\mathcal{B}_2}}(x_2)|^\lambda + \dots + |\omega_{\delta_{\mathcal{A}_h}}(x_2) - \omega_{\delta_{\mathcal{B}_h}}(x_2)|^\lambda \right)}{1 - (1/4)} \right] - 1 + \dots \\
&\quad + 2 \left[\frac{(1/g) \left(|\gamma_{\mathcal{A}_1}(x_n) - \gamma_{\mathcal{B}_1}(x_n)|^\lambda + |\gamma_{\mathcal{A}_2}(x_n) - \gamma_{\mathcal{B}_2}(x_n)|^\lambda + \dots + |\gamma_{\mathcal{A}_g}(x_n) - \gamma_{\mathcal{B}_g}(x_n)|^\lambda \right) + (1/g) \left(|\omega_{\gamma_{\mathcal{A}_1}}(x_n) - \omega_{\gamma_{\mathcal{B}_1}}(x_n)|^\lambda + |\omega_{\gamma_{\mathcal{A}_2}}(x_n) - \omega_{\gamma_{\mathcal{B}_2}}(x_n)|^\lambda + \dots + |\omega_{\gamma_{\mathcal{A}_g}}(x_n) - \omega_{\gamma_{\mathcal{B}_g}}(x_n)|^\lambda \right) + (1/h) \left(|\delta_{\mathcal{A}_1}(x_n) - \delta_{\mathcal{B}_1}(x_n)|^\lambda + |\delta_{\mathcal{A}_2}(x_n) - \delta_{\mathcal{B}_2}(x_n)|^\lambda + \dots + |\delta_{\mathcal{A}_h}(x_n) - \delta_{\mathcal{B}_h}(x_n)|^\lambda \right) + (1/h) \left(|\omega_{\delta_{\mathcal{A}_1}}(x_n) - \omega_{\delta_{\mathcal{B}_1}}(x_n)|^\lambda + |\omega_{\delta_{\mathcal{A}_2}}(x_n) - \omega_{\delta_{\mathcal{B}_2}}(x_n)|^\lambda + \dots + |\omega_{\delta_{\mathcal{A}_h}}(x_n) - \omega_{\delta_{\mathcal{B}_h}}(x_n)|^\lambda \right)}{1 - (1/4)} \right] - 1 \Big]^{(1/\lambda)}.
\end{aligned} \tag{17}$$

Now, as $\mathcal{A} = \mathcal{B} \iff \mu_{\mathcal{A}}(x_k) = \mu_{\mathcal{B}}(x_k)$ and $\nu_{\mathcal{A}}(x_k) = \nu_{\mathcal{B}}(x_k)$ for $k = 1, 2, \dots, n \iff \gamma_{\mathcal{A}_{\mathcal{J}}}(x_k)e^{i2\pi(\omega_{\gamma_{\mathcal{A}_{\mathcal{J}}}}(x_k))} = \gamma_{\mathcal{B}_{\mathcal{J}}}(x_k)e^{i2\pi(\omega_{\gamma_{\mathcal{B}_{\mathcal{J}}}}(x_k))}$ and $\delta_{\mathcal{A}_{\mathcal{J}}}(x_k)e^{i2\pi(\omega_{\delta_{\mathcal{A}_{\mathcal{J}}}}(x_k))} = \delta_{\mathcal{B}_{\mathcal{J}}}(x_k)e^{i2\pi(\omega_{\delta_{\mathcal{B}_{\mathcal{J}}}}(x_k))}$ for $k = 1, 2, \dots, n \iff \gamma_{\mathcal{A}_{\mathcal{J}}}(x_k) = \gamma_{\mathcal{B}_{\mathcal{J}}}(x_k)$, $\delta_{\mathcal{A}_{\mathcal{J}}}(x_k) = \delta_{\mathcal{B}_{\mathcal{J}}}(x_k)$,

$e^{i2\pi(\omega_{\gamma_{\mathcal{A}_{\mathcal{J}}}}(x_k))} = e^{i2\pi(\omega_{\gamma_{\mathcal{B}_{\mathcal{J}}}}(x_k))}$ and $e^{i2\pi(\omega_{\delta_{\mathcal{A}_{\mathcal{J}}}}(x_k))} = e^{i2\pi(\omega_{\delta_{\mathcal{B}_{\mathcal{J}}}}(x_k))}$ for $k = 1, 2, \dots, n$, then

$$\begin{aligned} \iff S_{\text{CD}}^1(\mathcal{A}, \mathcal{B}) &= \left[\frac{1}{n} \left[2^{1-0} - 1 + 2^{1-0} - 1 + \dots + 2^{1-0} - 1 \right] \right]^{(1/\lambda)} \\ \iff S_{\text{CD}}^1(\mathcal{A}, \mathcal{B}) &= 1. \end{aligned} \quad (18)$$

(3) We have

$$\begin{aligned} S_{\text{CD}}^1(\mathcal{A}, \mathcal{B}) &= \left[\frac{1}{n} \sum_{k=1}^n \left[2^{1-\left(\left(\frac{1}{g} \sum_{\mathcal{J}=1}^g \left| \gamma_{\mathcal{A}_{\mathcal{J}}}(x_k) - \gamma_{\mathcal{B}_{\mathcal{J}}}(x_k) \right|^\lambda \vee \left(\frac{1}{g} \sum_{\mathcal{J}=1}^g \left| \omega_{\gamma_{\mathcal{A}_{\mathcal{J}}}}(x_k) - \omega_{\gamma_{\mathcal{B}_{\mathcal{J}}}}(x_k) \right|^\lambda \right) \right. \right. \right. \\ &\quad \left. \left. \left. \vee \left(\frac{1}{h} \sum_{\mathcal{J}=1}^h \left| \delta_{\mathcal{A}_{\mathcal{J}}}(x_k) - \delta_{\mathcal{B}_{\mathcal{J}}}(x_k) \right|^\lambda \vee \left(\frac{1}{h} \sum_{\mathcal{J}=1}^h \left| \omega_{\delta_{\mathcal{A}_{\mathcal{J}}}}(x_k) - \omega_{\delta_{\mathcal{B}_{\mathcal{J}}}}(x_k) \right|^\lambda \right) \right) \right] \right]^{(1/\lambda)} \\ &= \left[\frac{1}{n} \sum_{k=1}^n \left[2^{1-\left(\left(\frac{1}{g} \sum_{\mathcal{J}=1}^g \left| \gamma_{\mathcal{B}_{\mathcal{J}}}(x_k) - \gamma_{\mathcal{A}_{\mathcal{J}}}(x_k) \right|^\lambda \vee \left(\frac{1}{g} \sum_{\mathcal{J}=1}^g \left| \omega_{\gamma_{\mathcal{B}_{\mathcal{J}}}}(x_k) - \omega_{\gamma_{\mathcal{A}_{\mathcal{J}}}}(x_k) \right|^\lambda \right) \right. \right. \right. \\ &\quad \left. \left. \left. \vee \left(\frac{1}{h} \sum_{\mathcal{J}=1}^h \left| \delta_{\mathcal{B}_{\mathcal{J}}}(x_k) - \delta_{\mathcal{A}_{\mathcal{J}}}(x_k) \right|^\lambda \vee \left(\frac{1}{h} \sum_{\mathcal{J}=1}^h \left| \omega_{\delta_{\mathcal{B}_{\mathcal{J}}}}(x_k) - \omega_{\delta_{\mathcal{A}_{\mathcal{J}}}}(x_k) \right|^\lambda \right) \right) \right] \right]^{(1/\lambda)} \\ &= S_{\text{CD}}^1(\mathcal{B}, \mathcal{A}). \end{aligned} \quad (19)$$

In the following definition, we defined another type of antilogarithmic-based generalized SM. $\square$

Definition 15. Let $\mathcal{A}$ and $\mathcal{B}$ be two CDHFS on set X . Then, the antilogarithmic-based generalized SM between $\mathcal{A}$ and $\mathcal{B}$ is given as

$$S_{CD}^2(\mathcal{A}, \mathcal{B}) = \left[\frac{1}{n} \sum_{k=1}^n 2^{1-(1/4) \left(\left((1/g) \sum_{\mathcal{J}=1}^g |\gamma_{\mathcal{A}_{\mathcal{J}}}(x_k) - \gamma_{\mathcal{B}_{\mathcal{J}}}(x_k)|^\lambda + (1/g) \sum_{\mathcal{J}=1}^g |\omega_{\gamma_{\mathcal{A}_{\mathcal{J}}}}(x_k) - \omega_{\gamma_{\mathcal{B}_{\mathcal{J}}}}(x_k)|^\lambda \right) + (1/h) \sum_{\mathcal{J}=1}^h |\delta_{\mathcal{A}_{\mathcal{J}}}(x_k) - \delta_{\mathcal{B}_{\mathcal{J}}}(x_k)|^\lambda + (1/h) \sum_{\mathcal{J}=1}^h |\bar{\omega}_{\delta_{\mathcal{A}_{\mathcal{J}}}}(x_k) - \bar{\omega}_{\delta_{\mathcal{B}_{\mathcal{J}}}}(x_k)|^\lambda \right) - 1 \right]^{(1/\lambda)}. \quad (20)$$

Theorem 2. Let A and B be two CDHES on set X . Then, the SM $S_{CD}^2(\mathcal{A}, \mathcal{B})$ hold the following axioms:

- (1) $0 \leq S_{CD}^2(\mathcal{A}, \mathcal{B}) \leq 1$
- (2) $S_{CD}^2(\mathcal{A}, \mathcal{B}) = 1$ if and only if $\mathcal{A} = \mathcal{B}$
- (3) $S_{CD}^2(\mathcal{A}, \mathcal{B}) = S_{CD}^2(\mathcal{B}, \mathcal{A})$

Proof

- (1) Since $(1/g) \sum_{\mathcal{J}=1}^g |\gamma_{\mathcal{A}_{\mathcal{J}}}(x_k) - \gamma_{\mathcal{B}_{\mathcal{J}}}(x_k)|^\lambda$, $(1/g) \sum_{\mathcal{J}=1}^g |\omega_{\gamma_{\mathcal{A}_{\mathcal{J}}}}(x_k) - \omega_{\gamma_{\mathcal{B}_{\mathcal{J}}}}(x_k)|^\lambda$, $(1/h) \sum_{\mathcal{J}=1}^h |\delta_{\mathcal{A}_{\mathcal{J}}}(x_k) - \delta_{\mathcal{B}_{\mathcal{J}}}(x_k)|^\lambda$, and $(1/h) \sum_{\mathcal{J}=1}^h |\bar{\omega}_{\delta_{\mathcal{A}_{\mathcal{J}}}}(x_k) - \bar{\omega}_{\delta_{\mathcal{B}_{\mathcal{J}}}}(x_k)|^\lambda \in [0, 1]$, then $(1/4) \left((1/g) \sum_{\mathcal{J}=1}^g |\gamma_{\mathcal{A}_{\mathcal{J}}}(x_k) - \gamma_{\mathcal{B}_{\mathcal{J}}}(x_k)|^\lambda + (1/g) \sum_{\mathcal{J}=1}^g |\omega_{\gamma_{\mathcal{A}_{\mathcal{J}}}}(x_k) - \omega_{\gamma_{\mathcal{B}_{\mathcal{J}}}}(x_k)|^\lambda + (1/h) \sum_{\mathcal{J}=1}^h |\delta_{\mathcal{A}_{\mathcal{J}}}(x_k) - \delta_{\mathcal{B}_{\mathcal{J}}}(x_k)|^\lambda + (1/h) \sum_{\mathcal{J}=1}^h |\bar{\omega}_{\delta_{\mathcal{A}_{\mathcal{J}}}}(x_k) - \bar{\omega}_{\delta_{\mathcal{B}_{\mathcal{J}}}}(x_k)|^\lambda \right) \in [0, 1]$; this implies that for $k = 1$, we obtain

$$2^{1-(1/4) \left(\left((1/g) \sum_{\mathcal{J}=1}^g |\gamma_{\mathcal{A}_{\mathcal{J}}}(x_1) - \gamma_{\mathcal{B}_{\mathcal{J}}}(x_1)|^\lambda + (1/g) \sum_{\mathcal{J}=1}^g |\omega_{\gamma_{\mathcal{A}_{\mathcal{J}}}}(x_1) - \omega_{\gamma_{\mathcal{B}_{\mathcal{J}}}}(x_1)|^\lambda \right) + (1/h) \sum_{\mathcal{J}=1}^h |\delta_{\mathcal{A}_{\mathcal{J}}}(x_1) - \delta_{\mathcal{B}_{\mathcal{J}}}(x_1)|^\lambda + (1/h) \sum_{\mathcal{J}=1}^h |\bar{\omega}_{\delta_{\mathcal{A}_{\mathcal{J}}}}(x_1) - \bar{\omega}_{\delta_{\mathcal{B}_{\mathcal{J}}}}(x_1)|^\lambda \right) - 1} \in [0, 1]. \quad (21)$$

For $k = 2$,

$$2^{1-(1/4) \left(\left((1/g) \sum_{\mathcal{J}=1}^g |\gamma_{\mathcal{A}_{\mathcal{J}}}(x_2) - \gamma_{\mathcal{B}_{\mathcal{J}}}(x_2)|^\lambda + (1/g) \sum_{\mathcal{J}=1}^g |\omega_{\gamma_{\mathcal{A}_{\mathcal{J}}}}(x_2) - \omega_{\gamma_{\mathcal{B}_{\mathcal{J}}}}(x_2)|^\lambda \right) + (1/h) \sum_{\mathcal{J}=1}^h |\delta_{\mathcal{A}_{\mathcal{J}}}(x_2) - \delta_{\mathcal{B}_{\mathcal{J}}}(x_2)|^\lambda + (1/h) \sum_{\mathcal{J}=1}^h |\bar{\omega}_{\delta_{\mathcal{A}_{\mathcal{J}}}}(x_2) - \bar{\omega}_{\delta_{\mathcal{B}_{\mathcal{J}}}}(x_2)|^\lambda \right) - 1} \in [0, 1]. \quad (22)$$

By continuing this process, we get

$$\begin{aligned}
 & \left[\sum_{k=1}^n 2^{1-(1/4)} \left((1/g) \sum_{\mathcal{J}=1}^g |\gamma_{\mathcal{A}_{\mathcal{J}}}(x_k) - \gamma_{\mathcal{B}_{\mathcal{J}}}(x_k)|^\lambda + (1/g) \sum_{\mathcal{J}=1}^g |\omega_{\gamma_{\mathcal{A}_{\mathcal{J}}}}(x_k) - \omega_{\gamma_{\mathcal{B}_{\mathcal{J}}}}(x_k)|^\lambda \right. \right. \\
 & \quad \left. \left. + (1/h) \sum_{\mathcal{J}=1}^h |\delta_{\mathcal{A}_{\mathcal{J}}}(x_k) - \delta_{\mathcal{B}_{\mathcal{J}}}(x_k)|^\lambda + (1/h) \sum_{\mathcal{J}=1}^h |\varpi_{\delta_{\mathcal{A}_{\mathcal{J}}}}(x_k) - \varpi_{\delta_{\mathcal{B}_{\mathcal{J}}}}(x_k)|^\lambda \right) - 1 \right] \in n[0, 1] \\
 & 0 \leq \sum_{k=1}^n \left[2^{1-(1/4)} \left((1/g) \sum_{\mathcal{J}=1}^g |\gamma_{\mathcal{A}_{\mathcal{J}}}(x_k) - \gamma_{\mathcal{B}_{\mathcal{J}}}(x_k)|^\lambda + (1/g) \sum_{\mathcal{J}=1}^g |\omega_{\gamma_{\mathcal{A}_{\mathcal{J}}}}(x_k) - \omega_{\gamma_{\mathcal{B}_{\mathcal{J}}}}(x_k)|^\lambda \right. \right. \\
 & \quad \left. \left. + (1/h) \sum_{\mathcal{J}=1}^h |\delta_{\mathcal{A}_{\mathcal{J}}}(x_k) - \delta_{\mathcal{B}_{\mathcal{J}}}(x_k)|^\lambda + (1/h) \sum_{\mathcal{J}=1}^h |\varpi_{\delta_{\mathcal{A}_{\mathcal{J}}}}(x_k) - \varpi_{\delta_{\mathcal{B}_{\mathcal{J}}}}(x_k)|^\lambda \right) - 1 \right] \leq n \\
 & 0 \leq \frac{1}{n} \sum_{k=1}^n \left[2^{1-(1/4)} \left((1/g) \sum_{\mathcal{J}=1}^g |\gamma_{\mathcal{A}_{\mathcal{J}}}(x_k) - \gamma_{\mathcal{B}_{\mathcal{J}}}(x_k)|^\lambda + (1/g) \sum_{\mathcal{J}=1}^g |\omega_{\gamma_{\mathcal{A}_{\mathcal{J}}}}(x_k) - \omega_{\gamma_{\mathcal{B}_{\mathcal{J}}}}(x_k)|^\lambda \right. \right. \\
 & \quad \left. \left. + (1/h) \sum_{\mathcal{J}=1}^h |\delta_{\mathcal{A}_{\mathcal{J}}}(x_k) - \delta_{\mathcal{B}_{\mathcal{J}}}(x_k)|^\lambda + (1/h) \sum_{\mathcal{J}=1}^h |\varpi_{\delta_{\mathcal{A}_{\mathcal{J}}}}(x_k) - \varpi_{\delta_{\mathcal{B}_{\mathcal{J}}}}(x_k)|^\lambda \right) - 1 \right] \leq 1 \tag{23} \\
 & 0 \leq \left[\frac{1}{n} \sum_{k=1}^n \left[2^{1-(1/4)} \left((1/g) \sum_{\mathcal{J}=1}^g |\gamma_{\mathcal{A}_{\mathcal{J}}}(x_k) - \gamma_{\mathcal{B}_{\mathcal{J}}}(x_k)|^\lambda + (1/g) \sum_{\mathcal{J}=1}^g |\omega_{\gamma_{\mathcal{A}_{\mathcal{J}}}}(x_k) - \omega_{\gamma_{\mathcal{B}_{\mathcal{J}}}}(x_k)|^\lambda \right. \right. \right. \\
 & \quad \left. \left. + (1/h) \sum_{\mathcal{J}=1}^h |\delta_{\mathcal{A}_{\mathcal{J}}}(x_k) - \delta_{\mathcal{B}_{\mathcal{J}}}(x_k)|^\lambda + (1/h) \sum_{\mathcal{J}=1}^h |\varpi_{\delta_{\mathcal{A}_{\mathcal{J}}}}(x_k) - \varpi_{\delta_{\mathcal{B}_{\mathcal{J}}}}(x_k)|^\lambda \right) - 1 \right] \right]^{(1/\lambda)} \leq 1 \\
 & 0 \leq S_{\text{CD}}^2(\mathcal{A}, \mathcal{B}) \leq 1.
 \end{aligned}$$

$$S_{\text{CD}}^2(\mathcal{A}, \mathcal{B}) = \left[\frac{1}{n} \sum_{k=1}^n 2^{1-(1/4) \left(\begin{aligned} & (1/g) \sum_{f=1}^g |\gamma_{\mathcal{A}_f}(x_k) - \gamma_{\mathcal{B}_f}(x_k)|^\lambda + (1/g) \sum_{f=1}^g |\omega_{\gamma_{\mathcal{A}_f}}(x_k) - \omega_{\gamma_{\mathcal{B}_f}}(x_k)|^\lambda \\ & +(1/h) \sum_{f=1}^h |\delta_{\mathcal{A}_f}(x_k) - \delta_{\mathcal{B}_f}(x_k)|^\lambda + (1/h) \sum_{f=1}^h |\bar{\omega}_{\delta_{\mathcal{A}_f}}(x_k) - \bar{\omega}_{\delta_{\mathcal{B}_f}}(x_k)|^\lambda \end{aligned} \right)} - 1 \right]^{(1/\lambda)}$$

$$\iff S_{\text{CD}}^2(\mathcal{A}, \mathcal{B}) = \left[\frac{1}{n} \sum_{k=1}^n 2^{1-(1/4) \left(\begin{aligned} & (1/g) \left(|\gamma_{\mathcal{A}_1}(x_k) - \gamma_{\mathcal{B}_1}(x_k)|^\lambda + |\gamma_{\mathcal{A}_2}(x_k) - \gamma_{\mathcal{B}_2}(x_k)|^\lambda + \dots + |\gamma_{\mathcal{A}_g}(x_k) - \gamma_{\mathcal{B}_g}(x_k)|^\lambda \right) + \\ & (1/g) \left(|\omega_{\gamma_{\mathcal{A}_1}}(x_k) - \omega_{\gamma_{\mathcal{B}_1}}(x_k)|^\lambda + |\omega_{\gamma_{\mathcal{A}_2}}(x_k) - \omega_{\gamma_{\mathcal{B}_2}}(x_k)|^\lambda + \dots + |\omega_{\gamma_{\mathcal{A}_g}}(x_k) - \omega_{\gamma_{\mathcal{B}_g}}(x_k)|^\lambda \right) + \\ & (1/h) \left(|\delta_{\mathcal{A}_1}(x_k) - \delta_{\mathcal{B}_1}(x_k)|^\lambda + |\delta_{\mathcal{A}_2}(x_k) - \delta_{\mathcal{B}_2}(x_k)|^\lambda + \dots + |\delta_{\mathcal{A}_h}(x_k) - \delta_{\mathcal{B}_h}(x_h)|^\lambda \right) + \\ & (1/h) \left(|\bar{\omega}_{\delta_{\mathcal{A}_1}}(x_k) - \bar{\omega}_{\delta_{\mathcal{B}_1}}(x_k)|^\lambda + |\bar{\omega}_{\delta_{\mathcal{A}_2}}(x_k) - \bar{\omega}_{\delta_{\mathcal{B}_2}}(x_k)|^\lambda + \dots + |\bar{\omega}_{\delta_{\mathcal{A}_h}}(x_k) - \bar{\omega}_{\delta_{\mathcal{B}_h}}(x_k)|^\lambda \right) \end{aligned} \right)} - 1 \right]^{(1/\lambda)}$$

$$\iff S_{\text{CD}}^2(\mathcal{A}, \mathcal{B}) = \left[\frac{1}{n} \sum_{k=1}^n 2^{-1 \left(\begin{aligned} & (1/g) \left(|\gamma_{\mathcal{A}_1}(x_1) - \gamma_{\mathcal{B}_1}(x_1)|^\lambda + |\gamma_{\mathcal{A}_2}(x_1) - \gamma_{\mathcal{B}_2}(x_1)|^\lambda + \dots + |\gamma_{\mathcal{A}_g}(x_1) - \gamma_{\mathcal{B}_g}(x_1)|^\lambda \right) + \\ & (1/g) \left(|\omega_{\gamma_{\mathcal{A}_1}}(x_1) - \omega_{\gamma_{\mathcal{B}_1}}(x_1)|^\lambda + |\omega_{\gamma_{\mathcal{A}_2}}(x_1) - \omega_{\gamma_{\mathcal{B}_2}}(x_1)|^\lambda + \dots + |\omega_{\gamma_{\mathcal{A}_g}}(x_1) - \omega_{\gamma_{\mathcal{B}_g}}(x_1)|^\lambda \right) + \\ & (1/h) \left(|\delta_{\mathcal{A}_1}(x_1) - \delta_{\mathcal{B}_1}(x_1)|^\lambda + |\delta_{\mathcal{A}_2}(x_1) - \delta_{\mathcal{B}_2}(x_1)|^\lambda + \dots + |\delta_{\mathcal{A}_h}(x_1) - \delta_{\mathcal{B}_h}(x_1)|^\lambda \right) + \\ & (1/h) \left(|\bar{\omega}_{\delta_{\mathcal{A}_1}}(x_1) - \bar{\omega}_{\delta_{\mathcal{B}_1}}(x_1)|^\lambda + |\bar{\omega}_{\delta_{\mathcal{A}_2}}(x_1) - \bar{\omega}_{\delta_{\mathcal{B}_2}}(x_1)|^\lambda + \dots + |\bar{\omega}_{\delta_{\mathcal{A}_h}}(x_1) - \bar{\omega}_{\delta_{\mathcal{B}_h}}(x_1)|^\lambda \right) \end{aligned} \right)} - 1 \right]^{(1/\lambda)}$$

$$+ 2^{1+(1/4) \left(\begin{aligned} & (1/g) \left(|\gamma_{\mathcal{A}_1}(x_2) - \gamma_{\mathcal{B}_1}(x_2)|^\lambda + |\gamma_{\mathcal{A}_2}(x_2) - \gamma_{\mathcal{B}_2}(x_2)|^\lambda + \dots + |\gamma_{\mathcal{A}_g}(x_2) - \gamma_{\mathcal{B}_g}(x_2)|^\lambda \right) + \\ & (1/g) \left(|\omega_{\gamma_{\mathcal{A}_1}}(x_2) - \omega_{\gamma_{\mathcal{B}_1}}(x_2)|^\lambda + |\omega_{\gamma_{\mathcal{A}_2}}(x_2) - \omega_{\gamma_{\mathcal{B}_2}}(x_2)|^\lambda + \dots + |\omega_{\gamma_{\mathcal{A}_g}}(x_2) - \omega_{\gamma_{\mathcal{B}_g}}(x_2)|^\lambda \right) + \\ & (1/h) \left(|\delta_{\mathcal{A}_1}(x_2) - \delta_{\mathcal{B}_1}(x_2)|^\lambda + |\delta_{\mathcal{A}_2}(x_2) - \delta_{\mathcal{B}_2}(x_2)|^\lambda + \dots + |\delta_{\mathcal{A}_h}(x_2) - \delta_{\mathcal{B}_h}(x_2)|^\lambda \right) + \\ & (1/h) \left(|\bar{\omega}_{\delta_{\mathcal{A}_1}}(x_2) - \bar{\omega}_{\delta_{\mathcal{B}_1}}(x_2)|^\lambda + |\bar{\omega}_{\delta_{\mathcal{A}_2}}(x_2) - \bar{\omega}_{\delta_{\mathcal{B}_2}}(x_2)|^\lambda + \dots + |\bar{\omega}_{\delta_{\mathcal{A}_h}}(x_2) - \bar{\omega}_{\delta_{\mathcal{B}_h}}(x_2)|^\lambda \right) \end{aligned} \right)} - 1 + \dots$$

$$+ 2^{1+(1/4) \left(\begin{aligned} & (1/g) \left(|\gamma_{\mathcal{A}_1}(x_n) - \gamma_{\mathcal{B}_1}(x_n)|^\lambda + |\gamma_{\mathcal{A}_2}(x_n) - \gamma_{\mathcal{B}_2}(x_n)|^\lambda + \dots + |\gamma_{\mathcal{A}_g}(x_n) - \gamma_{\mathcal{B}_g}(x_n)|^\lambda \right) + \\ & (1/g) \left(|\omega_{\gamma_{\mathcal{A}_1}}(x_n) - \omega_{\gamma_{\mathcal{B}_1}}(x_n)|^\lambda + |\omega_{\gamma_{\mathcal{A}_2}}(x_n) - \omega_{\gamma_{\mathcal{B}_2}}(x_n)|^\lambda + \dots + |\omega_{\gamma_{\mathcal{A}_g}}(x_n) - \omega_{\gamma_{\mathcal{B}_g}}(x_n)|^\lambda \right) + \\ & (1/h) \left(|\delta_{\mathcal{A}_1}(x_n) - \delta_{\mathcal{B}_1}(x_n)|^\lambda + |\delta_{\mathcal{A}_2}(x_n) - \delta_{\mathcal{B}_2}(x_n)|^\lambda + \dots + |\delta_{\mathcal{A}_h}(x_n) - \delta_{\mathcal{B}_h}(x_n)|^\lambda \right) + \\ & (1/h) \left(|\bar{\omega}_{\delta_{\mathcal{A}_1}}(x_n) - \bar{\omega}_{\delta_{\mathcal{B}_1}}(x_n)|^\lambda + |\bar{\omega}_{\delta_{\mathcal{A}_2}}(x_n) - \bar{\omega}_{\delta_{\mathcal{B}_2}}(x_n)|^\lambda + \dots + |\bar{\omega}_{\delta_{\mathcal{A}_h}}(x_n) - \bar{\omega}_{\delta_{\mathcal{B}_h}}(x_n)|^\lambda \right) \end{aligned} \right)} - 1 \Big]^{(1/\lambda)}.$$

Now, as $\mathcal{A} = \mathcal{B} \iff \mu_{\mathcal{A}}(x_k) = \mu_{\mathcal{B}}(x_k)$ and $\nu_{\mathcal{A}}(x_k) = \nu_{\mathcal{B}}(x_k)$ for $k = 1, 2, \dots, n \iff \gamma_{\mathcal{A}_{\mathcal{F}}}(x_k) e^{i2\pi(\omega_{\gamma_{\mathcal{A}_{\mathcal{F}}}}(x_k))} = \gamma_{\mathcal{B}_{\mathcal{F}}}(x_k) e^{i2\pi(\omega_{\gamma_{\mathcal{B}_{\mathcal{F}}}}(x_k))}$ and $\delta_{\mathcal{A}_{\mathcal{F}}}(x_k) e^{i2\pi(\omega_{\delta_{\mathcal{A}_{\mathcal{F}}}}(x_k))} = \delta_{\mathcal{B}_{\mathcal{F}}}(x_k) e^{i2\pi(\omega_{\delta_{\mathcal{B}_{\mathcal{F}}}}(x_k))}$ for $k = 1, 2, \dots, n \iff \gamma_{\mathcal{A}_{\mathcal{F}}}(x_k) = \gamma_{\mathcal{B}_{\mathcal{F}}}(x_k)$ and $\delta_{\mathcal{A}_{\mathcal{F}}}(x_k) = \delta_{\mathcal{B}_{\mathcal{F}}}(x_k)$ for $k = 1, 2, \dots, n$.

$(x_k), \delta_{\mathcal{A}_{\mathcal{F}}}(x_k) = \delta_{\mathcal{B}_{\mathcal{F}}}(x_k), e^{i2\pi(\omega_{\gamma_{\mathcal{A}_{\mathcal{F}}}}(x_k))} = e^{i2\pi(\omega_{\gamma_{\mathcal{B}_{\mathcal{F}}}}(x_k))}$ and $e^{i2\pi(\omega_{\delta_{\mathcal{A}_{\mathcal{F}}}}(x_k))} = e^{i2\pi(\omega_{\delta_{\mathcal{B}_{\mathcal{F}}}}(x_k))}$ for $k = 1, 2, \dots, n$, then

$$\begin{aligned} \iff S_{\text{CD}}^2(\mathcal{A}, \mathcal{B}) &= \left[\frac{1}{n} \left[2^{1-0} - 1 + 2^{1-0} - 1 + \dots + 2^{1-0} - 1 \right] \right]^{(1/\lambda)} \\ \iff S_{\text{CD}}^2(\mathcal{A}, \mathcal{B}) &= 1. \end{aligned} \quad (25)$$

(3) We have

$$\begin{aligned} S_{\text{CD}}^2(\mathcal{A}, \mathcal{B}) &= \left[\frac{1}{n} \sum_{k=1}^n 2^{1-(1/4) \left((1/g) \sum_{\mathcal{F}=1}^g |\gamma_{\mathcal{A}_{\mathcal{F}}}(x_k) - \gamma_{\mathcal{B}_{\mathcal{F}}}(x_k)|^\lambda + (1/g) \sum_{\mathcal{F}=1}^g |\omega_{\gamma_{\mathcal{A}_{\mathcal{F}}}}(x_k) - \omega_{\gamma_{\mathcal{B}_{\mathcal{F}}}}(x_k)|^\lambda \right. \right. \\ &\quad \left. \left. + (1/h) \sum_{\mathcal{F}=1}^h |\delta_{\mathcal{A}_{\mathcal{F}}}(x_k) - \delta_{\mathcal{B}_{\mathcal{F}}}(x_k)|^\lambda + (1/h) \sum_{\mathcal{F}=1}^h |\omega_{\delta_{\mathcal{A}_{\mathcal{F}}}}(x_k) - \omega_{\delta_{\mathcal{B}_{\mathcal{F}}}}(x_k)|^\lambda \right) - 1 \right]^{(1/\lambda)} \\ &= \left[\frac{1}{n} \sum_{k=1}^n 2^{1-(1/4) \left((1/g) \sum_{\mathcal{F}=1}^g |\gamma_{\mathcal{B}_{\mathcal{F}}}(x_k) - \gamma_{\mathcal{A}_{\mathcal{F}}}(x_k)|^\lambda + (1/g) \sum_{\mathcal{F}=1}^g |\omega_{\gamma_{\mathcal{B}_{\mathcal{F}}}}(x_k) - \omega_{\gamma_{\mathcal{A}_{\mathcal{F}}}}(x_k)|^\lambda \right. \right. \\ &\quad \left. \left. + (1/h) \sum_{\mathcal{F}=1}^h |\delta_{\mathcal{B}_{\mathcal{F}}}(x_k) - \delta_{\mathcal{A}_{\mathcal{F}}}(x_k)|^\lambda + (1/h) \sum_{\mathcal{F}=1}^h |\omega_{\delta_{\mathcal{B}_{\mathcal{F}}}}(x_k) - \omega_{\delta_{\mathcal{A}_{\mathcal{F}}}}(x_k)|^\lambda \right) - 1 \right]^{(1/\lambda)} \\ &= S_{\text{CD}}^2(\mathcal{B}, \mathcal{A}). \end{aligned} \quad (26)$$

Remark 1. If $\lambda = 1$, then the antilogarithmic-based generalized SMs between $\mathcal{A}$ and $\mathcal{B}$ become

□

$$\begin{aligned}
S_{CD}^1(\mathcal{A}, \mathcal{B}) &= \frac{1}{n} \sum_{k=1}^n \left[2 \left(\frac{1}{g} \sum_{j=1}^g |\gamma_{\mathcal{A}_j}(x_k) - \gamma_{\mathcal{B}_j}(x_k)|^\lambda \vee \frac{1}{g} \sum_{j=1}^g |\omega_{\gamma_{\mathcal{A}_j}}(x_k) - \omega_{\gamma_{\mathcal{B}_j}}(x_k)|^\lambda \right. \right. \\
&\quad \left. \left. \vee \frac{1}{h} \sum_{j=1}^h |\delta_{\mathcal{A}_j}(x_k) - \delta_{\mathcal{B}_j}(x_k)|^\lambda \vee \frac{1}{h} \sum_{j=1}^h |\omega_{\delta_{\mathcal{A}_j}}(x_k) - \omega_{\delta_{\mathcal{B}_j}}(x_k)|^\lambda \right) \right]^{-1}, \\
S_{CD}^2(\mathcal{A}, \mathcal{B}) &= \frac{1}{n} \sum_{k=1}^n \left[2 \left(\frac{1}{g} \sum_{j=1}^g |\gamma_{\mathcal{A}_j}(x_k) - \gamma_{\mathcal{B}_j}(x_k)|^\lambda + \frac{1}{g} \sum_{j=1}^g |\omega_{\gamma_{\mathcal{A}_j}}(x_k) - \omega_{\gamma_{\mathcal{B}_j}}(x_k)|^\lambda \right. \right. \\
&\quad \left. \left. + \frac{1}{h} \sum_{j=1}^h |\delta_{\mathcal{A}_j}(x_k) - \delta_{\mathcal{B}_j}(x_k)|^\lambda + \frac{1}{h} \sum_{j=1}^h |\omega_{\delta_{\mathcal{A}_j}}(x_k) - \omega_{\delta_{\mathcal{B}_j}}(x_k)|^\lambda \right) \right]^{-1}.
\end{aligned} \tag{27}$$

Definition 16. By using the CDHFs and their weight vector, the anti-logarithmic-based weighted generalized SMs between two CDHFSs $\mathcal{A}$ and $\mathcal{B}$ are given as

$$\begin{aligned}
S_{CDw}^1(\mathcal{A}, \mathcal{B}) &= \left[\sum_{k=1}^n w_k \left[2 \left(\frac{1}{g} \sum_{j=1}^g |\gamma_{\mathcal{A}_j}(x_k) - \gamma_{\mathcal{B}_j}(x_k)|^\lambda \vee \frac{1}{g} \sum_{j=1}^g |\omega_{\gamma_{\mathcal{A}_j}}(x_k) - \omega_{\gamma_{\mathcal{B}_j}}(x_k)|^\lambda \right. \right. \right. \\
&\quad \left. \left. \vee \frac{1}{h} \sum_{j=1}^h |\delta_{\mathcal{A}_j}(x_k) - \delta_{\mathcal{B}_j}(x_k)|^\lambda \vee \frac{1}{h} \sum_{j=1}^h |\omega_{\delta_{\mathcal{A}_j}}(x_k) - \omega_{\delta_{\mathcal{B}_j}}(x_k)|^\lambda \right) \right]^{-1} \right]^{(1/\lambda)}, \\
S_{CDw}^2(\mathcal{A}, \mathcal{B}) &= \left[\sum_{k=1}^n w_k \left[2 \left(\frac{1}{g} \sum_{j=1}^g |\gamma_{\mathcal{A}_j}(x_k) - \gamma_{\mathcal{B}_j}(x_k)|^\lambda + \frac{1}{g} \sum_{j=1}^g |\omega_{\gamma_{\mathcal{A}_j}}(x_k) - \omega_{\gamma_{\mathcal{B}_j}}(x_k)|^\lambda \right. \right. \right. \\
&\quad \left. \left. + \frac{1}{h} \sum_{j=1}^h |\delta_{\mathcal{A}_j}(x_k) - \delta_{\mathcal{B}_j}(x_k)|^\lambda + \frac{1}{h} \sum_{j=1}^h |\omega_{\delta_{\mathcal{A}_j}}(x_k) - \omega_{\delta_{\mathcal{B}_j}}(x_k)|^\lambda \right) \right]^{-1} \right]^{(1/\lambda)},
\end{aligned} \tag{28}$$

where $w = (w_1, w_2, \dots, w_n)^T$ represents the weight vector of each element x_k ($k = 1, 2, 3, \dots, n$) contained in CDHFS and the weight vector satisfies $w_k \in [0, 1]$ for each

$k = 1, 2, 3, \dots, n$, $\sum_{k=1}^n w_k = 1$. When we assume the weight vector to be $w = ((1/n), (1/n), \dots, (1/n))^T$, then the antilogarithmic-based weighted generalized SMs will

transform into antilogarithmic-based generalized SMs. Otherwise speaking, when $w_k = (1/n)$, $k = 1, 2, 3, \dots, n$, the $S_{CDw}^q(\mathcal{A}, \mathcal{B}) = S_{CD}^q(\mathcal{A}, \mathcal{B})$ $q = 1, 2$.

Remark 2. If $\lambda = 1$, then the antilogarithmic-based weighted generalized SM between $\mathcal{A}$ and $\mathcal{B}$ becomes

$$S_{CDw}^1(\mathcal{A}, \mathcal{B}) = \sum_{k=1}^n w_k \left[2 \left(\frac{(1/g) \sum_{j=1}^g |\gamma_{\mathcal{A}_j}(x_k) - \gamma_{\mathcal{B}_j}(x_k)|^\lambda \vee (1/g) \sum_{j=1}^g |\omega_{\gamma_{\mathcal{A}_j}}(x_k) - \omega_{\gamma_{\mathcal{B}_j}}(x_k)|^\lambda}{\vee (1/h) \sum_{j=1}^h |\delta_{\mathcal{A}_j}(x_k) - \delta_{\mathcal{B}_j}(x_k)|^\lambda \vee (1/h) \sum_{j=1}^h |\bar{\omega}_{\delta_{\mathcal{A}_j}}(x_k) - \bar{\omega}_{\delta_{\mathcal{B}_j}}(x_k)|^\lambda} \right)^{1-\lambda} - 1 \right],$$

$$S_{CDw}^2(\mathcal{A}, \mathcal{B}) = \sum_{k=1}^n w_k \left[2 \left(\frac{(1/g) \sum_{j=1}^g |\gamma_{\mathcal{A}_j}(x_k) - \gamma_{\mathcal{B}_j}(x_k)|^\lambda + (1/g) \sum_{j=1}^g |\omega_{\gamma_{\mathcal{A}_j}}(x_k) - \omega_{\gamma_{\mathcal{B}_j}}(x_k)|^\lambda}{+(1/h) \sum_{j=1}^h |\delta_{\mathcal{A}_j}(x_k) - \delta_{\mathcal{B}_j}(x_k)|^\lambda + (1/h) \sum_{j=1}^h |\bar{\omega}_{\delta_{\mathcal{A}_j}}(x_k) - \bar{\omega}_{\delta_{\mathcal{B}_j}}(x_k)|^\lambda} \right)^{1-(1/4)} - 1 \right]. \quad (29)$$

4.2. The Nonexponential-Based Generalized SMs for CDHFS. In this subsection, we defined some nonexponential-based generalized SMs for CDHFS.

Definition 17. Let $\mathcal{A}$ and $\mathcal{B}$ be two CDHFS on set X. Then, we defined following three nonexponential-based generalized SMs between $\mathcal{A}$ and $\mathcal{B}$ as

$$S_{CD}^3(\mathcal{A}, \mathcal{B}) = \frac{1}{n} \sum_{k=1}^n \left[\frac{1 - \left(\frac{(1/g) \sum_{j=1}^g |\gamma_{\mathcal{A}_j}(x_k) - \gamma_{\mathcal{B}_j}(x_k)|^\lambda \vee (1/g) \sum_{j=1}^g |\omega_{\gamma_{\mathcal{A}_j}}(x_k) - \omega_{\gamma_{\mathcal{B}_j}}(x_k)|^\lambda}{\vee (1/h) \sum_{j=1}^h |\delta_{\mathcal{A}_j}(x_k) - \delta_{\mathcal{B}_j}(x_k)|^\lambda \vee (1/h) \sum_{j=1}^h |\bar{\omega}_{\delta_{\mathcal{A}_j}}(x_k) - \bar{\omega}_{\delta_{\mathcal{B}_j}}(x_k)|^\lambda} \right)^{(1/\lambda)}}{1 + \left(\frac{(1/g) \sum_{j=1}^g |\gamma_{\mathcal{A}_j}(x_k) - \gamma_{\mathcal{B}_j}(x_k)|^\lambda \vee (1/g) \sum_{j=1}^g |\omega_{\gamma_{\mathcal{A}_j}}(x_k) - \omega_{\gamma_{\mathcal{B}_j}}(x_k)|^\lambda}{\vee (1/h) \sum_{j=1}^h |\delta_{\mathcal{A}_j}(x_k) - \delta_{\mathcal{B}_j}(x_k)|^\lambda \vee (1/h) \sum_{j=1}^h |\bar{\omega}_{\delta_{\mathcal{A}_j}}(x_k) - \bar{\omega}_{\delta_{\mathcal{B}_j}}(x_k)|^\lambda} \right)^{(1/\lambda)}} \right],$$

$$S_{CD}^4(\mathcal{A}, \mathcal{B}) = \frac{1}{n} \sum_{k=1}^n \frac{\left(\frac{(1/g) \sum_{j=1}^g |\gamma_{\mathcal{A}_j}(x_k) \wedge \gamma_{\mathcal{B}_j}(x_k)|^\lambda + (1/g) \sum_{j=1}^g |\omega_{\gamma_{\mathcal{A}_j}}(x_k) \wedge \omega_{\gamma_{\mathcal{B}_j}}(x_k)|^\lambda}{\vee (1/h) \sum_{j=1}^h |\delta_{\mathcal{A}_j}(x_k) \wedge \delta_{\mathcal{B}_j}(x_k)|^\lambda + (1/h) \sum_{j=1}^h |\bar{\omega}_{\delta_{\mathcal{A}_j}}(x_k) \wedge \bar{\omega}_{\delta_{\mathcal{B}_j}}(x_k)|^\lambda} \right)^{(1/\lambda)}}{\left(\frac{(1/g) \sum_{j=1}^g |\gamma_{\mathcal{A}_j}(x_k) \wedge \gamma_{\mathcal{B}_j}(x_k)|^\lambda + (1/g) \sum_{j=1}^g |\omega_{\gamma_{\mathcal{A}_j}}(x_k) \wedge \omega_{\gamma_{\mathcal{B}_j}}(x_k)|^\lambda}{\vee (1/h) \sum_{j=1}^h |\delta_{\mathcal{A}_j}(x_k) \wedge \delta_{\mathcal{B}_j}(x_k)|^\lambda + (1/h) \sum_{j=1}^h |\bar{\omega}_{\delta_{\mathcal{A}_j}}(x_k) \wedge \bar{\omega}_{\delta_{\mathcal{B}_j}}(x_k)|^\lambda} \right)^{(1/\lambda)}}}, \quad (30)$$

$$S_{CD}^5(\mathcal{A}, \mathcal{B}) = \frac{1}{n} \sum_{k=1}^n \left[\frac{\frac{1}{2g} \sum_{j=1}^g \eta_{cc} \left(\frac{(\gamma_{\mathcal{A}_j}(x_k) \wedge \gamma_{\mathcal{B}_j}(x_k))^\lambda}{(\gamma_{\mathcal{A}_j}(x_k) \vee \gamma_{\mathcal{B}_j}(x_k))^\lambda} + \frac{(\omega_{\gamma_{\mathcal{A}_j}}(x_k) \wedge \omega_{\gamma_{\mathcal{B}_j}}(x_k))^\lambda}{1 + (\omega_{\gamma_{\mathcal{A}_j}}(x_k) \vee \omega_{\gamma_{\mathcal{B}_j}}(x_k))^\lambda} \right)}{\frac{1}{2h} \sum_{j=1}^h \phi_{cc} \left(\frac{(\delta_{\mathcal{A}_j}(x_k) \wedge \delta_{\mathcal{B}_j}(x_k))^\lambda}{(\delta_{\mathcal{A}_j}(x_k) \vee \delta_{\mathcal{B}_j}(x_k))^\lambda} + \frac{(\bar{\omega}_{\delta_{\mathcal{A}_j}}(x_k) \wedge \bar{\omega}_{\delta_{\mathcal{B}_j}}(x_k))^\lambda}{1 + (\bar{\omega}_{\delta_{\mathcal{A}_j}}(x_k) \vee \bar{\omega}_{\delta_{\mathcal{B}_j}}(x_k))^\lambda} \right)} \right]^{(1/\lambda)}.$$

Where $\lambda > 0$ and $\eta_{cc}, \phi_{cc} \in [0, 1]$ such that $\eta_{cc} + \phi_{cc} = 1$.

Theorem 3. Let A and B be two CDHES on set X . Then, the SM $S_{CD}^3(\mathcal{A}, \mathcal{B})$ satisfies the following properties:

- (1) $0 \leq S_{CD}^3(\mathcal{A}, \mathcal{B}) \leq 1$
- (2) $S_{CD}^3(\mathcal{A}, \mathcal{B}) = 1$ if and only if $\mathcal{A} = \mathcal{B}$
- (3) $S_{CD}^3(\mathcal{A}, \mathcal{B}) = S_{CD}^3(\mathcal{B}, \mathcal{A})$

Proof

- (1) Since $(1/g) \sum_{j=1}^g |\gamma_{\mathcal{A}_j}(x_k) - \gamma_{\mathcal{B}_j}(x_k)|^\lambda \in [0, 1]$,
 $(1/g) \sum_{j=1}^g |\omega_{\gamma_{\mathcal{A}_j}}(x_k) - \omega_{\gamma_{\mathcal{B}_j}}(x_k)|^\lambda \in [0, 1]$, $(1/h)$

$\sum_{j=1}^h |\delta_{\mathcal{A}_j}(x_k) - \delta_{\mathcal{B}_j}(x_k)|^\lambda \in [0, 1]$, and $(1/h)$
 $\sum_{j=1}^h |\omega_{\delta_{\mathcal{A}_j}}(x_k) - \omega_{\delta_{\mathcal{B}_j}}(x_k)|^\lambda \in [0, 1]$, then $(1/g)$
 $\sum_{j=1}^g |\gamma_{\mathcal{A}_j}(x_k) - \gamma_{\mathcal{B}_j}(x_k)|^\lambda \vee (1/g) \sum_{j=1}^g |\omega_{\gamma_{\mathcal{A}_j}}(x_k)$
 $- \omega_{\gamma_{\mathcal{B}_j}}(x_k)|^\lambda \vee (1/h) \sum_{j=1}^h |\delta_{\mathcal{A}_j}(x_k) - \delta_{\mathcal{B}_j}(x_k)|^\lambda$
 $\vee (1/h) \sum_{j=1}^h |\omega_{\delta_{\mathcal{A}_j}}(x_k) - \omega_{\delta_{\mathcal{B}_j}}(x_k)|^\lambda \in [0, 1]$ and the
 denominator will always remain greater than the
 numerator, then for $k = 1$, we obtain

$$\left[\frac{1 - \left(\frac{(1/g) \sum_{j=1}^g |\gamma_{\mathcal{A}_j}(x_1) - \gamma_{\mathcal{B}_j}(x_1)|^\lambda \vee (1/g) \sum_{j=1}^g |\omega_{\gamma_{\mathcal{A}_j}}(x_1) - \omega_{\gamma_{\mathcal{B}_j}}(x_1)|^\lambda}{\vee (1/h) \sum_{j=1}^h |\delta_{\mathcal{A}_j}(x_1) - \delta_{\mathcal{B}_j}(x_1)|^\lambda \vee (1/h) \sum_{j=1}^h |\omega_{\delta_{\mathcal{A}_j}}(x_1) - \omega_{\delta_{\mathcal{B}_j}}(x_1)|^\lambda} \right)}{1 + \left(\frac{(1/g) \sum_{j=1}^g |\gamma_{\mathcal{A}_j}(x_1) - \gamma_{\mathcal{B}_j}(x_1)|^\lambda \vee (1/g) \sum_{j=1}^g |\omega_{\gamma_{\mathcal{A}_j}}(x_1) - \omega_{\gamma_{\mathcal{B}_j}}(x_1)|^\lambda}{\vee (1/h) \sum_{j=1}^h |\delta_{\mathcal{A}_j}(x_1) - \delta_{\mathcal{B}_j}(x_1)|^\lambda \vee (1/h) \sum_{j=1}^h |\omega_{\delta_{\mathcal{A}_j}}(x_1) - \omega_{\delta_{\mathcal{B}_j}}(x_1)|^\lambda} \right)} \right] \in [0, 1]. \quad (31)$$

For $k = 2$,

$$\left[\frac{1 - \left(\frac{(1/g) \sum_{j=1}^g |\gamma_{\mathcal{A}_j}(x_2) - \gamma_{\mathcal{B}_j}(x_2)|^\lambda \vee (1/g) \sum_{j=1}^g |\omega_{\gamma_{\mathcal{A}_j}}(x_2) - \omega_{\gamma_{\mathcal{B}_j}}(x_2)|^\lambda}{\vee (1/h) \sum_{j=1}^h |\delta_{\mathcal{A}_j}(x_2) - \delta_{\mathcal{B}_j}(x_2)|^\lambda \vee (1/h) \sum_{j=1}^h |\omega_{\delta_{\mathcal{A}_j}}(x_2) - \omega_{\delta_{\mathcal{B}_j}}(x_2)|^\lambda} \right)}{1 + \left(\frac{(1/g) \sum_{j=1}^g |\gamma_{\mathcal{A}_j}(x_2) - \gamma_{\mathcal{B}_j}(x_2)|^\lambda \vee (1/g) \sum_{j=1}^g |\omega_{\gamma_{\mathcal{A}_j}}(x_2) - \omega_{\gamma_{\mathcal{B}_j}}(x_2)|^\lambda}{\vee (1/h) \sum_{j=1}^h |\delta_{\mathcal{A}_j}(x_2) - \delta_{\mathcal{B}_j}(x_2)|^\lambda \vee (1/h) \sum_{j=1}^h |\omega_{\delta_{\mathcal{A}_j}}(x_2) - \omega_{\delta_{\mathcal{B}_j}}(x_2)|^\lambda} \right)} \right] \in [0, 1]. \quad (32)$$

By continuing this process, we get

$$\begin{aligned}
& \sum_{k=1}^n \left[\frac{1 - \left(\frac{(1/g) \sum_{j=1}^g |\gamma_{\mathcal{A}_j}(x_k) - \gamma_{\mathcal{B}_j}(x_k)|^\lambda \vee (1/g) \sum_{j=1}^g |\omega_{\gamma_{\mathcal{A}_j}}(x_k) - \omega_{\gamma_{\mathcal{B}_j}}(x_k)|^\lambda}{\vee (1/h) \sum_{j=1}^h |\delta_{\mathcal{A}_j}(x_k) - \delta_{\mathcal{B}_j}(x_k)|^\lambda \vee (1/h) \sum_{j=1}^h |\bar{\omega}_{\delta_{\mathcal{A}_j}}(x_k) - \bar{\omega}_{\delta_{\mathcal{B}_j}}(x_k)|^\lambda} \right)}{1 + \left(\frac{(1/g) \sum_{j=1}^g |\gamma_{\mathcal{A}_j}(x_k) - \gamma_{\mathcal{B}_j}(x_k)|^\lambda \vee (1/g) \sum_{j=1}^g |\omega_{\gamma_{\mathcal{A}_j}}(x_k) - \omega_{\gamma_{\mathcal{B}_j}}(x_k)|^\lambda}{\vee (1/h) \sum_{j=1}^h |\delta_{\mathcal{A}_j}(x_k) - \delta_{\mathcal{B}_j}(x_k)|^\lambda \vee (1/h) \sum_{j=1}^h |\bar{\omega}_{\delta_{\mathcal{A}_j}}(x_k) - \bar{\omega}_{\delta_{\mathcal{B}_j}}(x_k)|^\lambda} \right)} \right] \in n[0, 1] \\
& \Rightarrow 0 \leq \sum_{k=1}^n \left[\frac{1 - \left(\frac{(1/g) \sum_{j=1}^g |\gamma_{\mathcal{A}_j}(x_k) - \gamma_{\mathcal{B}_j}(x_k)|^\lambda \vee (1/g) \sum_{j=1}^g |\omega_{\gamma_{\mathcal{A}_j}}(x_k) - \bar{\omega}_{\gamma_{\mathcal{B}_j}}(x_k)|^\lambda}{\vee (1/h) \sum_{j=1}^h |\delta_{\mathcal{A}_j}(x_k) - \delta_{\mathcal{B}_j}(x_k)|^\lambda \vee (1/h) \sum_{j=1}^h |\bar{\omega}_{\delta_{\mathcal{A}_j}}(x_k) - \bar{\omega}_{\delta_{\mathcal{B}_j}}(x_k)|^\lambda} \right)}{1 + \left(\frac{(1/g) \sum_{j=1}^g |\gamma_{\mathcal{A}_j}(x_k) - \gamma_{\mathcal{B}_j}(x_k)|^\lambda \vee (1/g) \sum_{j=1}^g |\omega_{\gamma_{\mathcal{A}_j}}(x_k) - \bar{\omega}_{\gamma_{\mathcal{B}_j}}(x_k)|^\lambda}{\vee (1/h) \sum_{j=1}^h |\delta_{\mathcal{A}_j}(x_k) - \delta_{\mathcal{B}_j}(x_k)|^\lambda \vee (1/h) \sum_{j=1}^h |\bar{\omega}_{\delta_{\mathcal{A}_j}}(x_k) - \bar{\omega}_{\delta_{\mathcal{B}_j}}(x_k)|^\lambda} \right)} \right] \leq n \\
& \Rightarrow 0 \leq \frac{1}{n} \sum_{k=1}^n \left[\frac{1 - \left(\frac{(1/g) \sum_{j=1}^g |\gamma_{\mathcal{A}_j}(x_k) - \gamma_{\mathcal{B}_j}(x_k)|^\lambda \vee (1/g) \sum_{j=1}^g |\omega_{\gamma_{\mathcal{A}_j}}(x_k) - \bar{\omega}_{\gamma_{\mathcal{B}_j}}(x_k)|^\lambda}{\vee (1/h) \sum_{j=1}^h |\delta_{\mathcal{A}_j}(x_k) - \delta_{\mathcal{B}_j}(x_k)|^\lambda \vee (1/h) \sum_{j=1}^h |\bar{\omega}_{\delta_{\mathcal{A}_j}}(x_k) - \bar{\omega}_{\delta_{\mathcal{B}_j}}(x_k)|^\lambda} \right)}{1 + \left(\frac{(1/g) \sum_{j=1}^g |\gamma_{\mathcal{A}_j}(x_k) - \gamma_{\mathcal{B}_j}(x_k)|^\lambda \vee (1/g) \sum_{j=1}^g |\omega_{\gamma_{\mathcal{A}_j}}(x_k) - \bar{\omega}_{\gamma_{\mathcal{B}_j}}(x_k)|^\lambda}{\vee (1/h) \sum_{j=1}^h |\delta_{\mathcal{A}_j}(x_k) - \delta_{\mathcal{B}_j}(x_k)|^\lambda \vee (1/h) \sum_{j=1}^h |\bar{\omega}_{\delta_{\mathcal{A}_j}}(x_k) - \bar{\omega}_{\delta_{\mathcal{B}_j}}(x_k)|^\lambda} \right)} \right] \leq 1 \tag{33} \\
& \Rightarrow 0 \leq \left[\frac{1}{n} \sum_{k=1}^n \left[\frac{1 - \left(\frac{(1/g) \sum_{j=1}^g |\gamma_{\mathcal{A}_j}(x_k) - \gamma_{\mathcal{B}_j}(x_k)|^\lambda \vee (1/g) \sum_{j=1}^g |\omega_{\gamma_{\mathcal{A}_j}}(x_k) - \bar{\omega}_{\gamma_{\mathcal{B}_j}}(x_k)|^\lambda}{\vee (1/h) \sum_{j=1}^h |\delta_{\mathcal{A}_j}(x_k) - \delta_{\mathcal{B}_j}(x_k)|^\lambda \vee (1/h) \sum_{j=1}^h |\bar{\omega}_{\delta_{\mathcal{A}_j}}(x_k) - \bar{\omega}_{\delta_{\mathcal{B}_j}}(x_k)|^\lambda} \right)}{1 + \left(\frac{(1/g) \sum_{j=1}^g |\gamma_{\mathcal{A}_j}(x_k) - \gamma_{\mathcal{B}_j}(x_k)|^\lambda \vee (1/g) \sum_{j=1}^g |\omega_{\gamma_{\mathcal{A}_j}}(x_k) - \bar{\omega}_{\gamma_{\mathcal{B}_j}}(x_k)|^\lambda}{\vee (1/h) \sum_{j=1}^h |\delta_{\mathcal{A}_j}(x_k) - \delta_{\mathcal{B}_j}(x_k)|^\lambda \vee (1/h) \sum_{j=1}^h |\bar{\omega}_{\delta_{\mathcal{A}_j}}(x_k) - \bar{\omega}_{\delta_{\mathcal{B}_j}}(x_k)|^\lambda} \right)} \right] \right]^{(1/\lambda)} \leq 1 \\
& \Rightarrow S_{\text{CD}}^3(\mathcal{A}, \mathcal{B}).
\end{aligned}$$

(2) By Definition 17, we have

$$\begin{aligned}
 S_{\text{CD}}^3(\mathcal{A}, \mathcal{B}) & \left[\frac{1}{n} \sum_{k=1}^n \left[\frac{1 - \left((1/g) \sum_{\mathcal{F}=1}^g |\gamma_{\mathcal{A}_{\mathcal{F}}}(x_k) - \gamma_{\mathcal{B}_{\mathcal{F}}}(x_k)|^\lambda \vee (1/g) \sum_{\mathcal{F}=1}^g |\omega_{\gamma_{\mathcal{A}_{\mathcal{F}}}}(x_k) - \omega_{\gamma_{\mathcal{B}_{\mathcal{F}}}}(x_k)|^\lambda \right)}{\vee (1/h) \sum_{\mathcal{F}=1}^h |\delta_{\mathcal{A}_{\mathcal{F}}}(x_k) - \delta_{\mathcal{B}_{\mathcal{F}}}(x_k)|^\lambda \vee (1/h) \sum_{\mathcal{F}=1}^h |\bar{\omega}_{\delta_{\mathcal{A}_{\mathcal{F}}}}(x_k) - \bar{\omega}_{\delta_{\mathcal{B}_{\mathcal{F}}}}(x_k)|^\lambda} \right] \right]^{(1/\lambda)} \\
 & \iff S_{\text{CD}}^3(\mathcal{A}, \mathcal{B}) \left[\frac{1}{n} \sum_{k=1}^n \left[\frac{1 - \left((1/g) \left((|\gamma_{\mathcal{A}_1}(x_k) - \gamma_{\mathcal{B}_1}(x_k)|^\lambda + |\gamma_{\mathcal{A}_2}(x_k) - \gamma_{\mathcal{B}_2}(x_k)|^\lambda + \dots + |\gamma_{\mathcal{A}_g}(x_k) - \gamma_{\mathcal{B}_g}(x_k)|^\lambda) \vee \right. \right. \right. \\
 & \quad \left. \left. \left. (1/g) \left(|\omega_{\gamma_{\mathcal{A}_1}}(x_k) - \omega_{\gamma_{\mathcal{B}_1}}(x_k)|^\lambda + |\omega_{\gamma_{\mathcal{A}_2}}(x_k) - \omega_{\gamma_{\mathcal{B}_2}}(x_k)|^\lambda + \dots + |\omega_{\gamma_{\mathcal{A}_g}}(x_k) - \omega_{\gamma_{\mathcal{B}_g}}(x_k)|^\lambda \right) \right. \right. \right. \\
 & \quad \left. \left. \vee (1/h) \left((|\delta_{\mathcal{A}_1}(x_k) - \delta_{\mathcal{B}_1}(x_k)|^\lambda + |\delta_{\mathcal{A}_2}(x_k) - \delta_{\mathcal{B}_2}(x_k)|^\lambda + \dots + |\delta_{\mathcal{A}_h}(x_k) - \delta_{\mathcal{B}_h}(x_k)|^\lambda) \vee \right. \right. \right. \\
 & \quad \left. \left. \left. (1/h) \left(|\bar{\omega}_{\delta_{\mathcal{A}_1}}(x_k) - \bar{\omega}_{\delta_{\mathcal{B}_1}}(x_k)|^\lambda + |\bar{\omega}_{\delta_{\mathcal{A}_2}}(x_k) - \bar{\omega}_{\delta_{\mathcal{B}_2}}(x_k)|^\lambda + \dots + |\bar{\omega}_{\delta_{\mathcal{A}_h}}(x_k) - \bar{\omega}_{\delta_{\mathcal{B}_h}}(x_k)|^\lambda \right) \right) \right) \right] \\
 & \iff S_{\text{CD}}^3(\mathcal{A}, \mathcal{B}) = \frac{1}{n} \left[\frac{1 - \left((1/g) \left((|\gamma_{\mathcal{A}_1}(x_1) - \gamma_{\mathcal{B}_1}(x_1)|^\lambda + |\gamma_{\mathcal{A}_2}(x_1) - \gamma_{\mathcal{B}_2}(x_1)|^\lambda + \dots + |\gamma_{\mathcal{A}_g}(x_1) - \gamma_{\mathcal{B}_g}(x_1)|^\lambda) \vee \right. \right. \right. \\
 & \quad \left. \left. \left. (1/g) \left(|\omega_{\gamma_{\mathcal{A}_1}}(x_1) - \omega_{\gamma_{\mathcal{B}_1}}(x_1)|^\lambda + |\omega_{\gamma_{\mathcal{A}_2}}(x_1) - \omega_{\gamma_{\mathcal{B}_2}}(x_1)|^\lambda + \dots + |\omega_{\gamma_{\mathcal{A}_g}}(x_1) - \omega_{\gamma_{\mathcal{B}_g}}(x_1)|^\lambda \right) \right. \right. \right. \\
 & \quad \left. \left. \vee (1/h) \left((|\delta_{\mathcal{A}_1}(x_1) - \delta_{\mathcal{B}_1}(x_1)|^\lambda + |\delta_{\mathcal{A}_2}(x_1) - \delta_{\mathcal{B}_2}(x_1)|^\lambda + \dots + |\delta_{\mathcal{A}_h}(x_1) - \delta_{\mathcal{B}_h}(x_1)|^\lambda) \vee \right. \right. \right. \\
 & \quad \left. \left. \left. (1/h) \left(|\bar{\omega}_{\delta_{\mathcal{A}_1}}(x_1) - \bar{\omega}_{\delta_{\mathcal{B}_1}}(x_1)|^\lambda + |\bar{\omega}_{\delta_{\mathcal{A}_2}}(x_1) - \bar{\omega}_{\delta_{\mathcal{B}_2}}(x_1)|^\lambda + \dots + |\bar{\omega}_{\delta_{\mathcal{A}_h}}(x_1) - \bar{\omega}_{\delta_{\mathcal{B}_h}}(x_1)|^\lambda \right) \right) \right) \right]^{(1/\lambda)}
 \end{aligned}$$

(34)

$$\begin{aligned} \gamma_{\mathcal{B}_{\mathcal{F}}}(x_k), \quad \delta_{\mathcal{A}_{\mathcal{F}}}(x_k) &= \delta_{\mathcal{B}_{\mathcal{F}}}(x_k), \quad e^{i2\pi(\omega_{\gamma_{\mathcal{A}_{\mathcal{F}}}}(x_k))} = \\ e^{i2\pi(\omega_{\gamma_{\mathcal{B}_{\mathcal{F}}}}(x_k))} \quad \text{and} \quad e^{i2\pi(\omega_{\delta_{\mathcal{A}_{\mathcal{F}}}}(x_k))} &= e^{i2\pi(\omega_{\delta_{\mathcal{B}_{\mathcal{F}}}}(x_k))} \quad \text{for } k = \\ 1, 2, \dots, n, \text{ then} \end{aligned}$$

$$\Leftrightarrow S_{CD}^3(\mathcal{A}, \mathcal{B}) = \left[\frac{1}{n} [1 + 1 + \dots + 1] \right]^{(1/\lambda)} \quad (3) \text{ We have} \quad (35)$$

$$\Leftrightarrow S_{CD}^3(\mathcal{A}, \mathcal{B}) = 1.$$

$$S_{CD}^3(\mathcal{A}, \mathcal{B}) = \left[\frac{1}{n} \sum_{k=1}^n \frac{1 - \left(\frac{(1/g) \sum_{j=1}^g |\gamma_{\mathcal{A}_j}(x_k) - \gamma_{\mathcal{B}_j}(x_k)|^\lambda \vee (1/g) \sum_{j=1}^g |\omega_{\gamma_{\mathcal{A}_j}}(x_k) - \omega_{\gamma_{\mathcal{B}_j}}(x_k)|^\lambda}{\vee (1/h) \sum_{j=1}^h |\delta_{\mathcal{A}_j}(x_k) - \delta_{\mathcal{B}_j}(x_k)|^\lambda \vee (1/h) \sum_{j=1}^h |\bar{\omega}_{\delta_{\mathcal{A}_j}}(x_k) - \bar{\omega}_{\delta_{\mathcal{B}_j}}(x_k)|^\lambda} \right)}{1 + \left(\frac{(1/g) \sum_{j=1}^g |\gamma_{\mathcal{A}_j}(x_k) - \gamma_{\mathcal{B}_j}(x_k)|^\lambda \vee (1/g) \sum_{j=1}^g |\omega_{\gamma_{\mathcal{A}_j}}(x_k) - \omega_{\gamma_{\mathcal{B}_j}}(x_k)|^\lambda}{\vee (1/h) \sum_{j=1}^h |\delta_{\mathcal{A}_j}(x_k) - \delta_{\mathcal{B}_j}(x_k)|^\lambda \vee (1/h) \sum_{j=1}^h |\bar{\omega}_{\delta_{\mathcal{A}_j}}(x_k) - \bar{\omega}_{\delta_{\mathcal{B}_j}}(x_k)|^\lambda} \right)} \right]^{(1/\lambda)} \quad (36)$$

$$= \left[\frac{1}{n} \sum_{k=1}^n \frac{1 - \left(\frac{(1/g) \sum_{j=1}^g |\gamma_{\mathcal{B}_j}(x_k) - \gamma_{\mathcal{A}_j}(x_k)|^\lambda \vee (1/g) \sum_{j=1}^g |\omega_{\gamma_{\mathcal{B}_j}}(x_k) - \omega_{\gamma_{\mathcal{A}_j}}(x_k)|^\lambda}{\vee (1/h) \sum_{j=1}^h |\delta_{\mathcal{B}_j}(x_k) - \delta_{\mathcal{A}_j}(x_k)|^\lambda \vee (1/h) \sum_{j=1}^h |\bar{\omega}_{\delta_{\mathcal{B}_j}}(x_k) - \bar{\omega}_{\delta_{\mathcal{A}_j}}(x_k)|^\lambda} \right)}{1 + \left(\frac{(1/g) \sum_{j=1}^g |\gamma_{\mathcal{B}_j}(x_k) - \gamma_{\mathcal{A}_j}(x_k)|^\lambda \vee (1/g) \sum_{j=1}^g |\omega_{\gamma_{\mathcal{B}_j}}(x_k) - \omega_{\gamma_{\mathcal{A}_j}}(x_k)|^\lambda}{\vee (1/h) \sum_{j=1}^h |\delta_{\mathcal{B}_j}(x_k) - \delta_{\mathcal{A}_j}(x_k)|^\lambda \vee (1/h) \sum_{j=1}^h |\bar{\omega}_{\delta_{\mathcal{B}_j}}(x_k) - \bar{\omega}_{\delta_{\mathcal{A}_j}}(x_k)|^\lambda} \right)} \right]^{(1/\lambda)} \quad (36)$$

$$= S_{CD}^3(\mathcal{B}, \mathcal{A}).$$

Theorem 4. Let A and B be two CDHES on set X . Then, the SM $S_{CD}^4(\mathcal{A}, \mathcal{B})$ hold the following axioms:

- (1) $0 \leq S_{CD}^4(\mathcal{A}, \mathcal{B}) \leq 1$
- (2) $S_{CD}^4(\mathcal{A}, \mathcal{B}) = 1$ if and only if $\mathcal{A} = \mathcal{B}$
- (3) $S_{CD}^4(\mathcal{A}, \mathcal{B}) = S_{CD}^4(\mathcal{B}, \mathcal{A})$

Proof. Straight forward (similar to the proof of Theorem 3.), $\square$

Theorem 5. Let A and B be two CDHES on set X . Then, the SM $S_{CD}^5(\mathcal{A}, \mathcal{B})$ hold the following axioms:

- (1) $0 \leq S_{CD}^5(\mathcal{A}, \mathcal{B}) \leq 1$
- (2) $S_{CD}^5(\mathcal{A}, \mathcal{B}) = 1$ if and only if $\mathcal{A} = \mathcal{B}$
- (3) $S_{CD}^5(\mathcal{A}, \mathcal{B}) = S_{CD}^5(\mathcal{B}, \mathcal{A})$

Proof. Straight forward (similar to the proof of Theorem 3.), $\square$

Remark 3. If $\lambda = 1$, then nonexponential-based generalized SMs between $\mathcal{A}$ and $\mathcal{B}$ becomes

$$S_{CD}^3(\mathcal{A}, \mathcal{B}) = \left[\frac{1}{n} \sum_{k=1}^n \frac{1 - \left(\frac{(1/g) \sum_{j=1}^g |\gamma_{\mathcal{A}_j}(x_k) - \gamma_{\mathcal{B}_j}(x_k)|^\lambda \vee (1/g) \sum_{j=1}^g |\omega_{\gamma_{\mathcal{A}_j}}(x_k) - \omega_{\gamma_{\mathcal{B}_j}}(x_k)|^\lambda}{\vee (1/h) \sum_{j=1}^h |\delta_{\mathcal{A}_j}(x_k) - \delta_{\mathcal{B}_j}(x_k)|^\lambda \vee (1/h) \sum_{j=1}^h |\bar{\omega}_{\delta_{\mathcal{A}_j}}(x_k) - \bar{\omega}_{\delta_{\mathcal{B}_j}}(x_k)|^\lambda} \right)}{1 + \left(\frac{(1/g) \sum_{j=1}^g |\gamma_{\mathcal{A}_j}(x_k) - \gamma_{\mathcal{B}_j}(x_k)|^\lambda \vee (1/g) \sum_{j=1}^g |\omega_{\gamma_{\mathcal{A}_j}}(x_k) - \omega_{\gamma_{\mathcal{B}_j}}(x_k)|^\lambda}{\vee (1/h) \sum_{j=1}^h |\delta_{\mathcal{A}_j}(x_k) - \delta_{\mathcal{B}_j}(x_k)|^\lambda \vee (1/h) \sum_{j=1}^h |\bar{\omega}_{\delta_{\mathcal{A}_j}}(x_k) - \bar{\omega}_{\delta_{\mathcal{B}_j}}(x_k)|^\lambda} \right)} \right]^{(1/\lambda)},$$

$$\begin{aligned}
S_{CD}^4(\mathcal{A}, \mathcal{B}) &= \left[\frac{(1/g) \sum_{\mathcal{F}=1}^g |\gamma_{\mathcal{A}_{\mathcal{F}}}(x_k) \wedge \gamma_{\mathcal{B}_{\mathcal{F}}}(x_k)|^\lambda + (1/g) \sum_{\mathcal{F}=1}^g |\omega_{\gamma_{\mathcal{A}_{\mathcal{F}}}}(x_k) \wedge \omega_{\gamma_{\mathcal{B}_{\mathcal{F}}}}(x_k)|^\lambda}{(1/g) \sum_{\mathcal{F}=1}^g |\gamma_{\mathcal{A}_{\mathcal{F}}}(x_k) \vee \gamma_{\mathcal{B}_{\mathcal{F}}}(x_k)|^\lambda + (1/g) \sum_{\mathcal{F}=1}^g |\omega_{\gamma_{\mathcal{A}_{\mathcal{F}}}}(x_k) \vee \omega_{\gamma_{\mathcal{B}_{\mathcal{F}}}}(x_k)|^\lambda} \right. \\
&\quad \left. \frac{\vee (1/h) \sum_{\mathcal{F}=1}^h |\delta_{\mathcal{A}_{\mathcal{F}}}(x_k) \wedge \delta_{\mathcal{B}_{\mathcal{F}}}(x_k)|^\lambda + (1/h) \sum_{\mathcal{F}=1}^h |\bar{\omega}_{\delta_{\mathcal{A}_{\mathcal{F}}}}(x_k) \wedge \bar{\omega}_{\delta_{\mathcal{B}_{\mathcal{F}}}}(x_k)|^\lambda}{\vee (1/h) \sum_{\mathcal{F}=1}^h |\delta_{\mathcal{A}_{\mathcal{F}}}(x_k) \vee \delta_{\mathcal{B}_{\mathcal{F}}}(x_k)|^\lambda + (1/h) \sum_{\mathcal{F}=1}^h |\bar{\omega}_{\delta_{\mathcal{A}_{\mathcal{F}}}}(x_k) \vee \bar{\omega}_{\delta_{\mathcal{B}_{\mathcal{F}}}}(x_k)|^\lambda} \right]^{(1/\lambda)}, \\
S_{CD}^5(\mathcal{A}, \mathcal{B}) &= \frac{1}{n} \sum_{k=1}^n \left[\frac{1}{2g} \eta_{cc} \left(\frac{(\gamma_{\mathcal{A}_{\mathcal{F}}}(x_k) \wedge \gamma_{\mathcal{B}_{\mathcal{F}}}(x_k))}{\gamma_{\mathcal{A}_{\mathcal{F}}}(x_k) \vee \gamma_{\mathcal{B}_{\mathcal{F}}}(x_k)} + \frac{(\omega_{\gamma_{\mathcal{A}_{\mathcal{F}}}}(x_k) \wedge \omega_{\gamma_{\mathcal{B}_{\mathcal{F}}}}(x_k))}{1 + (\omega_{\gamma_{\mathcal{A}_{\mathcal{F}}}}(x_k) \vee \omega_{\gamma_{\mathcal{B}_{\mathcal{F}}}}(x_k))} \right) + \right. \\
&\quad \left. \frac{1}{2h} \phi_{cc} \left(\frac{(\delta_{\mathcal{A}_{\mathcal{F}}}(x_k) \wedge \delta_{\mathcal{B}_{\mathcal{F}}}(x_k))}{\delta_{\mathcal{A}_{\mathcal{F}}}(x_k) \vee \delta_{\mathcal{B}_{\mathcal{F}}}(x_k)} + \frac{(\bar{\omega}_{\delta_{\mathcal{A}_{\mathcal{F}}}}(x_k) \wedge \bar{\omega}_{\delta_{\mathcal{B}_{\mathcal{F}}}}(x_k))}{1 + (\bar{\omega}_{\delta_{\mathcal{A}_{\mathcal{F}}}}(x_k) \vee \bar{\omega}_{\delta_{\mathcal{B}_{\mathcal{F}}}}(x_k))} \right) \right]^{(1/\lambda)},
\end{aligned} \tag{37}$$

where $\eta_{cc}, \phi_{cc} \in [0, 1]$ such that $\eta_{cc} + \phi_{cc} = 1$.

Definition 18. Let the weighting vector of the elements in CDHFS. Then, the nonexponential-based weighted generalized SMs between two CDHFSs A and B are given as

$$\begin{aligned}
S_{CDw}^3(\mathcal{A}, \mathcal{B}) &= \sum_{k=1}^n w_k \left[\frac{1 - \left(\frac{(1/g) \sum_{\mathcal{F}=1}^g |\gamma_{\mathcal{A}_{\mathcal{F}}}(x_k) - \gamma_{\mathcal{B}_{\mathcal{F}}}(x_k)|^\lambda \vee (1/g) \sum_{\mathcal{F}=1}^g |\omega_{\gamma_{\mathcal{A}_{\mathcal{F}}}}(x_k) - \omega_{\gamma_{\mathcal{B}_{\mathcal{F}}}}(x_k)|^\lambda}{\vee (1/h) \sum_{\mathcal{F}=1}^h |\delta_{\mathcal{A}_{\mathcal{F}}}(x_k) - \delta_{\mathcal{B}_{\mathcal{F}}}(x_k)|^\lambda \vee (1/h) \sum_{\mathcal{F}=1}^h |\bar{\omega}_{\delta_{\mathcal{A}_{\mathcal{F}}}}(x_k) - \bar{\omega}_{\delta_{\mathcal{B}_{\mathcal{F}}}}(x_k)|^\lambda} \right)}{1 + \left(\frac{(1/g) \sum_{\mathcal{F}=1}^g |\gamma_{\mathcal{A}_{\mathcal{F}}}(x_k) - \gamma_{\mathcal{B}_{\mathcal{F}}}(x_k)|^\lambda \vee (1/g) \sum_{\mathcal{F}=1}^g |\omega_{\gamma_{\mathcal{A}_{\mathcal{F}}}}(x_k) - \omega_{\gamma_{\mathcal{B}_{\mathcal{F}}}}(x_k)|^\lambda}{\vee (1/h) \sum_{\mathcal{F}=1}^h |\delta_{\mathcal{A}_{\mathcal{F}}}(x_k) - \delta_{\mathcal{B}_{\mathcal{F}}}(x_k)|^\lambda \vee (1/h) \sum_{\mathcal{F}=1}^h |\bar{\omega}_{\delta_{\mathcal{A}_{\mathcal{F}}}}(x_k) - \bar{\omega}_{\delta_{\mathcal{B}_{\mathcal{F}}}}(x_k)|^\lambda} \right)} \right]^{(1/\lambda)}, \\
S_{CDw}^4(\mathcal{A}, \mathcal{B}) &= \sum_{k=1}^n w_k \left[\frac{(1/g) \sum_{\mathcal{F}=1}^g |\gamma_{\mathcal{A}_{\mathcal{F}}}(x_k) \wedge \gamma_{\mathcal{B}_{\mathcal{F}}}(x_k)|^\lambda + (1/g) \sum_{\mathcal{F}=1}^g |\omega_{\gamma_{\mathcal{A}_{\mathcal{F}}}}(x_k) \wedge \omega_{\gamma_{\mathcal{B}_{\mathcal{F}}}}(x_k)|^\lambda}{(1/g) \sum_{\mathcal{F}=1}^g |\gamma_{\mathcal{A}_{\mathcal{F}}}(x_k) \vee \gamma_{\mathcal{B}_{\mathcal{F}}}(x_k)|^\lambda + (1/g) \sum_{\mathcal{F}=1}^g |\omega_{\gamma_{\mathcal{A}_{\mathcal{F}}}}(x_k) \vee \omega_{\gamma_{\mathcal{B}_{\mathcal{F}}}}(x_k)|^\lambda} \right. \\
&\quad \left. \frac{\vee (1/h) \sum_{\mathcal{F}=1}^h |\delta_{\mathcal{A}_{\mathcal{F}}}(x_k) \wedge \delta_{\mathcal{B}_{\mathcal{F}}}(x_k)|^\lambda + (1/h) \sum_{\mathcal{F}=1}^h |\bar{\omega}_{\delta_{\mathcal{A}_{\mathcal{F}}}}(x_k) \wedge \bar{\omega}_{\delta_{\mathcal{B}_{\mathcal{F}}}}(x_k)|^\lambda}{\vee (1/h) \sum_{\mathcal{F}=1}^h |\delta_{\mathcal{A}_{\mathcal{F}}}(x_k) \vee \delta_{\mathcal{B}_{\mathcal{F}}}(x_k)|^\lambda + (1/h) \sum_{\mathcal{F}=1}^h |\bar{\omega}_{\delta_{\mathcal{A}_{\mathcal{F}}}}(x_k) \vee \bar{\omega}_{\delta_{\mathcal{B}_{\mathcal{F}}}}(x_k)|^\lambda} \right]^{(1/\lambda)}, \\
S_{CDw}^5(\mathcal{A}, \mathcal{B}) &= \sum_{k=1}^n \left[\frac{1}{2g} \eta_{cc} \left(\frac{(\gamma_{\mathcal{A}_{\mathcal{F}}}(x_k) \wedge \gamma_{\mathcal{B}_{\mathcal{F}}}(x_k))}{\gamma_{\mathcal{A}_{\mathcal{F}}}(x_k) \vee \gamma_{\mathcal{B}_{\mathcal{F}}}(x_k)} + \frac{(\omega_{\gamma_{\mathcal{A}_{\mathcal{F}}}}(x_k) \wedge \omega_{\gamma_{\mathcal{B}_{\mathcal{F}}}}(x_k))}{1 + (\omega_{\gamma_{\mathcal{A}_{\mathcal{F}}}}(x_k) \vee \omega_{\gamma_{\mathcal{B}_{\mathcal{F}}}}(x_k))} \right) + \right. \\
&\quad \left. \frac{1}{2h} \phi_{cc} \left(\frac{(\delta_{\mathcal{A}_{\mathcal{F}}}(x_k) \wedge \delta_{\mathcal{B}_{\mathcal{F}}}(x_k))}{\delta_{\mathcal{A}_{\mathcal{F}}}(x_k) \vee \delta_{\mathcal{B}_{\mathcal{F}}}(x_k)} + \frac{(\bar{\omega}_{\delta_{\mathcal{A}_{\mathcal{F}}}}(x_k) \wedge \bar{\omega}_{\delta_{\mathcal{B}_{\mathcal{F}}}}(x_k))}{1 + (\bar{\omega}_{\delta_{\mathcal{A}_{\mathcal{F}}}}(x_k) \vee \bar{\omega}_{\delta_{\mathcal{B}_{\mathcal{F}}}}(x_k))} \right) \right]^{(1/\lambda)},
\end{aligned} \tag{38}$$

where $w = (w_1, w_2, \dots, w_n)^T$ represents the weight vector of each element x_k ($k = 1, 2, 3, \dots, n$) contained in CDHFS

and the weight vector satisfies $w_k \in [0, 1]$ for each $k = 1, 2, 3, \dots, n$, $\sum_{k=1}^n w_k = 1$. If we choose the weight

vector in the form of $w = ((1/n), (1/n), \dots, (1/n))^T$, then the nonexponential-based weighted generalized SMs will transform into nonexponential-based generalized SMs. Otherwise speaking, when $w_k = (1/n)$, $k = 1, 2, 3, \dots, n$, the $S_{CDw}^q(\mathcal{A}, \mathcal{B}) = S_{CD}^q(\mathcal{A}, \mathcal{B})$ $q = 3, 4, 5$.

Remark 4. If $\lambda = 1$, then nonexponential-based weighted generalized SMs between $\mathcal{A}$ and $\mathcal{B}$ becomes

$$\begin{aligned}
 S_{CDw}^3(\mathcal{A}, \mathcal{B}) &= \sum_{k=1}^n w_k \left[\frac{1 - \left(\frac{(1/g) \sum_{j=1}^g |\gamma_{\mathcal{A}_j}(x_k) - \gamma_{\mathcal{B}_j}(x_k)|^\lambda \vee (1/g) \sum_{j=1}^g |\omega_{\gamma_{\mathcal{A}_j}}(x_k) - \omega_{\gamma_{\mathcal{B}_j}}(x_k)|^\lambda}{\vee (1/h) \sum_{j=1}^h |\delta_{\mathcal{A}_j}(x_k) - \delta_{\mathcal{B}_j}(x_k)|^\lambda \vee (1/h) \sum_{j=1}^h |\bar{\omega}_{\delta_{\mathcal{A}_j}}(x_k) - \bar{\omega}_{\delta_{\mathcal{B}_j}}(x_k)|^\lambda} \right)}{1 + \left(\frac{(1/g) \sum_{j=1}^g |\gamma_{\mathcal{A}_j}(x_k) - \gamma_{\mathcal{B}_j}(x_k)|^\lambda \vee (1/g) \sum_{j=1}^g |\omega_{\gamma_{\mathcal{A}_j}}(x_k) - \omega_{\gamma_{\mathcal{B}_j}}(x_k)|^\lambda}{\vee (1/h) \sum_{j=1}^h |\delta_{\mathcal{A}_j}(x_k) - \delta_{\mathcal{B}_j}(x_k)|^\lambda \vee (1/h) \sum_{j=1}^h |\bar{\omega}_{\delta_{\mathcal{A}_j}}(x_k) - \bar{\omega}_{\delta_{\mathcal{B}_j}}(x_k)|^\lambda} \right)} \right], \\
 S_{CDw}^4(\mathcal{A}, \mathcal{B}) &= \sum_{k=1}^n w_k \frac{\vee (1/h) \sum_{j=1}^h |\delta_{\mathcal{A}_j}(x_k) \wedge \delta_{\mathcal{B}_j}(x_k)|^\lambda + (1/h) \sum_{j=1}^h |\bar{\omega}_{\delta_{\mathcal{A}_j}}(x_k) \wedge \bar{\omega}_{\delta_{\mathcal{B}_j}}(x_k)|^\lambda}{(1/g) \sum_{j=1}^g |\gamma_{\mathcal{A}_j}(x_k) \vee \gamma_{\mathcal{B}_j}(x_k)|^\lambda + (1/g) \sum_{j=1}^g |\omega_{\gamma_{\mathcal{A}_j}}(x_k) \vee \omega_{\gamma_{\mathcal{B}_j}}(x_k)|^\lambda}, \\
 S_{CDw}^5(\mathcal{A}, \mathcal{B}) &= \sum_{k=1}^n w_k \left[\frac{\frac{1}{2g} \sum_{j=1}^g \eta_{cc} \left(\frac{(\gamma_{\mathcal{A}_j}(x_k) \wedge \gamma_{\mathcal{B}_j}(x_k))}{(\gamma_{\mathcal{A}_j}(x_k) \vee \gamma_{\mathcal{B}_j}(x_k))} + \frac{(\omega_{\gamma_{\mathcal{A}_j}}(x_k) \wedge \omega_{\gamma_{\mathcal{B}_j}}(x_k))}{1 + (\omega_{\gamma_{\mathcal{A}_j}}(x_k) \vee \omega_{\gamma_{\mathcal{B}_j}}(x_k))} \right) + \frac{1}{2h} \sum_{j=1}^h \phi_{cc} \left(\frac{(\delta_{\mathcal{A}_j}(x_k) \wedge \delta_{\mathcal{B}_j}(x_k))}{(\delta_{\mathcal{A}_j}(x_k) \vee \delta_{\mathcal{B}_j}(x_k))} + \frac{(\bar{\omega}_{\delta_{\mathcal{A}_j}}(x_k) \wedge \bar{\omega}_{\delta_{\mathcal{B}_j}}(x_k))}{1 + (\bar{\omega}_{\delta_{\mathcal{A}_j}}(x_k) \vee \bar{\omega}_{\delta_{\mathcal{B}_j}}(x_k))} \right)}{\left(\frac{(1/g) \sum_{j=1}^g |\gamma_{\mathcal{A}_j}(x_k) - \gamma_{\mathcal{B}_j}(x_k)|^\lambda \vee (1/g) \sum_{j=1}^g |\omega_{\gamma_{\mathcal{A}_j}}(x_k) - \omega_{\gamma_{\mathcal{B}_j}}(x_k)|^\lambda}{\vee (1/h) \sum_{j=1}^h |\delta_{\mathcal{A}_j}(x_k) - \delta_{\mathcal{B}_j}(x_k)|^\lambda \vee (1/h) \sum_{j=1}^h |\bar{\omega}_{\delta_{\mathcal{A}_j}}(x_k) - \bar{\omega}_{\delta_{\mathcal{B}_j}}(x_k)|^\lambda} \right)} \right].
 \end{aligned} \tag{39}$$

5. Applications

In this part of the article, we describe two applications, i.e., pattern recognition and medical diagnosis, to show the effectiveness and usefulness of the interpreted SMS. We applied the interpreted SMs to the environment of CDHFSs in pattern recognition and medical diagnosis.

5.1. Pattern Recognition. The instruments of similarity measures have applications in pattern classification. By using the known and unknown alternatives, we examine the similarity measures based on the investigated measures to examine the reliability and validity of the explored approaches. During this section, the similarity measures grew thus far in Section 4 are applied to a pattern recognition (building pattern recognition) downside wherever the category of an unknown

building substance has been assessed. The outcomes acquired utilizing the SMs of CDHFSs are then investigated for the depiction of the upsides of proposed SMs and the confinements of existing work. To clarify the marvel, the following example is discussed.

Example 3. Its clear that quantity of constructions by a company is directly proportional to the standard of building substances they use. Proper inspection of the building substance before construction is proof of good engineering standards. The construction substances to be used should be strictly checked before applying. The proper check and balance system of inspection authorizes the builders to use the right substances for constructions to improve the standard of their project. Let four known building substances $\mathcal{A}_j$ ($j = 1, 2, 3, 4$), which are given in the shape of CDHFSs, be as follows:

$$\begin{aligned}
\mathcal{A}_1 &= \left[\left(x_1, \left\{ \left\{ \begin{matrix} 0.1e^{i2\pi(0.2)}, 0.2e^{i2\pi(0.3)} \\ 0.4e^{i2\pi(0.5)} \end{matrix} \right\}, \left\{ \begin{matrix} 0.4e^{i2\pi(0.3)} \\ 0.3e^{i2\pi(0.2)} \end{matrix} \right\} \right\} \right), \left(x_2, \left\{ \left\{ 0.4e^{i2\pi(0.3)} \right\}, \left\{ 0.5e^{i2\pi(0.6)}, 0.2e^{i2\pi(0.3)} \right\} \right\} \right), \right. \\
&\quad \left. \left(x_3, \left\{ \left\{ 0.1e^{i2\pi(0.6)} \right\}, \left\{ 0.4e^{i2\pi(0.2)} \right\} \right\} \right), \left(x_4, \left\{ \left\{ 0.2e^{i2\pi(0.05)}, 0.65e^{i2\pi(0.75)} \right\}, \left\{ 0.15e^{i2\pi(0.05)} \right\} \right\} \right) \right], \\
\mathcal{A}_2 &= \left[\left(x_1, \left\{ \left\{ 0.15e^{i2\pi(0.25)} \right\}, \left\{ \begin{matrix} 0.75e^{i2\pi(0.6)} \\ 0.4e^{i2\pi(0.3)} \end{matrix} \right\} \right\} \right), \left(x_2, \left\{ \left\{ \begin{matrix} 0.4e^{i2\pi(0.35)}, 0.55e^{i2\pi(0.45)} \\ 0.3e^{i2\pi(0.6)} \end{matrix} \right\}, \left\{ \begin{matrix} 0.35e^{i2\pi(0.25)} \\ 0.2e^{i2\pi(0.3)} \end{matrix} \right\} \right\} \right), \right. \\
&\quad \left. \left(x_3, \left\{ \left\{ 0.4e^{i2\pi(0.5)} \right\}, \left\{ 0.4e^{i2\pi(0.35)} \right\} \right\} \right), \left(x_4, \left\{ \left\{ \begin{matrix} 0.33e^{i2\pi(0.6)} \\ 0.45e^{i2\pi(0.5)} \end{matrix} \right\}, \left\{ \begin{matrix} 0.5e^{i2\pi(0.2)} \\ 0.3e^{i2\pi(0.3)} \end{matrix} \right\} \right\} \right) \right], \\
\mathcal{A}_3 &= \left[\left(x_1, \left\{ \left\{ 0.7e^{i2\pi(0.5)}, 0.4e^{i2\pi(0.8)} \right\}, \left\{ 0.2e^{i2\pi(0.1)} \right\} \right\} \right), \left(x_2, \left\{ \left\{ 0.5e^{i2\pi(0.6)} \right\}, \left\{ 0.4e^{i2\pi(0.3)} \right\} \right\} \right), \\
&\quad \left(x_3, \left\{ \left\{ \begin{matrix} 0.3e^{i2\pi(0.4)}, 0.5e^{i2\pi(0.6)} \\ 0.4e^{i2\pi(0.2)} \end{matrix} \right\}, \left\{ \begin{matrix} 0.4e^{i2\pi(0.3)} \\ 0.3e^{i2\pi(0.2)} \end{matrix} \right\} \right\} \right), \left(x_4, \left\{ \left\{ 0.5e^{i2\pi(0.3)}, 0.7e^{i2\pi(0.6)} \right\}, \left\{ \begin{matrix} 0.1e^{i2\pi(0.1)} \\ 0.2e^{i2\pi(0.3)} \end{matrix} \right\} \right\} \right) \right], \\
\mathcal{A}_4 &= \left[\left(x_1, \left\{ \left\{ 0.2e^{i2\pi(0.3)}, 0.45e^{i2\pi(0.2)} \right\}, \left\{ 0.3e^{i2\pi(0.4)}, 0.1e^{i2\pi(0.5)} \right\} \right\} \right), \left(x_2, \left\{ \left\{ \begin{matrix} 0.7e^{i2\pi(0.45)} \\ 0.6e^{i2\pi(0.6)} \end{matrix} \right\}, \left\{ 0.2e^{i2\pi(0.4)} \right\} \right\} \right), \right. \\
&\quad \left. \left(x_3, \left\{ \left\{ 0.4e^{i2\pi(0.2)} \right\}, \left\{ 0.3e^{i2\pi(0.7)} \right\} \right\} \right), \left(x_4, \left\{ \left\{ 0.5e^{i2\pi(0.7)} \right\}, \left\{ 0.3e^{i2\pi(0.25)}, 0.2e^{i2\pi(0.15)} \right\} \right\} \right) \right].
\end{aligned} \tag{40}$$

Now, let an unknown building substance, which needs to be identified, be

$$\mathcal{A} = \left[\left(x_1, \left\{ \left\{ 0.8e^{i2\pi(0.6)}, 0.5e^{i2\pi(0.4)} \right\}, \left\{ 0.1e^{i2\pi(0.1)} \right\} \right\} \right), \left(x_2, \left\{ \left\{ 0.6e^{i2\pi(0.7)} \right\}, \left\{ 0.3e^{i2\pi(0.2)} \right\} \right\} \right), \right. \\
\left. \left(x_3, \left\{ \left\{ \begin{matrix} 0.5e^{i2\pi(0.6)}, 0.4e^{i2\pi(0.5)} \\ 0.7e^{i2\pi(0.4)} \end{matrix} \right\}, \left\{ \begin{matrix} 0.1e^{i2\pi(0.3)} \\ 0.2e^{i2\pi(0.1)} \end{matrix} \right\} \right\} \right), \left(x_4, \left\{ \left\{ 0.6e^{i2\pi(0.5)}, 0.3e^{i2\pi(0.4)} \right\}, \left\{ \begin{matrix} 0.3e^{i2\pi(0.2)} \\ 0.2e^{i2\pi(0.4)} \end{matrix} \right\} \right\} \right) \right].
\end{aligned} \tag{41}$$

The intention of this matter is to sort the unknown building substance $\mathcal{A}$ in one of the classifications $\mathcal{A}_{\mathcal{F}}$ ($\mathcal{F} = 1, 2, 3, 4$). For this, the interpreted antilogarithmic and nonexponential SMs are utilized to ascertain the similarity from $\mathcal{A}$ to $\mathcal{A}_{\mathcal{F}}$ ($\mathcal{F} = 1, 2, 3, 4$) and estimations are provided in Tables 1 and 2.

According to the above-computed estimations supplied in Table 1, we essentially mention that the similarity level between $\mathcal{A}$ and $\mathcal{A}_3$ is the best one as an infusion by all SMs. This defines that all interpreted SMs dole out the unknown building substance $\mathcal{A}$ to the known building substance $\mathcal{A}_3$, dependent on the principle of the greatest level of similarity. The positioning of the investigated antilogarithmic and nonexponential SMs between $\mathcal{A}$ and $\mathcal{A}_{\mathcal{F}}$ ($\mathcal{F} = 1, 2, 3, 4$) is likewise provided in Table 1. The graphical portrayal of the interpreted SMs between $\mathcal{A}$ and $\mathcal{A}_{\mathcal{F}}$ ($\mathcal{F} = 1, 2, 3, 4$) is demonstrated in Figure 1.

The heaviness of components has extraordinary noteworthy to consider genuine dynamic issues. By using the four attributes whose expressions are in the form of x_k ($k = 1, 2, 3, 4$) be $w_k(0.3, 0.2, 0.1, 0.4)$, respectively. At this point, the deciphered WSMs are utilized to decide the

similarity from $\mathcal{A}$ and $\mathcal{A}_{\mathcal{F}}$ ($\mathcal{F} = 1, 2, 3, 4$) and estimations are provided in Table 2.

According to the above-computed estimations supplied in Table 2, we essentially mention that the similarity level between $\mathcal{A}$ and $\mathcal{A}_3$ is the best one as an infusion by all WSMs. This defines that all interpreted WSMs dole out the unknown building substance $\mathcal{A}$ to the known building substance $\mathcal{A}_3$, dependent on the principle of the greatest level of similarity. The positioning of the investigated antilogarithmic and nonexponential SMs between $\mathcal{A}$ and $\mathcal{A}_{\mathcal{F}}$ ($\mathcal{F} = 1, 2, 3, 4$) is likewise provided in Table 2. The graphical portrayal of the interpreted SMs between $\mathcal{A}$ and $\mathcal{A}_{\mathcal{F}}$ ($\mathcal{F} = 1, 2, 3, 4$) is demonstrated in Figure 2.

5.2. Medical Diagnosis. Distinct diseases have various signs. The medical diagnosis trust in the casualty's signs to inspect what sort of disease the casualty has. The casualty's signs are a set of signs and unknown diseases will be a set of diagnostic diseases. The deciphered SMs are outlined by an after numerical example of medical diagnosis.

Example 4. Let a set of diagnoses be $\mathcal{D} = \{\mathcal{D}_1$ (Hepatitis A), $\mathcal{D}_2$ (Heart problem), $\mathcal{D}_3$ (Malaria), $\mathcal{D}_4$ (Flu)} and a set of signs be $X = \{x_1$ (Abdominal pain),

x_2 (Heart pain), x_3 (Fever), x_4 (Cough)}. The casualty's signs can be given in the shape of CDHFSs as follows:

$$\mathcal{P}(\text{Casualty}) = \left\{ \left(x_1, \left\{ \left\{ \begin{matrix} 0.6e^{i2\pi(0.8)} \\ 0.5e^{i2\pi(0.55)} \\ 0.8e^{i2\pi(0.45)} \end{matrix} \right\}, \left\{ \begin{matrix} 0.15e^{i2\pi(0.1)} \\ 0.1e^{i2\pi(0.15)} \\ 0.2e^{i2\pi(0.1)} \end{matrix} \right\} \right\} \right), \left(x_2, \left\{ \left\{ 0.2e^{i2\pi(0.3)} \right\} \left\{ \begin{matrix} 0.7e^{i2\pi(0.65)} \\ 0.3e^{i2\pi(0.5)} \end{matrix} \right\} \right\} \right), \right. \\ \left. \left(x_3, \left\{ \left\{ \begin{matrix} 0.65e^{i2\pi(0.35)} \\ 0.5e^{i2\pi(0.65)} \end{matrix} \right\}, \left\{ \begin{matrix} 0.25e^{i2\pi(0.15)} \\ 0.3e^{i2\pi(0.3)} \end{matrix} \right\} \right\} \right), \left(x_4, \left\{ \left\{ 0.55e^{i2\pi(0.45)} \right\}, \left\{ 0.45e^{i2\pi(0.25)} \right\} \right\} \right) \right\}. \quad (42)$$

The signs of each disease $\mathcal{D}_{\mathcal{F}}$ ($\mathcal{F} = 1, 2, 3, 4$) can be represented in CDHFSs as follows:

$$\begin{aligned} \mathcal{D}_1(\text{Hepatitis A}) &= \left\{ \left(x_1, \left\{ \left\{ \begin{matrix} 0.55e^{i2\pi(0.7)} \\ 0.6e^{i2\pi(0.65)} \\ 0.5e^{i2\pi(0.7)} \end{matrix} \right\}, \left\{ \begin{matrix} 0.1e^{i2\pi(0.25)} \\ 0.15e^{i2\pi(0.2)} \\ 0.35e^{i2\pi(0.3)} \end{matrix} \right\} \right\} \right), \left(x_2, \left\{ \left\{ 0.4e^{i2\pi(0.3)} \right\} \left\{ \begin{matrix} 0.5e^{i2\pi(0.45)} \\ 0.3e^{i2\pi(0.6)} \end{matrix} \right\} \right\} \right), \right. \\ &\quad \left. \left(x_3, \left\{ \left\{ \begin{matrix} 0.45e^{i2\pi(0.75)} \\ 0.6e^{i2\pi(0.55)} \end{matrix} \right\}, \left\{ \begin{matrix} 0.25e^{i2\pi(0.2)} \\ 0.3e^{i2\pi(0.15)} \end{matrix} \right\} \right\} \right), \left(x_4, \left\{ \left\{ 0.4e^{i2\pi(0.5)} \right\}, \left\{ 0.4e^{i2\pi(0.3)} \right\} \right\} \right) \right\}, \\ \mathcal{D}_2(\text{Heart problem}) &= \left\{ \left(x_1, \left\{ \left\{ 0.1e^{i2\pi(0.5)} \right\}, \left\{ \begin{matrix} 0.3e^{i2\pi(0.5)} \\ 0.2e^{i2\pi(0.1)} \end{matrix} \right\} \right\} \right), \left(x_2, \left\{ \left\{ \begin{matrix} 0.7e^{i2\pi(0.65)} \\ 0.6e^{i2\pi(0.6)} \\ 0.45e^{i2\pi(0.55)} \end{matrix} \right\}, \left\{ \begin{matrix} 0.2e^{i2\pi(0.15)} \\ 0.25e^{i2\pi(0.1)} \\ 0.25e^{i2\pi(0.3)} \end{matrix} \right\} \right\} \right), \right. \\ &\quad \left. \left(x_3, \left\{ \left\{ 0.15e^{i2\pi(0.05)} \right\}, \left\{ 0.3e^{i2\pi(0.15)} \right\} \right\} \right), \left(x_4, \left\{ \left\{ 0.05e^{i2\pi(0.35)} \right\}, \left\{ 0.25e^{i2\pi(0.25)} \right\} \right\} \right) \right\}, \\ \mathcal{D}_3(\text{Malaria}) &= \left\{ \left(x_1, \left\{ \left\{ 0.3e^{i2\pi(0.45)} \right\}, \left\{ 0.5e^{i2\pi(0.35)} \right\} \right\} \right), \left(x_2, \left\{ \left\{ 0.15e^{i2\pi(0.05)} \right\}, \left\{ 0.2e^{i2\pi(0.55)} \right\} \right\} \right), \right. \\ &\quad \left. \left(x_3, \left\{ \left\{ \begin{matrix} 0.5e^{i2\pi(0.6)} \\ 0.6e^{i2\pi(0.8)} \\ 0.45e^{i2\pi(0.7)} \end{matrix} \right\}, \left\{ \begin{matrix} 0.35e^{i2\pi(0.1)} \\ 0.2e^{i2\pi(0.15)} \end{matrix} \right\} \right\} \right), \left(x_4, \left\{ \left\{ \begin{matrix} 0.5e^{i2\pi(0.4)} \\ 0.6e^{i2\pi(0.7)} \end{matrix} \right\}, \left\{ \begin{matrix} 0.1e^{i2\pi(0.2)} \\ 0.3e^{i2\pi(0.25)} \end{matrix} \right\} \right\} \right) \right\}, \\ \mathcal{D}_4(\text{Flu}) &= \left\{ \left(x_1, \left\{ \left\{ 0.3e^{i2\pi(0.4)} \right\} \left\{ 0.35e^{i2\pi(0.2)} \right\} \right\} \right), \left(x_2, \left\{ \left\{ 0.4e^{i2\pi(0.2)} \right\} \left\{ \begin{matrix} 0.5e^{i2\pi(0.45)} \\ 0.45e^{i2\pi(0.6)} \end{matrix} \right\} \right\} \right), \right. \\ &\quad \left. \left(x_3, \left\{ \left\{ \begin{matrix} 0.7e^{i2\pi(0.5)} \\ 0.6e^{i2\pi(0.4)} \end{matrix} \right\}, \left\{ \begin{matrix} 0.2e^{i2\pi(0.4)} \\ 0.1e^{i2\pi(0.3)} \end{matrix} \right\} \right\} \right), \left(x_4, \left\{ \left\{ \begin{matrix} 0.4e^{i2\pi(0.7)} \\ 0.6e^{i2\pi(0.65)} \\ 0.75e^{i2\pi(0.55)} \end{matrix} \right\}, \left\{ \begin{matrix} 0.2e^{i2\pi(0.25)} \\ 0.15e^{i2\pi(0.1)} \\ 0.25e^{i2\pi(0.2)} \end{matrix} \right\} \right\} \right) \right\}. \end{aligned} \quad (43)$$

The intention of this matter is to sort the disease of the casualty $\mathcal{P}$ in one of the diseases, $\mathcal{D}_{\mathcal{F}}$ ($\mathcal{F} = 1, 2, 3, 4$). For this, the interpreted antilogarithmic and nonexponential SMs are utilized to ascertain the similarity from $\mathcal{P}$ to $\mathcal{D}_{\mathcal{F}}$ ($\mathcal{F} = 1, 2, 3, 4$), and estimations are provided in Tables 3 and 4.

According to the above-computed estimations supplied in Table 2, we essentially mention that the similarity level

between $\mathcal{P}$ and $\mathcal{D}_1$ is the best one as an infusion by all SMs. This defines that all interpreted SMs dole out that the casualty $\mathcal{P}$ has hepatitis A, dependent on the principle of the greatest level of similarity. The positioning of the investigated antilogarithmic and nonexponential SMs between $\mathcal{P}$ and $\mathcal{D}_{\mathcal{F}}$ ($\mathcal{F} = 1, 2, 3, 4$) is likewise provided in Table 3. The graphical portrayal of the interpreted SMs between $\mathcal{P}$ and $\mathcal{D}_{\mathcal{F}}$ ($\mathcal{F} = 1, 2, 3, 4$) is demonstrated in Figure 3.

TABLE 1: Calculation of explored SMs based on $\lambda = 1$ and $(\eta_{cc}, \phi_{cc}) = (0.5, 0.5)$.

Similarity measures	$\mathcal{A}_1$	$\mathcal{A}_2$	$\mathcal{A}_3$	$\mathcal{A}_4$	Ranking
$S_{CD}^1(\mathcal{A}, \mathcal{A}_{\mathcal{F}})$	0.4732	0.3968	0.7427	0.5277	$\mathcal{A}_3 \geq \mathcal{A}_4 \geq \mathcal{A}_1 \geq \mathcal{A}_2$
$S_{CD}^2(\mathcal{A}, \mathcal{A}_{\mathcal{F}})$	0.5228	0.649	0.8251	0.6735	$\mathcal{A}_3 \geq \mathcal{A}_4 \geq \mathcal{A}_2 \geq \mathcal{A}_1$
$S_{CD}^3(\mathcal{A}, \mathcal{A}_{\mathcal{F}})$	0.3886	0.4287	0.6712	0.4415	$\mathcal{A}_3 \geq \mathcal{A}_4 \geq \mathcal{A}_2 \geq \mathcal{A}_1$
$S_{CD}^4(\mathcal{A}, \mathcal{A}_{\mathcal{F}})$	0.3105	0.3715	0.6983	0.4156	$\mathcal{A}_3 \geq \mathcal{A}_4 \geq \mathcal{A}_2 \geq \mathcal{A}_1$
$S_{CD}^5(\mathcal{A}, \mathcal{A}_{\mathcal{F}})$	0.1288	0.1241	0.306	0.185	$\mathcal{A}_3 \geq \mathcal{A}_4 \geq \mathcal{A}_2 \geq \mathcal{A}_1$

TABLE 2: Calculation of explored WSMs based on $\lambda = 1$ and $(\eta_{cc}, \phi_{cc}) = (0.5, 0.5)$.

Similarity measures	$\mathcal{A}_1$	$\mathcal{A}_2$	$\mathcal{A}_3$	$\mathcal{A}_4$	Ranking
$S_{CDw}^1(\mathcal{A}, \mathcal{A}_{\mathcal{F}})$	0.485	0.4907	0.638	0.5016	$\mathcal{A}_3 \geq \mathcal{A}_4 \geq \mathcal{A}_2 \geq \mathcal{A}_1$
$S_{CDw}^2(\mathcal{A}, \mathcal{A}_{\mathcal{F}})$	0.5175	0.6979	0.7361	0.6346	$\mathcal{A}_3 \geq \mathcal{A}_2 \geq \mathcal{A}_4 \geq \mathcal{A}_1$
$S_{CDw}^3(\mathcal{A}, \mathcal{A}_{\mathcal{F}})$	0.3995	0.4213	0.5685	0.4238	$\mathcal{A}_3 \geq \mathcal{A}_4 \geq \mathcal{A}_2 \geq \mathcal{A}_1$
$S_{CDw}^4(\mathcal{A}, \mathcal{A}_{\mathcal{F}})$	0.3093	0.39	0.6133	0.4174	$\mathcal{A}_3 \geq \mathcal{A}_4 \geq \mathcal{A}_2 \geq \mathcal{A}_1$
$S_{CDw}^5(\mathcal{A}, \mathcal{A}_{\mathcal{F}})$	0.1257	0.1329	0.2499	0.1755	$\mathcal{A}_3 \geq \mathcal{A}_4 \geq \mathcal{A}_2 \geq \mathcal{A}_1$

FIGURE 1: The graphical portrayal of the interpreted SMs between $\mathcal{A}$ and $\mathcal{A}_{\mathcal{F}}$ ($\mathcal{F} = 1, 2, 3, 4$).

The heaviness of components has extraordinary noteworthy to consider genuine dynamic issues. By using the four attributes whose expressions are in the form of x_k ($k = 1, 2, 3, 4$) be w_k (0.3, 0.2, 0.1, 0.4), respectively. At this point, the deciphered WSMs are utilized to decide the similarity from $\mathcal{P}$ and $\mathcal{D}_{\mathcal{F}}$ ($\mathcal{F} = 1, 2, 3, 4$) and estimations are provided in Table 4.

According to the above-computed estimations supplied in Table 4, we essentially mention that the similarity level between $\mathcal{P}$ and $\mathcal{D}_3$ is the best one as an infusion by all WSMs. This defines that all interpreted WSMs dole out that the casualty $\mathcal{P}$ has hepatitis A, dependent on the principle of the greatest level of similarity. The positioning of the investigated antilogarithmic and nonexponential SMs between $\mathcal{P}$ and $\mathcal{D}_{\mathcal{F}}$ ($\mathcal{F} = 1, 2, 3, 4$) is likewise provided in Table 4. The graphical portrayal of the interpreted SMs between $\mathcal{P}$ and $\mathcal{D}_{\mathcal{F}}$ ($\mathcal{F} = 1, 2, 3, 4$) is demonstrated in Figure 4.

FIGURE 2: The graphical portrayal of the interpreted WSMs between $\mathcal{A}$ and $\mathcal{A}_{\mathcal{F}}$ ($\mathcal{F} = 1, 2, 3, 4$).

6. Comparison

In this section, we demonstrated the viability and points of interest of the deciphered SMs by contrasting some current SMs.

Example 5. Its clear that quantity of constructions by a company is directly proportional to the standard of building substances they use. Proper inspection of the building substance before construction is proof of good engineering standards. The construction substances to be used should be strictly checked before applying. The proper check and balance system of inspection authorizes the builders to use the right substances for constructions to improve the standard of their project. Let four known building substances $\mathcal{A}_{\mathcal{F}}$ ($\mathcal{F} = 1, 2, 3, 4$), which are given in the shape of DHFSs, be as follows:

TABLE 3: Calculation of explored SMs based on $\lambda = 1$ and $(\eta_{cc}, \phi_{cc}) = (0.5, 0.5)$.

Similarity measures	$\mathcal{D}_1$	$\mathcal{D}_2$	$\mathcal{D}_3$	$\mathcal{D}_4$	Ranking
$S_{CD}^1(\mathcal{P}, \mathcal{D}_{\mathcal{F}})$	0.757	0.287	0.4977	0.5695	$\mathcal{D}_1 \geq \mathcal{D}_4 \geq \mathcal{D}_3 \geq \mathcal{D}_2$
$S_{CD}^2(\mathcal{P}, \mathcal{D}_{\mathcal{F}})$	0.8528	0.6024	0.6482	0.7046	$\mathcal{D}_1 \geq \mathcal{D}_4 \geq \mathcal{D}_3 \geq \mathcal{D}_2$
$S_{CD}^3(\mathcal{P}, \mathcal{D}_{\mathcal{F}})$	0.6862	0.3088	0.4127	0.4927	$\mathcal{D}_1 \geq \mathcal{D}_4 \geq \mathcal{D}_3 \geq \mathcal{D}_2$
$S_{CD}^4(\mathcal{P}, \mathcal{D}_{\mathcal{F}})$	0.7593	0.2782	0.343	0.3488	$\mathcal{D}_1 \geq \mathcal{D}_4 \geq \mathcal{D}_3 \geq \mathcal{D}_2$
$S_{CD}^5(\mathcal{P}, \mathcal{D}_{\mathcal{F}})$	0.333	0.118	0.1068	0.1681	$\mathcal{D}_1 \geq \mathcal{D}_4 \geq \mathcal{D}_2 \geq \mathcal{D}_3$

TABLE 4: Calculation of explored WSMs based on $\lambda = 1$ and $(\eta_{cc}, \phi_{cc}) = (0.5, 0.5)$.

Similarity measures	$\mathcal{D}_1$	$\mathcal{D}_2$	$\mathcal{D}_3$	$\mathcal{D}_4$	Ranking
$S_{CDw}^1(\mathcal{P}, \mathcal{D}_{\mathcal{F}})$	0.7781	0.3427	0.4381	0.4283	$\mathcal{D}_1 \geq \mathcal{D}_3 \geq \mathcal{D}_4 \geq \mathcal{D}_2$
$S_{CDw}^2(\mathcal{P}, \mathcal{D}_{\mathcal{F}})$	0.8615	0.5789	0.5644	0.581	$\mathcal{D}_1 \geq \mathcal{D}_4 \geq \mathcal{D}_2 \geq \mathcal{D}_3$
$S_{CDw}^3(\mathcal{P}, \mathcal{D}_{\mathcal{F}})$	0.7107	0.2735	0.3624	0.3579	$\mathcal{D}_1 \geq \mathcal{D}_3 \geq \mathcal{D}_4 \geq \mathcal{D}_2$
$S_{CDw}^4(\mathcal{P}, \mathcal{D}_{\mathcal{F}})$	0.7732	0.3109	0.2778	0.2981	$\mathcal{D}_1 \geq \mathcal{D}_2 \geq \mathcal{D}_4 \geq \mathcal{D}_3$
$S_{CDw}^5(\mathcal{P}, \mathcal{D}_{\mathcal{F}})$	0.3602	0.1405	0.0855	0.0963	$\mathcal{D}_1 \geq \mathcal{D}_2 \geq \mathcal{D}_4 \geq \mathcal{D}_3$

$$\begin{aligned}
\mathcal{A}_1 &= \left\{ (x_1, \{\{0.1, 0.2, 0.4\}, \{0.4, 0.3\}\}), (x_2, \{\{0.4\}, \{0.5, 0.2\}\}), \right. \\
&\quad \left. (x_3, \{\{0.1\}, \{0.4\}\}), (x_4, \{\{0.2, 0.65\}, \{0.15\}\}) \right\}, \\
\mathcal{A}_2 &= \left\{ (x_1, \{\{0.15\}, \{0.75, 0.4\}\}), (x_2, \{\{0.4, 0.55, 0.3\}, \{0.35, 0.2\}\}), \right. \\
&\quad \left. (x_3, \{\{0.4\}, \{0.4\}\}), (x_4, \{\{0.33, 0.45\}, \{0.5, 0.3\}\}) \right\}, \\
\mathcal{A}_3 &= \left\{ (x_1, \{\{0.7, 0.4\}, \{0.2\}\}), (x_2, \{\{0.5\}, \{0.4\}\}), \right. \\
&\quad \left. (x_3, \{\{0.3, 0.5, 0.4\}, \{0.4, 0.3\}\}), (x_4, \{\{0.5, 0.7\}, \{0.1, 0.2\}\}) \right\}, \\
\mathcal{A}_4 &= \left\{ (x_1, \{\{0.2, 0.45\}, \{0.3, 0.1\}\}), (x_2, \{\{0.7, 0.6\}, \{0.2\}\}), \right. \\
&\quad \left. (x_3, \{\{0.4\}, \{0.3\}\}), (x_4, \{\{0.5\}, \{0.3, 0.2\}\}) \right\}.
\end{aligned} \tag{44}$$

Now, let an unknown building substance, which needs to be identified, be

$$\mathcal{A} = \left\{ (x_1, \{\{0.8, 0.5\}, \{0.1\}\}), (x_2, \{\{0.6\}, \{0.3\}\}), \right. \\
\left. (x_3, \{\{0.5, 0.4, 0.7\}, \{0.1, 0.2\}\}), (x_4, \{\{0.6, 0.3\}, \{0.3, 0.2\}\}) \right\}. \tag{45}$$

We transform the DHFSs in the shape of CDHFSs by taking $1 = e^0$ as follows:

$$\begin{aligned}
\mathcal{A}_1 &= \left\{ \left(x_1, \left\{ \left\{ \begin{matrix} 0.1e^{i2\pi(0.0)} & 0.2e^{i2\pi(0.0)} \\ & 0.4e^{i2\pi(0.0)} \end{matrix} \right\}, \left\{ \begin{matrix} 0.4e^{i2\pi(0.0)} \\ 0.3e^{i2\pi(0.0)} \end{matrix} \right\} \right\} \right), (x_2, \{\{0.4e^{i2\pi(0.0)}\}, \{0.5e^{i2\pi(0.0)}, 0.2e^{i2\pi(0.0)}\}\}), \right. \\
&\quad \left. (x_3, \{\{0.1e^{i2\pi(0.0)}\}, \{0.4e^{i2\pi(0.0)}\}\}), (x_4, \{\{0.2e^{i2\pi(0.0)}, 0.65e^{i2\pi(0.0)}\}, \{0.15e^{i2\pi(0.0)}\}\}) \right\}, \\
\mathcal{A}_2 &= \left\{ \left(x_1, \left\{ \{0.15\}, \left\{ \begin{matrix} 0.75e^{i2\pi(0.0)} \\ 0.4e^{i2\pi(0.0)} \end{matrix} \right\} \right\} \right), \left(x_2, \left\{ \left\{ \begin{matrix} 0.4e^{i2\pi(0.0)} & 0.55e^{i2\pi(0.0)} \\ & 0.3e^{i2\pi(0.0)} \end{matrix} \right\}, \left\{ \begin{matrix} 0.35e^{i2\pi(0.0)} \\ 0.2e^{i2\pi(0.0)} \end{matrix} \right\} \right\} \right), \right. \\
&\quad \left. (x_3, \{\{0.4e^{i2\pi(0.0)}\}, \{0.4e^{i2\pi(0.0)}\}\}), (x_4, \left\{ \left\{ \begin{matrix} 0.33e^{i2\pi(0.0)} \\ 0.45e^{i2\pi(0.0)} \end{matrix} \right\}, \left\{ \begin{matrix} 0.5e^{i2\pi(0.0)} \\ 0.3e^{i2\pi(0.0)} \end{matrix} \right\} \right\} \right) \right\},
\end{aligned}$$

FIGURE 3: The graphical portrayal of the interpreted SMs between $\mathcal{P}$ and $\mathcal{D}_J$ ($J = 1, 2, 3, 4$).

FIGURE 4: The graphical portrayal of the interpreted WSMs between $\mathcal{P}$ and $\mathcal{D}_J$ ($J = 1, 2, 3, 4$).

$$\begin{aligned}
 \mathcal{A}_3 &= \left\{ \left(x_1, \left\{ \left\{ 0.7e^{i2\pi(0.0)}, 0.4e^{i2\pi(0.0)} \right\}, \left\{ 0.2e^{i2\pi(0.0)} \right\} \right\} \right), \left(x_2, \left\{ \left\{ 0.5e^{i2\pi(0.0)} \right\}, \left\{ 0.4e^{i2\pi(0.0)} \right\} \right\} \right), \right. \\
 &\quad \left. \left(x_3, \left\{ \left\{ 0.3e^{i2\pi(0.0)}, 0.5e^{i2\pi(0.0)} \right\}, \left\{ 0.4e^{i2\pi(0.0)}, 0.4e^{i2\pi(0.0)} \right\} \right\} \right), \left(x_4, \left\{ \left\{ 0.5e^{i2\pi(0.0)}, 0.7e^{i2\pi(0.0)} \right\}, \left\{ 0.1, 0.2 \right\} \right\} \right) \right\}, \\
 \mathcal{A}_4 &= \left\{ \left(x_1, \left\{ \left\{ 0.2e^{i2\pi(0.0)}, 0.45e^{i2\pi(0.0)} \right\}, \left\{ 0.3e^{i2\pi(0.0)}, 0.1e^{i2\pi(0.0)} \right\} \right\} \right), \left(x_2, \left\{ \left\{ 0.7e^{i2\pi(0.0)}, 0.6e^{i2\pi(0.0)} \right\}, \left\{ 0.2e^{i2\pi(0.0)} \right\} \right\} \right), \right. \\
 &\quad \left. \left(x_3, \left\{ \left\{ 0.4e^{i2\pi(0.0)}, 0.3e^{i2\pi(0.0)} \right\}, \left\{ 0.5e^{i2\pi(0.0)}, 0.3e^{i2\pi(0.0)} \right\} \right\} \right), \left(x_4, \left\{ \left\{ 0.5e^{i2\pi(0.0)}, 0.3e^{i2\pi(0.0)} \right\}, \left\{ 0.2e^{i2\pi(0.0)} \right\} \right\} \right) \right\}, \\
 \mathcal{A} &= \left\{ \left(x_1, \left\{ \left\{ 0.8e^{i2\pi(0.0)}, 0.5e^{i2\pi(0.0)} \right\}, \left\{ 0.1e^{i2\pi(0.0)} \right\} \right\} \right), \left(x_2, \left\{ \left\{ 0.6e^{i2\pi(0.0)}, 0.3e^{i2\pi(0.0)} \right\} \right\} \right), \right. \\
 &\quad \left. \left(x_3, \left\{ \left\{ 0.5e^{i2\pi(0.0)}, 0.4e^{i2\pi(0.0)} \right\}, \left\{ 0.1e^{i2\pi(0.0)}, 0.7e^{i2\pi(0.0)} \right\} \right\} \right), \left(x_4, \left\{ \left\{ 0.6e^{i2\pi(0.0)}, 0.3e^{i2\pi(0.0)} \right\}, \left\{ 0.3e^{i2\pi(0.0)}, 0.2e^{i2\pi(0.0)} \right\} \right\} \right) \right\}.
 \end{aligned} \tag{46}$$

By using the information of Example 5, we examine the similarity measures between the family of unknown with the family of known to find the supremacy of the investigated approaches. The information in Example 5 presented in the shape of DHFSs. By some existing SMs for DHFSs, we detected the similarity between $\mathcal{A}$ and $\mathcal{A}_J$ ($J = 1, 2, 3, 4$) as appeared in Table 5. As we have $1 = e^0$, then the information given in Example 5 changed into CDHFSs. At this point through deciphered SMs, we detected the similarity between $\mathcal{A}$ and $\mathcal{A}_J$ ($J = 1, 2, 3, 4$) likewise appeared in Table 5. Three of our deciphered SMs ($S_{C\mathcal{D}}^1$, $S_{C\mathcal{D}}^4$, and $S_{C\mathcal{D}}^5$) demonstrated that unknown building substance $\mathcal{A}$ has a place with predefined building substance $\mathcal{A}_3$ on the grounds that the similarity among $\mathcal{A}$ and $\mathcal{A}_3$ is most prominent one and two of our deciphered SMs ($S_{C\mathcal{D}}^2$ and $S_{C\mathcal{D}}^3$) demonstrated that unknown building substance $\mathcal{A}$ has a place with predefined building substance $\mathcal{A}_4$ on the grounds that the similarity among $\mathcal{A}$ and $\mathcal{A}_4$ is most prominent one. Ranking of the deciphered and existing SMs is likewise introduced in Table 5. The graphical portrayal of the comparison of proposed and existing SMs is spoken in Figure 5.

Presently, we examine the comparison among deciphered and existing SMs for Example 3. For Example 3, the

information is given in the structure of CDHFSs. We realize that no SM exists in the writing to fathom this sort of information. The existing SMs are inadequate to discover the similarity among $\mathcal{A}$ and $\mathcal{A}_J$ ($J = 1, 2, 3, 4$) as appeared in Table 6. From Table 6, we see that the information presented in Example 3 is only feasible by the explored SMs. The deciphered SMs discover the similarity among $\mathcal{A}$ and $\mathcal{A}_J$ ($J = 1, 2, 3, 4$) as appeared in Table 6. Our deciphered SMs demonstrated that vague building material $\mathcal{A}$ has a place with the predetermined building material $\mathcal{A}_3$ on the grounds that the similarity among $\mathcal{A}$ and $\mathcal{A}_3$ is most noteworthy one. The positioning of the interpreted SMs is additionally introduced in Table 6. The geometrical expressions of the investigated measures are discussed in the form of Figure 6.

In this comparison, we contrasted our investigated SMs with some existing SMs characterized by Wang et al. [27]. From the above conversation, unmistakably, our proposed SMs can speak extra fuzzy information and put it comprehensively in circumstances in real-life issues. In the view of CDHFS, we explored the SMs; our SMs are progressively agreeable for genuine issues than the existing SMs and our SMs are broader than the existing SMs.

TABLE 5: Comparison between explored and existing SMs for Example 5 when $\lambda = 1$ and $(\eta_{cc}, \phi_{cc}) = (0.5, 0.5)$.

Method	Score value	Ranking
Wang et al. [27]	$s^1(\mathcal{A}, \mathcal{A}_1) = 0.7265, s^1(\mathcal{A}, \mathcal{A}_2) = 0.7506, s^1(\mathcal{A}, \mathcal{A}_3) = 0.8537, s^1(\mathcal{A}, \mathcal{A}_4) = 0.874$	$\mathcal{A}_4 \geq \mathcal{A}_3 \geq \mathcal{A}_2 \geq \mathcal{A}_1$
Wang et al. [27]	$s^2(\mathcal{A}, \mathcal{A}_1) = 0.525, s^2(\mathcal{A}, \mathcal{A}_2) = 0.62, s^2(\mathcal{A}, \mathcal{A}_3) = 0.775, s^2(\mathcal{A}, \mathcal{A}_4) = 0.7$	$\mathcal{A}_3 \geq \mathcal{A}_4 \geq \mathcal{A}_2 \geq \mathcal{A}_1$
Wang et al. [27]	$s^3(\mathcal{A}, \mathcal{A}_1) = 0.7261, s^3(\mathcal{A}, \mathcal{A}_2) = 0.6486, s^3(\mathcal{A}, \mathcal{A}_3) = 0.526, s^3(\mathcal{A}, \mathcal{A}_4) = 0.7031$	$\mathcal{A}_3 \geq \mathcal{A}_4 \geq \mathcal{A}_2 \geq \mathcal{A}_1$
Explored SM	$S_{C\mathcal{D}}^1(\mathcal{A}, \mathcal{A}_1) = 0.5362, S_{C\mathcal{D}}^1(\mathcal{A}, \mathcal{A}_2) = 0.3968, S_{C\mathcal{D}}^1(\mathcal{A}, \mathcal{A}_3) = 0.7887, S_{C\mathcal{D}}^1(\mathcal{A}, \mathcal{A}_4) = 0.6056$	$\mathcal{A}_3 \geq \mathcal{A}_4 \geq \mathcal{A}_2 \geq \mathcal{A}_1$
Explored SM	$S_{C\mathcal{D}}^2(\mathcal{A}, \mathcal{A}_1) = 0.7979, S_{C\mathcal{D}}^2(\mathcal{A}, \mathcal{A}_2) = 0.7972, S_{C\mathcal{D}}^2(\mathcal{A}, \mathcal{A}_3) = 0.8566, S_{C\mathcal{D}}^2(\mathcal{A}, \mathcal{A}_4) = 0.903$	$\mathcal{A}_4 \geq \mathcal{A}_3 \geq \mathcal{A}_1 \geq \mathcal{A}_2$
Explored SM	$S_{C\mathcal{D}}^3(\mathcal{A}, \mathcal{A}_1) = 0.4545, S_{C\mathcal{D}}^3(\mathcal{A}, \mathcal{A}_2) = 0.4582, S_{C\mathcal{D}}^3(\mathcal{A}, \mathcal{A}_3) = 0.5215, S_{C\mathcal{D}}^3(\mathcal{A}, \mathcal{A}_4) = 0.7257$	$\mathcal{A}_4 \geq \mathcal{A}_3 \geq \mathcal{A}_2 \geq \mathcal{A}_1$
Explored SM	$S_{C\mathcal{D}}^4(\mathcal{A}, \mathcal{A}_1) = 0.3035, S_{C\mathcal{D}}^4(\mathcal{A}, \mathcal{A}_2) = 0.3266, S_{C\mathcal{D}}^4(\mathcal{A}, \mathcal{A}_3) = 0.6849, S_{C\mathcal{D}}^4(\mathcal{A}, \mathcal{A}_4) = 0.479$	$\mathcal{A}_3 \geq \mathcal{A}_4 \geq \mathcal{A}_2 \geq \mathcal{A}_1$
Explored SM	$S_{C\mathcal{D}}^5(\mathcal{A}, \mathcal{A}_1) = 0.0916, S_{C\mathcal{D}}^5(\mathcal{A}, \mathcal{A}_2) = 0.0778, S_{C\mathcal{D}}^5(\mathcal{A}, \mathcal{A}_3) = 0.2215, S_{C\mathcal{D}}^5(\mathcal{A}, \mathcal{A}_4) = 0.1396$	$\mathcal{A}_3 \geq \mathcal{A}_4 \geq \mathcal{A}_1 \geq \mathcal{A}_2$

FIGURE 5: The graphical portrayal of the comparison of proposed and existing SMs for Example 5.

TABLE 6: Comparison between explored and existing SMs for Example 3 when $\lambda = 1$ and $(\eta_{cc}, \phi_{cc}) = (0.5, 0.5)$.

Method	Score value	Ranking
Wang et al. [27]	Failed	Failed
Wang et al. [27]	Failed	Failed
Wang et al. [27]	Failed	Failed
Explored SM	$S_{C\mathcal{D}}^1(\mathcal{A}, \mathcal{A}_1) = 0.4732, S_{C\mathcal{D}}^1(\mathcal{A}, \mathcal{A}_2) = 0.3968, S_{C\mathcal{D}}^1(\mathcal{A}, \mathcal{A}_3) = 0.5277, S_{C\mathcal{D}}^1(\mathcal{A}, \mathcal{A}_4) = 0.5277$	$\mathcal{A}_3 \geq \mathcal{A}_4 \geq \mathcal{A}_1 \geq \mathcal{A}_2$
Explored SM	$S_{C\mathcal{D}}^2(\mathcal{A}, \mathcal{A}_1) = 0.6228, S_{C\mathcal{D}}^2(\mathcal{A}, \mathcal{A}_2) = 0.649, S_{C\mathcal{D}}^2(\mathcal{A}, \mathcal{A}_3) = 0.8251, S_{C\mathcal{D}}^2(\mathcal{A}, \mathcal{A}_4) = 0.6735$	$\mathcal{A}_3 \geq \mathcal{A}_4 \geq \mathcal{A}_2 \geq \mathcal{A}_1$
Explored SM	$S_{C\mathcal{D}}^3(\mathcal{A}, \mathcal{A}_1) = 0.3885, S_{C\mathcal{D}}^3(\mathcal{A}, \mathcal{A}_2) = 0.429, S_{C\mathcal{D}}^3(\mathcal{A}, \mathcal{A}_3) = 0.6712, S_{C\mathcal{D}}^3(\mathcal{A}, \mathcal{A}_4) = 0.4415$	$\mathcal{A}_3 \geq \mathcal{A}_4 \geq \mathcal{A}_2 \geq \mathcal{A}_1$
Explored SM	$S_{C\mathcal{D}}^4(\mathcal{A}, \mathcal{A}_1) = 0.3105, S_{C\mathcal{D}}^4(\mathcal{A}, \mathcal{A}_2) = 0.3715, S_{C\mathcal{D}}^4(\mathcal{A}, \mathcal{A}_3) = 0.6983, S_{C\mathcal{D}}^4(\mathcal{A}, \mathcal{A}_4) = 0.4156$	$\mathcal{A}_3 \geq \mathcal{A}_4 \geq \mathcal{A}_2 \geq \mathcal{A}_1$
Explored SM	$S_{C\mathcal{D}}^5(\mathcal{A}, \mathcal{A}_1) = 0.1288, S_{C\mathcal{D}}^5(\mathcal{A}, \mathcal{A}_2) = 0.124, S_{C\mathcal{D}}^5(\mathcal{A}, \mathcal{A}_3) = 0.306, S_{C\mathcal{D}}^5(\mathcal{A}, \mathcal{A}_4) = 0.185$	$\mathcal{A}_3 \geq \mathcal{A}_4 \geq \mathcal{A}_2 \geq \mathcal{A}_1$

FIGURE 6: The graphical portrayal of the comparison of proposed and existing SMs for Example 3.

7. Conclusion

The aims of this article are to explore the novel approaches of complex dual hesitant fuzzy sets and their fundamental laws and also verify with the help of some numerical examples. The main purpose of this manuscript is discussed as follows:

- (1) The idea of complex dual hesitant fuzzy sets is found, which is the mixture of complex fuzzy sets and dual hesitant fuzzy sets to cope with complicated information in realistic issues.
- (2) The novel approach of complex interval-valued dual hesitant fuzzy set and its fundamental laws are also explored and also justified with the help of examples.
- (3) Some antilogarithmic and with-out exponential-based similarity measures and generalized similarity measures and their important characteristics are also explored.
- (4) These similarity measures are applied in the environment of pattern recognition and medical diagnosis to evaluate the proficiency and feasibility of the established measures.
- (5) To illustrate some numerical examples using the established measures to examine the reliability and validity of the proposed measures by comparing it with existing measures.
- (6) The advantages, comparative analysis, and graphical representation of the explored measures and existing measures are also discussed in detail.

In the future, there is a scope of extending the proposed approach to the different environments, such as complex Pythagorean fuzzy sets, linguistic features, decision making [28–34], its application to the various fields of the pattern recognition, and decision theory.

Data Availability

The data used in this article are artificial and hypothetical and anyone can use these data before prior permission by just citing this article.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

Acknowledgments

This paper was supported by “Algebra and Applications Research Unit, Faculty of Science, Prince of Songkla University.”

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