Research Article

Set-Valued SU-Type Fixed Point Theorems via Gauge Function with Applications

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Received 26 December 2020; Revised 21 March 2021; Accepted 25 March 2021; Published 19 May 2021

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1. Introduction and Preliminaries

The most publicized famous result in nonlinear analysis is Banach contraction principle, which made clear a systematic rule to find the fixed point of a given mapping on a metric space. So far, numerous authors have studied this classical result to examine the existence and uniqueness of a solution for different forms of contractive structure.

In 2014, Jeleli and Samet [1] introduced the concept of a new contraction known as the $\theta$-contraction, which generalizes the Banach contraction principle in a beautiful way.

In 2015, Khojasteh et al. [2] introduced simulation function. Recently, many researchers have proved fixed point theorems for Suzuki-type (SU) mappings in metric space [see [3, 4]].

Let $(X, \delta)$ be a metric space. For $\mu \in X$ and $\beta_1 \subseteq X$, let $\text{CL}(X)$ and $\text{CB}(X)$ denote the family of all nonempty closed subsets and the family of all nonempty closed bounded subsets of $X$. Design the Pompeiu–Hausdorff metric $H_d$ induced by $\delta$ on $\text{CB}(X)$ as

$$H_d(\beta_1, \beta_2) = \max \left\{ \sup_{\mu_1 \in \beta_1} \delta(\mu_1, \beta_2), \sup_{\mu_2 \in \beta_2} \delta(\mu_2, \beta_1) \right\},$$

for all $\beta_1, \beta_2 \in \text{CL}(X)$ and $\delta(\mu_1, \beta_1) = \inf \{ \delta(\mu_1, \mu_2): \mu_2 \in \beta_1 \}$. A point $\mu \in X$ is said to be a fixed point of $T: X \rightarrow \text{CB}(X)$, if $\mu \in T(\mu)$. If, for $\mu_0 \in X$, there exists a sequence $\{\mu_k\}$ in $X$ such that $\mu_k \in T(\mu_{k-1})$, then $O(T, \mu_0) = \{\mu_0, \mu_1, \mu_2, \ldots\}$ is said to be an orbit of $T: X \rightarrow \text{CB}(X)$. Mapping $f: X \rightarrow \mathbb{R}$ is said to be $T$-orbitally lower semi-continuous (o.l.s.c.), if a sequence $\{\mu_k\}$ in $O(T, \mu_0)$ and $\mu_k \rightarrow \gamma \Rightarrow f(\gamma) \leq \liminf f(\mu_k)$.

A multivalued mapping $T: X \rightarrow \text{CB}(X)$ is called a Nadler-contraction, if there exists $\gamma \in (0, 1)$ such that

$$H_d(T(\mu_1), T(\mu_2)) \leq \gamma \delta(\mu_1, \mu_2) \quad \text{for all} \quad \mu_1, \mu_2 \in X. \quad (2)$$

Nadler [5] obtained the multivalued version of the Banach contraction principle. Let $(X, \delta)$ be a complete metric space and $T: X \rightarrow \text{CL}(X)$ be a Nadler-contraction. Then, $T$ has a fixed point. Recently, Vetro [6] proved the following result to $\mu^*$.

Theorem 1. Let $(X, \delta)$ be a complete metric space and $T: X \rightarrow \text{CB}(X)$ be a multivalued mapping. Suppose that there exist $\theta \in \Xi$ and $k \in (0, 1)$ such that

$$\mu_1, \mu_2 \in X, H_d(T(\mu_1), T(\mu_2)) > 0 \Rightarrow \theta[H_d(T(\mu_1), T(\mu_2)) \leq \theta(\delta(\mu_1, \mu_2))]^k,$$

for all $\mu_1, \mu_2 \in X$. Then $T$ has a fixed point in $X$.
where $\Xi$ is the set of mapping $\theta: (0, \infty) \to (1, \infty)$ satisfying $(\theta_1 - \theta_2)$:

(i) $(\theta_1 - \theta_2)$ is nondecreasing and right-continuous.
(ii) $(\theta_n)$ For each $\{s_n\}$ in $(0, \infty)$, $\lim_{n \to \infty} \theta(s_n) = 1 \Rightarrow \lim_{n \to \infty} (s_n) = 0$.
(iii) $(\theta_n)$ There exist $r \in (0, 1)$ and $\mu \in (0, +\infty)$ such that $\lim_{r \to 0} \theta(s) - 1/s^r = \mu$. Then, $T$ has at least one fixed point.

Remark 1. Let $(X, \delta)$ be a metric space. If $T: X \to CB(X)$ is a multivalued mapping satisfying the above theorem, then

$$\ln \theta(H_d(T(\mu_1), T(\mu_2))) \leq k \ln \theta(H_d(\mu_1, \mu_2)) < \ln \theta(\delta(\mu_1, \mu_2)).$$

(4)

Since $\theta$ is nondecreasing, we obtain

$$H_d(T\mu_1, T\mu_2) < \delta(\mu_1, \mu_2), \quad \text{for all } \mu_1, \mu_2 \in X, T\mu_1 \neq T\mu_2.$$  

(5)

Example 1. The functions $\theta_1, \theta_2: (0, \infty) \to (1, \infty)$, defined by $\theta_1(r) = e^{\sqrt{r}}$ and $\theta_2(r) = 1 + \sqrt{r}$, are in $\Xi$.

Lemma 1 (see [6]). Let $(X, \delta)$ be a metric space and $\beta_1, \beta_2 \in CB(X)$ with $H_d(\beta_1, \beta_2) > 0$. Then, for all $k > 1$ and $\mu \in \beta_1$, there exists $\nu = \nu(\mu) \in \beta_2$ such that

$$\delta(\mu, \nu) < hH_d(\beta_1, \beta_2).$$

(6)

Example 2 (see [2]). For $j = 1, 2$, let $\bar{\theta}_j: \mathbb{R}^+ \to \mathbb{R}^+$ be continuous functions such that $\bar{\theta}_j(0) = 0$ and $\bar{\theta}_j(1) = 0$. Functions $\Gamma_j: \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}$ ($j = 1, 2$) are in $V$:

(i) $\Gamma_j(\mu_1, \mu_2) = \bar{\theta}_j(\mu_2) - \bar{\theta}_j(\mu_1)$ for all $\mu_1, \mu_2 \geq 0$.

(7)

Definition 1 (see [2]). Mapping $\Gamma: \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}$ is called a simulation function such that

(i) $\Gamma(0, 0) = 0$.
(ii) $\Gamma(\mu_1, \mu_2) < \mu_2 - \mu_1$ for all $\mu_1, \mu_2 > 0$.
(iii) If $\{\mu_{1n}\}, \{\mu_{2n}\} \in (0, \infty)$ such that $\lim_{n \to \infty} \mu_{1n} = \lim_{n \to \infty} \mu_{2n} > 0$, then

$$\limsup_{n \to \infty} \Gamma(\mu_{1n}, \mu_{2n}) < 0.$$  

(8)

Due to (T2), we have $\Gamma(\mu_1, \mu_2) < 0$ for all $\mu_1 > 0$. Here, the set $V$ satisfies the conditions (G1)–(G3).

Lemma 2 (see [6]). Let $(X, \delta)$ be a metric space, $\beta_1 \in CB(X)$, and $\mu \in X$. Then, for each $\epsilon > 0$, there exists $\nu \in \beta_2$ such that

$$\delta(\mu, \nu) \leq \delta(\mu, \beta_2) + \epsilon.$$  

(9)

Definition 2 (see [1]). Let $(X, \delta)$ be a metric space and $\Lambda$ be a nonempty subset of $X$, and $T: \Lambda \to CB(X)$ is known as $\alpha$-admissible, if there exists a mapping $\alpha: \Lambda \times \Lambda \to (0, \infty)$ such that

$$\alpha(\beta_1, \beta_2) \geq 1 \Rightarrow \alpha(\mu, v) \geq 1,$$  

(10)

for all $\mu \in T(\beta_1) \cap \Lambda$ and $v \in T(\beta_2) \cap \Lambda$.

Lemma 3 (see [7]). Suppose there is a point $\mu_0 \in \Lambda$ ($\Lambda$ is a closed subset of $X$) that satisfies

$$\delta(\mu_0, T(\mu_0)) \in \tilde{E},$$

(11)

and $\mu_0 \in \Lambda$ for some $\epsilon > 0$. Then, $\delta(\mu_0, T(\mu_0)) \in \tilde{E}$ where $\tilde{E}$ denotes an interval on $\mathbb{R}^+$ containing 0.

Definition 3 (see [7]). (inclusion ball) Suppose $\mu_0 \in \Lambda$ and $\delta(\mu_0, T(\mu_0)) \in \tilde{E}$. Then, for all $\mu_0$, $\mu$ which belongs to $\Lambda$, we define the closed-ball $B(\mu_1, \rho)$ with center $\mu_1$ and radius $\rho = \delta(\mu_1, T(\mu_1))$, where $\delta: \tilde{E} \to R_{+}$ is defined by (13).

Definition 4 (see [7]). Let $j \geq 1$, and $\eta: \tilde{E} \to \tilde{E}$ is known as a gauge function of order $j$ on $\tilde{E}$, if it satisfies the following conditions:

(i) $\eta(\lambda \mu) < l^j \eta(\mu)$ for all $\lambda \in (0, 1)$ and $\mu \in \tilde{E}$.
(ii) $\eta(\mu) < \mu$ for all $\mu \in \tilde{E} - \{0\}$.

Note that the first condition of Definition 4 is equivalent to $\eta(0) = 0$ and $\eta(\mu/\mu')$ is nondecreasing on $\tilde{E} - \{0\}$.

Definition 5 (see [7]). A gauge function $\eta: \tilde{E} \to \tilde{E}$ is said to be a Bianchini–Grandolfi gauge function on $\tilde{E}$ if

$$\sigma(\mu) = \sum_{i=0}^{\infty} \eta(\mu) < \infty,$$  

(12)

for all $\mu \in \tilde{E}$. Note that a Bianchini–Grandolfi gauge function also satisfies the following functional equation:

$$\sigma(\mu) = \sigma(\eta(\mu)) + \mu.$$  

(13)

2. Set-Valued $\theta_\eta$-Contraction

The first main definition of this exposition is as follows.
\textbf{Theorem 2.} Let \((X, \delta)\) be a complete metric space, \(\theta\) be a Bianchini–Grandolfo gauge function on \(E\). A mapping \(T: \Lambda \rightarrow CB(X)\) is known as set-valued \(SU_{\theta}\)-contraction, if there exists \(\theta \in \Xi\) such that for \(T(\mu) \cap \Lambda \neq \emptyset\),

\[
\frac{1}{2} \min \{\delta(\mu, T(\mu) \cap \Lambda), \delta(\nu, T(\nu) \cap \Lambda)\} < \delta(\mu, \nu), \quad (14)
\]

for all \(\mu \in \Lambda, \nu \in T(\mu) \cap \Lambda\) with \(\delta(\mu, \nu) \in \bar{E}\), where \(k \in (0, 1)\).

\begin{align*}
\Omega(\mu, \nu) &= \max \left\{ \delta(\mu, \nu), \delta(\mu, T(\mu)), \delta(\nu, T(\nu)), \frac{\delta(\mu, T(\nu)) + \delta(\nu, T(\mu))}{2} \right\}, \quad (16)
\end{align*}

\[
\frac{\rho[\hbar H_d(T(\mu) \cap \Lambda, T(\nu) \cap \Lambda)]}{\delta(\mu, \nu) \cap \Lambda} \leq \left[ \theta(\eta(\Omega(\mu, \nu))) \right]^k, \quad (15)
\]

where

\[
\theta(\mu, \nu) = \max \left\{ \delta(\mu, \nu), \delta(\mu, T(\mu)), \delta(\nu, T(\nu)), \frac{\delta(\mu, T(\nu)) + \delta(\nu, T(\mu))}{2} \right\},
\]

which implies that

\[
\frac{\Omega(\mu, \nu)}{\delta(\mu, \nu) \cap \Lambda} \leq \left[ \theta(\eta(\Omega(\mu, \nu))) \right]^k,
\]

where

\[
\delta(\mu_0, \mu_1) = \max \left\{ \delta(\mu_0, \mu_1), \delta(\mu_0, T(\mu_0)), \delta(\mu_1, T(\mu_0)), \frac{\delta(\mu_0, T(\mu_1)) + \delta(\mu_1, T(\mu_0))}{2} \right\},
\]

\[
\delta(\mu_0, \mu_1) \leq \max \left\{ \delta(\mu_0, \mu_1), \delta(\mu_1, T(\mu_1)), \frac{\delta(\mu_0, T(\mu_1))}{2} \right\}
\]

\[
\leq \max \left\{ \delta(\mu_0, \mu_1), \delta(\mu_1, T(\mu_1)) \right\},
\]

We claim that

\[
\theta(\delta(\mu_1, \mu_2)) \leq \theta[h_1 H_d(T(\mu_0) \cap \Lambda, T(\mu_1) \cap \Lambda)] \leq \left[ \theta(\delta(\mu_0, \mu_1)) \right]^k. \quad (23)
\]

Let \(\Phi = \max \{\delta(\mu_0, \mu_1), \delta(\mu_1, T(\mu_1))\}\). If \(\Phi = \delta(\mu_1, T(\mu_1))\), we have \(\mu_2 \in T(\mu_1) \cap \Lambda\), so we obtain

\[
\theta(\delta(\mu_1, \mu_2)) \leq \theta[h_1 H_d(T(\mu_0) \cap \Lambda, T(\mu_1) \cap \Lambda)] \leq \left[ \theta(\delta(\mu_1, \mu_2)) \right]^k, \quad (24)
\]

which is a contradiction. Thus, we conclude that \(\Phi = \delta(\mu_0, \mu_1)\). We assume that \(\delta(\mu_1, \mu_2) \neq 0\); otherwise, \(\mu_1\) is a fixed point of \(T\). From Remark 1, we have \(\delta(\mu_1, \mu_2) < \delta(\mu_0, \mu_1)\), and so \(\delta(\mu_1, \mu_2) \in \bar{E}\). Next, \(\mu_2 \in \bar{B}(\mu_0, \rho)\) because

\[
\delta(\mu_0, \mu_2) \leq \delta(\mu_0, \mu_1) + \delta(\mu_1, \mu_2) \leq \delta(\mu_0, \mu_1) + \eta(\delta(\mu_0, \mu_1)) \leq \delta(\mu_0, \mu_1) + \sigma(\eta(\delta(\mu_0, \mu_1))) = \sigma(\delta(\mu_0, \mu_1)) = \rho.
\]

\[
\theta(\mu_1, \mu_2) \leq \theta[h_1 H_d(T(\mu_0) \cap \Lambda, T(\mu_1) \cap \Lambda)] \leq \left[ \theta(\eta(\Omega(\mu, \nu))) \right]^k,
\]

where
Also, since
\[
\frac{1}{2} \min \{ \delta(\mu_1, T(\mu_1) \cap \Lambda), \delta(\mu_2, T(\mu_2) \cap \Lambda) \} < \delta(\mu_1, \mu_2),
\]
from (15), we get
\[
\theta[H_d(T(\mu_1) \cap \Lambda, T(\mu_2) \cap \Lambda)] \leq \left[ \theta(\eta(\delta(\mu_1, \mu_2))) \right]^k < \left[ \theta(\Omega(\mu_1, \mu_2)) \right]^k. \tag{27}
\]
Since \( \theta \) is right-continuous, there exists a real number \( h_2 > 1 \) such that
\[
\theta[h_2 H_d(T(\mu_1) \cap \Lambda, T(\mu_2) \cap \Lambda)] \leq \left[ \theta(\Omega(\mu_1, \mu_2)) \right]^k. \tag{28}
\]
Next, from
\[
\delta(\mu_2, T(\mu_2) \cap \Lambda) \leq \delta(H_d(T(\mu_1) \cap \Lambda, T(\mu_2) \cap \Lambda)),
\]
by Lemma 1, there exists \( \mu_3 \in T(\mu_2) \cap \Lambda \) such that
\[
\delta(\mu_2, \mu_3) \leq \delta h_2 H_d(T(\mu_1) \cap \Lambda, T(\mu_2) \cap \Lambda). \tag{29}
\]
By (28), this inequality gives that
\[
\theta(\delta(\mu_2, \mu_3)) \leq \theta[h_2 H_d(T(\mu_1) \cap \Lambda, T(\mu_2) \cap \Lambda)] \leq \left[ \theta(\Omega(\mu_1, \mu_2)) \right]^k \leq \left[ \theta(\Omega(\mu_0, \mu_1)) \right]^k.
\]
where

\[
\Omega(\mu_1, \mu_2) = \max \left\{ \delta(\mu_1, \mu_2), \delta(\mu_1, T(\mu_1)), \delta(\mu_2, T(\mu_2)), \frac{\delta(\mu_1, T(\mu_2)) + \delta(\mu_2, T(\mu_1))}{2} \right\}
\]

\[
\leq \max \left\{ \delta(\mu_1, \mu_2), \delta(\mu_2, T(\mu_2)), \frac{\delta(\mu_1, T(\mu_2))}{2} \right\}
\]

\[
\leq \max \{ \delta(\mu_1, \mu_2), \delta(\mu_2, T(\mu_2)) \}.
\]

We claim that
\[
\theta(\delta(\mu_2, \mu_3)) \leq \theta[h_2 H_d(T(\mu_1) \cap \Lambda, T(\mu_2) \cap \Lambda)] \leq \left[ \theta(\delta(\mu_0, \mu_1)) \right]^k \tag{32}
\]
Let
\[
\Phi = \max \{ \delta(\mu_1, \mu_2), \delta(\mu_2, T(\mu_2)) \}. \]
If \( \Phi = \delta(\mu_2, T(\mu_2)) \), we have \( \mu_3 \in T(\mu_2) \cap \Lambda \), so we obtain
\[
\theta(\delta(\mu_2, \mu_3)) \leq \theta[h_2 H_d(T(\mu_1) \cap \Lambda, T(\mu_2) \cap \Lambda)] \leq \left[ \theta(\delta(\mu_2, \mu_3)) \right]^k. \tag{33}
\]
which is a contradiction. Thus, we conclude that \( \Phi = \delta(\mu_1, \mu_2) \). We assume that \( \delta(\mu_2, \mu_3) \neq 0 \); otherwise, \( \mu_2 \) is a fixed point of \( T \). From Remark 1, we have \( \delta(\mu_2, \mu_3) \leq \delta(\mu_1, \mu_2) \), and so \( \delta(\mu_2, \mu_3) \in \mathcal{B} \). Also, we have \( \mu_3 \in \mathcal{B}(\mu_0, \rho) \), since
\[
\delta(\mu_0, \mu_3) \leq \delta(\mu_0, \mu_1) + \delta(\mu_1, \mu_2) + \delta(\mu_2, \mu_3)
\]
\[
\leq \delta(\mu_0, \mu_1) + \eta(\delta(\mu_0, \mu_1)) + \eta^2(\delta(\mu_0, \mu_1))
\]
\[
\leq \sum_{i=0}^{\infty} \eta^i(\delta(\mu_0, \mu_1)) = \sigma(\delta(\mu_0, \mu_1)) = \rho.
\tag{34}
\]
Continuing this setup, we have two sequences \( \{ \mu_1 \} \subset \mathcal{B}(\mu_0, \rho) \) and \( \{ h_i \} \subset (1, \infty) \) such that \( \mu_{i+1} \in T(\mu_i) \cap \Lambda \), \( \mu \neq \mu_{i+1} \) with \( \delta(\mu_{i+1}, \mu_{i+2}) \in \mathcal{B} \) and

\[
1 < \theta(\delta(\mu_{i+1}, \mu_{i+2})) \leq \theta[h_i H_d(T(\mu_{i-1}) \cap \Lambda, T(\mu_{i}) \cap \Lambda)] \leq \left[ \theta(\delta(\mu_{i-1}, \mu_{i})) \right]^k,
\tag{35}
\]
for all \( i \in \mathbb{N} \). Then,
\[
1 < \theta(\delta(\mu_{i+1}, \mu_{i+2})) \leq \theta(\delta(\mu_0, \mu_1))^k, \quad \text{for all } i \in \mathbb{N}, \tag{36}
\]
which gives that
\[
\lim_{i \to \infty} \theta(\delta(\mu_{i+1}, \mu_{i+2})) = 1,
\tag{37}
\]
and by \( (\theta_1) \), we have
\[
\lim_{i \to \infty} \delta(\mu_{i+1}, \mu_{i+2}) = 0. \tag{38}
\]
Next, we prove that \( \{ \mu_i \} \) is a Cauchy sequence in \( X \). Setting \( \delta_1 = \delta(\mu_{i+1}, \mu_{i+2}) \), from \( (\theta_3) \), there exist \( r \in (0, 1) \) and \( \mu \in (0, \infty) \) such that
\[
\lim_{i \to \infty} \frac{\theta(\delta_1) - 1}{\delta_1^r} = \mu. \tag{39}
\]
Take \( \lambda \in (0, \mu) \). From the definition of limit, there exists \( t_0 \in \mathbb{N} \) such that
\[
[\delta_1]_r^r \leq \lambda^{-1}[\theta(\delta_1) - 1], \quad \text{for all } i > t_0.
\tag{40}
\]
Using (36) and the above inequality,
\[
[\delta_1]_r^r \leq \lambda^{-1}[\theta(\delta_0)]^k - 1), \quad \text{for all } i > t_0. \tag{41}
\]
This implies that
\[
\lim_{i \to \infty} t[|\delta|]^i = \lim_{i \to \infty} t[d(\mu_i, \mu_{i+1})]^i = 0. \tag{42}
\]
Hence, there exists \( t_1 \in \mathbb{N} \) such that
\[
d(\mu_i, \mu_{i+1}) \leq \frac{1}{t_1^i} \quad \text{for all } i > t_1. \tag{43}
\]

Let \( p > t > t_1 \). Then, using the triangular inequality and (43), we get
\[
d(\mu_p, \mu_{t_1}) \leq \sum_{j=t}^{p-1} nd(\mu_j, \mu_{j+1}) \leq \sum_{j=t}^{p-1} \frac{1}{t_1^j} < \sum_{j=t}^{\infty} \frac{1}{t_1^j}. \tag{44}
\]

Owing to the convergence of the series \( \sum_{j=t}^{\infty} \frac{1}{t_1^j} \), \( \{\mu_i\} \) is a Cauchy sequence in \( B(\mu_0, \rho) \). Since \( B(\mu_0, \rho) \) is closed in \( X \), there exists \( \delta \in B(\mu_0, \rho) \) such that \( \mu_i \to \delta \). Note that \( \delta \in \Lambda \) because \( \mu_{i+1} \in T(\mu_i) \cap \Lambda \). Now, we claim that
\[
\frac{1}{2} \min\{d(\mu_i, T(\mu_i) \cap \Lambda), d(\delta, T(\delta) \cap \Lambda)\} < d(\mu_i, \delta), \tag{45}
\]
for all \( i \in \mathbb{N} \). Assume, on the contrary, that there exists \( i' \in \mathbb{N} \) such that
\[
\frac{1}{2} \min\{d(\mu_i, T(\mu_i) \cap \Lambda), d(\delta, T(\delta) \cap \Lambda)\} \geq d(\mu_i, \delta), \tag{46}
\]
and
\[
\frac{1}{2} \min\{d(\delta, T(\delta) \cap \Lambda), d(\mu_{i+1}, T(\mu_{i+1}) \cap \Lambda)\} \geq d(\mu_{i+1}, \delta). \tag{47}
\]

By (47), we have
\[
2d(\mu_i, \delta) \leq \min\{d(\mu_i, T(\mu_i) \cap \Lambda), d(\delta, T(\delta) \cap \Lambda)\} \leq \min\{d(\delta, T(\delta) \cap \Lambda), d(\delta, T(\delta) \cap \Lambda)\} \leq \min\{d(\delta, T(\delta) \cap \Lambda), d(\delta, T(\delta) \cap \Lambda)\} \leq [d(\mu_i, \delta) + d(\delta, T(\mu_i) \cap \Lambda)] < [d(\mu_i, \delta) + d(\delta, T(\mu_{i+1}) \cap \Lambda)] \leq [d(\mu_i, \delta) + d(\delta, T(\mu_{i+1}) \cap \Lambda)]. \tag{48}
\]

which implies that
\[
d(\mu_i, \delta) \leq d(\delta, \mu_{i+1}), \tag{50}
\]
which together with (38) gives
\[
d(\mu_i, \delta) \leq d(\delta, \mu_{i+1}) \leq \frac{1}{2} \min\{d(\delta, T(\delta) \cap \Lambda), d(\mu_{i+1}, T(\mu_{i+1}) \cap \Lambda)\}. \tag{51}
\]

Since
\[
\frac{1}{2} \min\{d(\mu_i, T(\mu_i) \cap \Lambda), d(\mu_{i+1}, T(\mu_{i+1}) \cap \Lambda)\} < d(\mu_i, \mu_{i+1}), \tag{52}
\]
from contractive condition (15), we have
\[
\theta(d(\mu_{i+1}, \mu_{i'+1})) \leq \theta[h_2H_d(T(\mu_i) \cap \Lambda, T(\mu_{i+1}) \cap \Lambda)] \leq \theta(\Omega(\mu_{i+1}, \mu_{i'+1}))^k, \tag{53}
\]
where
\[
\Omega(\mu_{i+1}, \mu_{i'+1}) = \max \left\{ \frac{\delta(\mu_{i+1}, \mu_{i'+1}), \delta(\mu_{i+1}, T(\mu_i)), \delta(\mu_{i+1}, T(\mu_{i+1}))}{2} \right\} \tag{54}
\]

We claim that
\[
\theta(d(\mu_{i+1}, \mu_{i'+1})) \leq \theta[h_2H_d(T(\mu_i) \cap \Lambda, T(\mu_{i+1}) \cap \Lambda)] \leq \theta(\Omega(\mu_{i+1}, \mu_{i'+1}))^k. \tag{55}
\]

Let \( \Delta = \max\{d(\mu_{i+1}, \mu_{i'+1}), d(\mu_{i+1}, \mu_{i'+1})\} \). If \( \Delta = d(\mu_{i+1}, \mu_{i'+1}) \). Since \( \mu_{i+1} \in T(\mu_i) \cap \Lambda \), we have
\[
\theta(d(\mu_{i+1}, \mu_{i'+1})) \leq \theta[h_2H_d(T(\mu_i) \cap \Lambda, T(\mu_{i+1}) \cap \Lambda)] \leq \theta(\Omega(\mu_{i+1}, \mu_{i'+1}))^k, \tag{56}
\]
which is a contradiction. Thus, we conclude that \( \Delta = d(\mu_{i+1}, \mu_{i'+1}) \). From Remark 1, we have
\[
d(\mu_{i+1}, \mu_{i'+1}) < d(\mu_i, \mu_{i+1}). \tag{57}
\]

From (38), (43), and (47), we obtain
\[
d(\mu_{i+1}, \mu_{i'+1}) < d(\mu_i, \mu_{i+1}) \leq \frac{1}{2} \min\{d(\delta, T(\delta) \cap \Lambda), d(\mu_{i+1}, T(\mu_{i+1}) \cap \Lambda)\} \leq \frac{1}{2} \min\{d(\delta, T(\delta) \cap \Lambda), d(\mu_{i+1}, T(\mu_{i+1}) \cap \Lambda)\} \leq \delta(\mu_{i+1}, T(\mu_{i+1}) \cap \Lambda), \tag{58}
\]
which is a contradiction. Hence, (45) holds true:
Also, we know that $\delta(\mu, \mu_{i+1}) \in \hat{E}$ for all $n$. Thus, from (15), we have
\[
\theta(\delta(\mu_{i+1}, T(\mu_{i+1}) \cap \Lambda)) \leq \theta[H_d(T(\mu) \cap \Lambda, T(\mu_{i+1}) \cap \Lambda)] \\
\leq [\theta(b(\Omega(\mu, \mu_{i+1})))]^k \\
< \theta(\Omega(\mu, \mu_{i+1})))^k,
\]
where
\[
\Omega(\mu, \mu_{i+1}) = \max \left\{ \frac{\delta(\mu_{i+1}, T(\mu_{i+1})), \delta(\mu_{i+1}, T(\mu_{i+1}))}{2} \right\}
\]
\[
\leq \max \left\{ \delta(\mu_{i+1}, \mu_{i+2}), \delta(\mu_{i+1}, \mu_{i+2}) \right\}
\]
\[
\leq \max[\delta(\mu_{i+1}, T(\mu_{i+1})), \delta(\mu_{i+1}, \mu_{i+2})].
\]
We claim that
\[
\theta(\delta(\mu_{i+1}, T(\mu_{i+1}) \cap \Lambda)) \leq [\theta(\Omega(\mu, \mu_{i+1})))]^k.
\]
(62)
\[
\delta(\mu_{i+1}, T(\mu_{i+1}) \cap \Lambda) < \delta(\mu_{i+1}, \mu_{i+1}).
\]
(63)
which is a contradiction. Thus, we obtain $\Phi = \delta(\mu_{i+1}, \mu_{i+1})$.
From Remark 1, we deduce
\[
\delta(\mu_{i+1}, T(\mu_{i+1}) \cap \Lambda) < \delta(\mu_{i+1}, \mu_{i+1}).
\]
(64)
Taking limit $i \rightarrow \infty$ in (64),
\[
\lim_{i \rightarrow \infty} \delta(\mu_{i+1}, T(\mu_{i+1}) \cap \Lambda) = 0.
\]
(65)
Since $g(\mu) = \delta(\mu, T(\mu) \cap \Lambda)$ is $T$-o.l.s.c at $\theta^*$, then
\[
\delta(\theta^*, T(\theta^*) \cap \Lambda) = g(\theta^*) \leq \liminf_{i \rightarrow \infty} g(\mu_{i+1})
\]
\[
= \liminf_{i \rightarrow \infty} \delta(\mu_{i+1}, T(\mu_{i+1}) \cap \Lambda) = 0.
\]
(66)
Since $T(\theta^*)$ is closed, we have $\theta^* \in T(\theta^*)$. Conversely, if $\theta^*$ is a fixed point of $T$, then $g(\theta^*) = 0 \leq \liminf_{i \rightarrow \infty} g(\mu_{i+1})$, since $\theta^* \in \Lambda$.
\[
\frac{1}{2} \min[\delta(\mu, T(\mu) \cap \Lambda), \delta(\theta^*, T(\theta^*) \cap \Lambda)] < \delta(\mu, \theta^*) \quad \text{for all } i \geq 2.
\]
(59)

Corollary 1. Let $(X, \delta)$ be a complete metric space, $\eta$ be a Bianchini–Grandolfi gauge function on an interval $\hat{E}$, and $T: \Lambda \rightarrow CB(X)$ be a given set-valued mapping. If $k \in (0, 1)$ and $T(\mu) \cap \Lambda \neq \emptyset$ exist,
\[
\frac{1}{2} \min[\delta(\mu, T(\mu) \cap \Lambda), \delta(\nu, T(\nu) \cap \Lambda)] < \delta(\mu, \nu),
\]
(67)
which implies that
\[
\theta[H_d(T(\mu) \cap \Lambda, T(\nu) \cap \Lambda)] \leq [\theta(\delta(\mu, \nu)))]^k,
\]
(68)
for all $\mu \in \Lambda$, $\nu \in T(\mu) \cap \Lambda$ with $\delta(\mu, \nu) \in \hat{E}$. Suppose $\mu_0 \in \Lambda$ such that $\delta(\mu_0, c^*) \in \hat{E}$ for some $c^* \in T(\mu_0) \cap \Lambda$. Then, there exists an orbit $\{\mu_n\}$ of $T$ in $\Lambda$ and $\theta^* \in \Lambda$ such that $\lim_{n \rightarrow \infty} \mu_n = \theta^*$. In addition, $\theta^*$ is a fixed point of $T$ if and only if the function $g(\mu): = \delta(\mu, T(\mu) \cap \Lambda)$ is $T$-o.l.s.c at point $\theta^*$.

Corollary 2. Let $(X, \delta)$ be a complete metric space, $\eta$ be a Bianchini–Grandolfi gauge function on an interval $\hat{E}$, and $T: \Lambda \rightarrow CB(X)$ be a given set-valued mapping. If $k \in (0, 1)$ and for $T(\mu) \cap \Lambda \neq \emptyset$ exist,
Theorem 3. Let \( X, \delta \) be a complete metric space, \( \eta \) be a Bianchini–Grandolfi gauge function on \( \overline{E} \), and \( T: \Lambda \to CB(X) \) be a given set-valued mapping. If \( \theta \in \Xi \) and \( k \in (0, 1) \) exist, then
\[
\frac{1}{2} \min \{ \delta (\mu, T(\mu) \cap \Lambda), \delta (\nu, T(\nu) \cap \Lambda) \} < \delta (\mu, \nu),
\]
implies that
\[
\sqrt{\theta [H_d(T(\mu), T(\nu))]} \leq k \sqrt{\eta (\delta (\mu, \nu))},
\]
for all \( \mu \in \Lambda, \nu \in T(\mu) \cap \Lambda, \) and \( \delta (\mu, \nu) \in \overline{E} \). In addition, suppose \( \mu_0 \in \Lambda \) such that \( \delta (\mu_0, c^*) \in \overline{E} \) for some \( c^* \in T(\mu_0) \cap \Lambda \). Then, there exists an orbit \( \{ \mu_i \} \) of \( T \) in \( \Lambda \), and \( 0^* \in \Lambda \) such that \( \lim_{i \to \infty} \mu_i = 0^* \) and \( 0^* \) is a fixed point of \( T \) if and only if the function \( g(\mu) = \delta (\mu, T(\mu) \cap \Lambda) \) is \( T \)-o.l.s.c at \( 0^* \).

Corollary 3. Let \( (X, \delta) \) be a complete metric space, \( \eta \) be a Bianchini–Grandolfi gauge function on \( \overline{E} \), and \( T: \Lambda \to CB(X) \) be a given set-valued mapping. If \( \theta \in \Xi \) and \( k \in (0, 1) \) exist, then
\[
\frac{1}{2} \min \{ \delta (\mu, T(\mu) \cap \Lambda), \delta (\nu, T(\nu) \cap \Lambda) \} < \delta (\mu, \nu),
\]
implies that
\[
\theta [H_d(T(\mu), T(\nu))] \leq \theta \left( \frac{|\mu - \nu|}{8} \right)
\]
for all \( \mu \in X, \nu \in T(\mu) \), and \( \delta (\mu, \nu) \in \overline{E} \). Suppose that \( \mu_0 \in X \) such that \( \delta (\mu_0, c^*) \in \overline{E} \) for some \( c^* \in T(\mu_0) \). Then, there exists an orbit \( \{ \mu_i \} \) of \( T \) in \( X \) that converges to the fixed point \( 0^* \), where \( 0^* \in \mathcal{F} = \{ \mu \in X: \delta (\mu, 0^*) \in \overline{E} \} \) of \( T \).

Example 3. Let \( X = [-10, \infty) \) be a usual metric \( \delta \) and let \( \overline{E} = [0, \infty) \). Mapping \( T: \Lambda \to CB(X) \) is defined as
\[
T(\mu) = \begin{cases} 
0, & \mu < 0, \\
\mu, & 0 \leq \mu \leq 4, \\
4, & \mu > 4.
\end{cases}
\]
Clearly, \( \frac{1}{2} \min \{ \delta (\mu, T(\mu) \cap \Lambda), \delta (\nu, T(\nu) \cap \Lambda) \} < \delta (\mu, \nu) \) if and only if \( \mu, \nu \in [0, 4] \). Let \( \mu_0 = 4 \); then, we have \( c^* = 1/2 \in T(\mu_0) \) such that \( \delta (\mu_0, c^*) \in \overline{E} \). First, we claim that \( T \) satisfies inequality (68) with setting \( \theta (r) = e^{\sqrt{2}r} \), \( \eta (r) = r/2 \), and \( k = 1/2 \). For \( \mu \in [0, 4] \) and \( \nu \in T(\mu) \), we obtain
\[
\Omega (\mu, \nu) = \max \left\{ \delta (\mu, \nu), \delta (\mu, T(\mu)), \frac{\delta (\mu, T(\nu)) + \delta (\nu, T(\mu))}{2} \right\}.
\]
for all \( \mu \in \Lambda, \nu \in T(\mu) \cap \Lambda \) with \( \delta (\mu, \nu) \in \overline{E} \).

Theorem 3. Let \( (X, \delta) \) be a complete metric space and \( T: \Lambda \to CB(X) \) be a set-valued SU-type \( \Gamma_\alpha \)-contraction such that the following conditions are satisfied:

(a) \( T \) is \( \alpha \)-admissible.

(b) There exists \( \mu_0 \in \Lambda \) with \( \delta (\mu_0, \mu_1) \in \overline{E} \) for some \( \mu_1 \in T(\mu_0) \cap \Lambda \) such that \( \alpha (\mu_0, \mu_1) \geq 1 \). Then, there exists an orbit \( \{ \mu_i \} \) of \( T \) in \( \Lambda \) and \( 0^* \in \Lambda \) such that \( \lim_{i \to \infty} \mu_i = 0^* \). In addition, \( 0^* \) is a fixed point of \( T \) if
\[
\frac{1}{2} \min \{ \delta (\mu_0, T(\mu_0) \cap \Lambda), \delta (\mu_1, T(\mu_1) \cap \Lambda) \} < \delta (\mu_0, \mu_1).
\]
In the case that \( \delta(\mu_0, \mu_1) = 0 \), \( \mu_0 \) is a fixed point of \( T \). Thus, we assume that \( \delta(\mu_0, \mu_1) \neq 0 \). Define \( \rho = \sigma(\delta(\mu_0, \mu_1)) \). From (13), we have \( \sigma(r) \geq r \). Hence, \( \delta(\mu_0, \mu_1) \leq \rho \), and so \( \mu_1 \in B(\mu_0, \rho) \). Since \( \alpha(\mu_0, \mu_1) \geq 1 \) and \( \delta(\mu_0, \mu_1) \in \bar{E} \), from (76) and (78), it follows that

\[
0 \leq [\alpha(\mu_0, \mu_1)H_d(T(\mu_0) \cap \Lambda, T(\mu_1) \cap \Lambda), \eta(\delta(\mu_0, \mu_1))] < \eta(\Omega(\mu_0, \mu_1)) - \alpha(\mu_0, \mu_1)H_d(T(\mu_0) \cap \Lambda, T(\mu_1) \cap \Lambda),
\]

which implies that

\[
\alpha(\mu_0, \mu_1)H_d(T(\mu_0) \cap \Lambda, T(\mu_1) \cap \Lambda) < \eta(\Omega(\mu_0, \mu_1)).
\]

We can choose \( \varepsilon_1 > 0 \) such that

\[
\alpha(\mu_0, \mu_1)H_d(T(\mu_0) \cap \Lambda, T(\mu_1) \cap \Lambda) + \varepsilon_1 \leq \eta(\Omega(\mu_0, \mu_1)).
\]

Thus,

\[
\delta(\mu_1, T(\mu_1) \cap \Lambda) + \varepsilon_1 \leq H_d(T(\mu_0) \cap \Lambda, T(\mu_1) \cap \Lambda) + \varepsilon_1 \leq a(\mu_0, \mu_1)H_d(T(\mu_0) \cap \Lambda, T(\mu_1) \cap \Lambda) + \varepsilon_1 \leq \eta(\Omega(\mu_0, \mu_1)).
\]

It follows from Lemma 2 that there exists \( \mu_2 \in T(\mu_1) \cap \Lambda \) such that

\[
\delta(\mu_1, \mu_2) \leq \delta(\mu_1, T(\mu_1) \cap \Lambda) + \varepsilon_1.
\]

From (82) and (83), we infer that

\[
\delta(\mu_1, \mu_2) \leq \eta(\Omega(\mu_0, \mu_1)),
\]

where

\[
\Omega(\mu_0, \mu_1) = \max\left\{ \delta(\mu_0, \mu_1), \delta(\mu_0, T(\mu_0)), \delta(\mu_1, T(\mu_1)), \left( \frac{\delta(\mu_0, T(\mu_1)) + \delta(\mu_1, T(\mu_0))}{2} \right) \right\}
\]

\[
\leq \max\left\{ \delta(\mu_0, \mu_1), \delta(\mu_1, T(\mu_1)), \frac{\delta(\mu_0, T(\mu_1))}{2} \right\}
\]

\[
\leq \max\{\delta(\mu_0, \mu_1), \delta(\mu_1, T(\mu_1))\}.
\]

We claim that

\[
\delta(\mu_1, \mu_2) \leq \eta(\Omega(\mu_0, \mu_1)).
\]

Let \( \Phi = \max\{\delta(\mu_0, \mu_1), \delta(\mu_1, T(\mu_1))\} \). If \( \Phi = \delta(\mu_1, T(\mu_1)) \), we have \( \mu_2 \in T(\mu_1) \cap \Lambda \), so we obtain

\[
\delta(\mu_1, \mu_2) \leq \eta(\Omega(\mu_0, \mu_1)),
\]

which is a contradiction. Thus, we obtain \( \Phi = \delta(\mu_0, \mu_1) \). We assume that \( \delta(\mu_1, \mu_2) \neq 0 \); otherwise, \( \mu_1 \) is a fixed point of \( T \). Since \( \delta(\mu_1, \mu_2) \leq \eta(\delta(\mu_0, \mu_1)) < \delta(\mu_0, \mu_1) \), we deduce that \( \delta(\mu_1, \mu_2) \in \bar{E} \). Next, \( \mu_2 \in B(\mu_0, \rho) \) because

\[
\delta(\mu_0, \mu_2) \leq \delta(\mu_0, \mu_1) + \delta(\mu_1, \mu_2)
\]

\[
\leq \delta(\mu_0, \mu_1) + \eta(\delta(\mu_0, \mu_1))
\]

\[
\leq \delta(\mu_0, \mu_1) + \sigma(\delta(\mu_0, \mu_1))
\]

\[
= \sigma(\delta(\mu_0, \mu_1)) = \rho.
\]

Because \( T \) is \( \alpha \)-admissible, \( \alpha(\mu_1, \mu_2) \geq 1 \). Also, since

\[
\frac{1}{2} \min\{\delta(\mu_1, \mu_2), \delta(\mu_2, \mu_1) \} \in \bar{E}
\]

from (76), we get

\[
0 \leq [\alpha(\mu_1, \mu_2)H_d(T(\mu_1) \cap \Lambda, T(\mu_2) \cap \Lambda), \eta(\Omega(\mu_1, \mu_2))] < \eta(\Omega(\mu_1, \mu_2)) - a(\mu_1, \mu_2)H_d(T(\mu_1) \cap \Lambda, T(\mu_2) \cap \Lambda).
\]

This implies that

\[
\alpha(\mu_1, \mu_2)H_d(T(\mu_1) \cap \Lambda, T(\mu_2) \cap \Lambda) < \eta(\Omega(\mu_1, \mu_2)).
\]

Now choose \( \varepsilon_2 > 0 \) such that

\[
\alpha(\mu_1, \mu_2)H_d(T(\mu_1) \cap \Lambda, T(\mu_2) \cap \Lambda) + \varepsilon_2 \leq \eta(\Omega(\mu_1, \mu_2)).
\]

Thus,

\[
\delta(\mu_2, T(\mu_2) \cap \Lambda) + \varepsilon_2 \leq H_d(T(\mu_1) \cap \Lambda, T(\mu_2) \cap \Lambda) + \varepsilon_2 \leq a(\mu_1, \mu_2)H_d(T(\mu_1) \cap \Lambda, T(\mu_2) \cap \Lambda)
\]

\[
+ \varepsilon_2 \leq \eta(\Omega(\mu_1, \mu_2)).
\]

It follows from Lemma 2 that there exists \( \mu_3 \in T(\mu_2) \cap \Lambda \) such that

\[
\delta(\mu_2, \mu_3) \leq \delta(\mu_2, T(\mu_2) \cap \Lambda) + \varepsilon_2.
\]

From (93) and (94),

\[
\delta(\mu_2, \mu_3) \leq \eta^2(\Omega(\mu_0, \mu_1)),
\]

where
\[
\Omega(\mu_1, \mu_2) = \max \left\{ \delta(\mu_1, \mu_2), \delta(\mu_1, T(\mu_1)), \delta(\mu_2, T(\mu_2)), \frac{\delta(\mu_1, T(\mu_2)) + \delta(\mu_2, T(\mu_1))}{2} \right\}
\]
\[
\leq \max \left\{ \delta(\mu_1, \mu_2), \delta(\mu_2, T(\mu_2)), \frac{\delta(\mu_1, T(\mu_2))}{2} \right\}
\]
\[
\leq \max \left\{ \delta(\mu_1, \mu_2), \delta(\mu_2, T(\mu_2)) \right\}.
\]

We claim that
\[
\delta(\mu_2, \mu_3) \leq \eta(\delta(\mu_1, \mu_2)).
\] (97)

Let \( \Phi = \max \{ \delta(\mu_1, \mu_2), \delta(\mu_2, T(\mu_2)) \} \). If \( \Phi = \delta(\mu_2, T(\mu_2)) \), since \( \mu_1 \in T(\mu_2) \cap \Lambda \), we have
\[
\delta(\mu_2, \mu_3) \leq \eta(\delta(\mu_1, \mu_2)),
\] (98)
which is a contradiction. Thus, we have \( \Phi = \delta(\mu_1, \mu_2) \). We assume that \( \delta(\mu_2, \mu_3) \neq 0 \); otherwise, \( \mu_2 \) is a fixed point of \( T \). From (95), we have \( \delta(\mu_2, \mu_3) \leq \delta(\mu_1, \mu_2) \), and so \( \delta(\mu_2, \mu_3) \in \hat{E} \). Also, we have \( \mu_3 \in \mathcal{B}(\mu_0, \rho) \), since
\[
\delta(\mu_0, \mu_3) \leq \delta(\mu_0, \mu_1) + \delta(\mu_1, \mu_2) + \delta(\mu_2, \mu_3) \leq \delta(\mu_0, \mu_1) + \eta(\delta(\mu_0, \mu_1)) + \eta^2(\delta(\mu_0, \mu_1)) \leq \sum_{i=0}^{\infty} \eta^i(\delta(\mu_0, \mu_1)) = \sigma(\delta(\mu_0, \mu_1)) = \rho.
\] (99)

Continuing this setup, we obtain a sequence \( \{ \mu_i \} \subset \mathcal{B}(\mu_0, \rho) \) such that \( \mu_{i+1} \in T(\mu_i) \cap \Lambda \), \( \mu_i \neq \mu_{i+1} \) with \( \alpha(\mu_i, \mu_{i+1}) \geq 1 \) and \( \delta(\mu_i, \mu_{i+1}) \in \hat{E} \) and
\[
\delta(\mu_i, \mu_{i+1}) \leq \eta^i(\delta(\mu_0, \mu_1)), \text{ for all } i \in \mathbb{N}.
\] (100)

For \( i, m \in \mathbb{N} \) with \( m > i \), by using the triangular inequality and (100), we get
\[
\delta(\mu_i, \mu_m) \leq \delta(\mu_i, \mu_{i+1}) + \delta(\mu_{i+1}, \mu_{i+2}) + \cdots + \delta(\mu_{m-1}, \mu_m) \leq \eta^i(\delta(\mu_0, \mu_1)) + \eta^{i+1}(\delta(\mu_0, \mu_1)) + \cdots + \eta^{m-1}(\delta(\mu_0, \mu_1)) \leq \sum_{j=i}^{m-1} \eta^j(\delta(\mu_0, \mu_1)) < \infty.
\] (101)

To show that \( \{ \mu_i \} \) is a Cauchy sequence in \( \mathcal{B}(\mu_0, \rho) \). Since \( \mathcal{B}(\mu_0, \rho) \) is closed in \( X \), there exists an \( \vartheta' \in \mathcal{B}(\mu_0, \rho) \) such that \( \mu_i \to \vartheta' \). Note that \( \vartheta' \in \Lambda \) because \( \mu_{i+1} \in T(\mu_i) \cap \Lambda \). By same argument of Theorem 2, we have
\[
\frac{1}{2^5} \min\{\delta(\mu_i, T(\mu_i) \cap \Lambda), \delta(\mu_{i+1}, T(\mu_{i+1}) \cap \Lambda)\} < \delta(\mu_i, \mu_{i+1}).
\] (102)

Also, we know that \( \alpha(\mu_i, \mu_{i+1}) \geq 1 \) and \( \delta(\mu_i, \mu_{i+1}) < \infty \) for all \( n \). Thus, from (76), we have
\[
0 \leq \Gamma \left[ \alpha(\mu_i, \mu_{i+1})H_d(T\mu_i \cap \Lambda, T\mu_{i+1} \cap \Lambda), \eta(\Omega(\mu_i, \mu_{i+1})) \right] < \eta(\Omega(\mu_i, \mu_{i+1})) - \alpha(\mu_i, \mu_{i+1})H_d(T\mu_i \cap \Lambda, T\mu_{i+1} \cap \Lambda),
\] (103)
which gives that
\[
\alpha(\mu_i, \mu_{i+1})H_d(T\mu_i \cap \Lambda, T\mu_{i+1} \cap \Lambda) < \eta(\Omega(\mu_i, \mu_{i+1})).
\] (104)

Since \( \mu_{i+1} \in T(\mu_i) \cap \Lambda \), from (100), we get
\[
\delta(\mu_{i+1}, T(\mu_i) \cap \Lambda) \leq \alpha(\mu_i, \mu_{i+1})H_d(T\mu_i \cap \Lambda, T\mu_{i+1} \cap \Lambda) \leq \eta(\Omega(\mu_i, \mu_{i+1})) < (\Omega(\mu_i, \mu_{i+1})),
\] (105)
where
\[
\Omega(\mu_i, \mu_{i+1}) = \max \left\{ \frac{\delta(\mu_i, T\mu_i), \delta(\mu_i, T\mu_{i+1})}{2} \right\}
\]
\[
\leq \max \left\{ \frac{\delta(\mu_i, \mu_{i+1}), \delta(\mu_{i+1}, T\mu_i)}{2} \right\}
\]
\[
\leq \max \{ \delta(\mu_i, \mu_{i+1}), \delta(\mu_{i+1}, \mu_{i+2}) \}.
\] (106)

We claim that
\[
\delta(\mu_{i+1}, T(\mu_i) \cap \Lambda) \leq \alpha(\mu_i, \mu_{i+1})H_d(T\mu_i \cap \Lambda, T\mu_{i+1} \cap \Lambda) \leq \eta(\Omega(\mu_i, \mu_{i+1})).
\] (107)

Let \( \Phi = \max\{\delta(\mu_i, \mu_{i+1}), \delta(\mu_{i+1}, \mu_{i+2})\} \). If \( \Phi = \delta(\mu_{i+1}, \mu_{i+2}) \), since \( \mu_{i+2} \in T(\mu_{i+1}) \cap \Lambda \), we have
\[
\delta(\mu_{i+1}, T(\mu_{i+1}) \cap \Lambda) \leq \alpha(\mu_i, \mu_{i+1})H_d(T\mu_{i+1} \cap \Lambda, T\mu_{i+2} \cap \Lambda) < \eta(\Omega(\mu_i, \mu_{i+1})),
\] (108)
which is a contradiction. Thus, we have \( \Phi = \delta(\mu_i, \mu_{i+1}) \).

Taking limit \( i \to \infty \) in (105), we obtain
\[ \lim_{r \to \infty} \delta(\mu_{r+1}, T(\mu_{r+1}) \cap \Lambda) = 0. \]  
(109)

Since \( g(\mu) = \delta(\mu, T(\mu) \cap \Lambda) \) is \( T \)-o.l.s.c at \( 8^* \), then
\[ \delta(\theta', T(\theta') \cap \Lambda) = g(\theta') \leq \liminf g(\mu_{r+1}) \]
\[ = \liminf \delta(\mu_{r+1}, T(\mu_{r+1}) \cap \Lambda) = 0. \]  
(110)

Since \( T\theta' \) is closed, \( \theta' \in T(\theta') \). Conversely, if \( \theta' \) is a fixed point of \( T \), then \( g(\theta') = 0 \leq \liminf g(\mu) \), since \( \theta' \in \Lambda \).

Taking \( \Gamma(r, s) = s - \int_0^r \zeta(t) dt \) for all \( r, s \geq 0 \), in Theorem 3, we obtain the following theorem.

**Corollary 4.** Let \( (X, \delta) \) be a complete metric space, \( \eta \) be a Bianchini–Grandolfi gauge function on an interval \( \bar{E} \), and \( T: \Lambda \to CB(X) \) be a given set-valued mapping. If \( T\mu \cap \Lambda \neq \emptyset \), then
\[ \frac{1}{2} \min \{ \delta(\mu, T(\mu) \cap \Lambda), \delta(\eta, T(\eta) \cap \Lambda) \} < \delta(\mu, \eta). \]  
(111)

which implies that
\[ \int_0^\infty \zeta(t) dt \leq \eta(\delta(\mu, \eta)), \]  
(112)
for all \( \mu \in \Lambda \), \( \eta \in T(\mu) \cap \Lambda \), and \( \delta(\mu, \eta) \in \bar{E} \), where \( \zeta: \mathbb{R}^+ \to \mathbb{R}^+ \) is a function such that \( \int_0^\infty \zeta(t) dt \) exists and \( \int_0^\infty \zeta(t) dt \geq \epsilon \) for all \( \epsilon > 0 \) such that the following holds:

(a) \( T \) is \( \alpha \)-admissible.

(b) There exists \( \mu_0, \mu_1 \in \Lambda \) with \( \delta(\mu_0, \mu_1) \in \bar{E} \) for some \( \mu_1 \in T(\mu_0) \cap \Lambda \) such that \( \alpha(\mu_0, \mu_1) \geq 1 \). Then, there exists an orbit \( \{ \mu_r \} \) of \( T \) in \( \Lambda \) and \( \theta' \in \Lambda \) such that \( \lim_{r \to \infty} \mu_r = \theta' \). In addition, \( \theta' \) is a fixed point of \( T \) if and only if the function \( g(\mu) = \delta(\mu, T(\mu) \cap \Lambda) \) is \( T \)-o.l.s.c at \( \theta' \).

**Corollary 5.** Let \( (X, \delta) \) be a complete metric space, \( \eta \) be a Bianchini–Grandolfi gauge function on an interval \( \bar{E} \), and \( T: X \to CB(X) \) be a given set-valued mapping. If \( \Gamma \in \mathcal{V} \) exists, then
\[ \frac{1}{2} \min \{ \delta(\mu, T(\mu) \cap \Lambda), \delta(\eta, T(\eta) \cap \Lambda) \} < \delta(\mu, \eta). \]  
(113)

for all \( \mu \in X \), \( \eta \in T(\mu) \), and \( \delta(\mu, \eta) \in \bar{E} \) such that the following holds:

(a) \( T \) is \( \alpha \)-admissible.

(b) There exists \( \mu_0 \in X \) with \( \delta(\mu_0, \mu_1) \in \bar{E} \) for some \( \mu_1 \in T(\mu_0) \) such that \( \alpha(\mu_0, \mu_1) \geq 1 \). Then, there exists an orbit \( \{ \mu_r \} \) of \( T \) in \( X \) that converges to the fixed point \( \theta' \in \mathcal{F} = \{ \mu \in X: \delta(\mu, \theta') \in \bar{E} \} \) of \( T \).

4. Application to Dynamical System

Dynamical system is connected to a multistage operation reduced for solving the following functional equation:
\[ T(\mu_1) = \sup_{\mu_2 \in H} [h(\mu_1, \mu_2) + \delta(\mu_1, \mu_2, T(\mu_1, \mu_2))] \]  
(114)

where
\[ l: \bar{B} \times H \to \bar{B}, \]
\[ h, h: \bar{B} \times H \to (-\infty, \infty), \]
\[ D, D: \bar{B} \times H \times (-\infty, \infty) \to (-\infty, \infty). \]

Assume that \( \bar{G}_1 \) and \( \bar{G}_2 \) are Banach spaces, \( \bar{B} \subset \bar{G}_1 \) is a state space, and \( H \subset \bar{G}_2 \) is a decision space. For more details, see [3]. Let \( B(\bar{B}) \) signify the set of all bounded real-valued functions on \( \bar{B} \). Choose an arbitrary point \( \sigma \in B(\bar{B}) \) defined as \( \| \sigma \| = \sup_{\sigma \in \bar{B}}|\sigma(r)| \). \( (B(\bar{B}), \| \|) \) endowed with the metric given by
\[ \delta(\mu_1, \mu_2) = \sup_{\sigma \in \bar{B}}|\mu_1(r) - \mu_2(r)|, \]  
(115)

for all \( \mu_1, \mu_2 \in B(\bar{B}) \), are BS. Define \( g: B(\bar{B}) \to B(\bar{B}) \) by
\[ g(\sigma)(r) = \max_{r \in H} \{ V(r, t, \omega_1[\mu_1(r)]) + f(\mu_1(r)) \}, \]  
(116)

for all \( \omega \in B(\bar{B}) \) and \( r \in \bar{B} \). Also,

\[ H_d[g(\omega_1)(r), g(\omega_2)(r)] = H_d[V^x(r, t, \omega_1[l(r, t)]) + f^x(r, t), V^y(r, t, \omega_2[l(r, t)]) + f^y(r, t)] \]
\[ \leq H_d[V^x(r, t, \omega_1[l(r, t)]), V^y(r, t, \omega_2[l(r, t)])]. \]  
(117)
and we have
\[
\left| V(r, t, \omega_1(r)) - V(r, t, \omega_2(r)) \right| \leq \left[ \left[ 1 + \sqrt{T} \left( |\omega_1(r) - \omega_2(r)| \right)^\alpha \right] - 1 \right]^2,
\]
for all $\omega_1, \omega_2 \in B(\bar{\beta})$, where $r \in \bar{\beta}$, $t \in V$ and $0 \leq \alpha < 1$.

**Theorem 4.** Let $\phi: B(\bar{\beta}) \to B(\bar{\beta})$ be a l.s.c mapping as defined in (117) such that the following conditions are satisfied:

* $V$ and $f$ are continuous and bounded.
* There exists an orbit $[\tilde{\omega}] \in \Lambda$ of $\phi$ and $c^* \in \Lambda$ such that $\lim_{r \to c^*} \tilde{\omega} = c^*$.

Then, functional (114) possesses a bounded solution.

**Proof.** Note that $(B(\bar{\beta}), \delta)$ is a complete metric, where $\delta(\mu_1, \mu_2)$ is the metric, as defined by (82). There exist $r \in \beta$, $t_1, t_2 \in V$ and $\psi: B(\bar{\beta}) \times B(\bar{\beta}) \to (0, \infty)$ such that
\[
\psi(\delta(\omega_1, \phi(\omega_1)) \cap \Lambda), \delta(\omega_1, \omega_2)) < 0, \omega_1, \omega_2 \in B(\bar{\beta}),
\]
and we have
\[
H_d[\phi_1(\omega_1)(r), \phi_1(\omega_2)(r)] \leq H_d[\psi^2(r, t, \omega_1(l(r, t))), \psi^2(r, t, \omega_2(l(r, t)))]
\]
\[
\leq \max_{r \in \bar{\beta}} \left\{ \max_{r \in \bar{\beta}} \left\{ \left[ \left[ 1 + \sqrt{T} \left( |\omega_1(r) - \omega_2(r)| \right)^\alpha \right] - 1 \right]^2 \right\} \right\}
\]
\[
\leq \left\{ \left[ 1 + \sqrt{T} \left( |\omega_1 - \omega_2| \right)^\alpha \right] - 1 \right\}.
\]

It implies that
\[
H_d[\phi_1(\omega_1)(r), \phi_1(\omega_2)(r)] \leq \left[ \left[ 1 + \sqrt{T} \left( |\omega_1 - \omega_2| \right)^\alpha \right] - 1 \right]^2.
\]

Owing to (123),
\[
1 + \sqrt{T} \left( |\omega_1 - \omega_2| \right)^\alpha \leq \left[ 1 + \sqrt{T} \left( |\omega_1 - \omega_2| \right)^\alpha \right] - 1.
\]

By $\theta \in \mathbb{Z}$ and $\theta(z) = 1 + \sqrt{z}$ with (124), we obtain
\[
\delta[H_d(\phi_1(\omega_1), \phi_1(\omega_2))]
\leq \left[ \theta(\delta(\omega_1, \omega_2)) \right] \text{ for all } \omega_1, \omega_2 \in B(\bar{\beta}).
\]

Furthermore, (1) - (3) are equivalent to (a) - (b) of Theorem 3. So, there exists a fixed point $c^* \in \Lambda$ in $B(\bar{\beta})$, which is a bounded solution of functional (117). \qed

### 4.1. An Application to Integral Inclusion

In this section, we consider the following set-valued integral inclusion:
\[
c(r) \in \kappa + U \int_{r_0}^r V(t, c(t))Vt,
\]
where $\kappa \subseteq (-\infty, \infty)$, $U$ is a bounded compact subset of $(-\infty, \infty)$, and $V(t, c(t))$ is l.s.c. Let $X = C(I)$ be the space of all continuous real-valued function and $C(I)$ is complete w.r.t the metric $\delta$, which defined by
\[
\delta(\mu_1, \mu_2) = \sup_{r \in I} \left| \mu_1(r) - \mu_2(r) \right|.
\]

Assume that there exists $\phi: B(\bar{\beta}) \to B(\bar{\beta})$ and $V: (-\infty, \infty) \times (-\infty, \infty) \to (-\infty, \infty)$ is continuous on
\[
R = \left\{ (r, \zeta): \left| r - r_0 \right| \leq \left( \frac{1}{\alpha_1} \right)^{1/2} \text{ and } |\zeta - \kappa| \leq \frac{1}{2\alpha_2} \right\},
\]

such that for
\[
\frac{1}{2} \min\{\delta(\mu_1, \phi(\mu_1) \cap \Lambda), \delta(\mu_2, \phi(\mu_2) \cap \Lambda)\} < \delta(\omega_1, \omega_2),
\]
\[
\mu_1, \mu_2 \geq 0,
\]
we have
\[
\left| V(r, c_1(r)) - V(r, c_2(r)) \right| \leq e^{\sqrt{\alpha_1/\alpha_2}} |c_1(r) - c_2(r)|^2,
\]
where $\alpha_2 = \max_{r \in \bar{\beta}} \left| r_0 \right|$, $0 < \alpha_1 \leq \alpha_2$, and $0 \leq \alpha < 1$.\head{Journal of Mathematics}
\[ |V(t, \zeta)| < \frac{1}{2\alpha_1} \left[ \frac{1}{\alpha_1} \right]^{1/2}. \]  

(131)  

Moreover, let \( \tilde{C} = \{ \zeta \in C(I) : \phi(\zeta, \kappa) \leq 1/2\alpha_3 \} \) be a closed subspace of \( C(I) \) and the operator \( \phi \) be defined by

\[ \phi(\zeta(\kappa)) \in \kappa + U \int_{r_0}^{r} V(t, \zeta(t))dt. \]  

(132)

Set \( V_x(r) = \int_{r_0}^{r} V(t, \zeta(t))dt. \) Note that

\[ \frac{d}{2} \frac{1}{\alpha_3} \left[ \frac{1}{\alpha_1} \right]^{1/2}. \]  

Consider

\[ H_d[\phi(\zeta_1(r)), \phi(\zeta_2(r))] = H_d[\kappa + UV_x(r), \kappa + UV_y(r)] \leq H_d[\kappa, \kappa] \]  

(133)

\[ = \max \left\{ \max_{\rho \in \rho(x)} \delta(\rho, UV_y(r)), \max_{\rho \in \rho(y)} \delta(\rho, UV_y(r)) \right\}. \]

Consider

\[ \max_{\rho \in \rho(x)} \delta(\rho, UV_y(r)) = \max_{\rho \in \rho(x)} \min_{\rho \in \rho(y)} \delta(\rho, \rho) \]

\[ = \max_{\rho \in \rho(x)} \min_{\rho \in \rho(y)} \delta(\rho, UV_y(r)) \]

\[ = \max_{\rho \in \rho(x)} \min_{\rho \in \rho(y)} \delta(\rho, \rho) \]

\[ \leq \max_{\rho \in \rho(x)} \min_{\rho \in \rho(y)} \delta(\rho, \rho) \]

\[ = \max_{\rho \in \rho(x)} \min_{\rho \in \rho(y)} \delta(\rho, \rho) \]

(134)

This implies that

\[ \max_{\rho \in \rho(x)} V(\rho, UV_y(r)) \leq \alpha_2 \sup_{\rho \in \rho} |V(r, \zeta_2(r)) - V(r, \zeta_1(r))| \]

(135)

**Theorem 5.** Let \( X = C(I) \) and \( \phi : (\tilde{C}, d) \rightarrow (D(\tilde{C}), H_d) \) be a l.s.c mapping. Suppose that the following assumptions hold:

(i) \( \phi \) is defined for all \( \zeta \in \tilde{C} \).

(ii) \( \phi(\zeta(r)) \) is a CS of \( \tilde{C} \) for all \( \zeta \in \tilde{C} \).

Then, owing to (127)–(135), integral (126) has a solution on \( I \).

**Proof.** Let \( \kappa \in I \). Then, \( |\kappa - r_0| \leq 1/2\alpha_3 \). Hence, we have

\[ |\zeta(\kappa) - \kappa| \leq 1/2\alpha_3. \]

If \( (\kappa, \zeta(\kappa)) \in ( \infty, \infty) \), the integral equation in (132) exists. Since \( \kappa \in ( \infty, \infty) \) is continuous, \( \kappa \) is defined for all \( \kappa \in \tilde{C} \). Next, let \( \theta(\kappa) \in \phi(\zeta(\kappa)) \). Then, \( \theta(\kappa) = \kappa + \Pi \) for \( \Pi \in U \):

\[ \theta(\kappa) = \kappa + \Pi \]

Thus, \( |\theta(\kappa) - \kappa| \leq 1/2\alpha_3 \) for each \( \theta(\kappa) \in \phi(\zeta(\kappa)) \). So, \( \phi(\zeta(\kappa)) \) is a subset of \( \tilde{C} \). Now, let \( \{ \zeta \} \subseteq \phi(\zeta(\kappa)) \); then, \( \zeta = \kappa + \Pi \) for \( \Pi \in U \). Since \( U \) is compact, there exists subsequence \( \Pi \) convergent to \( \Pi \) such that \( \Pi \) converges to \( \Pi \) in \( U \). Let \( \Pi = \kappa + \Pi \), then,
\[ d(\tilde{u}_t, \tilde{u}) = \sup_{r \in I} \left| \tilde{u}_t - \tilde{u} \right| V_x(r) \]
\[ \leq \left| \tilde{u}_t - t \tilde{u} \right| \sup_{r \in I} V_x(r) \rightarrow 0, \text{ as } t^* \rightarrow \infty. \]  

(137)

Hence, \( \phi(\zeta) \) is a CS of \( \hat{C} \) for all \( \zeta \in \hat{C} \). Next,
\[ |V(r, \zeta_1(r)) - V(r, \zeta_2(r))| \leq \int_{t_0}^{t'} |V(t, \zeta_1(t)) - V(t, \zeta_2(t))| dt \]
\[ \leq e^{\alpha t} \int_{t_0}^{t'} \left| \zeta_1(t) - \zeta_2(t) \right|^2 dt \]
\[ = e^{\alpha t} \int_{t_0}^{t'} \left| \zeta_1(t) - \zeta_2(t) \right|^2 dt \]
\[ \leq e^{\alpha t} \int_{t_0}^{t'} \left( \zeta_1(t) - \zeta_2(t) \right) \left( \zeta_2(t) - \zeta_1(t) \right) dt \]
\[ = e^{\alpha t} \int_{t_0}^{t'} \left( \zeta_1(t) - \zeta_2(t) \right) \left( \zeta_2(t) - \zeta_1(t) \right) dt \]
\[ = e^{\alpha t} \int_{t_0}^{t'} \left( \zeta_1(t) - \zeta_2(t) \right) \left( \zeta_2(t) - \zeta_1(t) \right) dt \]
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\[ = e^{\alpha t} \int_{t_0}^{t'} \left( \zeta_1(t) - \zeta_2(t) \right) \left( \z_