Research Article

Classification of Upper Bound Sequences of Local Fractional Metric Dimension of Rotationally Symmetric Hexagonal Planar Networks

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The term metric or distance of a graph plays a vital role in the study to check the structural properties of the networks such as complexity, modularity, centrality, accessibility, connectivity, robustness, clustering, and vulnerability. In particular, various metrics or distance-based dimensions of different kinds of networks are used to resolve the problems in different strata such as in security to find a suitable place for fixing sensors for security purposes. In the field of computer science, metric dimensions are most useful in aspects such as image processing, navigation, pattern recognition, and integer programming problem. Also, metric dimensions play a vital role in the field of chemical engineering, for example, the problem of drug discovery and the formation of different chemical compounds are resolved by means of some suitable metric dimension algorithm. In this paper, we take rotationally symmetric and hexagonal planar networks with all possible faces. We find the sequences of the local fractional metric dimensions of proposed rotationally symmetric and planar networks. Also, we discuss the boundedness of sequences of local fractional metric dimensions. Moreover, we summarize the sequences of local fractional metric dimension by means of their graphs.

1. Introduction

The subject graph theory has a vital role in computer science, medical, chemical, networking, image processing, navigation, pattern recognition, drug discovery, chemical formation, sensors fixing, circuit designing, and many other fields. In graph theory, the term distance or metric plays a crucial role in solving many problems, in which we want to set the minimum distance between different objects to fixed objects, for example, if we want to fix a sensor for recognition of object, then we will fix the sensor to minimum distance from each object for better result. Moreover, when we check the structural properties of the networks such as complexity, modularity, centrality, accessibility, connectivity, robustness, clustering, and vulnerability, then metric or distance is very useful.

A graph or network is linked between two sets: one of them is a set of nodes and the other one is a set of edges. The set of vertices is denoted by $V(G)$, and $E(G)$ represents a set of edges which is a subset of $V(G) \times V(G)$. The distance or metric between two vertices x and y is defined as the number of minimum edges between x and y ; it is denoted as $d(x, y)$. For more details and insights into the nexus of graph theory, we refer the readers to [1–3].

A subset $S = \{s_1, s_2, \dots, s_t\}$ of $V(G)$ is known as resolving set if $a \in V(G)$, with k -tuple metric form represented as $r(x|S) = (d(x, s_1), d(x, s_2), \dots, d(x, s_k))$. Then, S will be a resolving set with k elements if $r(x|S) \neq r(y|S)$ for each pair of distinct vertices x and y . The minimum number set S is known as the basis of G . The cardinality of the set S is called the metric dimension of G ; it is denoted by $\beta(G)$ in [4, 5].

Slatter investigated the term resolving set of connected graphs by using the notion of locating set [6–8]. Harary and Melter [9] investigated individually the aforementioned terminologies and proclaimed them as the metric dimension of a graph. A family of graphs is bounded (constant) with respect to metric dimension if each graph in this family has bounded (constant) metric dimension, respectively. The metric dimension of path and cycle has been investigated in [10]. In [4], it is shown that Peterson, anti-prism, and circulates networks have constant metric dimension. The metric dimension of wheel and Jahangir graphs was found in [11]. In [12], the authors have shown that the generalized Peterson network $P(n, 3)$ has a bounded metric dimension. Liu et al. discussed many aspects of graph theory such as indices, metric dimension domination number, and so on in [13–21].

In 2000, Chartrand et al. found the solution of integer programming problem (IPP) by means of metric dimension [4]. For a more accurate solution of integer programming problem (IPP), Currie and Oellermann introduced the concept of fractional metric dimension in [22]. Fehr et al. found an optimal solution of a certain linear programming relaxation problem by means of fractional metric dimension [23]. Liu et al. discussed many interesting results on the fractional metric dimension (FMD) in [20]. Many results of fractional metric dimension on several networks such as networks obtained from Cartesian, hierarchical, corona, lexicographic, and comb products of connected networks have been discussed in [4, 20, 24, 25]. The fractional metric dimension of the generalized Jahangir network was computed by Liu et al. in [26]. The local fractional metric dimensions of planar networks were computed in [27]. In 2019, Aisyah et al. introduced the concept of local fractional metric dimension (LFMD) and computed it for the corona product of two graphs [28]. Many researchers discussed this area in [5, 6, 8, 29–31]. A new family of graphs based on totient, super totient, and hyper totient numbers was proposed in [32–36].

In this paper, we choose the family of rotationally symmetric hexagonal planar networks with all possible faces

and then find their sequences of local fractional metric dimension (LFMD). Moreover, we summarize sequences of local fractional metric dimensions by means of discrete graphs. This paper proceeds in the following sequence. Section 2 gives the preliminaries. In Section 3, we define the vertex and edge sets of rotationally symmetric hexagonal planar networks and their subsections; we find the sequences of these networks by introducing lemmas and theorems. In Section 4, we give the boundedness table of sequences of local fractional metric dimension (LFMD) and also plot their discrete graphs.

2. Preliminaries

Let G be a connected graph. A pair of vertices $\{s, t\} \subseteq V(G)$ is a resolved set by a vertex u if $d(s, u) \neq d(t, u)$. The set of resolving neighborhood of pair $\{s, t\}$ is defined as $\mathcal{R} = \{u \in V(G) \mid d(s, u) \neq d(t, u)\}$. Let G be a connected graph of order n . A function $\vartheta: V(G) \rightarrow [0, 1]$ is termed as resolving function of G if $\vartheta(\mathcal{R}\{s, t\}) \geq 1, \forall s, t \in V(G)$, where

$$\vartheta(\mathcal{R}\{s, t\}) = \sum_{u \in \mathcal{R}\{s, t\}} \vartheta(u). \quad (1)$$

A resolving function ϑ' of G is called a minimal resolving function if there exists $u \in V(G)$ which is not a resolving function of G and if any other function $\vartheta: V(G) \rightarrow [0, 1]$ with $\vartheta \leq \vartheta', \vartheta(u) \neq \vartheta'(u)$. The fractional metric dimension (FMD) of the graph G is defined as $\dim_f(G) = \min\{|\vartheta'|: \vartheta' \text{ is the minimal resolving function of } G\}$, where

$$|\vartheta'| = \sum_{u \in V(G)} \vartheta'(u). \quad (2)$$

The resolving function will be a local resolving function if $\vartheta'(\mathcal{R}\{s, t\}) \geq 1$ and the fractional metric dimension becomes a local fractional metric dimension (LFMD) if we assume the pair of adjacent vertices only; it is denoted by $\dim_{lf}(G)$ [20, 28]. The LFMD of given graph in Figure 1 is $\dim_{lf}(G) \leq (12/7)$.

The local resolving neighborhoods are

$$\begin{aligned} \mathcal{R}\{x_7x_{13}\} &= V(G) - \{x_2, x_5, x_8, x_{12}\}, \\ \mathcal{R}\{x_1x_7\} &= V(G) - \{x_{10}, x_{14}, x_{15}, x_{16}, x_{20}, x_{21}\}, \\ \mathcal{R}\{x_9x_{15}\} &= V(G) - \{x_1, x_4, x_8, x_{10}\}, \\ \mathcal{R}\{x_7x_6\} &= V(G) - \{x_{10}, x_{16}, x_{17}, x_{18}, x_{22}, x_{23}\}, \\ \mathcal{R}\{x_8x_{14}\} &= V(G) - \{x_3, x_6, x_7, x_9\}, \\ \mathcal{R}\{x_3x_9\} &= V(G) - \{x_{12}, x_{16}, x_{17}, x_{18}, x_{22}, x_{23}\}, \\ \mathcal{R}\{x_{12}x_{18}\} &= V(G) - \{x_1, x_4, x_7, x_{11}\}, \\ \mathcal{R}\{x_2x_9\} &= V(G) - \{x_{12}, x_{13}, x_{14}, x_{18}, x_{19}, x_{24}\}, \\ \mathcal{R}\{x_{11}x_{17}\} &= V(G) - \{x_2, x_6, x_{10}, x_{12}\}, \\ \mathcal{R}\{x_2x_8\} &= V(G) - \{x_{11}, x_{15}, x_{16}, x_{17}, x_{21}, x_{22}\}, \end{aligned}$$

FIGURE 1: Hexagonal planar graph G with $\mathcal{D}\text{im}_{lf}(G) \leq (12/7)$.

$$\begin{aligned}
\mathcal{R}\{x_{10}x_{16}\} &= V(G) - \{x_2, x_5, x_9, x_{11}\}, \\
\mathcal{R}\{x_1x_8\} &= V(G) - \{x_{11}, x_{13}, x_{17}, x_{18}, x_{23}, x_{24}\}, \\
\mathcal{R}\{x_{13}x_{18}\} &= V(G) - \{x_3, x_6, x_{21}, x_{24}\}, \\
\mathcal{R}\{x_6x_{12}\} &= V(G) - \{x_9, x_{13}, x_{14}, x_{15}, x_{19}, x_{20}\}, \\
\mathcal{R}\{x_{18}x_{17}\} &= V(G) - \{x_2, x_5, x_{20}, x_{23}\}, \\
\mathcal{R}\{x_5x_{12}\} &= V(G) - \{x_9, x_{15}, x_{16}, x_{17}, x_{21}, x_{22}\}, \\
\mathcal{R}\{x_{16}x_{17}\} &= V(G) - \{x_1, x_4, x_9, x_{22}\}, \\
\mathcal{R}\{x_5x_{11}\} &= V(G) - \{x_8, x_{13}, x_{14}, x_{18}, x_{19}, x_{24}\}, \\
\mathcal{R}\{x_{15}x_{16}\} &= V(G) - \{x_3, x_6, x_{21}, x_{24}\}, \\
\mathcal{R}\{x_4x_{11}\} &= V(G) - \{x_8, x_{14}, x_{15}, x_{16}, x_{20}, x_{21}\}, \\
\mathcal{R}\{x_{14}x_{15}\} &= V(G) - \{x_2, x_5, x_{20}, x_{23}\}, \\
\mathcal{R}\{x_4x_{10}\} &= V(G) - \{x_7, x_{13}, x_{17}, x_{18}, x_{23}, x_{24}\}, \\
\mathcal{R}\{x_{13}x_{14}\} &= V(G) - \{x_1, x_4, x_{19}, x_{22}\}, \\
\mathcal{R}\{x_3x_{10}\} &= V(G) - \{x_7, x_{13}, x_{14}, x_{15}, x_{19}, x_{20}\}, \\
\mathcal{R}\{x_{13}x_{24}\} &= \mathcal{R}\{x_{15}x_{21}\} = V(G) - \{x_4, x_5, x_{10}, x_{11}, x_{12}, x_{16}, x_{17}, x_{18}, x_{22}, x_{23}\}, \\
\mathcal{R}\{x_{18}x_{24}\} &= \mathcal{R}\{x_{16}x_{21}\} = V(G) - \{x_1, x_2, x_7, x_8, x_9, x_{13}, x_{14}, x_{15}, x_{19}, x_{20}\}, \\
\mathcal{R}\{x_{18}x_{23}\} &= \mathcal{R}\{x_{14}x_{20}\} = V(G) - \{x_3, x_4, x_9, x_{10}, x_{11}, x_{15}, x_{16}, x_{17}, x_{21}, x_{22}\}, \\
\mathcal{R}\{x_{17}x_{23}\} &= \mathcal{R}\{x_{15}x_{20}\} = V(G) - \{x_1, x_6, x_7, x_8, x_{12}, x_{13}, x_{14}, x_{18}, x_{19}, x_{24}\}, \\
\mathcal{R}\{x_{17}x_{22}\} &= \mathcal{R}\{x_{13}x_{19}\} = V(G) - \{x_2, x_3, x_8, x_9, x_{10}, x_{14}, x_{15}, x_{16}, x_{20}, x_{21}\}, \\
\mathcal{R}\{x_{16}x_{22}\} &= \mathcal{R}\{x_{14}x_{19}\} = V(G) - \{x_5, x_6, x_7, x_{11}, x_{12}, x_{13}, x_{17}, x_{18}, x_{23}, x_{24}\}.
\end{aligned} \tag{3}$$

The cardinality of each of the local resolving neighborhoods is either 20, 16, or 14. Since 14 is less than the

cardinality of all other resolving neighborhoods, we construct a mapping $\vartheta: V(G) \rightarrow [0, 1]$ as

$$\vartheta(x_i) = \vartheta(x_j) = \frac{1}{14}, \quad i \in \{13, 16, 17, 18\}, j \in \{22, 23, 24\}. \quad (4)$$

Since 14 is a minimal cardinal number, there must exist a minimal resolving function ϑ' such that $|\vartheta'| \leq |\vartheta|$. Hence,

$$\mathcal{D}im_{lf}(G) \leq \sum_{i=1}^{24} \frac{1}{14} = \frac{12}{7}. \quad (5)$$

3. Rotationally Symmetric Hexagonal Planar Networks

In this section, we define hexagonal circular ladder networks. In Figure 2, the different hexagonal networks are given with faces.

The vertex set and edge set of hexagonal circular ladder networks are defined as

$$\begin{aligned} V(H_n^j) &= \{s_i, t_i, u_i, v_i | 1 \leq i \leq n\}, \quad j = 1, 2, 3, 4, 5, \\ E(H_n^1) &= \{s_i t_i, t_i u_i, u_i v_i, u_i v_{i-1}, s_i t_{i+1} | 1 \leq i \leq n, t_{n+1} = t_1, v_0 = v_n\}, \\ E(H_n^2) &= \{s_i t_i, t_i u_i, u_i v_i, u_i v_{i-1}, s_i t_{i+1}, s_i v_i | 1 \leq i \leq n, t_{n+1} = t_1, v_0 = v_n\}, \\ E(H_n^3) &= \{s_i t_i, t_i u_i, u_i v_i, u_i v_{i-1}, s_i t_{i+1}, u_i u_{i+1} | 1 \leq i \leq n, t_{n+1} = t_1, u_{n+1} = u_1, v_0 = v_n\}, \\ E(H_n^4) &= \{s_i t_i, t_i u_i, u_i v_i, u_i v_{i-1}, s_i t_{i+1}, s_i u_i | 1 \leq i \leq n, t_{n+1} = t_1, v_0 = v_n\}, \\ E(H_n^5) &= \{s_i t_i, t_i u_i, u_i v_i, u_i v_{i-1}, s_i t_{i+1}, t_i u_{i+1} | 1 \leq i \leq n, t_{n+1} = t_1, u_{n+1} = u_1, v_0 = v_n\}, \end{aligned} \quad (6)$$

and hexagonal circular ladder networks H_n^i , $1 \leq i \leq 5$, are shown in Figure 3.

The vertex set and edge set of hexagonal circular ladder networks H_n^i , $6 \leq i \leq 11$, are defined as

$$\begin{aligned} V(H_n^j) &= \{s_i, t_i, u_i, v_i | 1 \leq i \leq n\}, \quad j = 6, 7, 8, 9, 10, 11, \\ E(H_n^6) &= \{s_i t_i, t_i u_i, u_i v_i, u_i v_{i-1}, s_i t_{i+1}, s_i v_i, s_i u_i | 1 \leq i \leq n, t_{n+1} = t_1, v_0 = v_n\}, \\ E(H_n^7) &= \{s_i t_i, t_i u_i, u_i v_i, u_i v_{i-1}, s_i t_{i+1}, t_i t_{i+1}, t_i u_{i+1} | 1 \leq i \leq n, t_{n+1} = t_1, u_{n+1} = u_1, v_0 = v_n\}, \\ E(H_n^8) &= \{s_i t_i, t_i u_i, u_i v_i, u_i v_{i-1}, s_i t_{i+1}, t_i v_i, t_i t_{i+1} | 1 \leq i \leq n, t_{n+1} = t_1, v_0 = v_n\}, \\ E(H_n^9) &= \{s_i t_i, t_i u_i, u_i v_i, u_i v_{i-1}, s_i t_{i+1}, t_i v_i, t_i u_{i+1} | 1 \leq i \leq n, t_{n+1} = t_1, u_{n+1} = u_1, v_0 = v_n\}, \\ E(H_n^{10}) &= \{s_i t_i, t_i u_i, u_i v_i, u_i v_{i-1}, s_i t_{i+1}, u_i u_{i+1}, t_i t_{i+1} | 1 \leq i \leq n, t_{n+1} = t_1, u_{n+1} = u_1, v_0 = v_n\}, \\ E(H_n^{11}) &= \{s_i t_i, t_i u_i, u_i v_i, u_i v_{i-1}, s_i t_{i+1}, t_i v_i, s_i u_{i+1} | 1 \leq i \leq n, t_{n+1} = t_1, u_{n+1} = u_1, v_0 = v_n\}, \end{aligned} \quad (7)$$

and hexagonal circular ladder networks H_n^i , $6 \leq i \leq 11$, are shown in Figure 4.

The vertex set and edge set of hexagonal circular ladder networks H_n^i , $12 \leq i \leq 17$, are defined as

$$\begin{aligned} V(H_n^j) &= \{s_i, t_i, u_i, v_i | 1 \leq i \leq n\}, \quad j = 12, 13, 14, 15, 16, 17, \\ E(H_n^{12}) &= \{s_i t_i, t_i u_i, u_i v_i, u_i v_{i-1}, s_i t_{i+1}, s_i u_i, u_i u_{i+1}, s_i u_{i+1} | 1 \leq i \leq n, t_{n+1} = t_1, u_{n+1} = u_1, v_0 = v_n\}, \\ E(H_n^{13}) &= \{s_i t_i, t_i u_i, u_i v_i, u_i v_{i-1}, s_i t_{i+1}, t_i t_{i+1}, u_i u_{i+1}, t_i u_{i+1} | 1 \leq i \leq n, t_{n+1} = t_1, u_{n+1} = u_1, v_0 = v_n\}, \\ E(H_n^{14}) &= \{s_i t_i, t_i u_i, u_i v_i, u_i v_{i-1}, s_i t_{i+1}, t_i v_i, s_i v_i, s_i u_{i+1} | 1 \leq i \leq n, t_{n+1} = t_1, u_{n+1} = u_1, v_0 = v_n\}, \\ E(H_n^{15}) &= \{s_i t_i, t_i u_i, u_i v_i, u_i v_{i-1}, s_i t_{i+1}, t_i v_i, t_i u_{i+1}, s_i u_{i+1} | 1 \leq i \leq n, t_{n+1} = t_1, u_{n+1} = u_1, v_0 = v_n\}, \\ E(H_n^{16}) &= \{s_i t_i, t_i u_i, u_i v_i, u_i v_{i-1}, s_i t_{i+1}, s_i u_i, s_i v_i, s_i u_{i+1} | 1 \leq i \leq n, t_{n+1} = t_1, u_{n+1} = u_1, v_0 = v_n\}, \\ E(H_n^{17}) &= \{s_i t_i, t_i u_i, u_i v_i, u_i v_{i-1}, s_i t_{i+1}, t_i t_{i+1}, u_i t_{i+1}, v_i t_{i+1} | 1 \leq i \leq n, t_{n+1} = t_1, u_{n+1} = u_1, v_0 = v_n\}, \end{aligned} \quad (8)$$

FIGURE 2: Hexagonal planar graphs with 2, 3, 4, and 5 faces.

FIGURE 3: Rotationally symmetric hexagonal planar graphs with 2 and 3 faces.

FIGURE 4: Rotationally symmetric hexagonal planar graphs with 4 faces.

and hexagonal circular ladder networks H_n^i , $12 \leq i \leq 17$, are shown in Figure 5.

3.1. LFMD of Rotationally Symmetric Hexagonal Planar Networks with 2 and 3 Faces. In this section, we find the sequences of upper bounds of local fractional metric dimension of rotationally symmetric hexagonal planar networks with 2 and 3 faces.

Javaid et. al. [37] proved that if a network is a bipartite, then its local fractional metric should be one. Since our proposed networks H_n^m , $m = 1, 2, 5$, for each n are bipartite, we have the following result.

Theorem 1. *The local fractional metric dimension of rotationally symmetric hexagonal planar network H_n^m , $m = 1, 2, 5$, is 1. That is, $\dim_{lf}(H_n^m) = 1$.*

Lemma 1. *Let H_n^3 be a hexagonal planar circular network for $n \geq 5$ with $n \equiv 1 \pmod{2}$.*

- (1) Then, for $1 \leq k \leq n$, $|\mathcal{R}_k| = |\mathcal{R}\{u_k v_k\}| = |\mathcal{R}\{u_k v_{k-1}\}| = 2(n+1)$, with $v_0 = v_n$ and $|\cup_{k=1}^n \mathcal{R}_k| = 4n$.
- (2) $|\mathcal{R}_k| < |\mathcal{R}\{s_k t_k\}|$ and $|\mathcal{R}\{s_k t_k\} \cap \cup_{k=1}^n \mathcal{R}_k| \geq |\mathcal{R}_k|$.
- (3) $|\mathcal{R}_k| < |\mathcal{R}\{s_k t_{k+1}\}|$ and $|\mathcal{R}\{s_k t_{k+1}\} \cap \cup_{k=1}^n \mathcal{R}_k| \geq |\mathcal{R}_k|$ with $t_{n+1} = t_1$.
- (4) $|\mathcal{R}_k| < |\mathcal{R}\{t_k u_k\}|$ and $|\mathcal{R}\{t_k u_k\} \cap \cup_{k=1}^n \mathcal{R}_k| \geq |\mathcal{R}_k|$.
- (5) $|\mathcal{R}_k| < |\mathcal{R}\{u_k u_{k+1}\}|$ and $|\mathcal{R}\{u_k u_{k+1}\} \cap \cup_{k=1}^n \mathcal{R}_k| \geq |\mathcal{R}_k|$ with $u_{n+1} = u_1$.

Proof

- (1) The set of local resolving neighborhoods of $u_k v_k$ and $u_k v_{k-1}$ is $\mathcal{R}\{u_k v_k\} = V(H_n^3) - \{s_r | r \equiv k+1, k+2\}$,

FIGURE 5: Rotationally symmetric hexagonal planar graphs with 5 faces.

- $\dots, k-1 + (n+1/2)(\text{mod } n)\} \cup \{t_r | r \equiv k+1, k+2, \dots, k-1 + (n+1/2)(\text{mod } n)\} \cup \{u_r | r \equiv k+1, k+2, \dots, k-1 + (n+1/2)(\text{mod } n)\} \cup \{v_r | r \equiv k+1, k+2, \dots, k-1 + (n+1/2)(\text{mod } n)\}$ and $\mathcal{R}\{u_k v_{k-1}\} = V(H_n^3) - \{s_r | r \equiv k + (n-1/2), k+1 + (n-1/2), \dots, k+n-2 (\text{mod } n)\} \cup \{t_r | r \equiv k+1 + (n-1/2), k+2 + (n-1/2), \dots, k+n-1 (\text{mod } n)\} \cup \{u_r | r \equiv k+1 + (n-1/2), k+2 + (n-1/2), \dots, k+n-1 (\text{mod } n)\} \cup \{v_r | r \equiv k + (n-1/2), k+1 + (n-1/2), \dots, k+n-2 (\text{mod } n)\}$. Clearly, $\cup_{k=1}^n \mathcal{R}_k = V(H_n^3)$. Therefore, $|\cup_{k=1}^n \mathcal{R}_k| = 4n$.
- (2) The set of local resolving neighborhoods of $s_k t_k$ is $\mathcal{R}\{s_k t_k\} = V(H_n^3) - \{s_r | r \equiv k+3, k+4, \dots, k+(n+1/2)-1 (\text{mod } n), n \geq 7\} \cup \{t_r | r \equiv k+3, k+4, \dots, k+(n+1/2)-1 (\text{mod } n), n \geq 7\} \cup \{u_r | r \equiv k+1, k+2, \dots, k+(n+1/2)-1 (\text{mod } n), n \geq 5\} \cup \{v_r | r \equiv k+1, k+2, \dots, k+(n+1/2)-1 (\text{mod } n), n \geq 5\}$. Hence, $|\mathcal{R}\{s_k t_k\}| = 2n+6$. Since $|\mathcal{R}_k| = 2(n+1) < |\mathcal{R}\{s_k t_k\}| = 2n+6$, $|\mathcal{R}\{s_k t_k\} \cap \cup_{k=1}^n \mathcal{R}_k| = 2n+6 \geq |\mathcal{R}_k|$.
- (3) The set of local resolving neighborhoods of $s_k t_{k+1}$, where $t_{n+1} = t_1$, is $\mathcal{R}\{s_k t_{k+1}\} = V(H_n^3) - \{s_r | r \equiv k+1 + (n-1/2), k+2 + (n-1/2), \dots, k+n-3 (\text{mod } n), n \geq 7\} \cup \{t_r | r \equiv k+2 + (n-1/2), k+3 + (n-1/2), \dots, k+n-2 (\text{mod } n), n \geq 7\} \cup \{u_r | r \equiv k (\text{mod } n), r \equiv k+2 + (n-1/2), k+3 + (n-1/2), \dots, k+n-1 (\text{mod } n)\} \cup \{v_r | r \equiv k+1 + (n-1/2), k+2 + (n-1/2), \dots, k+n-1 (\text{mod } n)\}$. Hence, $|\mathcal{R}\{s_k t_{k+1}\}| = 2n+6$. Since $|\mathcal{R}_k| = 2(n+1) < |\mathcal{R}\{s_k t_{k+1}\}| = 2n+6$, $|\mathcal{R}\{s_k t_{k+1}\} \cap \cup_{k=1}^n \mathcal{R}_k| = 2n+6 \geq |\mathcal{R}_k|$.
- (4) The set of local resolving neighborhoods of $t_k u_k$ is $\mathcal{R}\{t_k u_k\} = V(H_n^3) - \{s_r | r \equiv k+1 (\text{mod } n), r \equiv k+n-2 (\text{mod } n)\} \cup \left\{ \begin{array}{l} t_r | r \equiv k+1 (\text{mod } n), \\ r \equiv k+n-1 (\text{mod } n) \end{array} \right\}$. Hence,

$|\mathcal{R}\{t_k u_k\}| = 4(n-1)$. Since $|\mathcal{R}_k| = 2(n+1) < |\mathcal{R}\{t_k u_k\}| = 4(n-1)$, $|\mathcal{R}\{t_k u_k\} \cap \bigcup_{k=1}^n \mathcal{R}_k| = 4(n-1) \geq |\mathcal{R}_k|$.

- (5) The set of local resolving neighborhoods of $u_k u_{k+1}$, where $u_{n+1} = u_1$, is $\mathcal{R}\{u_k u_{k+1}\} = V(H_n^3) - \{s_r | r \equiv k \pmod{n}\} \cup \{t_r | r \equiv k + (n+3/2) - 1 \pmod{n}\} \cup \{u_r | r \equiv k + (n+3/2) - 1 \pmod{n}\} \cup \{v_r | r \equiv k \pmod{n}\}$. Hence, $|\mathcal{R}\{u_k u_{k+1}\}| = 4(n-1)$. Since $|\mathcal{R}_k| = 2$

$$(n+1) < |\mathcal{R}\{u_k u_{k+1}\}| = 4(n-1), \quad |\mathcal{R}\{u_k u_{k+1}\} \cap \bigcup_{k=1}^n \mathcal{R}_k| = 4(n-1) \geq |\mathcal{R}_k|. \quad \square$$

Theorem 2. If H_n^3 is a hexagonal planar circular network for $n \geq 3$ with $n \equiv 1 \pmod{2}$, then $\mathcal{D}\text{im}_{lf}(H_n^m) \leq (2n/n+1)$.

Proof. The local resolving neighborhoods are

Case (a): for $n = 3$, $1 \leq k \leq 3$, by Lemma 1, we have

$$\begin{aligned} \overline{R_1} &= \mathcal{R}\{u_k v_k\} = V(H_3^3) - \{s_r | r \equiv k+1 \pmod{3}\} \cup \{t_r | r \equiv k+1 \pmod{3}\} \\ &\quad \cup \{u_r | r \equiv k+1 \pmod{3}\} \cup \{v_r | r \equiv k+1 \pmod{3}\}, \\ \overline{R_2} &= \mathcal{R}\{u_k v_{k-1}\} = V(H_3^3) - \{s_r | r \equiv k+1 \pmod{3}\} \cup \{t_r | r \equiv k+2 \pmod{3}\} \\ &\quad \cup \{u_r | r \equiv k+2 \pmod{3}\} \cup \{v_r | r \equiv k+1 \pmod{3}\}, \\ \overline{R_3} &= \mathcal{R}\{u_k u_{k+1}\} = V(H_3^3) - \{s_r | r \equiv k \pmod{3}\} \cup \{t_r | r \equiv k+2 \pmod{3}\} \cup \{u_r | r \equiv k+2 \pmod{3}\} \\ &\quad \cup \{v_r | r \equiv k \pmod{3}\}, \\ R_1 &= \mathcal{R}\{s_k t_k\} = V(H_3^3) - \{u_r | r \equiv k+1 \pmod{3}\} \cup \{v_r | r \equiv k+1 \pmod{3}\}, \\ R_2 &= \mathcal{R}\{s_k t_{k+1}\} = V(H_3^3) - \{u_r | r \equiv k \pmod{3}\} \cup \{v_r | r \equiv k+1 \pmod{3}\}, \\ R_3 &= \mathcal{R}\{t_k u_k\} = V(H_3^3) - \{s_r | r \equiv k+1 \pmod{3}\} \cup \{t_r | r \equiv k+1 \pmod{3}, r \equiv k+2 \pmod{3}\}. \end{aligned} \quad (9)$$

The cardinality of each of the local resolving neighborhoods $\overline{R_1}, \overline{R_2}, \overline{R_3}, R_1, R_2$, and R_3 for each $k = 1, 2, 3$ is 8, 8, 8, 10, 10, and 9, respectively. Since cardinality of $\overline{R_1}, \overline{R_2}$, and $\overline{R_3}$ is 8 which is less than the cardinality of all other resolving neighborhoods and $\bigcup_{i=1}^3 R_i = V(H_3^3)$, this implies that $|\overline{R_j} \cap \bigcup_{i=1}^3 R_i| \geq |R_i|$. Now, we construct a mapping $\vartheta: V(H_3^3) \rightarrow [0, 1]$ as

$$\vartheta(u_k) = \vartheta(v_{k-1}) = \vartheta(v_k) = \vartheta(u_{k+1}) = \frac{1}{8}. \quad (10)$$

Since 8 is a very small cardinal number, there must exist a minimal resolving function ϑ' such that $|\vartheta'| \leq |\vartheta|$. Hence,

$$\mathcal{D}\text{im}_{lf}(H_3^3) \leq \sum_{i=1}^{12} \frac{1}{8} = \frac{3}{2}. \quad (11)$$

Case (b): for $n = 5$, $1 \leq k \leq 5$, by Lemma 2, we have

$$\begin{aligned} \overline{R_1} &= \mathcal{R}\{u_k v_k\} = V(H_5^3) - \{s_r | r \equiv k+1, k+2 \pmod{5}\} \cup \{t_r | r \equiv k+1, k+2 \pmod{5}\} \cup \\ &\quad \cdot \{u_r | r \equiv k+1, k+2 \pmod{5}\} \cup \{v_r | r \equiv k+1, k+2 \pmod{5}\}, \\ \overline{R_2} &= \mathcal{R}\{u_k v_{k-1}\} = V(H_5^3) - \{s_r | r \equiv k+2, k+3 \pmod{5}\} \cup \{t_r | r \equiv k+3, k+4 \pmod{5}\} \cup \\ &\quad \cdot \{u_r | r \equiv k+3, k+4 \pmod{5}\} \cup \{v_r | r \equiv k+2, k+3 \pmod{5}\}, \\ R_1 &= \mathcal{R}\{s_k t_k\} = V(H_5^3) - \{u_r | r \equiv k+1, k+2 \pmod{5}\} \cup \{v_r | r \equiv k+1, k+2 \pmod{5}\}, \\ R_2 &= \mathcal{R}\{s_k t_{k+1}\} = V(H_5^3) - \{u_r | r \equiv k \pmod{5}, r \equiv k+4 \pmod{5}\} \cup \{v_r | r \equiv k+3, k+4 \pmod{5}\}, \\ R_3 &= \mathcal{R}\{t_k u_k\} = V(H_5^3) - \{s_r | r \equiv k+1 \pmod{5}, r \equiv k+3 \pmod{5}\} \cup \{t_r | r \equiv k+1 \pmod{5}, r \equiv k+4 \pmod{5}\}, \\ R_4 &= \mathcal{R}\{u_k u_{k+1}\} = V(H_5^3) - \left\{ s_r | r \equiv k \pmod{5}, \cup \{t_r | r \equiv k+3 \pmod{5}\} \right. \\ &\quad \left. \cup \{u_r | r \equiv k+3 \pmod{5}\} \cup \{v_r | r \equiv k \pmod{5}\} \right\}. \end{aligned} \quad (12)$$

The cardinality of each of the local resolving neighborhoods $\overline{R_1}, \overline{R_2}, R_1, R_2, R_3$, and R_4 for each $1 \leq k \leq 5$ is

12, 12, 16, 16, 16, and 16, respectively. Since cardinality of $\overline{R_1}$ and $\overline{R_2}$ is 12 which is less than the cardinality of all

other resolving neighborhoods and $\bigcup_{i=1}^4 R_i = V(H_5^3)$, this implies that $|\overline{R}_j \cap \bigcup_{i=1}^4 R_i| \geq |R_i|$. Now, we construct a mapping $\vartheta: V(H_5^3) \rightarrow [0, 1]$ as

$$\vartheta(u_k) = \vartheta(v_{k-1}) = \vartheta(v_k) = \frac{1}{12}. \quad (13)$$

Since 12 is a very small cardinal number, there must exist a minimal resolving function ϑ' such that $|\vartheta'| \leq |\vartheta|$. Hence,

$$\mathcal{Dim}_{lf}(H_5^3) \leq \sum_{i=1}^{20} \frac{1}{12} = \frac{5}{3}. \quad (14)$$

Case (c): for $n = 7$, $1 \leq k \leq 7$, by Lemma 2, we have

$$\begin{aligned} \overline{R}_1 &= \mathcal{R}\{u_k v_k\} = V(H_7^3) - \{s_r | r \equiv k+1, k+2, k+3 \pmod{7}\} \cup \{t_r | r \equiv k+1, k+2, k+3 \pmod{7}\} \\ &\quad \cup \{u_r | r \equiv k+1, k+2, k+3 \pmod{7}\} \cup \{v_r | r \equiv k+1, k+2, k+3 \pmod{7}\}, \\ \overline{R}_2 &= \mathcal{R}\{u_k v_{k-1}\} = V(H_7^3) - \{s_r | r \equiv k+3, k+4, k+5 \pmod{7}\} \cup \{t_r | r \equiv k+4, k+5, k+6 \pmod{7}\} \\ &\quad \cup \{u_r | r \equiv k+4, k+5, k+6 \pmod{7}\} \cup \{v_r | r \equiv k+3, k+4, k+5 \pmod{7}\}, \\ R_1 &= \mathcal{R}\{s_k t_k\} = V(H_7^3) - \{s_r | r \equiv k+3 \pmod{7}\} \cup \{t_r | r \equiv k+3 \pmod{7}\} \\ &\quad \cup \{u_r | r \equiv k+1, k+2, k+3 \pmod{7}\} \cup \{v_r | r \equiv k+1, k+2, k+3 \pmod{7}\}, \\ R_2 &= \mathcal{R}\{s_k t_{k+1}\} = V(H_7^3) - \{s_r | r \equiv k+3 \pmod{7}\} \cup \{t_r | r \equiv k+5 \pmod{7}\} \\ &\quad \cup \{u_r | r \equiv k \pmod{7}, r \equiv k+5, k+6 \pmod{7}\} \cup \{v_r | r \equiv k+4, k+5, k+6 \pmod{7}\}, \\ R_3 &= \mathcal{R}\{t_k u_k\} = V(H_7^3) - \{s_r | r \equiv k+1 \pmod{7}, r \equiv k+5 \pmod{7}\} \cup \\ &\quad \cdot \{t_r | r \equiv k+1 \pmod{7}, r \equiv k+6 \pmod{7}\}, \\ R_4 &= \mathcal{R}\{u_k u_{k+1}\} = V(H_7^3) - \left\{ \begin{array}{l} s_r | r \equiv k \pmod{7}, \cup \{t_r | r \equiv k+4 \pmod{7}\} \\ \cup \{u_r | r \equiv k+4 \pmod{7}\} \cup \{v_r | r \equiv k \pmod{7}\} \end{array} \right\}. \end{aligned} \quad (15)$$

The cardinality of each of the local resolving neighborhoods $\overline{R}_1, \overline{R}_2, R_1, R_2, R_3$, and R_4 for each $1 \leq k \leq 7$ is 16, 16, 20, 20, 24, and 24, respectively. Since cardinality of $\overline{R}_1$ and $\overline{R}_2$ is 16 which is less than the cardinality of all other resolving neighborhoods and $\bigcup_{i=1}^4 R_i = V(H_7^3)$, this implies that $|\overline{R}_j \cap \bigcup_{i=1}^4 R_i| \geq |R_i|$. Now, we construct a mapping $\vartheta: V(H_7^3) \rightarrow [0, 1]$ as

$$\vartheta(u_k) = \vartheta(v_{k-1}) = \vartheta(v_k) = \frac{1}{16}. \quad (16)$$

Since 16 is a very small cardinal number, there must exist a minimal resolving function ϑ' such that $|\vartheta'| \leq |\vartheta|$. Hence,

$$\mathcal{Dim}_{lf}(H_7^3) \leq \sum_{i=1}^{28} \frac{1}{16} = \frac{7}{4}. \quad (17)$$

Case (d): for $n \geq 9$, $1 \leq k \leq n$, by Lemma 2, we have

$$\begin{aligned} \overline{R}_1 &= \mathcal{R}\{u_k v_k\} = V(H_n^3) - \left\{ s_r | r \equiv k+1, k+2, \dots, k-1 + \frac{n+1}{2} \pmod{n} \right\} \cup \\ &\quad \cdot \left\{ t_r | r \equiv k+1, k+2, \dots, k-1 + \frac{n+1}{2} \pmod{n} \right\} \cup \left\{ u_r | r \equiv k+1, k+2, \dots, k-1 + \frac{n+1}{2} \pmod{n} \right\} \\ &\quad \cup \left\{ v_r | r \equiv k+1, k+2, \dots, k-1 + \frac{n+1}{2} \pmod{n} \right\}, \\ \overline{R}_2 &= \mathcal{R}\{u_k v_{k-1}\} = V(H_n^3) - \left\{ s_r | r \equiv k + \frac{n-1}{2}, k+1 + \frac{n-1}{2}, \dots, k+n-2 \pmod{n} \right\} \cup \\ &\quad \cdot \left\{ t_r | r \equiv k+1 + \frac{n-1}{2}, k+2 + \frac{n-1}{2}, \dots, k+n-1 \pmod{n} \right\} \cup \end{aligned}$$

$$\begin{aligned}
& \cdot \left\{ u_r | r \equiv k+1 + \frac{n-1}{2}, k+2 + \frac{n-1}{2}, \dots, k+n-1 \pmod{n} \right\} \cup \\
& \cdot \left\{ v_r | r \equiv k + \frac{n-1}{2}, k+1 + \frac{n-1}{2}, \dots, k+n-2 \pmod{n} \right\}, \\
R_1 = \mathcal{R}\{s_k t_k\} &= V(H_n^3) - \left\{ s_r | r \equiv k+3, k+4, \dots, k + \frac{n+1}{2} - 1 \pmod{n}, n \geq 7 \right\} \cup \\
& \cdot \left\{ t_r | r \equiv k+3, k+4, \dots, k + \frac{n+1}{2} - 1 \pmod{n}, n \geq 7 \right\} \cup \\
& \cdot \left\{ u_r | r \equiv k+1, k+2, \dots, k + \frac{n+1}{2} - 1 \pmod{n}, n \geq 5 \right\} \cup \\
& \cdot \left\{ v_r | r \equiv k+1, k+2, \dots, k + \frac{n+1}{2} - 1 \pmod{n}, n \geq 5 \right\}, \\
R_2 = \mathcal{R}\{s_k t_{k+1}\} &= V(H_n^3) - \left\{ s_r | r \equiv k+1 + \frac{n-1}{2}, k+2 + \frac{n-1}{2}, \dots, k+n-3 \pmod{n}, n \geq 7 \right\} \cup \\
& \cdot \left\{ t_r | r \equiv k+2 + \frac{n-1}{2}, k+3 + \frac{n-1}{2}, \dots, k+n-2 \pmod{n}, n \geq 7 \right\} \cup \\
& \cdot \left\{ u_r | r \equiv k \pmod{n}, r \equiv k+2 + \frac{n-1}{2}, k+3 + \frac{n-1}{2}, \dots, k+n-1 \pmod{n} \right\} \cup \\
& \cdot \left\{ v_r | r \equiv k+1 + \frac{n-1}{2}, k+2 + \frac{n-1}{2}, \dots, k+n-1 \pmod{n} \right\}, \\
R_3 = \mathcal{R}\{t_k u_k\} &= V(H_n^3) - \{s_r | r \equiv k+1 \pmod{n}, r \equiv k+n-2 \pmod{n}\} \cup \\
& \cdot \{t_r | r \equiv k+1 \pmod{n}, r \equiv k+n-1 \pmod{n}\}, \\
R_4 = \mathcal{R}\{u_k u_{k+1}\} &= V(H_n^3) - \{s_r | r \equiv k \pmod{n}\} \cup \left\{ t_r | r \equiv k + \frac{n+3}{2} - 1 \pmod{n} \right\} \cup \\
& \cdot \left\{ u_r | r \equiv k + \frac{n+3}{2} - 1 \pmod{n} \right\} \cup \{v_r | r \equiv k \pmod{n}\}.
\end{aligned} \tag{18}$$

The cardinality of each of the local resolving neighborhoods $\overline{R_1}, \overline{R_2}, R_1, R_2, R_3$, and R_4 for each $1 \leq k \leq n$ is $2(n+1), 2(n+1), 2n+6, 2n+6, 4(n-1)$, and $4(n-1)$, respectively. Since cardinality of $\overline{R_1}$ and $\overline{R_2}$ is $2(n+1)$ which is less than the cardinality of all other resolving neighborhoods and $\bigcup_{i=1}^4 R_i = V(H_n^3)$, this implies that $|\overline{R_j} \cap \bigcup_{i=1}^4 R_i| \geq |R_i|$. Now, we construct a mapping $\vartheta: V(H_n^3) \longrightarrow [0, 1]$ as

$$\vartheta(u_k) = \vartheta(v_{k-1}) = \vartheta(v_k) = \frac{1}{2(n+1)}. \tag{19}$$

Since $2(n+1)$ is a very small cardinal number, there must exist a minimal resolving function ϑ' such that $|\vartheta'| \leq |\vartheta|$. Hence,

$$\mathcal{D}\text{im}_{lf}(H_n^3) \leq \sum_{i=1}^{4n} \frac{1}{2(n+1)} = \frac{2n}{n+1}. \tag{20}$$

□

Lemma 2. Let H_n^4 be a hexagonal planar circular network for $n \geq 3$ with $n \equiv 1 \pmod{2}$.

- (1) Then, for $1 \leq k \leq n$, $|\mathcal{R}_k| = |\mathcal{R}\{t_k u_k\}| = |\mathcal{R}\{s_k u_k\}| = 2n+1$, and $|\bigcup_{k=1}^n \mathcal{R}_k| = 4n$.
- (2) $|\mathcal{R}_k| < |\mathcal{R}\{s_k t_k\}|$ and $|\mathcal{R}\{s_k t_k\} \cap \bigcup_{k=1}^n \mathcal{R}_k| \geq |\mathcal{R}_k|$.
- (3) $|\mathcal{R}_k| < |\mathcal{R}\{s_k t_{k+1}\}|$ and $|\mathcal{R}\{s_k t_{k+1}\} \cap \bigcup_{k=1}^n \mathcal{R}_k| \geq |\mathcal{R}_k|$ with $t_{n+1} = t_1$.
- (4) $|\mathcal{R}_k| < |\mathcal{R}\{u_k v_k\}|$ and $|\mathcal{R}\{u_k v_k\} \cap \bigcup_{k=1}^n \mathcal{R}_k| \geq |\mathcal{R}_k|$.
- (5) $|\mathcal{R}_k| < |\mathcal{R}\{u_k v_{k-1}\}|$ and $|\mathcal{R}\{u_k v_{k-1}\} \cap \bigcup_{k=1}^n \mathcal{R}_k| \geq |\mathcal{R}_k|$ with $v_0 = v_n$.

Theorem 3. If H_n^4 is a hexagonal planar circular network for $n \geq 3$ with $n \equiv 1 \pmod{2}$, then $\mathcal{D}\text{im}_{lf}(H_n^m) \leq (4n/2n+1)$.

3.2. LFMD of Rotationally Symmetric Hexagonal Planar Networks with 4 Faces. In this section, we find the sequences of upper bounds of local fractional metric dimension of rotationally symmetric hexagonal planar networks with 4 faces.

Lemma 3. Let H_n^6 be a hexagonal planar circular network for $n \geq 3$ with $n \equiv 1 \pmod{2}$.

- (1) Then, for $1 \leq k \leq n$, $|\mathcal{R}_k| = |\mathcal{R}\{s_k u_k\}| = 2n$, and $|\cup_{k=1}^n \mathcal{R}_k| = 4n$.
- (2) $|\mathcal{R}_k| < |\mathcal{R}\{s_k t_k\}|$ and $|\mathcal{R}\{s_k t_k\} \cap \cup_{k=1}^n \mathcal{R}_k| \geq |\mathcal{R}_k|$.
- (3) $|\mathcal{R}_k| < |\mathcal{R}\{s_k t_{k+1}\}|$ and $|\mathcal{R}\{s_k t_{k+1}\} \cap \cup_{k=1}^n \mathcal{R}_k| \geq |\mathcal{R}_k|$ with $t_{n+1} = t_1$.
- (4) $|\mathcal{R}_k| < |\mathcal{R}\{u_k v_k\}|$ and $|\mathcal{R}\{u_k v_k\} \cap \cup_{k=1}^n \mathcal{R}_k| \geq |\mathcal{R}_k|$.
- (5) $|\mathcal{R}_k| < |\mathcal{R}\{u_k v_{k-1}\}|$ and $|\mathcal{R}\{u_k v_{k-1}\} \cap \cup_{k=1}^n \mathcal{R}_k| \geq |\mathcal{R}_k|$ with $v_0 = v_n$.
- (6) $|\mathcal{R}_k| < |\mathcal{R}\{t_k u_k\}|$ and $|\mathcal{R}\{t_k u_k\} \cap \cup_{k=1}^n \mathcal{R}_k| \geq |\mathcal{R}_k|$.
- (7) $|\mathcal{R}_k| < |\mathcal{R}\{s_k v_k\}|$ and $|\mathcal{R}\{s_k v_k\} \cap \cup_{k=1}^n \mathcal{R}_k| \geq |\mathcal{R}_k|$.

Proof

- (1) The set of local resolving neighborhoods of $s_k u_k$ is $\mathcal{R}\{s_k u_k\} = V(H_n^6) - \{s_r | r \equiv k+1 + (n-1/2), k+2 + (n-1/2), \dots, k+n-1 \pmod{n}\} \cup \{t_r | r \equiv k \pmod{n}, r \equiv k+1 + (n-1/2), k+2 + (n-1/2), \dots, k+n-1 \pmod{n}\} \cup \{u_r | r \equiv k+1, k+2, \dots, k+(n-1/2) \pmod{n}\} \cup \{v_r | r \equiv k, k+1, \dots, k-1 + (n+1/2) \pmod{n}\}$. Clearly, $|\cup_{k=1}^n \mathcal{R}_k| = V(H_n^6)$. Therefore, $|\cup_{k=1}^n \mathcal{R}_k| = 4n$.
- (2) The set of local resolving neighborhoods of $s_k t_k$ is $\mathcal{R}\{s_k t_k\} = V(H_n^6) - \{u_r | r \equiv k \pmod{n}\} \cup \{v_r | r \equiv k - (n-1/2) \pmod{n}, r \equiv k+n-1 \pmod{n}\}$. Hence, $|\mathcal{R}\{s_k t_k\}| = 4n-3$. Since $|\mathcal{R}_k| = 2n < |\mathcal{R}\{s_k t_k\}| = 4n-3$, $|\mathcal{R}\{s_k t_k\} \cap \cup_{k=1}^n \mathcal{R}_k| = 4n-3 \geq |\mathcal{R}_k|$.
- (3) The set of local resolving neighborhoods of $s_k t_{k+1}$ with $t_{k+1} = t_1$ is $\mathcal{R}\{s_k t_{k+1}\} = V(H_n^6) - \{u_r | r \equiv k + (n-1/2) + 1 \pmod{n}\}$. Hence, $|\mathcal{R}\{s_k t_{k+1}\}| = 4n-1$. Since $|\mathcal{R}_k| = 2n < |\mathcal{R}\{s_k t_{k+1}\}| = 4n-1$, $|\mathcal{R}\{s_k t_{k+1}\} \cap \cup_{k=1}^n \mathcal{R}_k| = 4n-1 \geq |\mathcal{R}_k|$.

- (4) The set of local resolving neighborhoods of $u_k v_k$ is $\mathcal{R}\{u_k v_k\} = V(H_n^6) - \{s_r | r \equiv k \pmod{n}\} \cup \{t_r | r \equiv k+1 \pmod{n}, r \equiv k + (n+1/2) \pmod{n}\}$. Hence, $|\mathcal{R}\{u_k v_k\}| = 4n-3$. Since $|\mathcal{R}_k| = 2n < |\mathcal{R}\{u_k v_k\}| = 4n-3$, $|\mathcal{R}\{u_k v_k\} \cap \cup_{k=1}^n \mathcal{R}_k| = 4n-3 \geq |\mathcal{R}_k|$.
- (5) The set of local resolving neighborhoods of $u_k v_{k-1}$ with $v_n = v_0$ is $\mathcal{R}\{u_k v_{k-1}\} = V(H_n^6) - \{s_r | r \equiv k + (n-1/2) \pmod{n}\}$. Hence, $|\mathcal{R}\{u_k v_{k-1}\}| = 4n-1$. Since $|\mathcal{R}_k| = 2n < |\mathcal{R}\{u_k v_{k-1}\}| = 4n-1$, $|\mathcal{R}\{u_k v_{k-1}\} \cap \cup_{k=1}^n \mathcal{R}_k| = 4n-1 \geq |\mathcal{R}_k|$.
- (6) The set of local resolving neighborhoods of $t_k u_k$ is $\mathcal{R}\{t_k u_k\} = V(H_n^6) - \{s_r | r \equiv k, k+1, \dots, k-1 + (n+1/2) \pmod{n}\} \cup \{t_r | r \equiv k+1, k+2, \dots, k+(n-1/2) \pmod{n}\} \cup \{u_r | r \equiv k+1 + (n-1/2), k+2 + (n-1/2), \dots, k+n-1 \pmod{n}\} \cup \{v_r | r \equiv k + (n-1/2), k+1 + (n-1/2), \dots, k+n-2 \pmod{n}\}$. Hence, $|\mathcal{R}\{t_k u_k\}| = 2n+1$. Since $|\mathcal{R}_k| = 2n < |\mathcal{R}\{t_k u_k\}| = 2n+1$, $|\mathcal{R}\{t_k u_k\} \cap \cup_{k=1}^n \mathcal{R}_k| = 2n+1 \geq |\mathcal{R}_k|$.
- (7) The set of local resolving neighborhoods of $s_k v_k$ is $\mathcal{R}\{s_k v_k\} = V(H_n^6) - \{s_r | r \equiv k+1, k+2, \dots, k+(n-1/2) \pmod{n}\} \cup \{t_r | r \equiv k+2, k+3, \dots, k+(n+1/2) \pmod{n}\} \cup \{v_r | r \equiv k+1 + (n-1/2), k+2 + (n-1/2), \dots, k+n-1 \pmod{n}\} \cup \{u_r | r \equiv k \pmod{n}, r \equiv k+1 + (n-1/2), k+2 + (n-1/2), \dots, k+n-1 \pmod{n}\}$. Hence, $|\mathcal{R}\{s_k v_k\}| = 2n+1$. Since $|\mathcal{R}_k| = 2n < |\mathcal{R}\{s_k v_k\}| = 2n+1$, $|\mathcal{R}\{s_k v_k\} \cap \cup_{k=1}^n \mathcal{R}_k| = 2n+1 \geq |\mathcal{R}_k|$. $\square$

Theorem 4. If H_n^6 is a hexagonal planar circular network for $n \geq 3$ with $n \equiv 1 \pmod{2}$, then $\dim_{lf}(H_n^6) \leq 2$.

Proof. When $n \geq 3$ with $n \equiv 1 \pmod{2}$, then by Lemma 3, the local resolving neighborhoods are

$$\begin{aligned}
 \overline{R_1} &= \mathcal{R}\{s_k u_k\} = V(H_n^6) - \left\{s_r | r \equiv k+1 + \frac{n-1}{2}, k+2 + \frac{n-1}{2}, \dots, k+n-1 \pmod{n}\right\} \cup \\
 &\quad \cdot \left\{t_r | r \equiv k \pmod{n}, r \equiv k+1 + \frac{n-1}{2}, k+2 + \frac{n-1}{2}, \dots, k+n-1 \pmod{n}\right\} \cup \\
 &\quad \cdot \left\{u_r | r \equiv k+1, k+2, \dots, k + \frac{n-1}{2} \pmod{n}\right\} \cup \left\{v_r | r \equiv k, k+1, \dots, k-1 + \frac{n+1}{2} \pmod{n}\right\}, \\
 R_1 &= \mathcal{R}\{s_k t_k\} = V(H_n^6) - \{u_r | r \equiv k \pmod{n}\} \cup \left\{v_r | r \equiv k - \frac{n-1}{2} \pmod{n}, r \equiv k+n-1 \pmod{n}\right\}, \\
 R_2 &= \mathcal{R}\{s_k t_{k+1}\} = V(H_n^6) - \left\{u_r | r \equiv k + \frac{n-1}{2} + 1 \pmod{n}\right\}, \\
 R_3 &= \mathcal{R}\{u_k v_k\} = V(H_n^6) - \{s_r | r \equiv k \pmod{n}\} \cup \left\{t_r | r \equiv k+1 \pmod{n}, r \equiv k + \frac{n+1}{2} \pmod{n}\right\}, \\
 R_4 &= \mathcal{R}\{u_k v_{k-1}\} = V(H_n^6) - \left\{s_r | r \equiv k + \frac{n-1}{2} \pmod{n}\right\}, \\
 R_5 &= \mathcal{R}\{t_k u_k\} = V(H_n^6) - \{s_r | r \equiv k, k+1 \pmod{4}\} \cup \{t_r | r \equiv k+1, k+2 \pmod{4}\} \cup
 \end{aligned}$$

$$\begin{aligned}
& \cdot \{v_r | r \equiv k+2 \pmod{4}\} \cup \{u_r | r \equiv k+2, k+3 \pmod{4}\}, \\
R_6 = \mathcal{R}\{s_k v_k\} = & V(H_n^6) - \left\{s_r | r \equiv k+1, k+2, \dots, k + \frac{n-1}{2} \pmod{n}\right\} \cup \\
& \cdot \left\{t_r | r \equiv k+2, k+3, \dots, k + \frac{n+1}{2} \pmod{n}\right\} \cup \left\{v_r | r \equiv k+1 + \frac{n-1}{2}, k+2 + \frac{n-1}{2}, \dots, k+n-1 \pmod{n}\right\} \\
& \cup \left\{u_r | r \equiv k \pmod{n}, r \equiv k+1 + \frac{n-1}{2}, k+2 + \frac{n-1}{2}, \dots, k+n-1 \pmod{n}\right\}.
\end{aligned} \tag{21}$$

The cardinality of each of the local resolving neighborhoods $\overline{R}_1, R_1, R_2, R_3, R_4, R_5$, and R_6 for each $1 \leq k \leq n$ is $2n, 4n-3, 4n-1, 4n-3, 4n-1, 2n+1$, and $2n+1$, respectively. Since cardinality of $\overline{R}_1$ is $2n$ which is less than the cardinality of all other resolving neighborhoods and $\cup_{i=1}^6 R_i = V(H_n^6)$, this implies that $|\overline{R}_j \cap \cup_{i=1}^6 R_i| \geq |R_i|$. Now, we construct a mapping $\vartheta: V(H_n^6) \rightarrow [0, 1]$ as

$$\vartheta(u_k) = \vartheta(s_k) = \frac{1}{2n}. \tag{22}$$

Since $2n$ is a minimal cardinal number, there must exist a minimal resolving function ϑ' such that $|\vartheta'| \leq |\vartheta|$. Hence,

$$\text{Dim}_{lf}(H_n^6) \leq \sum_{i=1}^{4n} \frac{1}{2n} = 2. \tag{23}$$

□

Lemma 4. Let H_n^7 be a hexagonal planar circular network for $n \geq 5$ with $n \equiv 1 \pmod{2}$.

- (1) Then, for $1 \leq k \leq n$, $|\mathcal{R}_k| = |\mathcal{R}\{s_k t_k\}| = |\mathcal{R}\{s_k t_{k+1}\}| = |\mathcal{R}\{t_k u_k\}| = |\mathcal{R}\{t_k u_{k+1}\}| = 2n+3$ with $t_{n+1} = t_1$, $u_{n+1} = u_1$ and $|\cup_{k=1}^n \mathcal{R}_k| = 4n$.
- (2) $|\mathcal{R}_k| < |\mathcal{R}\{u_k v_k\}|$ and $|\mathcal{R}\{u_k v_k\} \cap \cup_{k=1}^n \mathcal{R}_k| \geq |\mathcal{R}_k|$.
- (3) $|\mathcal{R}_k| < |\mathcal{R}\{u_k v_{k-1}\}|$ and $|\mathcal{R}\{u_k v_{k-1}\} \cap \cup_{k=1}^n \mathcal{R}_k| \geq |\mathcal{R}_k|$ with $v_n = v_0$.
- (4) $|\mathcal{R}_k| < |\mathcal{R}\{t_k t_{k+1}\}|$ and $|\mathcal{R}\{t_k t_{k+1}\} \cap \cup_{k=1}^n \mathcal{R}_k| \geq |\mathcal{R}_k|$, $t_{n+1} = t_1$.

Proof

- (1) The set of local resolving neighborhoods of $s_k t_k$, $s_k t_{k+1}$, $t_k u_k$ and $t_k u_{k+1}$ with $u_{n+1} = u_1, t_{n+1} = t_1$ is $\mathcal{R}\{s_k t_k\} = V(H_n^7) - \{s_r | r \equiv k+1, k+2, \dots, k + (n+1/2) - 1 \pmod{n}\} \cup \{t_r | r \equiv k+1, k+2, \dots, k + (n+1/2) - 1 \pmod{n}\} \cup \{v_r | r \equiv k+2, k+3, \dots, k + (n+3/2) - 1 \pmod{n}\} \cup \{u_r | r \equiv k+2, k+3, \dots, k + (n+1/2) - 1 \pmod{n}\}$, $\mathcal{R}\{s_k t_{k+1}\} = V(H_n^7) - \{s_r | r \equiv k+1 + (n-1/2), k+2 + (n-1/2), \dots, k+n-1 \pmod{n}\} \cup \{t_r | r \equiv k \pmod{n}, r \equiv k+2 + (n-1/2), k+3 + (n-1/2), \dots, k+n-1 \pmod{n}\} \cup \{u_r | r \equiv k \pmod{n}, r \equiv k+2 + (n-1/2), k+3 + (n-1/2), \dots, k+n-1 \pmod{n}\} \cup \{v_r | r \equiv k+2 + (n-1/2), k+3 + (n-1/2), \dots, k+n-1 \pmod{n}\}$, $\mathcal{R}\{t_k u_k\} = V(H_n^7) - \{s_r | r \equiv k + (n-1/2), k+1 + (n-1/2), \dots, k+n-2 \pmod{n}\} \cup \{t_r | r \equiv k + (n-1/2) + 1, k+2 + (n-1/2), \dots, k+n-1 \pmod{n}\} \cup \{u_r | r \equiv k+1 +$

$(n-1/2), k+2 + (n-1/2), \dots, k+n-1 \pmod{n}\} \cup \{v_r | r \equiv k+1 + (n-1/2), k+2 + (n-1/2), \dots, k+n-2 \pmod{n}\}$, and $\mathcal{R}\{t_k u_{k+1}\} = V(H_n^7) - \{s_r | r \equiv k+1, k+2, \dots, k + (n+1/2) - 1 \pmod{n}\} \cup \{t_r | r \equiv k+1, k+2, \dots, k + (n+1/2) - 1 \pmod{n}\} \cup \{u_r | r \equiv k+2, k+3, \dots, k + (n+3/2) - 1 \pmod{n}\} \cup \{v_r | r \equiv k+2, k+3, \dots, k + (n+1/2) - 1 \pmod{n}\}$. Clearly, $\cup_{k=1}^n \mathcal{R}_k = V(H_n^7)$. Therefore, $|\cup_{k=1}^n \mathcal{R}_k| = 4n$.

- (2) The set of local resolving neighborhoods of $u_k v_k$ is $\mathcal{R}\{u_k v_k\} = V(H_n^7) - \{s_r | r \equiv k+1, k+2, \dots, k + (n-1/2) - 1 \pmod{n}\} \cup \{t_r | r \equiv k+1, k+2, \dots, k + (n-1/2) - 1 \pmod{n}\} \cup \{u_r | r \equiv k+2, k+3, \dots, k + (n+1/2) - 1 \pmod{n}\} \cup \{v_r | r \equiv k+2, k+3, \dots, k + (n+1/2) - 1 \pmod{n}\}$. Hence, $|\mathcal{R}\{u_k v_k\}| = 2n+5$. Since $|\mathcal{R}_k| = 2n+3 < |\mathcal{R}\{u_k v_k\}| = 2n+5$, $|\mathcal{R}\{u_k v_k\} \cap \cup_{k=1}^n \mathcal{R}_k| = 2n+3 \geq |\mathcal{R}_k|$.
- (3) The set of local resolving neighborhoods of $u_k v_{k-1}$ is $\mathcal{R}\{u_k v_{k-1}\} = V(H_n^7) - \{s_r | r \equiv k + (n-1/2), k+1 + (n-1/2), k+2 + (n-1/2), \dots, k+n-3 \pmod{n}\} \cup \{t_r | r \equiv k + (n-1/2), k+1 + (n-1/2), \dots, k+n-2 \pmod{n}\} \cup \{u_r | r \equiv k+1 + (n-1/2), k+2 + (n-1/2), \dots, k+n-2 \pmod{n}\} \cup \{v_r | r \equiv k + (n-1/2), k+1 + (n-1/2), \dots, k+n-3 \pmod{n}\}$. Hence, $|\mathcal{R}\{u_k v_{k-1}\}| = 2n+5$. Since $|\mathcal{R}_k| = 2n+3 < |\mathcal{R}\{u_k v_{k-1}\}| = 2n+5$, $|\mathcal{R}\{u_k v_{k-1}\} \cap \cup_{k=1}^n \mathcal{R}_k| = 2n+3 \geq |\mathcal{R}_k|$.
- (4) The set of local resolving neighborhoods of $t_k t_{k+1}$, with $v_{n+1} = v_1$, is $\mathcal{R}\{t_k t_{k+1}\} = V(H_n^7) - \{s_r | r \equiv k \pmod{n}\} \cup \{t_r | r \equiv k + (n+3/2) - 1 \pmod{n}\} \cup \{u_r | r \equiv k+1 \pmod{n}\} \cup \{v_r | r \equiv k \pmod{n}, r \equiv k+1 \pmod{n}, r \equiv k + (n+3/2) - 1 \pmod{n}\}$. Hence, $|\mathcal{R}\{t_k t_{k+1}\}| = 4n-6$. Since $|\mathcal{R}_k| = 2n+3 < |\mathcal{R}\{t_k t_{k+1}\}| = 4n-6$, $|\mathcal{R}\{t_k t_{k+1}\} \cap \cup_{k=1}^n \mathcal{R}_k| = 2n+3 \geq |\mathcal{R}_k|$. □

Theorem 5. If H_n^7 is a hexagonal planar circular network, then

$$\text{Dim}_{lf}(H_n^7) \leq \begin{cases} 2, & n = 3, \\ \frac{4n}{2n+3}, & n \geq 5. \end{cases} \tag{24}$$

Proof. The local resolving neighborhoods are

Case (a): for $n = 3, 1 \leq k \leq 3$, by Lemma 4,

$$\begin{aligned}
\overline{R_1} &= \mathcal{R}\{t_k t_{k+1}\} = V(H_3^6) - \{s_r | r \equiv k \pmod{3}\} \cup \left\{t_r | r \equiv k + \frac{n+3}{2} - 1 \pmod{3}\right\} \cup \\
&\quad \cdot \{u_r | r \equiv k+1 \pmod{3}\} \cup \left\{v_r | r \equiv k, k+1, k + \frac{n+3}{2} - 1 \pmod{3}\right\}, \\
R_1 &= \mathcal{R}\{s_k t_k\} = V(H_3^6) - \{s_r | r \equiv k+1 \pmod{3}\} \cup \{t_r | r \equiv k+1 \pmod{3}\} \cup \{u_r | r \equiv k+2 \pmod{3}\}, \\
R_2 &= \mathcal{R}\{s_k t_{k+1}\} = V(H_3^6) - \{s_r | r \equiv k+2 \pmod{3}\} \cup \{t_r | r \equiv k \pmod{3}\} \cup \{u_r | r \equiv k \pmod{3}\}, \\
R_3 &= \mathcal{R}\{t_k u_k\} = V(H_3^6) - \{s_r | r \equiv k+1 \pmod{3}\} \cup \{t_r | r \equiv k+2 \pmod{3}\} \cup \{u_r | r \equiv k+2 \pmod{3}\}, \\
R_4 &= \mathcal{R}\{t_k u_{k+1}\} = V(H_3^6) - \{s_r | r \equiv k+1 \pmod{3}\} \cup \{t_r | r \equiv k+1 \pmod{3}\} \cup \{u_r | r \equiv k+2 \pmod{3}\}, \\
R_5 &= \mathcal{R}\{u_k v_k\} = V(H_3^6) - \{s_r | r \equiv k+1 \pmod{3}\} \cup \{t_r | r \equiv k+1, k+2 \pmod{3}\} \cup \\
&\quad \cdot \{u_r | r \equiv k+2 \pmod{3}\} \cup \{v_r | r \equiv k+2 \pmod{3}\}, \\
R_6 &= \mathcal{R}\{u_k v_{k+1}\} = V(H_3^6) - \{t_r | r \equiv k+1 \pmod{3}\}.
\end{aligned} \tag{25}$$

The cardinality of each of the local resolving neighborhoods $\overline{R_1}, R_1, R_2, R_3, R_4, R_5$, and R_6 for each $k = 1, 2, 3$ is 6, 9, 9, 9, 9, 11, and 11, respectively. Since cardinality of $\overline{R_1}$ is 6 which is less than the cardinality of all other resolving neighborhoods and $\cup_{i=1}^6 R_i = V(H_3^6)$, this implies that $|\overline{R_j} \cap \cup_{i=1}^6 R_i| \geq |R_i|$. Now, we construct a mapping $\vartheta: V(H_3^6) \rightarrow [0, 1]$ as

$$\vartheta(t_k) = \vartheta(t_{k+1}) = \frac{1}{6}. \tag{26}$$

Since 6 is a minimal cardinal number, there must exist a minimal resolving function ϑ' such that $|\vartheta'| \leq |\vartheta|$. Hence,

$$\dim_{lf}(H_3^6) \leq \sum_{i=1}^{12} \frac{1}{6} = 2. \tag{27}$$

Case (b): for $n \geq 5$, $1 \leq k \leq 5$, by Lemma 8, we have

$$\begin{aligned}
\overline{R_1} &= \mathcal{R}\{s_k t_k\} = V(H_n^7) - \left\{s_r | r \equiv k+1, k+2, \dots, k + \frac{n+1}{2} - 1 \pmod{n}\right\} \cup \\
&\quad \cdot \left\{t_r | r \equiv k+1, k+2, \dots, k + \frac{n+1}{2} - 1 \pmod{n}\right\} \cup \left\{u_r | r \equiv k+2, k+3, \dots, k + \frac{n+3}{2} - 1 \pmod{n}\right\} \\
&\quad \cup \left\{v_r | r \equiv k+2, k+3, \dots, k + \frac{n+1}{2} - 1 \pmod{n}\right\}, \\
\overline{R_2} &= \mathcal{R}\{s_k t_{k+1}\} = V(H_n^7) - \left\{s_r | r \equiv k+1 + \frac{n-1}{2}, k+2 + \frac{n-1}{2}, \dots, k+n-1 \pmod{n}\right\} \cup \\
&\quad \cdot \left\{t_r | r \equiv k \pmod{n}, r \equiv k+2 + \frac{n-1}{2}k+3 + \frac{n-1}{2}, \dots, k+n-1 \pmod{n}\right\} \cup \\
&\quad \cdot \left\{u_r | r \equiv k \pmod{n}, r \equiv k+2 + \frac{n-1}{2}, k+3 + \frac{n-1}{2}, \dots, k+n-1 \pmod{n}\right\} \cup \\
&\quad \cdot \left\{v_r | r \equiv k+2 + \frac{n-1}{2}, k+3 + \frac{n-1}{2}, \dots, k+n-1 \pmod{n}\right\}, \\
\overline{R_3} &= \mathcal{R}\{t_k u_k\} = V(H_n^7) - \left\{s_r | r \equiv k + \frac{n-1}{2}, k+1 + \frac{n-1}{2}, \dots, k+n-2 \pmod{n}\right\} \cup
\end{aligned}$$

$$\begin{aligned}
& \cdot \left\{ t_r | r \equiv k + \frac{n-1}{2} + 1, k + 2 + \frac{n-1}{2}, \dots, k + n - 1 \pmod{n} \right\} \cup \\
& \cdot \left\{ u_r | r \equiv k + 1 + \frac{n-1}{2}, k + 2 + \frac{n-1}{2}, \dots, k + n - 1 \pmod{n} \right\} \cup \\
& \cdot \left\{ v_r | r \equiv k + 1 + \frac{n-1}{2}, k + 2 + \frac{n-1}{2}, \dots, k + n - 2 \pmod{n} \right\}, \\
\overline{R_4} &= \mathcal{R}\{t_k u_{k+1}\} = V(H_n^7) - \left\{ s_r | r \equiv k + 1, k + 2, \dots, k + \frac{n+1}{2} - 1 \pmod{n} \right\} \cup \\
& \cdot \left\{ t_r | r \equiv k + 1, k + 2, \dots, k + \frac{n+1}{2} - 1 \pmod{n} \right\} \cup \left\{ u_r | r \equiv k + 2, k + 3, \dots, k + \frac{n+3}{2} - 1 \pmod{n} \right\} \\
& \cup \left\{ v_r | r \equiv k + 2, k + 3, \dots, k + \frac{n+1}{2} - 1 \pmod{n} \right\}, \\
R_1 &= \mathcal{R}\{u_k v_k\} = V(H_n^7) - \left\{ s_r | r \equiv k + 1, k + 2, \dots, k + \frac{n-1}{2} - 1 \pmod{n} \right\} \cup \\
& \cdot \left\{ t_r | r \equiv k + 1, k + 2, \dots, k + \frac{n-1}{2} - 1 \pmod{n} \right\} \cup \left\{ u_r | r \equiv k + 2, k + 3, \dots, k + \frac{n+1}{2} - 1 \pmod{n} \right\} \\
& \cup \left\{ v_r | r \equiv k + 2, k + 3, \dots, k + \frac{n+1}{2} - 1 \pmod{n} \right\}, \\
R_2 &= \mathcal{R}\{u_k v_{k-1}\} = V(H_n^7) - \left\{ s_r | r \equiv k + \frac{n-1}{2}, k + 1 + \frac{n-1}{2}, k + 2 + \frac{n-1}{2}, \dots, k + n - 3 \pmod{n} \right\} \\
& \cup \left\{ t_r | r \equiv k + \frac{n-1}{2}, k + 1 + \frac{n-1}{2}, \dots, k + n - 2 \pmod{n} \right\} \cup \\
& \cdot \left\{ u_r | r \equiv k + 1 + \frac{n-1}{2}, k + 2 + \frac{n-1}{2}, \dots, k + n - 2 \pmod{n} \right\} \\
& \cup \left\{ v_r | r \equiv k + \frac{n-1}{2}, k + 1 + \frac{n-1}{2}, \dots, k + n - 3 \pmod{n} \right\}, \\
R_3 &= \mathcal{R}\{t_k t_{k+1}\} = V(H_n^7) - \{s_r | r \equiv k \pmod{n}\} \cup \left\{ t_r | r \equiv k + \frac{n+3}{2} - 1 \pmod{n} \right\} \\
& \cup \{u_r | r \equiv k + 1 \pmod{n}\} \cup \left\{ v_r | r \equiv k \pmod{n}, r \equiv k + 1 \pmod{n}, r \equiv k + \frac{n+3}{2} - 1 \pmod{n} \right\}. \tag{28}
\end{aligned}$$

The cardinality of each of the local resolving neighborhoods $\overline{R_1}, \overline{R_2}, \overline{R_3}, \overline{R_4}, R_1, R_2$, and R_3 for each $1 \leq k \leq n$ is $2n+3, 2n+3, 2n+3, 2n+3, 2n+5$, and $4n-6$, respectively. Since cardinality of $\overline{R_i}, 1 \leq i \leq 4$ is $2n+3$ which is less than the cardinality of all other resolving neighborhoods and $\bigcup_{i=1}^3 R_i = V(H_n^7)$, this implies that $|\overline{R_j} \cap \bigcup_{i=1}^3 R_i| \geq |R_i|$. Now, we construct a mapping $\vartheta: V(H_n^7) \longrightarrow [0, 1]$ as

$$\vartheta(t_k) = \vartheta(s_k) = \vartheta(t_{k+1}) = \vartheta(u_k) = \vartheta(u_{k+1}) = \frac{1}{2n+3}. \tag{29}$$

Since $2n+3$ is a minimal cardinal number, there must exist a minimal resolving function ϑ' such that $|\vartheta'| \leq |\vartheta|$. Hence,

$$\dim_{lf}(H_n^7) \leq \sum_{i=1}^{4n} \frac{1}{2n+3} = \frac{4n}{2n+3}. \tag{30}$$

□

Lemma 5. Let H_n^8 be a hexagonal planar circular network for $n \geq 5$ with $n \equiv 1 \pmod{2}$.

- (1) Then, for $1 \leq k \leq n$, $|\mathcal{R}_k| = |\mathcal{R}\{u_k v_k\}| = 8$ and $|\bigcup_{k=1}^n \mathcal{R}_k| = 4n$ with $u_{n+1} = u_1$.
- (2) $|\mathcal{R}_k| < |\mathcal{R}\{s_k t_k\}|$ and $|\mathcal{R}\{s_k t_k\} \cap \bigcup_{k=1}^n \mathcal{R}_k| \geq |\mathcal{R}_k|$.
- (3) $|\mathcal{R}_k| < |\mathcal{R}\{t_k u_k\}|$ and $|\mathcal{R}\{t_k u_k\} \cap \bigcup_{k=1}^n \mathcal{R}_k| \geq |\mathcal{R}_k|$.
- (4) $|\mathcal{R}_k| < |\mathcal{R}\{t_k t_{k+1}\}|$ and $|\mathcal{R}\{t_k t_{k+1}\} \cap \bigcup_{k=1}^n \mathcal{R}_k| \geq |\mathcal{R}_k|$, with $t_{n+1} = t_1$.

- (5) $|\mathcal{R}_k| < |\mathcal{R}\{t_k v_k\}|$ and $|\mathcal{R}\{t_k v_k\} \cap \cup_{k=1}^n \mathcal{R}_k| \geq |\mathcal{R}_k|$.
 (6) $|\mathcal{R}_k| < |\mathcal{R}\{u_k v_{k-1}\}|$ and $|\mathcal{R}\{u_k v_{k-1}\} \cap \cup_{k=1}^n \mathcal{R}_k| \geq |\mathcal{R}_k|$, with $v_n = v_0$.
 (7) $|\mathcal{R}_k| < |\mathcal{R}\{s_k t_{k+1}\}|$ and $|\mathcal{R}\{s_k t_{k+1}\} \cap \cup_{k=1}^n \mathcal{R}_k| \geq |\mathcal{R}_k|$, with $t_{k+1} = t_1$.

Theorem 6. If H_n^8 is a hexagonal planar circular network, then

$$\mathcal{D}\text{im}_{lf}(H_n^8) \leq \begin{cases} 2, & n = 3, \\ \frac{n}{2}, & n \geq 5. \end{cases} \quad (31)$$

Lemma 6. Let H_n^9 be a hexagonal planar circular network for $n \geq 3$ with $n \equiv 1 \pmod{2}$.

- (1) Then, for $1 \leq k \leq n$, $|\mathcal{R}_k| = |\mathcal{R}\{t_k v_k\}| = 4$, and $|\cup_{k=1}^n \mathcal{R}_k| = 4n$.
 (2) $|\mathcal{R}_k| < |\mathcal{R}\{t_k u_k\}|$ and $|\mathcal{R}\{t_k u_k\} \cap \cup_{k=1}^n \mathcal{R}_k| \geq |\mathcal{R}_k|$.
 (3) $|\mathcal{R}_k| < |\mathcal{R}\{u_k v_{k-1}\}|$ and $|\mathcal{R}\{u_k v_{k-1}\} \cap \cup_{k=1}^n \mathcal{R}_k| \geq |\mathcal{R}_k|$ with $v_n = u_0$.
 (4) $|\mathcal{R}_k| < |\mathcal{R}\{u_k v_k\}|$ and $|\mathcal{R}\{u_k v_k\} \cap \cup_{k=1}^n \mathcal{R}_k| \geq |\mathcal{R}_k|$.
 (5) $|\mathcal{R}_k| < |\mathcal{R}\{s_k t_{k+1}\}|$ and $|\mathcal{R}\{s_k t_{k+1}\} \cap \cup_{k=1}^n \mathcal{R}_k| \geq |\mathcal{R}_k|$, with $t_{n+1} = t_1$.
 (6) $|\mathcal{R}_k| < |\mathcal{R}\{s_k t_k\}|$ and $|\mathcal{R}\{s_k t_k\} \cap \cup_{k=1}^n \mathcal{R}_k| \geq |\mathcal{R}_k|$.
 (7) $|\mathcal{R}_k| < |\mathcal{R}\{t_k u_{k+1}\}|$ and $|\mathcal{R}\{t_k u_{k+1}\} \cap \cup_{k=1}^n \mathcal{R}_k| \geq |\mathcal{R}_k|$, with $u_{n+1} = u_1$.

Proof

- (1) The set of local resolving neighborhoods of $t_k v_k$ is $\mathcal{R}\{t_k v_k\} = V(H_n^9) - \{s_r | r \equiv k+1, k+2, \dots, k+n-2 \pmod{n}\} \cup \{t_r | r \equiv k+1, k+2, \dots, k+n-2 \pmod{n}\} \cup \{u_r | r \equiv k, k+1, k+2, \dots, k+n-1 \pmod{n}\} \cup \{v_r | r \equiv k \pmod{n}\}$

$$\begin{aligned} \overline{R_1} &= \mathcal{R}\{t_k v_k\} = V(H_n^9) - \{s_r | r \equiv k+1, k+2, \dots, k+n-2 \pmod{n}\} \cup \\ &\quad \cdot \{t_r | r \equiv k+1, k+2, \dots, k+n-2 \pmod{n}\} \cup \{u_r | r \equiv k, k+1, k+2, \dots, k+n-1 \pmod{n}\} \cup \\ &\quad \cdot \{v_r | r \equiv k+1, k+2, \dots, k+n-1 \pmod{n}\}, \\ R_1 &= \mathcal{R}\{s_k t_{k+1}\} = V(H_n^9) - \{v_r | r \equiv k+1 \pmod{n}\}, \\ R_2 &= \mathcal{R}\{s_k t_k\} = V(H_n^9) - \{v_r | r \equiv k \pmod{n}\}, \\ R_3 &= \mathcal{R}\{t_k u_k\} = V(H_n^9) - \{s_r | r \equiv k \pmod{n}, r \equiv k+n-1 \pmod{n}\} \cup \{t_r | r \equiv k \pmod{n}\}, \\ R_4 &= \mathcal{R}\{t_k u_{k+1}\} = V(H_n^9) - \{v_r | r \equiv k \pmod{n}\}, \\ R_5 &= \mathcal{R}\{u_k v_{k-1}\} = V(H_n^9) - \{v_r | r \equiv k+n-2 \pmod{n}, r \equiv k+n-1 \pmod{n}\} \cup \{t_r | r \equiv k+n-1 \pmod{n}\}, \\ R_6 &= \mathcal{R}\{u_k v_k\} = V(H_n^9) - \{v_r | r \equiv k \pmod{n}, r \equiv k+n-1 \pmod{n}\} \cup \{t_r | r \equiv k+n-1 \pmod{n}\}. \end{aligned} \quad (32)$$

The cardinality of each of the local resolving neighborhoods $\overline{R_1}, \overline{R_2}, R_1, R_2, R_3, R_4, R_5$, and R_6 for each $1 \leq k \leq n$

+1, $k+2, \dots, k+n-1 \pmod{n}$. Clearly, $|\cup_{k=1}^n \mathcal{R}_k| = |V(H_n^9)|$. Therefore, $|\cup_{k=1}^n \mathcal{R}_k| = 4n$.

- (2) The set of local resolving neighborhoods of $s_k t_{k+1}$ is $\mathcal{R}\{s_k t_{k+1}\} = V(H_n^9) - \{v_r | r \equiv k+1 \pmod{n}\}$. Hence, $|\mathcal{R}\{s_k t_{k+1}\}| = 4n-1$. Since $|\mathcal{R}_k| = 4 < |\mathcal{R}\{s_k t_{k+1}\}| = 4n-1$, $|\mathcal{R}\{s_k t_{k+1}\} \cap \cup_{k=1}^n \mathcal{R}_k| = 4n-1 \geq |\mathcal{R}_k|$.
 (3) The set of local resolving neighborhoods of $s_k t_k$ is $\mathcal{R}\{s_k t_k\} = V(H_n^9) - \{v_r | r \equiv k \pmod{n}\}$. Hence, $|\mathcal{R}\{s_k t_k\}| = 4n-1$. Since $|\mathcal{R}_k| = 4 < |\mathcal{R}\{s_k t_k\}| = 4n-1$, $|\mathcal{R}\{s_k t_k\} \cap \cup_{k=1}^n \mathcal{R}_k| = 4n-1 \geq |\mathcal{R}_k|$.
 (4) The set of local resolving neighborhoods of $t_k u_k$ is $\mathcal{R}\{t_k u_k\} = V(H_n^9) - \{s_r | r \equiv k \pmod{n}, r \equiv k+n-1 \pmod{n}\} \cup \{t_r | r \equiv k \pmod{n}\}$. Hence, $|\mathcal{R}\{t_k u_k\}| = 4n-1$. Since $|\mathcal{R}_k| = 4 < |\mathcal{R}\{t_k u_k\}| = 4n-1$, $|\mathcal{R}\{t_k u_k\} \cap \cup_{k=1}^n \mathcal{R}_k| = 4n-1 \geq |\mathcal{R}_k|$.
 (5) The set of local resolving neighborhoods of $t_k u_{k+1}$ is $\mathcal{R}\{t_k u_{k+1}\} = V(H_n^9) - \{v_r | r \equiv k \pmod{n}\}$. Hence, $|\mathcal{R}\{t_k u_{k+1}\}| = 4n-1$. Since $|\mathcal{R}_k| = 4 < |\mathcal{R}\{t_k u_{k+1}\}| = 4n-1$, $|\mathcal{R}\{t_k u_{k+1}\} \cap \cup_{k=1}^n \mathcal{R}_k| = 4n-1 \geq |\mathcal{R}_k|$.
 (6) The set of local resolving neighborhoods of $u_k v_{k-1}$ is $\mathcal{R}\{u_k v_{k-1}\} = V(H_n^9) - \{v_r | r \equiv k+n-2 \pmod{n}, r \equiv k+n-1 \pmod{n}\} \cup \{t_r | r \equiv k+n-1 \pmod{n}\}$. Hence, $|\mathcal{R}\{u_k v_{k-1}\}| = 4n-3$. Since $|\mathcal{R}_k| = 4 < |\mathcal{R}\{u_k v_{k-1}\}| = 4n-3$, $|\mathcal{R}\{u_k v_{k-1}\} \cap \cup_{k=1}^n \mathcal{R}_k| = 4n-3 \geq |\mathcal{R}_k|$.
 (7) The set of local resolving neighborhoods of $u_k v_k$ is $\mathcal{R}\{u_k v_k\} = V(H_n^9) - \{v_r | r \equiv k \pmod{n}, r \equiv k+n-1 \pmod{n}\} \cup \{t_r | r \equiv k+n-1 \pmod{n}\}$. Hence, $|\mathcal{R}\{u_k v_k\}| = 4n-3$. Since $|\mathcal{R}_k| = 4 < |\mathcal{R}\{u_k v_k\}| = 4n-3$, $|\mathcal{R}\{u_k v_k\} \cap \cup_{k=1}^n \mathcal{R}_k| = 4n-3 \geq |\mathcal{R}_k|$. $\square$

Theorem 7. If H_n^9 is a hexagonal planar circular network for $n \geq 3$ with $n \equiv 1 \pmod{2}$, then $\mathcal{D}\text{im}_{lf}(H_n^9) \leq n$.

Proof. When $n \geq 3$ with $n \equiv 1 \pmod{2}$, by Lemma 6, the local resolving neighborhoods are

is 4, $4n-1, 4n-1, 4n-1, 4n-1, 4n-1, 4n-3$, and $4n-3$, respectively. Since cardinality of $\overline{R_1}$ is 4 which is less than the

cardinality of all other resolving neighborhoods and $\cup_{i=1}^6 R_i = V(H_n^9)$, this implies that $|\bar{R}_j \cap \cup_{i=1}^6 R_i| \geq |R_j|$. Now, we construct a mapping $\vartheta: V(H_n^9) \rightarrow [0, 1]$ as

$$\vartheta(t_k) = \vartheta(v_k) = \frac{1}{4}. \quad (33)$$

Since 4 is a minimal cardinal number, there must exist a minimal resolving function ϑ' such that $|\vartheta'| \leq |\vartheta|$. Hence,

$$\mathcal{D}\text{im}_{lf}(H_n^9) \leq \sum_{i=1}^{4n} \frac{1}{4} = n. \quad (34)$$

□

Lemma 7. Let H_n^{10} be a hexagonal planar circular network for $n \geq 3$ with $n \equiv 1 \pmod{2}$.

- (1) Then, for $1 \leq k \leq n$, $|\mathcal{R}_k| = |\mathcal{R}\{t_k t_{k+1}\}| = |\mathcal{R}\{u_k v_{k-1}\}| = |\mathcal{R}\{u_k v_k\}| = |\mathcal{R}\{s_k t_{k+1}\}| = |\mathcal{R}\{s_k t_k\}| = |\mathcal{R}\{u_k u_{k+1}\}| = 2n+2$, with $t_{n+1} = t_1$, $v_n = v_0$, $u_{n+1} = u_1$, and $|\cup_{k=1}^n \mathcal{R}_k| = 4n$.
- (2) $|\mathcal{R}_k| < |\mathcal{R}\{t_k u_k\}|$ and $|\mathcal{R}\{t_k u_k\} \cap \cup_{k=1}^n \mathcal{R}_k| \geq |\mathcal{R}_k|$.

Theorem 8. If H_n^{10} is a hexagonal planar circular network for $n \geq 3$ with $n \equiv 1 \pmod{2}$, then $\mathcal{D}\text{im}_{lf}(H_n^{10}) \leq (2n/n+1)$.

Lemma 8. Let H_n^{11} be a hexagonal planar circular network for $n \geq 3$ with $n \equiv 1 \pmod{2}$.

- (1) Then, for $1 \leq k \leq n$, $|\mathcal{R}_k| = |\mathcal{R}\{t_k u_k\}| = 2n$, and $|\cup_{k=1}^n \mathcal{R}_k| = 4n$.
- (2) $|\mathcal{R}_k| < |\mathcal{R}\{u_k v_{k-1}\}|$ and $|\mathcal{R}\{u_k v_{k-1}\} \cap \cup_{k=1}^n \mathcal{R}_k| \geq |\mathcal{R}_k|$ with $v_n = v_0$.
- (3) $|\mathcal{R}_k| < |\mathcal{R}\{u_k v_k\}|$ and $|\mathcal{R}\{u_k v_k\} \cap \cup_{k=1}^n \mathcal{R}_k| \geq |\mathcal{R}_k|$.
- (4) $|\mathcal{R}_k| < |\mathcal{R}\{s_k t_{k+1}\}|$ and $|\mathcal{R}\{s_k t_{k+1}\} \cap \cup_{k=1}^n \mathcal{R}_k| \geq |\mathcal{R}_k|$, with $t_{k+1} = t_1$.
- (5) $|\mathcal{R}_k| < |\mathcal{R}\{s_k t_k\}|$ and $|\mathcal{R}\{s_k t_k\} \cap \cup_{k=1}^n \mathcal{R}_k| \geq |\mathcal{R}_k|$.
- (6) $|\mathcal{R}_k| < |\mathcal{R}\{t_k v_k\}|$ and $|\mathcal{R}\{t_k v_k\} \cap \cup_{k=1}^n \mathcal{R}_k| \geq |\mathcal{R}_k|$.
- (7) $|\mathcal{R}_k| < |\mathcal{R}\{s_k u_{k+1}\}|$ and $|\mathcal{R}\{s_k u_{k+1}\} \cap \cup_{k=1}^n \mathcal{R}_k| \geq |\mathcal{R}_k|$, with $u_{k+1} = u_1$.

Theorem 9. If H_n^{11} is a hexagonal planar circular network for $n \geq 3$ with $n \equiv 1 \pmod{2}$, then $\mathcal{D}\text{im}_{lf}(H_n^{11}) \leq 2$.

3.3. LFMD of Rotationally Symmetric Hexagonal Planar Networks with 5 Faces. In this section, we find the sequences of upper bounds of local fractional metric dimension of rotationally symmetric hexagonal planar networks with 5 faces.

Lemma 9. Let H_n^{12} be a hexagonal planar circular network for $n \geq 5$ with $n \equiv 1 \pmod{2}$.

- (1) Then for $1 \leq k \leq n$, $|\mathcal{R}_k| = |\mathcal{R}\{s_k t_k\}| = |\mathcal{R}\{s_k t_{k+1}\}| = |\mathcal{R}\{u_k v_k\}| = |\mathcal{R}\{u_k v_{k-1}\}| = |\mathcal{R}\{s_k u_k\}| = |\mathcal{R}\{s_k u_{k+1}\}| = 2(n+1)$, with $u_{n+1} = u_1$, $t_{n+1} = t_1$, $v_n = v_0$, and $|\cup_{k=1}^n \mathcal{R}_k| = 4n$.

- (2) $|\mathcal{R}_k| < |\mathcal{R}\{t_k u_k\}|$ and $|\mathcal{R}\{t_k u_k \cap \cup_{k=1}^n \mathcal{R}_k\}| \geq |\mathcal{R}_k|$.
- (3) $|\mathcal{R}_k| < |\mathcal{R}\{u_k u_{k+1}\}|$ and $|\mathcal{R}\{u_k v_k \cap \cup_{k=1}^n \mathcal{R}_k\}| \geq |\mathcal{R}_k|$, with $u_{n+1} = u_1$.

Proof

- (1) The set of local resolving neighborhoods of $s_k t_k$, $s_k t_{k+1}$, $u_k v_k$, $u_k v_{k-1}$, $s_k u_k$, and $s_k u_{k+1}$ with $u_{k+1} = u_1$, $t_{k+1} = t_1$ is $\mathcal{R}\{s_k t_k\} = V(H_n^{12}) - \{s_r | r \equiv k+1 + (n-1/2), k+2 + (n-1/2), \dots, k+n-2 \pmod{n}\} \cup \{t_r | r \equiv k+1 + (n-1/2), k+2 + (n-1/2), \dots, k+n-2 \pmod{n}\} \cup \{u_r | r \equiv k \pmod{n}, r \equiv k+1 + (n-1/2), k+2 + (n-1/2), \dots, k+n-1 \pmod{n}\} \cup \{v_r | r \equiv k \pmod{n}, r \equiv k+1 + (n-1/2), k+2 + (n-1/2), \dots, k+n-1 \pmod{n}\} \cup \{s_r | r \equiv k+2, k+3, \dots, k+(n-1/2) \pmod{n}\} \cup \{t_r | r \equiv k+3, k+4, \dots, k+(n+1/2) \pmod{n}\} \cup \{u_r | r \equiv k+1, k+2, \dots, k+(n+1/2) \pmod{n}\} \cup \{v_r | r \equiv k, k+1, \dots, k+(n-1/2) \pmod{n}\}$, $\mathcal{R}\{s_k t_{k+1}\} = V(H_n^{12}) - \{s_r | r \equiv k+1, k+2, \dots, k+(n-1/2) \pmod{n}\} \cup \{t_r | r \equiv k+1, k+2, \dots, k+(n-1/2) \pmod{n}\} \cup \{u_r | r \equiv k+1, k+2, \dots, k+(n-1/2) \pmod{n}\} \cup \{v_r | r \equiv k+1, k+2, \dots, k+(n-1/2) \pmod{n}\}$, $\mathcal{R}\{u_k v_k\} = V(H_n^{12}) - \{s_r | r \equiv k+1, k+2, \dots, k+(n-1/2) \pmod{n}\} \cup \{t_r | r \equiv k+1, k+2, \dots, k+(n-1/2) \pmod{n}\} \cup \{u_r | r \equiv k+1, k+2, \dots, k+(n-1/2) \pmod{n}\} \cup \{v_r | r \equiv k+1, k+2, \dots, k+(n-1/2) \pmod{n}\}$, $\mathcal{R}\{u_k v_{k-1}\} = V(H_n^{12}) - \{s_r | r \equiv k+(n-1/2), k+1 + (n-1/2), \dots, k+n-2 \pmod{n}\} \cup \{t_r | r \equiv k+1 + (n-1/2), k+2 + (n-1/2), \dots, k+n-1 \pmod{n}\} \cup \{u_r | r \equiv k+1 + (n-1/2), k+2 + (n-1/2), \dots, k+n-1 \pmod{n}\} \cup \{v_r | r \equiv k+(n-1/2), k+1 + (n-1/2), \dots, k+n-2 \pmod{n}\}$, $\mathcal{R}\{s_k u_k\} = V(H_n^{12}) - \{s_r | r \equiv k+1, k+2, \dots, k+(n-1/2) \pmod{n}\} \cup \{t_r | r \equiv k+1, k+2, \dots, k+(n-1/2) \pmod{n}\} \cup \{u_r | r \equiv k+1, k+2, \dots, k+(n-1/2) \pmod{n}\} \cup \{v_r | r \equiv k+1, k+2, \dots, k+(n-1/2) \pmod{n}\}$, and $\mathcal{R}\{s_k u_{k+1}\} = V(H_n^{12}) - \{s_r | r \equiv k+1 + (n-1/2), k+2 + (n-1/2), \dots, k+n-1 \pmod{n}\} \cup \{t_r | r \equiv k+1 \pmod{n}, r \equiv k+2 + (n-1/2), k+3 + (n-1/2), \dots, k+n-1 \pmod{n}\} \cup \{u_r | r \equiv k \pmod{n}, r \equiv k+2 + (n-1/2), k+3 + (n-1/2), \dots, k+n-1 \pmod{n}\} \cup \{v_r | r \equiv k+1 + (n-1/2), k+2 + (n-1/2), \dots, k+n-1 \pmod{n}\}$. Clearly, $|\cup_{k=1}^n \mathcal{R}_k| = 4n$. Therefore, $|\cup_{k=1}^n \mathcal{R}_k| = 4n$.
- (2) The set of local resolving neighborhoods of $t_k u_k$ is $\mathcal{R}\{t_k u_k\} = V(H_n^{12}) - \{s_r | r \equiv k \pmod{n}, r \equiv k+n-1 \pmod{n}\} \cup \{t_r | r \equiv k+1 \pmod{n}, r \equiv k+n-1 \pmod{n}\}$. Hence, $|\mathcal{R}\{t_k u_k\}| = 4(n-1)$. Since $|\mathcal{R}_k| = 2(n+1) < |\mathcal{R}\{t_k u_k\}| = 4(n-1)$, $|\mathcal{R}\{t_k u_k\} \cap \cup_{k=1}^n \mathcal{R}_k| = 4(n-1) \geq |\mathcal{R}_k|$.
- (3) The set of local resolving neighborhoods of $u_k u_{k+1}$ with $u_{n+1} = u_1$ is $\mathcal{R}\{u_k u_{k+1}\} = V(H_n^{12}) - \{s_r | r \equiv k \pmod{n}\} \cup \{t_r | r \equiv k+(n+1/2) \pmod{n}\} \cup \{u_r | r \equiv k+(n+1/2) \pmod{n}\} \cup \{v_r | r \equiv k \pmod{n}\}$. Hence, $|\mathcal{R}\{u_k u_{k+1}\}| = 4(n-1)$. Since $|\mathcal{R}_k| = 2(n+1) < |\mathcal{R}\{u_k u_{k+1}\}| = 4(n-1)$, $|\mathcal{R}\{u_k u_{k+1}\} \cap \cup_{k=1}^n \mathcal{R}_k| = 4(n-1) \geq |\mathcal{R}_k|$. □

Theorem 10. If H_n^{12} is a hexagonal planar circular network with $n \equiv 1 \pmod{2}$, then

$$\mathcal{D}\mathbf{im}_{lf}(H_n^{12}) \leq \begin{cases} \frac{3}{2}, & n = 3, \\ \frac{2n}{n+1}, & n \geq 5. \end{cases} \quad (35)$$

Proof. The local resolving neighborhoods are

Case (a): for $n = 3$, $1 \leq k \leq 3$, by Lemma 9, we have

$$\begin{aligned} \overline{R_1} &= \mathcal{R}\{s_k t_k\} = V(H_3^{12}) - \{u_r | r \equiv k, k+2 \pmod{3}\} \cup \{v_r | r \equiv k, k+2 \pmod{3}\}, \\ \overline{R_2} &= \mathcal{R}\{s_k t_{k+1}\} = V(H_3^{12}) - \{u_r | r \equiv k+1, k+2 \pmod{3}\} \cup \{v_r | r \equiv k, k+1 \pmod{3}\}, \\ \overline{R_3} &= \mathcal{R}\{u_k v_k\} = V(H_3^{12}) - \{s_r | r \equiv k+1 \pmod{3}\} \cup \{t_r | r \equiv k+1 \pmod{3}\} \cup \\ &\quad \cdot \{u_r | r \equiv k+1 \pmod{3}\} \cup \{v_r | r \equiv k+1 \pmod{3}\}, \\ \overline{R_4} &= \mathcal{R}\{u_k v_{k-1}\} = V(H_3^{12}) - \{s_r | r \equiv k+1 \pmod{3}\} \cup \{t_r | r \equiv k+2 \pmod{3}\} \cup \\ &\quad \cdot \{u_r | r \equiv k+2 \pmod{3}\} \cup \{v_r | r \equiv k+1 \pmod{3}\}, \\ \overline{R_5} &= \mathcal{R}\{s_k u_k\} = V(H_3^{12}) - \{s_r | r \equiv k+1 \pmod{3}\} \cup \{t_r | r \equiv k \pmod{3}\} \cup \{u_r | r \equiv k+1 \pmod{3}\} \\ &\quad \cup \{v_r | r \equiv k+1 \pmod{3}\}, \\ \overline{R_6} &= \mathcal{R}\{s_k u_{k+1}\} = V(H_3^{12}) - \{s_r | r \equiv k+2 \pmod{3}\} \cup \{t_r | r \equiv k+1 \pmod{3}\} \cup \\ &\quad \cdot \{u_r | r \equiv k \pmod{3}\} \cup \{v_r | r \equiv k+2 \pmod{3}\}, \\ R_1 &= \mathcal{R}\{t_k u_k\} = V(H_3^{12}) - \{s_r | r \equiv k, k+2 \pmod{3}\} \cup \{t_r | r \equiv k+1, k+2 \pmod{3}\}, \\ R_2 &= \mathcal{R}\{u_k u_{k+1}\} = V(H_3^{12}) - \{s_r | r \equiv k \pmod{3}\} \cup \{t_r | r \equiv k+2 \pmod{3}\} \cup \\ &\quad \cdot \{u_r | r \equiv k+2 \pmod{3}\} \cup \{v_r | r \equiv k \pmod{3}\}. \end{aligned} \quad (36)$$

The cardinality of each of the local resolving neighborhoods $\overline{R_1}, \overline{R_2}, \overline{R_3}, \overline{R_4}, \overline{R_5}, \overline{R_6}, R_1$, and R_2 for each $k = 1, 2, 3$ is 8, 8, 8, 8, 8, 8, and 8, respectively. Now, we construct a mapping $\vartheta: V(H_3^{12}) \rightarrow [0, 1]$ as

$$\begin{aligned} \vartheta(s_k) &= \vartheta(u_k) = \vartheta(v_k) = \vartheta(t_k) = \vartheta(u_{k+1}) \\ &= \vartheta(t_{k+1}) = \vartheta(v_{k-1}) = \frac{1}{8}. \end{aligned} \quad (37)$$

Since 8 is a minimal cardinal number, there must exist a minimal resolving function ϑ' such that $|\vartheta'| \leq |\vartheta|$. Hence,

$$\mathcal{D}\mathbf{im}_{lf}(H_3^{12}) \leq \sum_{i=1}^{12} \frac{1}{8} = \frac{3}{2}. \quad (38)$$

Case (b): for $n \geq 5$, $1 \leq k \leq n$, by Lemma 9, we have

$$\begin{aligned} \overline{R_1} &= \mathcal{R}\{s_k t_k\} = V(H_n^{12}) - \left\{s_r | r \equiv k+1 + \frac{n-1}{2}, k+2 + \frac{n-1}{2}, \dots, k+n-2 \pmod{n}\right\} \cup \\ &\quad \cdot \left\{t_r | r \equiv k+1 + \frac{n-1}{2}, k+2 + \frac{n-1}{2}, \dots, k+n-2 \pmod{n}\right\} \cup \\ &\quad \cdot \left\{u_r | r \equiv k \pmod{n}, r \equiv k+1 + \frac{n-1}{2}, k+2 + \frac{n-1}{2}, \dots, k+n-1 \pmod{n}\right\} \cup \\ &\quad \cdot \left\{v_r | r \equiv k \pmod{n}, r \equiv k+1 + \frac{n-1}{2}, k+2 + \frac{n-1}{2}, \dots, k+n-1 \pmod{n}\right\}, \end{aligned}$$

$$\begin{aligned}
\overline{R_2} &= \mathcal{R}\{s_k t_{k+1}\} = V(H_n^{12}) - \left\{s_r | r \equiv k+2, k+3, \dots, k + \frac{n-1}{2} \pmod{n}\right\} \cup \\
&\quad \cdot \left\{t_r | r \equiv k+3, k+4, \dots, k + \frac{n+1}{2} \pmod{n}\right\} \cup \left\{u_r | r \equiv k+1, k+2, \dots, k + \frac{n+1}{2} \pmod{n}\right\} \\
&\quad \cup \left\{v_r | r \equiv k, k+1, \dots, k + \frac{n-1}{2} \pmod{n}\right\}, \\
\overline{R_3} &= \mathcal{R}\{u_k v_k\} = V(H_n^{12}) - \left\{s_r | r \equiv k+1, k+2, \dots, k + \frac{n-1}{2} \pmod{n}\right\} \cup \\
&\quad \cdot \left\{t_r | r \equiv k+1, k+2, \dots, k + \frac{n-1}{2} \pmod{n}\right\} \cup \left\{u_r | r \equiv k+1, k+2, \dots, k + \frac{n-1}{2} \pmod{n}\right\} \\
&\quad \cup \left\{v_r | r \equiv k+1, k+2, \dots, k + \frac{n-1}{2} \pmod{n}\right\}, \\
\overline{R_4} &= \mathcal{R}\{u_k v_{k-1}\} = V(H_n^{12}) - \left\{s_r | r \equiv k + \frac{n-1}{2}, k+1 + \frac{n-1}{2}, \dots, k+n-2 \pmod{n}\right\} \cup \\
&\quad \cdot \left\{t_r | r \equiv k+1 + \frac{n-1}{2}, k+2 + \frac{n-1}{2}, \dots, k+n-1 \pmod{n}\right\} \cup \\
&\quad \cdot \left\{u_r | r \equiv k+1 + \frac{n-1}{2}, k+2 + \frac{n-1}{2}, \dots, k+n-1 \pmod{n}\right\} \cup \\
&\quad \cdot \left\{v_r | r \equiv k + \frac{n-1}{2}, k+1 + \frac{n-1}{2}, \dots, k+n-2 \pmod{n}\right\}, \\
\overline{R_5} &= \mathcal{R}\{s_k u_k\} = V(H_n^{12}) - \left\{s_r | r \equiv k+1, k+2, \dots, k + \frac{n-1}{2} \pmod{n}\right\} \cup \\
&\quad \cdot \left\{t_r | r \equiv k \pmod{n}, r \equiv k+2, k+3, \dots, k + \frac{n-1}{2} \pmod{n}\right\} \cup \left\{u_r | r \equiv k+1, k+2, \dots, k + \frac{n-1}{2} \pmod{n}\right\} \\
&\quad \cup \left\{v_r | r \equiv k+1, k+2, \dots, k + \frac{n-1}{2} \pmod{n}\right\}, \\
\overline{R_6} &= \mathcal{R}\{s_k u_{k+1}\} = V(H_n^{12}) - \left\{s_r | r \equiv k+1 + \frac{n-1}{2}, k+2 + \frac{n-1}{2}, \dots, k+n-1 \pmod{n}\right\} \cup \\
&\quad \cdot \left\{t_r | r \equiv k+1 \pmod{n}, r \equiv k+2 + \frac{n-1}{2}, k+3 + \frac{n-1}{2}, \dots, k+n-1 \pmod{n}\right\} \cup \\
&\quad \cdot \left\{u_r | r \equiv k \pmod{n}, r \equiv k+2 + \frac{n-1}{2}, k+3 + \frac{n-1}{2}, \dots, k+n-1 \pmod{n}\right\} \cup \\
&\quad \cdot \left\{v_r | r \equiv k+1 + \frac{n-1}{2}, k+2 + \frac{n-1}{2}, \dots, k+n-1 \pmod{n}\right\}, \\
R_1 &= \mathcal{R}\{t_k u_k\} = V(H_n^{12}) - \{s_r | r \equiv k \pmod{n}, r \equiv k+n-1 \pmod{n}\} \cup \\
&\quad \cdot \{t_r | r \equiv k+1 \pmod{n}, r \equiv k+n-1 \pmod{n}\}, \\
R_2 &= \mathcal{R}\{u_k u_{k+1}\} = V(H_n^{12}) - \{s_r | r \equiv k \pmod{n}\} \cup \left\{t_r | r \equiv k + \frac{n+1}{2} \pmod{n}\right\} \cup \\
&\quad \cdot \left\{u_r | r \equiv k + \frac{n+1}{2} \pmod{n}\right\} \cup \{v_r | r \equiv k \pmod{n}\}.
\end{aligned} \tag{39}$$

The cardinality of each of the local resolving neighborhoods $\overline{R_1}, \overline{R_2}, \overline{R_3}, \overline{R_4}, \overline{R_5}, \overline{R_6}, R_1$, and R_2 for each

$1 \leq k \leq n$ is $2(n+1), 2(n+1), 2(n+1), 2(n+1), 2(n+1), 2(n+1), 4(n-1)$, and $4(n-1)$, respectively. Since

cardinality of $\overline{R}_i$, $1 \leq i \leq 6$ is $2(n+1)$ which is less than the cardinality of all other resolving neighborhoods and $\cup_{i=1}^2 R_i = V(H_n^{12})$, this implies that $|\overline{R}_j \cap \cup_{i=1}^2 R_i| \geq |R_i|$. Now, we construct a mapping $\vartheta: V(H_n^{12}) \rightarrow [0, 1]$ as

$$\begin{aligned} \vartheta(s_k) &= \vartheta(u_k) = \vartheta(v_k) = \vartheta(t_k) = \vartheta(u_{k+1}) \\ &= \vartheta(t_{k+1}) = \vartheta(v_{k-1}) = \frac{1}{2(n+1)}. \end{aligned} \quad (40)$$

Since $2(n+1)$ is a minimal cardinal number, there must exist a minimal resolving function ϑ' such that $|\vartheta'| \leq |\vartheta|$. Hence,

$$\text{Dim}_{lf}(H_n^{12}) \leq \sum_{i=1}^{4n} \frac{1}{2(n+1)} = \frac{2n}{n+1}. \quad (41)$$

□

Lemma 10. Let H_n^{13} be a hexagonal planar circular network for $n \geq 5$ with $n \equiv 1 \pmod{2}$.

- (1) Then, for $1 \leq k \leq n$, $|\mathcal{R}_k| = |\mathcal{R}\{t_k u_k\}| = |\mathcal{R}\{t_k u_{k+1}\}| = 2(n+1)$, with $u_1 = u_{n+1}$ and $|\cup_{k=1}^n \mathcal{R}_k| = 4n$.
- (2) $|\mathcal{R}_k| < |\mathcal{R}\{s_k t_k\}|$ and $|\mathcal{R}\{s_k t_k\} \cap \cup_{k=1}^n \mathcal{R}_k| \geq |\mathcal{R}_k|$.
- (3) $|\mathcal{R}_k| < |\mathcal{R}\{u_k v_k\}|$ and $|\mathcal{R}\{u_k v_k\} \cap \cup_{k=1}^n \mathcal{R}_k| \geq |\mathcal{R}_k|$.
- (4) $|\mathcal{R}_k| < |\mathcal{R}\{t_k t_{k+1}\}|$ and $|\mathcal{R}\{t_k t_{k+1}\} \cap \cup_{k=1}^n \mathcal{R}_k| \geq |\mathcal{R}_k|$, with $t_{n+1} = t_1$.
- (5) $|\mathcal{R}_k| < |\mathcal{R}\{u_k u_{k+1}\}|$ and $|\mathcal{R}\{u_k u_{k+1}\} \cap \cup_{k=1}^n \mathcal{R}_k| \geq |\mathcal{R}_k|$, with $u_{n+1} = u_1$.
- (6) $|\mathcal{R}_k| < |\mathcal{R}\{s_k t_{k+1}\}|$ and $|\mathcal{R}\{s_k t_{k+1}\} \cap \cup_{k=1}^n \mathcal{R}_k| \geq |\mathcal{R}_k|$, with $t_1 = t_{n+1}$.
- (7) $|\mathcal{R}_k| < |\mathcal{R}\{u_k v_{k-1}\}|$ and $|\mathcal{R}\{u_k v_{k-1}\} \cap \cup_{k=1}^n \mathcal{R}_k| \geq |\mathcal{R}_k|$, with $v_0 = v_n$.

Theorem 11. If H_n^{13} is a hexagonal planar circular network with $n \equiv 1 \pmod{2}$, then

$$\text{Dim}_{lf}(H_n^{13}) \leq \begin{cases} 2, & n = 3, \\ \frac{2n}{n+1}, & n \geq 5. \end{cases} \quad (42)$$

Lemma 11. Let H_n^{14} be a hexagonal planar circular network for $n \geq 3$ with $n \equiv 1 \pmod{2}$.

- (1) Then, for $1 \leq k \leq n$, $|\mathcal{R}_k| = |\mathcal{R}\{t_k u_k\}| = |\mathcal{R}\{s_k u_{k+1}\}| = |\mathcal{R}\{s_k v_k\}| = |\mathcal{R}\{t_k v_k\}| = 2n$, with $u_1 = u_{n+1}$ and $|\cup_{k=1}^n \mathcal{R}_k| = 4n$.
- (2) $|\mathcal{R}_k| < |\mathcal{R}\{u_k v_k\}|$ and $|\mathcal{R}\{u_k v_k\} \cap \cup_{k=1}^n \mathcal{R}_k| \geq |\mathcal{R}_k|$.
- (3) $|\mathcal{R}_k| < |\mathcal{R}\{s_k t_k\}|$ and $|\mathcal{R}\{s_k t_k\} \cap \cup_{k=1}^n \mathcal{R}_k| \geq |\mathcal{R}_k|$.
- (4) $|\mathcal{R}_k| < |\mathcal{R}\{s_k t_{k+1}\}|$ and $|\mathcal{R}\{s_k t_{k+1}\} \cap \cup_{k=1}^n \mathcal{R}_k| \geq |\mathcal{R}_k|$, with $t_1 = t_{n+1}$.
- (5) $|\mathcal{R}_k| < |\mathcal{R}\{u_k v_{k-1}\}|$ and $|\mathcal{R}\{u_k v_{k-1}\} \cap \cup_{k=1}^n \mathcal{R}_k| \geq |\mathcal{R}_k|$, with $v_0 = v_n$.

Theorem 12. If H_n^{14} is a hexagonal planar circular network for $n \geq 3$ with $n \equiv 1 \pmod{2}$, then $\text{Dim}_{lf}(H_n^{14}) \leq 2$.

Lemma 12. Let H_n^{15} be a hexagonal planar circular network for $n \geq 3$ with $n \equiv 1 \pmod{2}$.

- (1) Then, for $1 \leq k \leq n$, $|\mathcal{R}_k| = |\mathcal{R}\{s_k u_{k+1}\}| = |\mathcal{R}\{t_k v_k\}| = 4$ and $|\cup_{k=1}^n \mathcal{R}_k| = 4n$ with $u_{n+1} = u_1$.
- (2) $|\mathcal{R}_k| < |\mathcal{R}\{s_k t_k\}|$ and $|\mathcal{R}\{s_k t_k\} \cap \cup_{k=1}^n \mathcal{R}_k| \geq |\mathcal{R}_k|$.
- (3) $|\mathcal{R}_k| < |\mathcal{R}\{t_k u_k\}|$ and $|\mathcal{R}\{t_k u_k\} \cap \cup_{k=1}^n \mathcal{R}_k| \geq |\mathcal{R}_k|$.
- (4) $|\mathcal{R}_k| < |\mathcal{R}\{u_k v_k\}|$ and $|\mathcal{R}\{u_k v_k\} \cap \cup_{k=1}^n \mathcal{R}_k| \geq |\mathcal{R}_k|$.
- (5) $|\mathcal{R}_k| < |\mathcal{R}\{t_k u_{k+1}\}|$ and $|\mathcal{R}\{t_k u_{k+1}\} \cap \cup_{k=1}^n \mathcal{R}_k| \geq |\mathcal{R}_k|$, with $u_{n+1} = u_1$.
- (6) $|\mathcal{R}_k| < |\mathcal{R}\{s_k t_{k+1}\}|$ and $|\mathcal{R}\{s_k t_{k+1}\} \cap \cup_{k=1}^n \mathcal{R}_k| \geq |\mathcal{R}_k|$, with $t_1 = t_{n+1}$.
- (7) $|\mathcal{R}_k| < |\mathcal{R}\{u_k v_{k-1}\}|$ and $|\mathcal{R}\{u_k v_{k-1}\} \cap \cup_{k=1}^n \mathcal{R}_k| \geq |\mathcal{R}_k|$ with $v_0 = v_n$.

Proof

- (1) The set of local resolving neighborhoods of $s_k u_{k+1}$ and $t_k v_k$ with $u_{n+1} = u_1$ is $\mathcal{R}\{s_k u_{k+1}\} = V(H_n^{15}) - \{s_r | r \equiv k+1, k+2, \dots, k+n-1 \pmod{n}\} \cup \{t_r | r \equiv k, k+1, \dots, k+n-1 \pmod{n}\} \cup \{u_r | r \equiv k \pmod{n}, r \equiv k+2, k+3, \dots, k+n-1 \pmod{n}\} \cup \{v_r | r \equiv k+2, k+3, \dots, k+n-1 \pmod{n}\}$ and $\mathcal{R}\{t_k v_k\} = V(H_n^{15}) - \{s_r | r \equiv k+1, k+2, \dots, k+n-2 \pmod{n}\} \cup \{t_r | r \equiv k+1, k+2, \dots, k+n-1 \pmod{n}\} \cup \{u_r | r \equiv k, k+1, \dots, k+n-1 \pmod{n}\} \cup \{v_r | r \equiv k+1, k+2, \dots, k+n-1 \pmod{n}\}$. Clearly, $|\cup_{k=1}^n \mathcal{R}_k| = V(H_n^{15})$. Therefore, $|\cup_{k=1}^n \mathcal{R}_k| = 4n$.
- (2) The set of local resolving neighborhoods of $s_k t_k$ is $\mathcal{R}\{s_k t_k\} = V(H_n^{15}) - \{u_r | r \equiv k+1 \pmod{n}\} \cup \{v_r | r \equiv k+1 \pmod{n}\}$. Hence, $|\mathcal{R}\{s_k t_k\}| = 2(2n-1)$. Since $|\mathcal{R}_k| = 4 < |\mathcal{R}\{s_k t_k\}| = 2(2n-1)$, $|\mathcal{R}\{s_k t_k\} \cap \cup_{k=1}^n \mathcal{R}_k| = 2(2n-1) \geq |\mathcal{R}_k|$.
- (3) The set of local resolving neighborhoods of $t_k u_k$ is $\mathcal{R}\{t_k u_k\} = V(H_n^{15}) - \{s_r | r \equiv k+n-1 \pmod{n}\} \cup \{v_r | r \equiv k \pmod{n}\}$. Hence, $|\mathcal{R}\{t_k u_k\}| = 2(2n-1)$. Since $|\mathcal{R}_k| = 4 < |\mathcal{R}\{t_k u_k\}| = 2(2n-1)$, $|\mathcal{R}\{t_k u_k\} \cap \cup_{k=1}^n \mathcal{R}_k| = 2(2n-1) \geq |\mathcal{R}_k|$.
- (4) The set of local resolving neighborhoods of $u_k v_k$ is $\mathcal{R}\{u_k v_k\} = V(H_n^{15}) - \{s_r | r \equiv k \pmod{n}\} \cup \{t_r | r \equiv k \pmod{n}\}$. Hence, $|\mathcal{R}\{u_k v_k\}| = 2(2n-1)$. Since $|\mathcal{R}_k| = 4 < |\mathcal{R}\{u_k v_k\}| = 2(2n-1)$, $|\mathcal{R}\{u_k v_k\} \cap \cup_{k=1}^n \mathcal{R}_k| = 2(2n-1) \geq |\mathcal{R}_k|$.
- (5) The set of local resolving neighborhoods of $t_k u_{k+1}$, with $u_{n+1} = u_1$ is $\mathcal{R}\{t_k u_{k+1}\} = V(H_n^{15}) - \{s_r | r \equiv k \pmod{n}\} \cup \{v_r | r \equiv k \pmod{n}\}$. Hence, $|\mathcal{R}\{t_k u_{k+1}\}| = 2(2n-1)$. Since $|\mathcal{R}_k| = 4 < |\mathcal{R}\{t_k u_{k+1}\}| = 2(2n-1)$, $|\mathcal{R}\{t_k u_{k+1}\} \cap \cup_{k=1}^n \mathcal{R}_k| = 2(2n-1) \geq |\mathcal{R}_k|$.
- (6) The set of local resolving neighborhoods of $s_k t_{k+1}$ with $t_{n+1} = t_1$ is $\mathcal{R}\{s_k t_{k+1}\} = V(H_n^{15}) - \{u_r | r \equiv k+1 \pmod{n}\} \cup \{v_r | r \equiv k \pmod{n}\}$. Hence, $|\mathcal{R}\{s_k t_{k+1}\}| = 2(2n-1)$.

TABLE 1: Sequences of local fractional metric dimension of rotationally symmetric hexagonal planar networks.

Network	Upper bound of $\mathcal{D}\text{im}_{lf}$
H_n^1, H_n^2, H_n^5	1
$H_n^3, H_n^{10}, H_n^{17}$	$(2n/n + 1)$
H_n^4	$(4n/2n + 1)$
$H_n^6, H_n^{11}, H_n^{14}$	2
H_n^7	$\begin{cases} 2, & n = 3; \\ (4n/2n + 3), & n \geq 5 \end{cases}$
H_n^8	$\begin{cases} 2, & n = 3; \\ (n/2), & n \geq 5 \end{cases}$
$H_n^9, H_n^{15}, H_n^{16}$	n
H_n^{12}	$\begin{cases} (3/2), & n = 3; \\ (2n/n + 1), & n \geq 5 \end{cases}$
H_n^{13}	$\begin{cases} 2, & n = 3; \\ (2n/n + 1), & n \geq 5 \end{cases}$

$$= 2(2n - 1). \text{ Since } |\mathcal{R}_k| = 4 < |\mathcal{R}\{s_k t_k\}| = 2(2n - 1), \\ |\mathcal{R}\{s_k t_k\} \cap \cup_{k=1}^n \mathcal{R}_k| = 2(2n - 1) \geq |\mathcal{R}_k|.$$

- (7) The set of local resolving neighborhoods of $u_k v_{k-1}$ with $v_{n-1} = v_0$ is $\mathcal{R}\{u_k v_{k+1}\} = V(H_n^{15}) - \{s_r | r \equiv k + n - 2 \pmod{n}\} \cup \{t_r | r \equiv k + n - 1 \pmod{n}\}$. Hence, $|\mathcal{R}\{u_k v_{k+1}\}| = 2(2n - 1)$. Since $|\mathcal{R}_k| = 4 < |\mathcal{R}\{u_k v_{k+1}\}| = 2(2n - 1)$, $|\mathcal{R}\{u_k v_{k+1}\} \cap \cup_{k=1}^n \mathcal{R}_k| = 2(2n - 1) \geq |\mathcal{R}_k|$. $\square$

Theorem 13. If H_n^{15} is a hexagonal planar circular network for $n \geq 3$ with $n \equiv 1 \pmod{2}$, then $\mathcal{D}\text{im}_{lf}(H_n^{15}) \leq n$.

Proof. When $n \geq 3$ with $n \equiv 1 \pmod{2}$, by Lemma 12, the local resolving neighborhoods are

$$\begin{aligned} \overline{R_1} &= \mathcal{R}\{s_k u_{k+1}\} = V(H_n^{15}) - \{s_r | r \equiv k + 1, k + 2, \dots, k + n - 1 \pmod{n}\} \cup \{t_r | r \equiv k, k + 1, \dots, k + n - 1 \pmod{n}\} \\ &\quad \cup \{u_r | r \equiv k \pmod{n}, r \equiv k + 2, k + 3, \dots, k + n - 1 \pmod{n}\} \cup \{v_r | r \equiv k + 2, k + 3, \dots, k + n - 1 \pmod{n}\}, \\ \overline{R_2} &= \mathcal{R}\{t_k v_k\} = V(H_n^{15}) - \{s_r | r \equiv k + 1, k + 2, \dots, k + n - 2 \pmod{n}\} \cup \{t_r | r \equiv k + 1, k + 2, \dots, k + n - 1 \pmod{n}\} \\ &\quad \cup \{u_r | r \equiv k, k + 1, \dots, k + n - 1 \pmod{n}\} \cup \{v_r | r \equiv k + 1, k + 2, \dots, k + n - 1 \pmod{n}\}, \\ R_1 &= \mathcal{R}\{s_k t_k\} = V(H_n^{15}) - \{u_r | r \equiv k + 1 \pmod{n}\} \cup \{v_r | r \equiv k + 1 \pmod{n}\}, \\ R_2 &= \mathcal{R}\{t_k u_k\} = V(H_n^{15}) - \{s_r | r \equiv k + n - 1 \pmod{n}\} \cup \{v_r | r \equiv k \pmod{n}\}, \\ R_3 &= \mathcal{R}\{u_k v_k\} = V(H_n^{15}) - \{s_r | r \equiv k \pmod{n}\} \cup \{t_r | r \equiv k \pmod{n}\}, \\ R_4 &= \mathcal{R}\{t_k u_{k+1}\} = V(H_n^{15}) - \{s_r | r \equiv k \pmod{n}\} \cup \{v_r | r \equiv k \pmod{n}\}, \\ R_5 &= \mathcal{R}\{s_k t_{k+1}\} = V(H_n^{15}) - \{u_r | r \equiv k + 1 \pmod{n}\} \cup \{v_r | r \equiv k \pmod{n}\}, \\ R_6 &= \mathcal{R}\{u_k v_{k-1}\} = V(H_n^{15}) - \{s_r | r \equiv k + n - 2 \pmod{n}\} \cup \{t_r | r \equiv k + n - 1 \pmod{n}\}. \end{aligned} \tag{43}$$

The cardinality of each of the local resolving neighborhoods $\overline{R_1}, \overline{R_2}, R_1, R_2, R_3, R_4, R_5$, and R_6 for each $1 \leq k \leq n$ is 4, 4, $4n - 2$, $4n - 2$, $4n - 2$, $4n - 2$, $4n - 2$, and $4n - 2$, respectively. Since cardinality of $\overline{R_i}$, $1 \leq i \leq 2$ is 4 which is less than the cardinality of all other resolving neighborhoods and $\cup_{i=1}^6 R_i = V(H_n^{15})$, this implies that $|\overline{R_j} \cap \cup_{i=1}^6 R_i| \geq |R_i|$. Now, we construct a mapping $\vartheta: V(H_n^{15}) \rightarrow [0, 1]$ as

$$\vartheta(s_k) = \vartheta(u_{k+1}) = \vartheta(v_k) = \vartheta(t_k) = \frac{1}{4}. \tag{44}$$

Since 4 is a minimal cardinal number, there must exist a minimal resolving function ϑ' such that $|\vartheta'| \leq |\vartheta|$. Hence,

$$\mathcal{D}\text{im}_{lf}(H_n^{15}) \leq \sum_{i=1}^{4n} \frac{1}{4} = n. \tag{45}$$

$\square$

Lemma 13. Let H_n^{16} be a hexagonal planar circular network for $n \geq 3$ with $n \equiv 1 \pmod{2}$.

- (1) Then, for $1 \leq k \leq n$, $|\mathcal{R}_k| = |\mathcal{R}\{t_k u_k\}| = |\mathcal{R}\{s_k v_k\}| = 4$ and $|\cup_{k=1}^n \mathcal{R}_k| = 4n$.
- (2) $|\mathcal{R}_k| < |\mathcal{R}\{s_k t_k\}|$ and $|\mathcal{R}\{s_k t_k\} \cap \cup_{k=1}^n \mathcal{R}_k| \geq |\mathcal{R}_k|$.
- (3) $|\mathcal{R}_k| < |\mathcal{R}\{u_k v_k\}|$ and $|\mathcal{R}\{u_k v_k\} \cap \cup_{k=1}^n \mathcal{R}_k| \geq |\mathcal{R}_k|$.
- (4) $|\mathcal{R}_k| < |\mathcal{R}\{s_k u_{k+1}\}|$ and $|\mathcal{R}\{s_k u_{k+1}\} \cap \cup_{k=1}^n \mathcal{R}_k| \geq |\mathcal{R}_k|$, with $u_{n+1} = u_1$.
- (5) $|\mathcal{R}_k| < |\mathcal{R}\{s_k u_k\}|$ and $|\mathcal{R}\{s_k u_k\} \cap \cup_{k=1}^n \mathcal{R}_k| \geq |\mathcal{R}_k|$.
- (6) $|\mathcal{R}_k| < |\mathcal{R}\{s_k t_{k+1}\}|$ and $|\mathcal{R}\{s_k t_{k+1}\} \cap \cup_{k=1}^n \mathcal{R}_k| \geq |\mathcal{R}_k|$, with $t_{n+1} = t_1$.
- (7) $|\mathcal{R}_k| < |\mathcal{R}\{u_k v_{k-1}\}|$ and $|\mathcal{R}\{u_k v_{k-1}\} \cap \cup_{k=1}^n \mathcal{R}_k| \geq |\mathcal{R}_k|$, with $v_n = v_0$.

FIGURE 6: Graph of sequences of rotationally symmetric hexagonal planar networks H_n^i , $i = 1, 2, 5, 6, 11, 14$, and their comparison.

FIGURE 7: Continued.

FIGURE 7: Graph of sequences of rotationally symmetric hexagonal planar networks H_n^i , $i = 8, 9, 15, 16$, and their comparison.

FIGURE 8: Graph of sequences of rotationally symmetric hexagonal planar networks H_n^i , $i = 3, 4, 7, 10, 12, 13, 17$, and their comparison.

Theorem 14. If H_n^{16} is a hexagonal planar circular network for $n \geq 3$ with $n \equiv 1 \pmod{2}$, then $\mathcal{D}\text{im}_{lf}(H_n^{16}) \leq n$.

Lemma 14. Let H_n^{17} be a hexagonal planar circular network for $n \geq 3$ with $n \equiv 1 \pmod{2}$.

- (1) Then, for $1 \leq k \leq n$, $|\mathcal{R}_k| = |\mathcal{R}\{s_k t_k\}| = |\mathcal{R}\{s_k t_{k+1}\}| = |\mathcal{R}\{t_k u_k\}| = |\mathcal{R}\{u_k v_k\}| = |\mathcal{R}\{u_k v_{k-1}\}| = |\mathcal{R}\{t_{k+1} u_k\}| = 2(n+1)$, with $t_1 = t_{n+1}$, $u_1 = u_{n+1}$, $v_0 = v_n$, and $|\cup_{k=1}^n \mathcal{R}_k| = 4n$.
- (2) $|\mathcal{R}_k| < |\mathcal{R}\{t_k t_{k+1}\}|$ and $|\mathcal{R}\{t_k t_{k+1}\} \cap \cup_{k=1}^n \mathcal{R}_k| \geq |\mathcal{R}_k|$ with $t_{n+1} = t_1$.
- (3) $|\mathcal{R}_k| < |\mathcal{R}\{v_k t_{k+1}\}|$ and $|\mathcal{R}\{v_k t_{k+1}\} \cap \cup_{k=1}^n \mathcal{R}_k| \geq |\mathcal{R}_k|$ with $t_{n+1} = t_1$.

Theorem 15. If H_n^{17} is a hexagonal planar circular network for $n \geq 3$ with $n \equiv 1 \pmod{2}$, then $\mathcal{D}\text{im}_{lf}(H_n^{17}) \leq (2n/n+1)$.

4. Table of Sequences of LFMD of Rotationally Symmetric Hexagonal Planar Networks and Their Comparison via Graphs

In this section, we propose the table of sequences of local fractional metric dimension of rotationally symmetric hexagonal planar networks and their comparison via plotting the graphs of these sequences (Table 1).

The boundedness comparison of sequences of local fractional metric dimension of rotationally symmetric hexagonal planar networks are shown in Figures 6–8.

5. Conclusion

In this paper, we discuss the local fractional metric dimension of rotationally symmetric hexagonal planar networks. The sequences of their upper bounds are investigated. The sequences of upper bounds of each rotationally symmetric hexagonal planar network are plotted for compression of their bounds. We can see easily that there are two types of sequences: bounded as well as unbounded. In future work, researchers can find domination and spectral number of rotationally symmetric hexagonal planar networks.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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