Hindawi
Journal of Mathematics
Volume 2023, Article ID 9815043, 1 page
https://doi.org/10.1155/2023/9815043

Retraction

Retracted: General Complex-Valued Overlap Functions

Journal of Mathematics

Received 10 October 2023; Accepted 10 October 2023; Published 11 October 2023

This article has been retracted by Hindawi following an investigation undertaken by the publisher [1]. This investigation has uncovered evidence of one or more of the following indicators of systematic manipulation of the publication process:

(1) Discrepancies in scope
(2) Discrepancies in the description of the research reported
(3) Discrepancies between the availability of data and the research described
(4) Inappropriate citations
(5) Incoherent, meaningless and/or irrelevant content included in the article
(6) Peer-review manipulation

The presence of these indicators undermines our confidence in the integrity of the article's content and we cannot, therefore, vouch for its reliability. Please note that this notice is intended solely to alert readers that the content of this article is unreliable. We have not investigated whether authors were aware of or involved in the systematic manipulation of the publication process.

Wiley and Hindawi regrets that the usual quality checks did not identify these issues before publication and have since put additional measures in place to safeguard research integrity.

We wish to credit our own Research Integrity and Research Publishing teams and anonymous and named external researchers and research integrity experts for contributing to this investigation.

The corresponding author, as the representative of all authors, has been given the opportunity to register their agreement or disagreement to this retraction. We have kept a record of any response received.

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[1] Y. Chen, L. Bi, B. Hu, and S. Dai, "General Complex-Valued Overlap Functions," *Journal of Mathematics*, vol. 2021, Article ID 6613730, 6 pages, 2021.

Hindawi
Journal of Mathematics
Volume 2021, Article ID 6613730, 6 pages
https://doi.org/10.1155/2021/6613730

Research Article

General Complex-Valued Overlap Functions

Ying Chen,[1] Lvqing Bi,[2] Bo Hu,[3] and Songsong Dai [1]

[1] *School of Electronics and Information Engineering, Taizhou University, Taizhou 318000, China*
[2] *School of Electronics and Communication Engineering, Guangxi Colleges and Universities Key Laboratory of Complex System Optimization and Big Data Processing, Yulin Normal University, Yulin 537000, China*
[3] *School of Mechanical and Electrical Engineering, Guizhou Normal University, Guiyang 550025, China*

Correspondence should be addressed to Songsong Dai; ssdai@stu.xmu.edu.cn

Received 29 November 2020; Revised 4 January 2021; Accepted 11 January 2021; Published 20 January 2021

Academic Editor: Sami Ullah Khan

Overlap function is a special type of aggregation function which measures the degree of overlapping between different classes. Recently, complex fuzzy sets have been successfully applied in many applications. In this paper, we extend the concept of overlap functions to the complex-valued setting. We introduce the notions of complex-valued overlap, complex-valued 0-overlap, complex-valued 1-overlap, and general complex-valued overlap functions, which can be regarded as the generalizations of the concepts of overlap, 0-overlap, 1-overlap, and general overlap functions, respectively. We study some properties of these complex-valued overlap functions and their construction methods.

1. Introduction

Bustince et al. [1] introduced the concept of overlap function in order to express the overlapping degree between two different classes. Overlap functions are a special type of aggregation functions [2] that are used in many applications such as image processing [1, 3], classification [4, 5], and decision making [6, 7]. It has gained a rapid development with various forms. The concepts of Archimedean, general, 0-overlap, 1-overlap, n-dimensional, interval-valued overlap functions have been proposed [8–11]. Various properties including migrativity, distributivity, idempotency, and homogeneity of overlap functions have been investigated [7, 8, 12–17]. The additive generators [11, 18] and multiplicative generators [19] of overlap functions have been given. Implications derived from overlap functions have been studied [20, 21].

Ramot et al. [22, 23] introduced the concept of complex fuzzy sets. It is an effective tool to handle uncertainty and periodicity simultaneously. It has been successfully applied in signal processing [23–25], image processing [26], time series forecasting [27–30], and decision making [31, 32]. Different measures including distance, similarity, and entropy of complex fuzzy sets have been proposed [33–38]. Various properties including δ-equality, parallelity, orthogonality, and rotational invariance of complex fuzzy sets have been investigated [39–42].

Complex fuzzy sets have been successfully applied in many different fields. In some cases, overlapping degree is needed for complex-valued information of two or more objects. In this paper, we extend traditional real-valued overlap functions to complex-valued overlap function. The starting point is that complex-valued overlap differs from other real-valued overlap functions. For example, $e^{j\cdot x}$ ($j = \sqrt{-1}$) is a periodic function and negative operation $(-)$ is closed in the range of complex unit circle. These features may lead to special properties and construction methods of complex-valued overlap functions and provide a good issue for generation of overlap functions. As far as we know, nowadays, there are no corresponding discussions to propose the complex-valued overlap functions. Therefore, in this paper, from the theoretical point of view, we propose the definitions and construction methods of complex-valued overlap functions.

This paper is organized as follows. In Section 2, we recall the concepts of overlap functions. In Section 3, we introduce complex-valued overlap functions and their properties. In Section 4, we propose some construction methods of complex-valued overlap functions. Conclusions are given in Section 5.

2. Preliminaries

In this section, we recall some concepts of bivariate overlap functions and n-dimensional overlap functions, which are largely studied [1, 10, 11].

2.1. Overlap Functions

Definition 1 (see [1]). A mapping O: $[0,1]^2 \longrightarrow [0,1]$ is an overlap function if, for all $a, b \in [0,1]$, it is commutative, nondecreasing, and continuous and satisfies the following conditions:

(O1) $O(a,b) = 0$ if and only if $ab = 0$;
(O2) $O(a,b) = 1$ if and only if $ab = 1$.

As introduced in [11], a mapping O: $[0,1]^2 \longrightarrow [0,1]$ is a 0-overlap function if we loose the condition (O1) to (O1') $ab = 0 \Rightarrow O(a,b) = 0$ without changing any other condition.

Similarly, a mapping O: $[0,1]^2 \longrightarrow [0,1]$ is a 0-overlap function if we loose the condition (O2) to (O2') $ab = 1 \Rightarrow O(a,b) = 1$ without changing any other condition.

Definition 2 (see [10]). A mapping O_n: $[0,1]^n \longrightarrow [0,1]$ is a n-dimensional overlap function if, for all $a_1, \ldots, a_n \in [0,1]$, it is commutative, nondecreasing, and continuous and satisfies the following conditions:

(O_n1) $O_n(a_1, \ldots, a_n) = 0$ if and only if $\prod_{i=1}^n a_i = 0$;
(O_n2) $O_n(a_1, \ldots, a_n) = 1$ if and only if $\prod_{i=1}^n a_i = 1$.

As introduced in [11], a mapping O_n: $[0,1]^n \longrightarrow [0,1]$ is an n-dimensional 0-overlap function if we loose condition (O_n1) to (O_n1') $\prod_{i=1}^n a_i = 0 \Rightarrow O_n(a_1, \ldots, a_n) = 0$ without changing any other condition.

Analogously, a mapping O_n: $[0,1]^n \longrightarrow [0,1]$ is an n-dimensional 1-overlap function if we loose condition (O_n2) to (O_n2') $\prod_{i=1}^n a_i = 1 \Rightarrow O_n(a_1, \ldots, a_n) = 1$without changing any other condition.

Based on the concepts of n-dimensional 0-overlap and 1-overlap functions, the general overlap functions are defined as follows:

Definition 3 (see [10]). A mapping O_n: $[0,1]^n \longrightarrow [0,1]$ is an n-dimensional general overlap function if, for all $a_1, \ldots, a_n \in [0,1]$, it is commutative, nondecreasing, and continuous and satisfies the following conditions:

(GO_n1) if $\prod_{i=1}^n a_i = 0$, then $O_n(a_1, \ldots, a_n) = 0$;
(GO_n2) if $\prod_{i=1}^n a_i = 1$, then $O_n(a_1, \ldots, a_n) = 1$.

3. N-Dimensional Complex-Valued Overlap Functions

Let $\mathbf{D} = \{\alpha \in \mathbb{C} \| \alpha | t \leq n1\}$, then we define n-dimensional complex-valued overlap functions.

Definition 4. A mapping CO_n: $\mathbf{D}^n \longrightarrow \mathbf{D}$ is an n-dimensional complex-valued overlap function if, for all $a_1, \ldots, a_n \in \mathbf{D}$, it is commutative and continuous and satisfies the following conditions:

(CO_n1) $CO_n(a_1, \ldots, a_n) = 0$ if and only if $\prod_{i=1}^n a_i = 0$;
(CO_n2) $CO_n(a_1, \ldots, a_n) = 1$ if and only if $\prod_{i=1}^n a_i = 1$;
(CO_n3) CO_n is amplitude monotonic in the first component: $|CO_n(a, a_2, \ldots, a_n) \leq |CO_n(b, a_2, \ldots, a_n)|$ when $|a| \leq |b|$.

Since CO_n is commutative, n-dimensional complex-valued overlap functions also are amplitude monotonic in any other component based on (GCO_n3). Obviously, these conditions are analogous to those of Definition 1. When the domain is limited to [0,1], n-dimensional complex-valued overlap function reduces to n-dimensional real-valued overlap function of Definition 1.

Example 1. Nevertheless, there are mappings that are overlap functions in the domain [0,1] but are not complex-valued overlap functions. The function f: $\mathbf{D}^2 \longrightarrow \mathbf{D}$ given by

$$f(a,b) = ab\frac{a+b}{2} \tag{1}$$

is an overlap function but not a complex-valued overlap function.

There are many types of real-valued overlap functions. Similarly, we extend the concept of 0-overlap and 1-overlap functions to n-dimensional complex-valued overlap functions.

A mapping CO_n: $\mathbf{D}^n \longrightarrow \mathbf{D}$ is an n-dimensional complex-valued 0-overlap function if we loose condition (CO_n1) to (CO_n1') $\prod_{i=1}^n a_i = 0 \Rightarrow CO_n(a_1, \ldots, a_n) = 0$ without changing any other condition.

A mapping CO_n: $\mathbf{D}^n \longrightarrow \mathbf{D}$ is an n-dimensional 1-overlap complex-valued function if we loose condition (CO_n2) to (CO_n2') $\prod_{i=1}^n a_i = 1 \Rightarrow CO_n(a_1, \ldots, a_n) = 1$ without changing any other condition.

Based on the concepts of n-dimensional complex-valued 0-overlap and 1-overlap functions, we define the concept of n-dimensional general complex-valued overlap functions

Definition 5. A mapping CO_n: $\mathbf{D}^n \longrightarrow \mathbf{D}$ is an n-dimensional general complex-valued overlap function if, for all $a_1, \ldots, a_n \in \mathbf{D}$, it is commutative and continuous and satisfies the following conditions:

(GCO_n1) if $\prod_{i=1}^n a_i = 0$, then $CO_n(a_1, \ldots, a_n) = 0$;
(GCO_n2) if $\prod_{i=1}^n a_i = 1$, then $CO_n(a_1, \ldots, a_n) = 1$;

GCO$_n$ CO$_n$ is amplitude monotonic in the first component: $|\mathrm{CO}_n(a, a_2, \ldots, a_n) \leq |\mathrm{CO}_n(b, a_2, \ldots, a_n)|$ when $|a| \leq |b|$.

The relations between n-dimensional complex-valued overlap functions, complex-valued 0-overlap functions, complex-valued 1-overlap functions, and general complex-valued overlap functions are shown in Figure 1. Asmus et al. [17] gave the relations between n-dimensional interval-valued overlap functions, interval-valued 0-overlap functions, interval-valued 1-overlap functions, and general interval-valued overlap functions, which are similar to that of Figure 1.

Clearly, we have the following.

Proposition 1. *If a mapping $g\colon \mathbf{D}^n \longrightarrow \mathbf{D}$ is either an n-dimensional complex-valued overlap, complex-valued 0-overlap, or complex-valued 1-overlap function, then g is also a general complex-valued overlap function.*

We give some examples of these complex-valued overlap functions to demonstrate their relations.

Example 2. The function $\pi\colon \mathbf{D}^2 \longrightarrow \mathbf{D}$ given by

$$\pi(a, b) = a \cdot b \tag{2}$$

is a complex-valued overlap function.

Example 3. The function $\pi_2\colon \mathbf{D}^2 \longrightarrow \mathbf{D}$ given by

$$\pi_2(a, b) = a^2 \cdot b^2 \tag{3}$$

is a general complex-valued overlap function. Moreover, it is a complex-valued 1-overlap function but not a complex-valued 0-overlap function.

Example 4. The function $g\colon \mathbf{D}^2 \longrightarrow \mathbf{D}$ given by

$$g(a, b) = \min(|a|, |b|) \cdot a \cdot b \tag{4}$$

is a general complex-valued overlap function. Moreover, it is a complex-valued 1-overlap function but not a complex-valued 0-overlap function.

Example 5. The function $h\colon \mathbf{D}^2 \longrightarrow \mathbf{D}$ given by

$$h(a, b) = \max(|a| + |b| - 1, 0) \cdot a \cdot b \tag{5}$$

is a general complex-valued overlap function. Moreover, it is a complex-valued 0-overlap function but not a complex-valued 1-overlap function.

Note that the class of overlap functions is convex. But the convex sun is not amplitude monotonic [?], then the class of complex-valued overlap functions is not convex.

Negative operation (−) is closed in the range of complex unit circle, but is not closed in [0,1]. Then, we have the following properties only for complex-valued overlap functions.

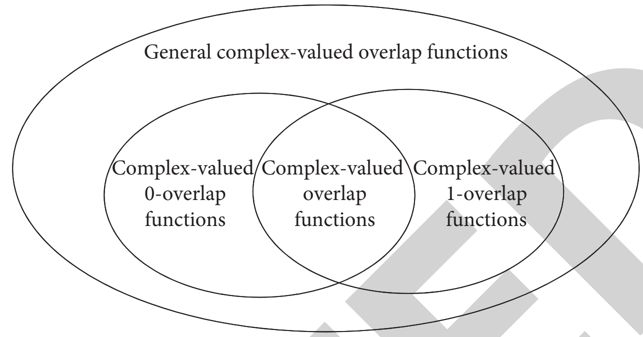

Figure 1: Relations between complex-valued overlap functions, complex-valued 0-overlap functions, complex-valued 1-overlap functions, and general complex-valued overlap functions.

Definition 6. We say the complex-valued overlap function CO: $\mathbf{D}^n \longrightarrow \mathbf{D}$ satisfies the self-duality property, if it satisfies

$$\mathrm{CO}(a_1, \ldots, a_n) = -\mathrm{CO}(-a_1, -a_2, \ldots, -a_n), \tag{6}$$

for any $a_1, \ldots, a_n \in \mathbf{D}$.

Definition 7. We say the complex-valued overlap function CO: $\mathbf{D}^n \longrightarrow \mathbf{D}$ is symmetric with respect to the point 0, if it satisfies

$$\mathrm{CO}(a_1, \ldots, a_n) = \mathrm{CO}(-a_1, -a_2, \ldots, -a_n), \tag{7}$$

for any $a_1, \ldots, a_n \in \mathbf{D}$.

There are complex-valued overlap functions satisfying the abovementioned properties.

Example 8. The function $g\colon \mathbf{D}^n \longrightarrow \mathbf{D}$ given by

$$g(a_1, \ldots, a_n) = \max\left(\sum_{i=1}^{n} |a_i| - n + 1, 0\right) \cdot \prod_{i=1}^{n} a_i \tag{8}$$

is a complex-valued overlap function. Interestingly, it satisfies the self-duality property when n is an odd number. It is symmetric with respect to the point 0 when n is an even number.

4. Construction of Complex-Valued Overlap Functions

We assume that the complex numbers are used in the form of exponent, i.e., $a \in \mathbf{D}$ is of the form $r_a e^{j\theta_a}$, where $j = \sqrt{-1}$, the amplitude term $r_a \in \mathbb{R}$, and the phase term $\theta_a \in [0, 2\pi)$. In order to let the phase term within valid range, we compute the least positive residue modulo 2π of the phase term when it is out of range. For simplicity, we omit the symbol $(\bmod 2\pi)$.

Proposition 2. *If a mapping $f\colon \mathbf{D}^n \longrightarrow \mathbf{D}$ is n-dimensional complex-valued overlap (complex-valued 0-overlap, complex-valued 1-overlap, or general complex-valued overlap) function is expressed as*

$$f(a_1,\ldots,a_n) = g\left(r_{a_1}, r_{a_2},\ldots,r_{a_n}\right)e^{jh\left(\theta_{a_1},\theta_{a_2},\ldots\theta_{a_n}\right)}, \quad (9)$$

then the function g is an n-dimensional overlap (0-overlap, 1-overlap, or general overlap) function.

Theorem 1. *If the function $g\colon [0,1]^n \longrightarrow [0,1]$ is an n-dimensional overlap function, the function $h\colon [0,2\pi)^n \longrightarrow [0,2\pi)$ satisfies the following properties:*

(i) h is commutative;

(ii) $\sum_{i=1}^n \theta_{a_i} = 0$ if and only if $h(\theta_{a_1},\ldots,\theta_{a_n}) = 0$;

(iii) h is continuous.

then, the function $f\colon \mathbf{D}^n \longrightarrow \mathbf{D}$ defined by equation (9) is an n-dimensional complex-valued overlap (complex-valued 0-overlap, complex-valued 1-overlap, or general complex-valued overlap) function.

Proof. It is immediate that f is commutative, amplitude monotonic, and continuous, since g is nondecreasing, and g and h are both commutative and continuous.

(CO$_n$1): ($\Rightarrow$) if $f(a_1,\ldots,a_n) = g(r_{a_1}, r_{a_2},\ldots r_{a_n})e^{jh}(\theta_{a_1},\theta_{a_2},\ldots\theta_{a_n}) = 0$, then $g(r_{a_1}, r_{a_2},\ldots r_{a_n}) = 0$. Then, $\prod_{i=1}^n r_{a_i} = 0$ since g is an overlap function. Then, $\prod_{i=1}^n a_i = \prod_{i=1}^n r_{a_i}\cdot e^{j(\sum_{i=1}^n \theta_{a_i})} = 0\cdot e^{j(\sum_{i=1}^n \theta_{a_i})} = 0$.
(CO$_n$1): ($\Leftarrow$) if $\prod_{i=1}^n a_i = \prod_{i=1}^n r_{a_i}\cdot e^{j(\sum_{i=1}^n \theta_{a_i})} = 0$, this means $\prod_{i=1}^n r_{a_i} = 0$, then $g(r_{a_1}, r_{a_2},\ldots r_{a_n}) = 0$ since g is an overlap function. Then, $f(a_1,\ldots,a_n) = g(r_{a_1}, r_{a_2},\ldots r_{a_n})e^{jh(\theta_{a_1},\theta_{a_2},\ldots\theta_{a_n})} = 0\cdot e^{jh(\theta_{a_1},\theta_{a_2},\ldots\theta_{a_n})} = 0$.
(CO$_n$2): ($\Rightarrow$) if $f(a_1,\ldots,a_n) = g(r_{a_1}, r_{a_2},\ldots r_{a_n})e^{jh}(\theta_{a_1},\theta_{a_2},\ldots\theta_{a_n}) = 1$, then $g(r_{a_1}, r_{a_2},\ldots r_{a_n}) = 1$ and $h(\theta_{a_1},\theta_{a_2},\ldots\theta_{a_n}) = 0$. Then, $\prod_{i=1}^n r_{a_i} = 1$ since g is an overlap function, and $\sum_{i=1}^n \theta_{a_i} = 0$ since h satisfies (ii). Then, $\prod_{i=1}^n a_i = \prod_{i=1}^n r_{a_i}\cdot e^{j(\sum_{i=1}^n \theta_{a_i})} = 1\cdot e^{j0} = 1$.
(CO$_n$2): ($\Leftarrow$) if $\prod_{i=1}^n a_i = \prod_{i=1}^n r_{a_i}\cdot e^{j(\sum_{i=1}^n \theta_{a_i})} = 1$, this means $\prod_{i=1}^n r_{a_i} = 1$ and $\sum_{i=1}^n \theta_{a_i} = 0$, then $g(r_{a_1}, r_{a_2},\ldots r_{a_n}) = 1$ since g is an overlap function, and $h(\theta_{a_1},\theta_{a_2},\ldots\theta_{a_n}) = 0$ since h satisfies (ii). Then, $f(a_1,\ldots,a_n) = g(r_{a_1}, r_{a_2},\ldots r_{a_n})e^{jh(\theta_{a_1},\theta_{a_2},\ldots\theta_{a_n})} = 1\cdot e^{j0} = 1$. □ □

Theorem 2. *If the function $g\colon [0,1]^n \longrightarrow [0,1]$ is an n-dimensional 0-overlap function, the function $h\colon [0,2\pi)^n \longrightarrow [0,2\pi)$ satisfies the following properties:*

(i) h is commutative;

(ii) $\sum_{i=1}^n \theta_{a_i} = 0$ if and only if $h(\theta_{a_1},\ldots,\theta_{a_n}) = 0$;

(iii) h is continuous.

Then, the function $f\colon \mathbf{D}^n \longrightarrow \mathbf{D}$ defined by equation (9) is an n-dimensional complex-valued 0-overlap function.

Proof. Analogous to the proof of Theorem 1. □

Theorem 3. *If the function $g\colon [0,1]^n \longrightarrow [0,1]$ is an n-dimensional 1-overlap (or general overlap) function, the function $h\colon [0,2\pi)^n \longrightarrow [0,2\pi)$ satisfies the following properties:*

(i) h is commutative;

(ii) If $\sum_{i=1}^n \theta_{a_i} = 0$, then $h(\theta_{a_1},\ldots,\theta_{a_n}) = 0$;

(iii) h is continuous.

Then, the function $f\colon \mathbf{D}^n \longrightarrow \mathbf{D}$ defined by equation (9) is an n-dimensional complex-valued 1-overlap (or general complex-valued overlap) function.

Proof. Analogous to the proof of Theorem 1. □

There are several construction methods of (general) overlap functions. Here, we consider the construction of (general) complex-valued overlap functions. If the n-dimensional complex-valued function $f\colon \mathbf{D}^n \longrightarrow \mathbf{D}$ is defined by equation (9), then we can easily see that it is a key step to construct the function $h\colon [0,2\pi)^n \longrightarrow [0,2\pi)$, which satisfies the condition (ii) of Theorem 2 (or 3).

Now, we give some examples of bivariate functions $h\colon [0,2\pi)^2 \longrightarrow [0,2\pi)$ satisfying condition (ii) of Theorem 2 (or 3).

Example 6. The function $h_1\colon [0,2\pi)^2 \longrightarrow [0,2\pi)$ given by

$$h_1(a,b) = a+b \quad (10)$$

satisfies condition (ii) of Theorem 2. The function $h_2\colon [0,2\pi)^2 \longrightarrow [0,2\pi)$ given by

$$h_2(a,b) = -a-b \quad (11)$$

satisfies condition (ii) of Theorem 2. The function $h_3\colon [0,2\pi)^2 \longrightarrow [0,2\pi)$ given by

$$h_3(a,b) = 2(a+b) \quad (12)$$

satisfies condition (ii) of Theorem 3. But it does not satisfy condition (ii) of Theorem 2.

Note that we omit the operation of $\bmod 2\pi$. If $a = b = (\pi/2)$, then $h_3(a,b) = 2(a+b) = 2\pi = 0$, but $a+b = \pi \neq 0$. So, h_3 does not satisfy condition (ii) of Theorem 2. In general, we have the following results.

Example 7. The function $h_4\colon [0,2\pi)^2 \longrightarrow [0,2\pi)$ given by

$$h_4(a,b) = k(a+b), \quad k = \pm 2, \pm 3,\ldots, \quad (13)$$

satisfies condition (ii) of Theorem 3. But it does not satisfy condition (ii) of Theorem 2.

Based on results of complex-valued overlap functions, we give the following examples.

Example 8. The function $h_{n,p,k}\colon \mathbf{D}^n \longrightarrow \mathbf{D}$ given by

$$h_{n,p,k} = \left(r_{a_1}\cdot r_{a_2},\ldots,r_{a_n}\right)^p\cdot e^{jk\left(\theta_{a_1}+\theta_{a_2}+\cdots+\theta_{a_n}\right)} \quad (14)$$

is a complex-valued overlap function when $p>0$ and $k = \pm 1$.

The function $h_{n,p,k}$: $\mathbf{D}^n \longrightarrow \mathbf{D}$ given by

$$h_{n,L,k} = \max\left(\sum_{i=1}^{n} r_{a_i} - n + 1, 0\right) \cdot e^{jk\left(\theta_{a_1}+\theta_{a_2}+\cdots+\theta_{a_n}\right)} \tag{15}$$

is a complex-valued 0-overlap function when $k = \pm 1$.

The function $h_{n,p,k}$: $\mathbf{D}^n \longrightarrow \mathbf{D}$ given by

$$h_{n,\wedge,k} = \left(\min_{i=1}^{n} r_{a_i}\right) \cdot e^{jk\left(\theta_{a_1}+\theta_{a_2}+\cdots+\theta_{a_n}\right)} \tag{16}$$

is a complex-valued 1-overlap function when $k = \pm 2, \pm 3, \ldots$.

5. Conclusions

In this paper, we introduced the concepts of complex-valued overlap, complex-valued 0-overlap, complex-valued 1-overlap, and general complex-valued overlap functions. We gave the relationship between them and studied their properties. Different from the traditional real-valued overlap functions, we added the following properties for complex-valued overlap functions since the domain of each variable is the unit disk of complex plane:

$$\begin{aligned} f(a_1, \ldots, a_n) = -f(-a_1, -a_2 \ldots, -a_n), f(a_1, \ldots, a_n) \\ = f(-a_1, -a_2 \ldots, -a_n). \end{aligned} \tag{17}$$

Then, we presented some construction methods for complex-valued overlap functions. Because of the periodicity of exponential function e^{jx}, our method includes the construction of a continuous, commutative function h: $[0, 2\pi)^n \longrightarrow [0, 2\pi)$ satisfying the following property:

$$\sum_{i=1}^{n} \theta_{a_i} = 0 \quad \text{if and only if } h\left(\theta_{a_1}, \ldots, \theta_{a_n}\right) = 0. \tag{18}$$

Of course, we should note that complex-valued overlap functions have many differences with the traditional real-valued overlap functions. Some interesting properties are useful for complex-valued overlap functions but they do not appear in traditional real-valued overlap functions. As further works, we intend to investigate these special properties of complex-valued overlap functions.

Complex-valued overlap functions can be viewed as a special class of complex fuzzy aggregation functions which have been widely used in many application fields. How to apply the complex-valued overlap functions is another problem of interest.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

Acknowledgments

This research was funded by the National Science Foundation of China under Grant no. 62006168 and Zhejiang Provincial Natural Science Foundation of China under Grant no. LQ21A010001.

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